# On the Stability of the Quenched State in Mean Field Spin Glass Models

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**Abstract**

While the Gibbs states of spin glass models have been noted to have an erratic dependence on temperature, one may expect the mean over the disorder to produce a continuously varying "quenched state". The assumption of such continuity in temperature implies that in the infinite volume limit the state is stable under a class of deformations of the Gibbs measure. The condition is satisfied by the Parisi Ansatz, along with an even broader stationarity property. The stability conditions have equivalent expressions as marginal additivity of the quenched free energy. Implications of the continuity assumption include constraints on the overlap distribution, which are expressed as the vanishing of the expectation value for an infinite collection of multi-overlap polynomials. The polynomials can be computed with the aid of a *real*-replica calculation in which the number of replicas is taken to zero.

**Key words:** Mean field, spin glass, quenched state, overlap distribution, replicas.

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## 1. Introduction

We consider here the quenched state of the Sherrington-Kirkpatrick (SK) spin glass model, and discuss some stationarity properties which seem to emerge in the infinite volume limit.

The SK spin glass model [1] has spin variables $\sigma_i = \pm 1$, $i = 1, \ldots, N$, interacting via the Hamiltonian

$$H_N(\sigma, J) \;=\; -\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} J_{i,j} \sigma_i \sigma_j \;, \tag{1.1}$$

with $J_{i,j}$ independent normal Gaussian variables.

Sampling repeatedly spin configurations, distributed independently relative to a common Gibbs state, one obtains a random matrix of overlaps, defined for pairs of spin configurations $\sigma$ and $\sigma'$ as:

$$q_{\sigma,\sigma'} \;=\; \frac{1}{N} \sum_i \sigma_i \sigma'_i \;. \tag{1.2}$$

The quenched free energy, at a given temperature $T = (k\beta)^{-1}$, is determined in this model by the joint distribution of such overlaps for arbitrary number of "replicas" (copies of the spin system subject to the same random interaction).

The purported solution of the model via the Parisi ansatz [2] has a number of remarkable stability properties. Our purpose is to discuss some of these, starting from elementary continuity assumptions without assuming the validity of the proposed solution. Some results in a similar direction were previously obtained by F. Guerra [3].

Following are the main observations presented here.

1) We identify a stability condition, in the sense of invariance of the quenched state in the thermodynamic limit under a broad class of deformations, which is satisfied by the state corresponding to the Parisi solution.

2) We show that a restricted version of the stability condition is a consequence of a property which make good physical sense, namely the continuity of the quenched ensemble as function of the temperature.

3) The restricted stability of the quenched state implies the vanishing of expectation values for a family of overlap multireplica polynomials. This accounts for some of the relations, though far from all, found in the Parisi solution.

4) We ask whether the stability condition singles the Parisi family of states (the GREM models in the terminology of [4]). A considerably simplified version of such a question (concerning the characterization of the the REM states) has a positive answer [5] (in preparation).

## 2. Invariance of the Parisi solution.

Let us introduce the notation employed here for the quenched state, and state a remarkable invariance property of the Parisi solution.

### *2.a Notation for Quenched Ensembles*

In disordered spin systems (in equilibrium) we encounter two distinct random structures: the spins, distributed according to the Gibbs distribution, and the random couplings (and/or random fields and other parameters) which affect the Gibbs state, making it into a random measure. We denote here by $< - >$, and in case of possible confusion by $< - >_J$, the expectation value over the spins averaged with respect to the Gibbs state. An average over the couplings is denoted by $Av(-)$. The combined *quenched average* is a double average, denoted below by $E(-)$, over the spins and the disorder (whose probability distribution is not affected [in the quenched case] by the response of the spin system to the random Hamiltonian).

Quantities of interest include:

$$\begin{aligned} E\left(q^2_{\sigma,\sigma'}\right) \quad &= \quad \frac{1}{N^2}\sum_{i,j} Av\left( <\sigma_i\sigma_j>^2_J \right) \;=\; Av\left( <\sigma_1\sigma_2>^2_J \right) + O(\frac{1}{N}) \ , \\ \text{and} \qquad & \sqrt{N} Av\left(<\sigma_1\sigma_2> \;\; J_{2,3} \;\; <\sigma_3\sigma_1>\right) \ . \end{aligned} \tag{2.1}$$

The second example is seen among other terms in $\frac{\partial}{\partial\beta}E\left(q^2_{\sigma,\sigma'}\right)$. Through integration by parts over the normalized gaussian variables $J$.

$$Av\left(Jf(J)\right) \;=\; Av\left(\frac{\partial}{\partial J}f(J)\right) \ , \tag{2.2}$$

such expressions can be reduced to averages of products of the overlaps $q_{\sigma,\sigma'}$ over a number of replicas.

In order not to overburden the notation we resist introducing the average over a direct sum of different number of replicas. Instead, we denote somewhat flexibly an average over their suitable collections (indexed by $j$) by the symbol

$$\ll - \gg = \otimes_j < - >^{(j)}, \tag{2.3}$$

and we let

$$E(-) = Av\left(\ll - \gg\right) \ . \tag{2.4}$$

Finally, as will be seen next, we shall add to the objects under consideration certain Gaussian random fields, indexed by the spin configuration. We extend the symbols $Av(-)$ and $E(-)$ to include averages over all such fields.

*2.b Invariance of the Parisi solution*

Let $h(\sigma)$ and $K(\sigma)$ be two Gaussian random fields, defined over the spin configuration, which are mutually independent, and independent of the random couplings (i.e., independent of the Hamiltonian), with the covariances

$$Av\,(h(\sigma)h(\sigma')) = q_{\sigma,\sigma'} \tag{2.5}$$

$$Av\,(K(\sigma)K(\sigma')) = q^2_{\sigma,\sigma'}\ . \tag{2.6}$$

Such quantities are motivated below: $h(\sigma)$ as the cavity field associated with an increase in $N$, and $K(\sigma)$ as representing the change in the action corresponding to an increase in the temperature.

The Parisi solution has the property that quenched averages are not affected by the addition to the action of terms of the form $F(K(\sigma), h(\sigma))$, where $F(\cdot,\cdot)$ is any smooth bounded function.

To express the above stated property, let us denote the deformed states as

$$\begin{aligned} < - >_{F(K,h)} &:= \frac{< - \exp\{F(K,h)\} >}{< \exp\{F(K,h)\}) >}\ , \\ \ll - \gg_{F(K,h)} &= \otimes_i < - >^{(i)}_{F(K,h)}\ , \end{aligned} \tag{2.7}$$

and

$$E_{F(K,h)}(-) = Av(\ll - \gg_{F(K,h)})\ . \tag{2.8}$$

**Claim 2.1** *For the Parisi solution, in the infinite volume limit at any temperature:*

$$E_{F(K,h)}(-) = E(-), \tag{2.9}$$

*when the expectation value functionals are restricted to quantities which do not involve the deformation fields $K$ and $h$.*

We shall not verify this statement here (The reader is invited to do so from the solution, which is discussed in [2] and references therein) but instead discuss the origin and consequences of a somewhat restricted invariance of this kind.

## 3. Continuity in the temperature and stability under deformation

The broad stability of the quenched state expressed by eq. (2.9) has not yet been rigorously derived for the SK model. We shall now find that a somewhat restricted version of this condition follows from a natural continuity assumption.

There is a significant difference between the spin–glass and the ferromagnetic spin models in the effect of a change in the temperature on the equilibrium state. Reduction in the temperature amounts to increased coercion towards the low energy states of the system. If the ground state is unique, it is natural to expect the equilibrium state to vary continuously at low temperatures. When there are only few ground states, one may expect some discontinuities (as in the Pirogov-Sinai theory [6]). However, when there is a high multiplicity of competing low energy states the result may be quite different. Indeed it is reported that for a given realization of the random Hamiltonian, the equilibrium state has a very erratic dependence on the temperature. Nevertheless it may seem reasonable to expect that with the average over the disorder, the quenched state might vary continuously with $\beta$.

To illuminate the consequences of the continuity assumption, let us note that due to the addition law for independent Gaussian variables, the Gibbs factor determining the equilibrium state at the inverse–temperature $\beta + \Delta\beta$ can be presented as a sum of two independent terms:

$$(\beta + \Delta\beta) H(\sigma, J) \stackrel{\mathcal{D}}{=} \beta H(\sigma, J) + \delta(\beta) H(\sigma, \tilde{J}) \; , \tag{3.1}$$

where $A \stackrel{\mathcal{D}}{=} B$ means that $A$ and $B$ have equal probability distributions, $\{J, \tilde{J}\}$ are two independent sets of couplings, and

$$\delta(\beta) = \sqrt{2\beta\Delta\beta + (\Delta\beta)^2} \tag{3.2}$$

With the action cast in the form eq. (3.1), the modified state is seen to incorporate the effects of a strong term (of the order of the volume) pulling in some randomly chosen directions, when the main term itself has many competing states. The assumption of the continuity of the quenched state appears now as less obvious, and it should therefore carry some notable consequences.

A related observation can be made by considering the effects of deformation of the state through the addition of a Gaussian field of the type $K(\sigma)$ (eq. (2.6)). An easy computation based on the fact that

$$H(\sigma, J) \stackrel{\mathcal{D}}{=} \sqrt{N} K(\sigma) \tag{3.3}$$

shows that the effect is equivalent in a change of the order $O(\frac{1}{N})$ in the temperature:

$$E^{(N,\beta)}_{\lambda K}(-) = E^{(N, \beta + \frac{\lambda^2}{2N\beta})}(-) \;\; , \tag{3.4}$$

where the superscripts refer to the size and the inverse–temperature, the subscript indicates a deformation of the state in the sense of eq. (2.7) and, as for the eq. (2.9) the equality is understood when the two measures are restricted to the quantities independent from the deformation variable $K$.

In the limit $N \to \infty$, the change in the temperature on the right side vanishes. This immediately leads to the following observation.

**Proposition 3.1** *If, over a certain temperature range, the states $E^{(N,\beta)}(-)$ are uniformly continuous in beta, as $N \to \infty$, and the infinite–volume limit exist for the quenched state then the limiting measures $E^{(\beta)}(-)$ are stable under linear deformations in $\lambda K$, in the sense:*

$$E^{(\beta)}\,(-) \;=\; E^{(\beta)}_{\lambda K}\,(-)\,. \tag{3.5}$$

*restricted to the algebra of observables generated by all the random variables but the deformation field $K$*

Let us note that the assumptions made above imply also another stationarity principle: the quenched state would be invariant under the deformation induced by $\ln 2\cosh \beta h$. To see that, compare the state of $N+1$ particles with that of $N$. The trace over the "last" spin yields for expectation values of functions of the "first" $N$ spins

$$E^{(N+1,\beta)}\,(-) \;=\; E^{(N,\tilde\beta)}_{\ln 2\cosh \tilde\beta h}\,(-)\ , \tag{3.6}$$

where $\tilde\beta = \sqrt{\frac{N}{N+1}}\beta$ and $h(\sigma)$ is a Gaussian field with covariance $q_{\sigma,\sigma'}$. Under the stated assumptions, in the thermodynamical limit the previous relation becomes:

$$E\,(-) \;=\; E_{\ln 2\cosh \beta h}\,(-)\ . \tag{3.7}$$

## 4. A logarithmic relation expressing the stability property

The stability condition which follows from the above discussed continuity assumption is indeed found among the properties of the Parisi solution, along with the more sweeping stability property expressed in eq. (2.9). In this section we cast the broader invariance property in the form of a logarithmic relation, which expresses an additivity property for the marginal increments in the quenched free energy.

**Definition 4.1** *We say that a random system, in the quenched state $Av\,(\ll - \gg)$, has marginally-additive free energy if for any finite collection of independent Gaussian fields*

$K^{(1)}, K^{(2)}, \ldots, K^{(l)}$ *with the covariance eq. (2.6), and any smooth polynomially bounded functions* $F_1, F_2, \ldots, F_l$

$$Av \ln < \exp(\sum_{i=1}^{l} F_i(K^{(i)})) >= \sum_{i=1}^{l} Av \ln < \exp F_i(K^{(i)}) > \tag{4.1}$$

Our main observation is that the above marginal additivity of the quenched free energy is equivalent to the *stability* of the quenched state expressed by

$$E\,(-) \ = E_{F(K)}\,(-) \ , \tag{4.2}$$

where $F$ is an arbitrary smooth bounded function, and the equation means equality of the expectation values of quantities involving functions of the spins, random couplings, and auxiliary Gaussian fields – other than the deforming field $K$.

Let us note that the expectation values of quantities involving any of the above, can be evaluated by first integrating over the extraneous Gaussian variables ($K$). Using Wick's formula, this integration produces expressions involving the overlaps among arbitrary number of replicas, as in:

$$\begin{aligned} E\,(K_1\,K_2\,K_2'\,K_3') \ &= \ Av\,(< K >< KK' >< K' >) \\ &= \ E(q_{1,2}^2 q_{2,3}^2)\ , \end{aligned} \tag{4.3}$$

where we defined $K_i = K(\sigma^{(i)})$

Conversely, the averages of polynomials in replica overlaps, as $q_{1,2}^2 q_{2,3}^2$ in the above expression, can be easily expressed through the expectation values of suitable products of independent copies of the $K$ field. Based on this observation one can see that the measure $E(-)$ is completely determined by the distribution of the overlaps.

Clearly the stability implies the marginal additivity property (of the free energy). To prove the converse, we need to show that

$$Av\left[< G(K) >_{F(K')}\right]^n \ = \ Av\,[< G(K) >]^n \ , \tag{4.4}$$

for any integer $n$ and polynomial function $G$. (The full statement takes a bit more general form – involving products with different functions $G$ for the different copies of the spin system, however by the polarization argument there is no loss of generality in taking the same function $G$ for all the $n$ replicas.) Let us note also that, by an elementary approximation argument, it suffices to prove eq. (4.4) for bounded functions $G$.

The logarithmic property (4.1) implies that for all $\varepsilon$ the following function vanishes

$$\varphi(\varepsilon) \equiv= Av \ln \frac{< \exp(\varepsilon G + F) >}{< \exp(\varepsilon G) >< \exp(F) >} \ = \ 0\ . \tag{4.5}$$

For bounded $G$ the function $\varphi(\varepsilon)$ is analytical in a strip containing the real axis, and the logarithmic property eq. (4.1) is equivalent to the vanishing of all the derivatives of $\varphi(\varepsilon)$ at $\varepsilon = 0$. By an inductive argument, these conditions imply eq. (4.4): first we observe that

$$\varphi'(\varepsilon)|_0 = Av(< G >_F) - Av(< G >) = 0, \tag{4.6}$$

which is the stability for the first power. The second derivative gives

$$\varphi''(\varepsilon)|_0 = Av(< G >_F^2) - Av(< G^2 >_F) - Av(< G >^2) + Av(< G^2 >) = 0\,. \tag{4.7}$$

Thus

$$\begin{aligned} Av(< G >_F^2) - Av(< G >^2) &= Av(< G^2 >_F) - Av(< G^2 >) \\ &= 0\,, \end{aligned} \tag{4.8}$$

where the last equality is by the first order equation, eq. (4.6), applied to the smooth function $G^2$. Continuing in this fashion one may see that if stability is fulfilled up to power $n$ it is fulfilled for power $n+1$.

**Remark 4.2** *The truncated expectations (cumulants) of order p are generally defined by*

$$< -; p >^T = \frac{\partial^p}{\partial \lambda^p} \ln < \exp(\lambda -) > |_{\lambda=0}\,. \tag{4.9}$$

*In these terms the logarithmic relation is equivalent to:*

$$Av < \sum_{i=1}^{l} F_i(K^{(i)}); p >^T = Av \sum_{i=1}^{l} < F_i(K^{(i)}); p >^T \tag{4.10}$$

*for every integer p. (The implication eq. (4.1) $\Rightarrow$ eq. (4.10) is obvious. In the other direction the proof can be based on the analyticity argument indicated above.) It might be noted that an equation like eq. (4.10) cannot possibly hold without the average* Av, *unless the Gibbs state is typically supported on a narrow collection of configurations over which the overlap function takes only the value $q_{\sigma,\sigma'} = 1$.*

Equations (4.1) and (4.10) have natural counterparts for the more limited stability of the quenched state, expressed by eq. (3.5). In that case $F_i(K^{(i)})$ need by replaced by $\lambda_i K^{(i)}$, for $i = 2, 3, \ldots$ (thought $F_1$ may still be left arbitrary.)

It is an interesting open question whether there are states stable in the limited sense which do not fulfill the stronger stability condition.

## 5. Overlap polynomials with zero average

We now turn to some of the implications of the stability condition eq. (3.5) which was shown to follow from the continuity assumption. Since the free energy and its derivatives are determined through the distribution of the overlaps, it is natural to ask what consequences does the stability property have in those terms. As we shall see next, the implications include a family of relations expressed as the vanishing of the expectation value of an infinite collection of overlap polynomials. ¿From a combinatorial point of view, expressions with vanishing expectation are constructed by applying a certain operation to graphs representing overlap polynomials.

We shall use the notation encountered already in eq. (4.3), where $q_{1,2}$ indicates the overlap between two spin configurations sampled from two different copies of the system, one in replica 1 and the other in the replica 2, subject to the same random interaction. Products of such terms can be represented by labeled graphs, introduced below. The expectation value does not depend on the particular labeling of the different replicas, for instance

$$E(q_{1,2}^2 q_{2,3}^4 q_{1,4}^2) = E(q_{1,2}^2 q_{2,3}^2 q_{3,4}^4) \ . \tag{5.1}$$

Let now illustrate some consequences of the stability condition, starting from the easy example

$$Av(< K >^2_{\lambda K'}) = Av(< K >^2). \tag{5.2}$$

The first derivative with respect to lambda gives

$$2Av(< K >_{\lambda K'} (< KK' >_{\lambda K'} - < K >_{\lambda K'} < K' >_{\lambda K'})) = 0 \tag{5.3}$$

which vanishes, for the trivial reason of parity, at $\lambda = 0$. On the other hand, the second derivative yields

$$\begin{aligned} 2Av(< KK' >^2_{\lambda K'} & -4 < K >_{\lambda K'} < K' >_{\lambda K'} < KK' >_{\lambda K'} +3 < K >^2_{\lambda K'} < K' >^2_{\lambda K'} \\ & + < K >_{\lambda K'} < KK'^2 >_{\lambda K'} - < K >^2_{\lambda K'} < K'^2 >_{\lambda K'}) = 0. \end{aligned} \tag{5.4}$$

At $\lambda = 0$ the last two terms cancel and the expression reduces to:

$$2E(q_{1,2}^4) - 8E(q_{1,2}^2 q_{2,3}^2) + 6E(q_{1,2}^2 q_{3,4}^2) = 0 \ . \tag{5.5}$$

As was mentioned already, the stability property is satisfied by the Parisi solution, and hence this relation, as well as those of higher order derived below, are satisfied there. The particular case of, eq. (5.5) was recently derived (for almost every *beta*) without any assumptions in ref. ([3]). One may note, that eq. (5.5) is also the lowest non-trivial identity of those listed in eq. (4.10) (corresponding to $p = 4$, $F(K) \equiv K$).

The previous relation tells us that the polynomial

$$q_{1,2}^4 - 4q_{1,2}^2 q_{2,3}^2 + 3q_{1,2}^2 q_{3,4}^2 \tag{5.6}$$

has zero average. Let us present now a systematic approach for the derivation of other such conditions.

One may use a graphical representation in which a monomial of the form $q_{1,2}^2 q_{2,3}$ is identified with a graph whose vertices are the replica indices $\{1, 2, 3\}$ and the edges correspond to the overlaps, $q_{i,j}$. Such a graph will be indicated by the symbol $(1, 2)^2(2, 3)$. Furthermore, we shall consider also products involving an additional Gaussian field $(K)$. The graphical representation of that factor is a half-edge, represented by a singleton. I.e., $(1, 2)(2)$ and $(1, 2)(3)$ correspond to $q_{1,2}K_2$ and $q_{1,2}K_3$.

We shall use a product "$\cdot$" which acts in the space of graphs as *composition* combined with *contraction*, where possible, of the two unpaired legs. The notion may be clarified by the following examples:

$$\begin{aligned}(1,2)\cdot(1,2)(2) &= (1,2)^2(2) \\ (1,2)(3)\cdot(4) &= (1,2)(3,4) \end{aligned} \tag{5.7}$$

(5.8)

Terms of the form $(1, 1)$ can be omitted, since in our case $q_{1,1} = 1$.

The above product turns out to be commutative but not associative. The *order* of a graph is defined as $2 \times$ *number of edges*, with half-edges counting as $1/2$. Let $W_k$ denote the space of formal linear combination of graphs of order $k$. For $G \in W_k$ we denote by $O_G$ the corresponding element of the overlap algebra.

We define $\delta : W_k \to W_{k+1}$ as the linear operator which acts on single graphs by

$$\delta G = \sum_{v \in V(G)} \delta_v G, \tag{5.9}$$

where $V(G)$ is the set of vertices of $G$, and

$$\delta_v G = G \cdot (v) - G \cdot (\tilde{v}), \tag{5.10}$$

where $\tilde{v}$ is a new vertex not belonging to $G$. E.g., $\delta_1(1, 2) = (1, 2)(1) - (1, 2)(3)$, $\delta_1(1, 2)(1) = (1, 2)(1) \cdot (1) - (1, 2)(1) \cdot (3) = (1, 2) - (1, 2)(1, 3)$.

Following is the pertinent observation.

**Proposition 5.1** *For any measure $E(-)$, of the structure seen in eq. (2.4),*

$$\frac{\partial^2}{\partial \lambda^2} E_{\lambda K}(Q_G)|_{\lambda=0} = E(Q_{\delta^2 G}) \ , \tag{5.11}$$

*for all the elements $Q_G$ of the overlap algebra.*

The proof is given below. A direct consequence is:

**Proposition 5.2** *If the measure $E(-)$ is invariant under linear deformations, in the sense of eq. (3.5), then for all the elements $O_G$ of the overlap algebra:*

$$E(O_{\delta^2 G}) = 0 \, . \tag{5.12}$$

The proof of Proposition 5.11 is straightforward. The operation $\delta$ is the graphical counterpart of the usual derivative with respect to the parameter $\lambda$ in the Boltzmann weight (where it appears in $\lambda K$). Such a derivative produces a *truncated correlation* expressed in the rule (5.10). The (5.9) is nothing but the Leibnitz rule for derivative of products. The first differentiation produces a sum of monomials, each containing an unpaired centered Gaussian variable ($K$) of zero mean. The second derivative produces another unpaired variable, which is contracted with the previous one via the Wick rule. (This contraction motivates the product introduced above.) The fact that the second derivative has a zero average is a nontrivial consequence of the stability.

Following is a related statement, obtained by combining eq. (3.4) with eq. (3.5).

**Proposition 5.3** *In the SK model, at finite values of $N$,*

$$\frac{\partial}{\beta\partial\beta}E_N(Q_G) = N\frac{\partial^2}{\partial\lambda^2}E_{N,\lambda K}(Q_G)|_{\lambda=0} = NE_N(Q_{\delta^2 G}), \tag{5.13}$$

*for every element of the overlap algebra. In particular, if the quenched state is locally differentiable in $\beta$ uniformly in $N$, then the expectation values of quantities of the form $Q_{\delta^2 G}$ vanish at the rate $O(1/N)$ or faster.*

The last statement supplements the previous assertion with a somewhat stronger conclusion (suggesting a numerical test), which is derived under a stronger assumption.

Identical result holds for other mean field spin glass models, with the $p$-spin interaction Hamiltonian ([7])

$$H(\sigma, J) = -\sum_{1\le i_1<\cdots<i_p\le N} J_{i_1,\cdots,i_p}\sigma_{i_1}\cdots\sigma_{i_p}, \tag{5.14}$$

where $J_{i_1,\cdots,i_p}$ are Gaussian variables, rescaled so that the Hamiltonian covariance is

$$AvH(\sigma, J)H(\sigma', J) = Nq^p_{\sigma,\sigma'} \, . \tag{5.15}$$

(Under the above calling, $H$ reaches values of order $N$. The corresponding choice for the field $K$ is Gaussian with the covariance $Av(K(\sigma)K(\sigma')) = q^p_{\sigma,\sigma'}$. The definition of $\delta$ is unchanged.

## 6. Computation with real replicas

In this section we give several characterizations of the overlap polynomials of the form $Q_{\delta^2 G}$, for which we saw that stability condition implies zero mean.

The main result is a formula which permits to compute the polynomials from a quadratic expression in the number $r$ of *real*-replicas, evaluated at $r = 0$. To state it, let $M_r = \sum_{i \neq j=1}^{r} q_{i,j}^2$, for all integers $r \geq 1$. Let $E(-)$ be an expectation value functional, on the algebra of overlaps, which depends only on the graph structure of the overlap monomials (i.e., is independent of the choice of labels). Then the quantity $E(Q_G M_r)$ is quadratic in $r$. In the following proposition, we refer to the *polynomial extension* of this function to all real $r$.

**Proposition 6.1** *For any expectation value functional $E(-)$, as above,*

$$E(Q_{\delta^2 G}) = E(Q_G M_r)|_{r=0}. \tag{6.1}$$

*where the quantity $E(Q_G M_r)$ is first computed for $r$ large enough so that all the indices appearing in $Q_G$ do appear also in $M_r$ and $r > |G| + 1$.*

To illustrate the statement, let us take: $G = q_{1,2}^2$. In this case, the left side of eq. (6.1) is given by eq. (5.5) and the right side is:

$$E(q_{1,2}^2 M_r) = 2E(q_{1,2}^4) + 4(r-2)E(q_{1,2}^2 q_{2,3}^2) + (r-2)(r-3)E(q_{1,2}^2 q_{3,4}^2)\ . \tag{6.2}$$

The two coincide at $r = 0$ (defined by polynomial extension).

The proof proceeds through the explicit computation of the left and right sides of eq. (6.1).

**Lemma 6.2** *If the number of vertices in $G$ is $l$ then*

$$\delta^2 G = \sum_{v \neq v'} G \cdot (v, v') - 2l \sum_{v} G \cdot (v, \tilde{v}) + l(l+1) G \cdot (\tilde{v}, \tilde{v}'), \tag{6.3}$$

*where $v$ and $v'$ are summed over the set of vertices of $G$ and $\tilde{v}$ and $\tilde{v}'$ denote a pair of added vertices.*

This is a rather explicit expression for the polynomials corresponding to a given graph $G$. Two examples are:

$$\begin{aligned} \delta^2(1,2)(3,4) \;&=\; 4(1,2)^2(3,4) \;+\; 8(1,2)(2,3)(3,4) - \\ &\quad -32(1,2)(2,3)(4,5) \;+\; 20(1,2)(3,4)(5,6)\ , \end{aligned} \tag{6.4}$$

and

$$\delta^2(1,2)(2,3) = 4(1,2)^2(2,3) + 2(1,2)(2,3)(3,1) - 12(1,2)(2,3)(3,4) + \\ +12(1,2)(2,3)(4,5) - 6(1,2)(2,3)(2,4)\ . \tag{6.5}$$

**Remark** In the above example $\delta^2 G$ is a polynomial expression with integer coefficients whose sum is zero. That property is shared by $\delta^2 G$ of arbitrary monomials $G$.

To prove the lemma we note that by the definition of $\delta$

$$\delta G = \sum_v G \cdot (v) - lG \cdot (\tilde{v})\ . \tag{6.6}$$

Applying this rule twice

$$\delta^2 G = \sum_{v,v'} G \cdot (v,v') - l \sum_v G \cdot (v,\tilde{v}) - l \sum_{v'} G \cdot (v',\tilde{v}) - l \sum_{v'} G \cdot (\tilde{v},\tilde{v}) \tag{6.7}$$
$$+l(l+1) G \cdot (\tilde{v},\tilde{v}'),$$

which coincides with eq. (6.3) since $(v,v) = 1$ for every $v$.

**Lemma 6.3** *If all the replica indices appearing in $Q_G$ are contained in the entries of the matrix $M_r$ and $r > l+1$ then*

$$E(Q_G M_r) = \sum_{\substack{v,v' \in V(G) \\ v \neq v'}} E(Q_{G\cdot(v,v')}) + 2(r-l) \sum_{v \in V(G)} E(Q_{G\cdot(v,\tilde{v})}) \\ + (r-l)(r-l-1) E(Q_{G\cdot(\tilde{v},\tilde{v}')}). \tag{6.8}$$

This formula is an elementary consequence of the fact that the measure $E(-)$ depends only on the isomorphism type of the graph associated to a given overlap monomial. The first sum on the right side of eq. (6.8) corresponds to those overlap terms in $M_r$ which involve only the replicas appearing in $G$, the other two sums are split according to whether the number of vertices not in $G$ is 1 or 2.

The two previous lemmas prove Proposition 6.1. ∎

## 7. Comments

We have seen that elementary continuity assumptions on the quenched state, imply a stability property for the infinite volume limit of the SK spin glass models. A particular implication, translated in terms of the overlap distribution, is that in mean-field models the joint probability distribution of the overlap meets an infinite family of conditions, expressed as the vanishing of the expectation value for a family of multi-overlap polynomials. We also saw a finite volume condition expressed as a decay rate for the expectation values of suitable quantities.

These observations are consistent with the Parisi theory. However, the family of identities discussed here does not yet permit the reconstruction of the joint probability distribution from that of a single overlap, as is the case under the Parisi Ansatz.

I. Kondor and M. Mezard pointed out that within the replica-symmetry- breaking approach the vanishing of the expectation values of the polynomials discussed here ($\delta^2 G$) requires only the so called "replica–equivalence" assumption, which says that in the matrix $Q$ (defined in [2]) each row is a permutation of any other. However, *replica–equivalence* does not appear to yield a broader class of states than the Parisi Ansatz (M. Mezard - private communication).

An interesting question is whether the stability property is the stationarity condition for some variational principle. This is related to the main question which emerges at this point, which is whether stability implies the GREM state structure [4]. We study a restricted version of this question in a separate paper [5].

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