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# Invariant tori of intermediate dimensions in Hamiltonian systems: A brief survey

In the present paper, we survey recent results on the existence and the structure of Cantor families of invariant tori of dimensions $p > n$ in a neighborhood of families of invariant $n$-tori in Hamiltonian systems with $d \geq p$ degrees of freedom.

## 1. Whitney-smooth families of invariant tori

Invariant submanifolds of the phase space are one of the main subjects of research in Hamiltonian dynamics as well as in the general theory of dynamical systems. Of all the invariant manifolds of Hamiltonian systems, invariant tori (and asymptotic surfaces to tori) are best studied and occur most frequently. This is due, in the long run, to the fact that any connected compact Abelian finite-dimensional Lie group is a torus. Invariant tori of Hamiltonian systems possess a number of remarkable properties under examination in the KAM (Kolmogorov–Arnold–Moser) theory. The aim of the present survey is to discuss the recent achievements pertaining to one of these properties, namely, the so called excitation of elliptic normal modes. For simplicity, in the sequel all the systems will be autonomous and real-analytic (many of the results presented below can be generalized *mutatis mutandis* to infinitely differentiable and even finitely smooth systems).

To start with, we will list some main properties of invariant tori in Hamiltonian flows. The relevant bibliography and/or proofs are given, e.g., in recent works [1]–[8] which contain surveys for various branches of Hamiltonian dynamics.

**A.** A "typical" (in the sense to be explained below) invariant $n$-torus of a "typical" analytic Hamiltonian system with $n + m$ degrees of freedom ($n \geq 0$, $m \geq 0$) is analytic itself and *isotropic* (i.e., the restriction of the symplectic structure to this torus vanishes), while the dynamics in a neighborhood of the torus is extremely regular *in the approximation linear with respect to the distance from the torus.* To be more precise, in a neighborhood of the torus, one can introduce the coordinates $\varphi \in \mathbf{T}^n = (\mathbf{R}/2\pi\mathbf{Z})^n$, $I \in \mathbf{R}^n$, $z \in \mathbf{R}^{2m}$ ($I$ and $z$ range near the origins) in which the torus itself is given by the equations $\{I = 0, z = 0\}$, the symplectic structure takes the form

$$\textstyle\sum_{i=1}^n dI_i \wedge d\varphi_i + \sum_{j=1}^m dz_j \wedge dz_{j+m}, \tag{1}$$

and the Hamilton function takes the form

$$H = c + \langle \omega, I \rangle + \tfrac{1}{2}\langle z, Bz \rangle + O(|I|^2 + |z|^3).$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors, $c \in \mathbf{R}$, $\omega \in \mathbf{R}^n$, and $B$ is a symmetric $2m \times 2m$ matrix (independent of $\varphi$), $\det B \neq 0$, the frequencies $\omega_1, \ldots, \omega_n$ satisfying the standard strong incommensurability condition: there exist positive constants $\tau_0$ and $\gamma_0$ such that

$$|\langle k, \omega \rangle| \geq \gamma_0 |k|^{-\tau_0} \tag{2}$$

for all $k \in \mathbf{Z}^n \setminus \{0\}$. The equations of motion in the coordinates $(\varphi, I, z)$ have the form

$$\dot{\varphi} = \omega + O(|I| + |z|^2), \quad \dot{I} = O(|I|^2 + |z|^3), \quad \dot{z} = \Omega z + O(|I|^2 + |z|^2). \tag{3}$$

Here $\Omega = J_m B$, whereas $J_m$ is the matrix of the canonical skew-symmetric product in $\mathbf{R}^{2m}$:

$$J_m = \begin{pmatrix} 0 & -E_m \\ E_m & 0 \end{pmatrix}, \tag{4}$$

$E_m$ being the identity $m \times m$ matrix. Without loss of generality one can set $c = 0$.

Remark 1. An analytic function of $\varphi$, $I$, $z$ equal to $O(|I|^2 + |z|^3)$ can be represented as a sum of terms of the form $I_u I_v \chi$, $I_u z_v z_w \chi$, and $z_u z_v z_w \chi$ where $\chi$ are analytic functions of $\varphi$, $I$, $z$.

Remark 2. The flow on the initial torus is *parallel* (the equations of motion can be reduced to the form $\dot{\varphi} = \omega$ with a constant vector $\omega$), *ergodic* (the frequencies $\omega_1, \ldots, \omega_n$ are independent over rationals) and, moreover, *Diophantine* (the infinite system of inequalities (2) is met). Ergodic parallel flows on tori are otherwise said to be *quasi-periodic.* Invariant tori carrying quasi-periodic (or Diophantine) flows are sometimes themselves said to be quasi-periodic (respectively Diophantine).

REMARK 3. In coordinates $(I, z)$, the matrix of the variational equation (of order $2m+n$) along the torus under consideration has the form

$$\begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix}.$$

In particular, it is independent of $\varphi$. This property is called *reducibility* of the torus. The term "Floquet torus" is also used [6, 7]. The torus is said to be *elliptic* if all the eigenvalues of matrix $\Omega$ are purely imaginary, and is said to be *hyperbolic* if all the eigenvalues of $\Omega$ lie outside the imaginary axis.

REMARK 4. In fact, any quasi-periodic invariant torus of a Hamiltonian system is isotropic automatically provided that the symplectic structure is exact (determines the zero cohomology class) [7, 9].

REMARK 5. Hamiltonian systems on manifolds with a non-exact symplectic structure admit also Diophantine invariant tori of dimension *greater* than the number of degrees of freedom. As a rule, those tori are *coisotropic* (their tangent space at any point contains its skew-orthogonal complement). We will not consider such tori in the present paper.

**B.** Reducible Diophantine isotropic invariant $n$-tori of "typical" Hamiltonian systems with $n+m$ degrees of freedom are not isolated in the phase space (for $n>0$) but are organized into *Whitney-smooth* $n$-parameter families. The parameter $\xi$ labeling the tori ranges in some Cantor set $\Xi \subset \mathbf{R}^n$ of positive Lebesgue measure, the $2n$-dimensional Hausdorff measure of the union of all the tori $T_\xi$ of a given family being positive. The coordinate functions $\varphi_\xi : M \to \mathbf{T}^n$, $I_\xi : M \to \mathbf{R}^n$, $z_\xi : M \to \mathbf{R}^{2m}$ on the phase space $M$,[1] that normalize (in the first approximation) the flow near the torus $T_\xi$, can be treated as functions defined on $M \times \Xi$. Whitney-smoothness means that these functions are extendible to functions defined on the whole space $M \times \mathbf{R}^n$, analytic in the first argument $\zeta \in M$ and infinitely differentiable in the second argument $\xi \in \mathbf{R}^n$. In particular, the tori $T_\xi$ themselves, the frequency vectors $\omega = \omega(\xi)$, and the matrices $\Omega = \Omega(\xi)$ depend on $\xi$ in a Whitney-smooth fashion.

REMARK. For $n \le 1$, the $n$-parameter families of invariant $n$-tori in question are analytic rather than merely Whitney-smooth. For $n=0$, these families are isolated equilibrium points, whereas for $n=1$, they are one-parameter families of close trajectories (one can take, e.g., the value of the Hamilton function as a parameter).

[1] to be more precise, on some neighborhood of the union of the tori

Hamiltonian systems with $n+m$ degrees of freedom that possess Whitney-smooth $n$-parameter families of reducible Diophantine isotropic invariant $n$-tori constitute a set with non-empty interior in the functional space of all the Hamiltonian systems with $n+m$ degrees of freedom [6, 7]. It is this property of Hamiltonian dynamics that is called the "typicality" of such systems in the present paper. It is the *families of tori* of this form rather than individual tori that are the elementary "bricks" the whole complex of quasi-periodic motions in Hamiltonian systems consists of.

**C.** In spite of quasi-periodicity and reducibility of each invariant $n$-torus $T_\xi$ belonging to a Whitney-smooth $n$-parameter family ($\xi \in \Xi \subset \mathbf{R}^n$), the dynamics in a neighborhood of the family as a whole is extremely complicated.

1) In the space between the tori, there are other families of invariant tori of smaller dimensions $p$ ($1 \le p \le n-1$), so called *cantori* (invariant sets of Cantor structure), and chaotic motion zones which surround $p$-tori and cantori (if $n > 1$). Roughly speaking, the tori of dimensions $p < n$ correspond to those values of $\xi \in \mathbf{R}^n \setminus \Xi$ for which the frequencies $\omega_1(\xi), \ldots, \omega_n(\xi)$ are rationally dependent,[2] whereas the cantori correspond to those values of $\xi \in \mathbf{R}^n \setminus \Xi$ for which these frequencies are independent, but "not sufficiently strongly". However, the arrangement of families of tori of smaller dimensions between tori of the initial family has been studied for $m = 0$ only ($n+m$ being the number of degrees of freedom) and that of cantori, in the case $m = 0$, $n = 2$ only (with a few exceptions).

2) If tori $T_\xi$ are hyperbolic then each of them possesses two $(n+m)$-dimensional Lagrangian analytic asymptotic surfaces, the attracting one and the repelling one (recall that a *Lagrangian* submanifold is an isotropic submanifold whose dimension is equal to the number of degrees of freedom). The asymptotic surfaces attached to different tori intersect transversely and constitute an intricate net.

3) Any torus $T = T_\xi$ is surrounded by a zone where the dynamics is very close to integrable one. Suppose that among the eigenvalues of matrix $\Omega$ in system (3), there are $\nu$ pairs ($0 \le \nu \le m$) of purely imaginary numbers (ellipticity of the torus corresponds to the case $\nu = m$ and hyperbolicity, to the case $\nu = 0$). Then generically for each $r$, $0 \le r \le \nu$, torus $T$ is adjoined by $\nu!/r!(\nu-r)!$ Whitney-smooth $(n+r)$-parameter families of $(n+r)$-dimensional reducible Diophantine isotropic invariant tori. Those families can be said to be associated with $T$. The "size" of the spaces between the tori in each family is *exponentially*

[2] To be more precise, $n-p$ frequencies are rational combinations of the remaining $p$ frequencies which are strongly rationally independent.

*small* with respect to the distance from the initial torus $T$ (for such exponentially small effects, analyticity of the Hamiltonian system is very essential).

All the families of invariant tori in a given system constitute a complicated hierarchical structure with infinitely many levels. For instance, each torus of a "secondary" family associated with a torus $T$ of the initial ("chief") family is adjoined by "tertiary" families of tori, and so on (cf. [10]). Transverse intersections of the asymptotic surfaces to hyperbolic tori (of one and the same family or of different families) have proven to be the main geometrical obstacle to integrability of Hamiltonian systems and one of the main sources of chaos in Hamiltonian dynamics.

If $\nu > 0$ then in an arbitrarily small neighborhood of the $n$-torus $T$, there are invariant tori of dimension $n+\nu$ as well as those of all the intermediate dimensions $n+1, \dots, n+\nu-1$. It is the existence of such tori of dimensions greater than $n$ that is called the *excitation of the elliptic normal modes* of torus $T$. In the sequel, we will consider two approaches to studying the families of "excited" tori of dimensions $p > n$. These approaches differ mainly in the genericity conditions imposed on the initial family of $n$-tori.

## 2. The formal normal form of the Hamilton function in a neighborhood of an invariant torus

For simplicity, suppose first that the initial $n$-torus $T$ is elliptic, and $\pm i\omega_1^N, \dots, \pm i\omega_m^N$ are the eigenvalues of matrix $\Omega$ in (3). The numbers $\omega_1, \dots, \omega_n$ are called the *intrinsic* frequencies (or just frequencies) of torus $T$, while the numbers $\omega_1^N, \dots, \omega_m^N$ are called the *normal* frequencies of torus $T$ [6, 7] (the superscript $N$ is from the word "normal"). If the complete collection of frequencies $(\omega, \omega^N)$ is Diophantine:

$$|\langle k, \omega\rangle + \langle k^N, \omega^N\rangle| \geq \gamma_1(|k| + |k^N|)^{-\tau_1}$$

for all $k \in \mathbf{Z}^n$, $k^N \in \mathbf{Z}^m$, $|k| + |k^N| > 0$, then under an appropriate choice of the signs of numbers $\omega_1^N, \dots, \omega_m^N$, the Hamilton function $H$ can be reduced in a neighborhood of torus $T$ to a *Birkhoff-like normal form*

$$H = \langle \omega, I\rangle + \langle \omega^N, I^N\rangle + F(I, I^N) \tag{5}$$

by a formal canonical transformation. Here $I^N \in \mathbf{R}_+^m$, $I_j^N = \frac{1}{2}(z_j^2 + z_{j+m}^2)$ for $1 \leq j \leq m$, and $F$ is a formal power series without constant and linear terms [8, 11] ($\mathbf{R}_+^m$ being the space of vectors of length $m$ with non-negative components). In the notation $z_j = \sqrt{2I_j^N} \cos\varphi_j^N$,

$z_{j+m} = \sqrt{2I_j^N} \sin \varphi_j^N$ ($\varphi^N \in \mathbf{T}^m$), the symplectic structure (1) takes the form

$$\textstyle\sum_{i=1}^n dI_i \wedge d\varphi_i + \sum_{j=1}^m dI_j^N \wedge d\varphi_j^N,$$

so that the equations of motion afforded by Hamilton function (5) have the form

$$\dot{I} = 0, \quad \dot{I}^N = 0, \quad \dot{\varphi} = \partial H / \partial I, \quad \dot{\varphi}^N = \partial H / \partial I^N.$$

Each manifold

$$\Bigl\{ I = C = \text{const} \in \mathbf{R}^n, \;\; I^N = C^N = \text{const} \in \mathbf{R}_+^m \Bigr\} \tag{6}$$

is a formal invariant torus in our system of dimension $n + r$ where $r$ is the number of *positive* components of vector $C^N$ ($0 \le r \le m$). The flow on torus (6) is parallel, and the torus itself is reducible, elliptic and isotropic. Assume for definiteness that the first $r$ components of vector $C^N$ are positive while the remaining components are equal to zero. One easily sees that the intrinsic $n + r$ frequencies of torus (6) are

$$\omega_i + \partial F / \partial I_i, \quad 1 \le i \le n, \qquad \omega_j^N + \partial F / \partial I_j^N, \quad 1 \le j \le r,$$

whereas the normal $m - r$ frequencies are

$$\omega_j^N + \partial F / \partial I_j^N, \quad r + 1 \le j \le m$$

(of course, the derivatives are taken at the point $(C, C^N)$). Torus $T$ corresponds to the values $C = 0$, $C^N = 0$. Its neighborhood is a formal union of invariant tori (6) of dimensions $n, n + 1, \ldots, n + m$.

We wonder what will persist in this picture if one takes into account the divergence of the transformation that reduces the Hamilton function to form (5).

## 3. Excitation of elliptic normal modes of an invariant torus

Consider the general case of torus $T$ with an arbitrary number $\nu$ ($0 \le \nu \le m$) of pairs of purely imaginary eigenvalues of matrix $\Omega$ in (3). Fix an arbitrary integer $r$ in the interval $0 \le r \le \nu$ and $r$ pairs $\pm i\omega_1^N, \ldots, \pm i\omega_r^N$ of purely imaginary eigenvalues of matrix $\Omega$. Our problem is to construct reducible Diophantine isotropic invariant $(n+r)$-tori with intrinsic frequencies close to $\omega_1, \ldots, \omega_n, \omega_1^N, \ldots, \omega_r^N$ in a neighborhood of $n$-torus $T$. Introduce the notation

$$(\omega_1, \ldots, \omega_n, \omega_1^N, \ldots, \omega_r^N) = \omega^{\text{all}} \in \mathbf{R}^{n+r},$$

so that

$$\omega_i^{\mathrm{all}} = \omega_i, \quad 1 \le i \le n, \qquad \omega_{n+j}^{\mathrm{all}} = \omega_j^N, \quad 1 \le j \le r.$$

For the sequel, we have to point out exactly what norms in $\mathbf{R}^L$ and $\mathbf{Z}^L$ we are considering. Under $|\cdot|$, we will always understand the $l_1$-norm:

$$|x| = |x_1| + \cdots + |x_L|. \tag{7}$$

Besides, we will also use the Euclidean norm $l_2$ to be denoted by $\|\cdot\|$:

$$\|x\|^2 = x_1^2 + \cdots + x_L^2. \tag{8}$$

Assume all the eigenvalues $\pm i\omega_1^N, \ldots, \pm i\omega_r^N, \pm\lambda_1, \ldots, \pm\lambda_{m-r}$ of matrix $\Omega$ to be pairwise distinct and to satisfy, together with the frequencies $\omega_1, \ldots, \omega_n$, the following condition: there exist positive constants $\tau$ and $\gamma$ such that

$$|i\langle k^{\mathrm{all}}, \omega^{\mathrm{all}}\rangle + \langle l, \lambda\rangle| \ge \gamma |k^{\mathrm{all}}|^{-\tau} \tag{9}$$

for all $k^{\mathrm{all}} \in \mathbf{Z}^{n+r} \setminus \{0\}$, $l \in \mathbf{Z}^{m-r}$, $|l| \le 2$. Then, under an appropriate choice of the signs of numbers $\omega_1^N, \ldots, \omega_r^N$, for any $R > 0$ the Hamilton function $H$ can be reduced to the form

$$\begin{aligned} H \;=\; & \langle \omega^{\mathrm{all}}, I^{\mathrm{all}}\rangle + \tfrac{1}{2}\langle z^{\mathrm{part}}, [B^{\mathrm{part}} + f(I^{\mathrm{all}})] z^{\mathrm{part}}\rangle + F(I^{\mathrm{all}}) + \\ & O(|z^{\mathrm{part}}|^3) + U(\varphi, I, z) \end{aligned} \tag{10}$$

by an ($R$-dependent) canonical transformation analytic in a neighborhood of torus $T$ [8, 11].[3] Here $I^{\mathrm{all}} \in \mathbf{R}^{n+r}$,

$$I_i^{\mathrm{all}} = I_i, \quad 1 \le i \le n, \qquad I_{n+j}^{\mathrm{all}} = \tfrac{1}{2}(z_j^2 + z_{j+m}^2) \ge 0, \quad 1 \le j \le r;$$

$z^{\mathrm{part}} = (z_{r+1}, \ldots, z_m, z_{m+r+1}, \ldots, z_{2m}) \in \mathbf{R}^{2(m-r)}$; $B^{\mathrm{part}}$ is an $R$-independent symmetric matrix of order $2(m-r)$ such that the eigenvalues of the matrix $J_{m-r}B^{\mathrm{part}}$ are $\pm\lambda_1, \ldots, \pm\lambda_{m-r}$ ($J_{m-r}$ is defined by formula (4) with $m$ having been replaced by $m-r$); $f$ is an analytic mapping of a neighborhood of the origin in $\mathbf{R}^{n+r}$ to the space of symmetric matrices of order $2(m-r)$, and $f(0) = 0$; $F$ is a power series without constant and linear terms convergent in a neighborhood of the origin in $\mathbf{R}^{n+r}$; the remainder $U$ satisfies the inequality

$$|U(\varphi, I, z)| < c(R) \quad \text{for} \quad |I| \le R^2, \;\; |z| \le R,$$

where $c(R)$ is *exponentially small* as $R \to 0$. The mappings $F$ and $f$ can be chosen to be polynomial.

[3]This transformation depends on $R$ discontinuously, but piecewise analytically.

Denote by $\pm\Lambda_1(I^{\rm all}),\ldots,\pm\Lambda_{m-r}(I^{\rm all})$ the eigenvalues of the matrix

$$J_{m-r}[B^{\rm part}+f(I^{\rm all})],$$

so that $\Lambda(0)=\lambda$. For sufficiently small $I^{\rm all}$, there are exactly $\nu-r$ purely imaginary numbers among $m-r$ numbers $\Lambda_1(I^{\rm all}),\ldots,\Lambda_{m-r}(I^{\rm all})$.

To construct invariant $(n+r)$-tori near torus $T$, one needs the nondegeneracy condition on normal form (10):

$$\det K\neq 0, \quad \text{where} \quad K=\frac{\partial^2 F}{(\partial I^{\rm all})^2}(0), \tag{11}$$

and the nonresonance condition

$$\mathrm{Im}\left(\frac{\partial\langle l,\Lambda\rangle}{\partial I^{\rm all}}(0)K^{-1}\right)\notin \mathbf{Z}^{n+r} \quad \text{for all} \quad l\in\mathbf{Z}^{m-r}, \;\; 1\leq|l|\leq 2, \quad i\langle l,\lambda\rangle\in\mathbf{R} \tag{12}$$

(we treat $\partial\langle l,\Lambda\rangle/\partial I^{\rm all}$ as a row-vector of length $n+r$). Although the functions $F$ and $f$ depend on $R$, conditions (11) and (12) involve the quadratic terms of $F$ and the linear terms of $f$ only and are $R$-independent.

Theorem 1 [8, 11]. *Let all the eigenvalues of matrix $\Omega$ be pairwise distinct, the Diophantine condition* (9)*, the nondegeneracy condition* (11)*, and the nonresonance condition* (12) *being satisfied. Then there exists a Cantor set $A\subset\mathbf{R}^{n+r}$ possessing the following three properties.*

*1) All the points $a\in A$ are Diophantine.*

*2) For each $a\in A$, the Hamiltonian system under consideration has a reducible Diophantine isotropic invariant $(n+r)$-torus with the intrinsic frequency vector equal to $a$. The distance between this torus and the initial $n$-torus $T$ tends to $0$ as $a\to\omega^{\rm all}$, and the $2(n+r)$-dimensional Hausdorff measure of the union of all the $(n+r)$-tori lying in any neighborhood of $T$ is positive.*

*3) Having set*

$$W(R)=\Big\{a\in\mathbf{R}^{n+r} \;:\; |a-\omega^{\rm all}|\leq R^2,\; a=\omega^{\rm all}+\frac{\partial F}{\partial I^{\rm all}}(I^{\rm all}) \text{ for some } I^{\rm all}\in\mathbf{R}^{n+r},\; I^{\rm all}_j\geq 0 \text{ for } n+1\leq j\leq n+r\Big\}$$

*for any sufficiently small $R>0$, one has*

$$\mathrm{mes}(W(R)\setminus A)\leq c_1\exp\left[-c_2R^{-2/(\tau+1)}\right], \tag{13}$$

*where $\tau$ is the exponent entering inequalities* (9), *positive constants $c_1$ and $c_2$ do not depend on $R$, and* mes *denotes the Lebesgue measure in $\mathbf{R}^{n+r}$.*

Condition (11) means that the mapping

$$I^{\mathrm{all}} \mapsto \omega^{\mathrm{all}} + \frac{\partial F}{\partial I^{\mathrm{all}}}(I^{\mathrm{all}})$$

is a local diffeomorphism at the origin. We can conclude that through torus $T$, there passes a Cantor $(n+r)$-parameter family of reducible Diophantine isotropic invariant $(n+r)$-tori with frequencies close to $\omega_1^{\mathrm{all}}, \ldots, \omega_{n+r}^{\mathrm{all}}$, these tori "condensing" exponentially while approaching the initial torus $T$. As a matter of fact, the family of $(n+r)$-tori obtained is Whitney-smooth.

Inequalities (9) involve Diophantine conditions on the numbers $\omega_1^N, \ldots, \omega_r^N$. For $n \geq 1$, these conditions are not restrictive because the initial torus $T$ belongs to a Whitney-smooth $n$-parameter family of invariant $n$-tori, and generically almost all the tori in this family meet condition (9). On the other hand, for $n = 0$ torus $T$ is just an isolated equilibrium point, and the Diophantine condition on the normal frequencies $\omega_1^N, \ldots, \omega_r^N$ is no longer a genericity condition: for $r > 1$ one can make those frequencies, e.g., commensurable by an arbitrarily small perturbation of the Hamilton function.

The excitation of elliptic normal modes of an equilibrium point of a Hamiltonian system has been considered in a much larger number of works (see, e.g., [12, 13]) and has been studied much thoroughly than the excitation of normal modes of an invariant torus of arbitrary dimension. It turns out that if among the eigenvalues of the linearization of a Hamiltonian system at an equilibrium point 0, there are $\nu \geq 1$ pairs of purely imaginary numbers, then generically this system possesses reducible Diophantine isotropic invariant tori of all the dimensions $r = 1, 2, \ldots, \nu$ in any neighborhood of 0 (for $r = 1$ this assertion is the classical Lyapunov center theorem which does not require the KAM theory methods). To prove this statement, one does not need the corresponding collections of eigenfrequencies to be Diophantine, there suffice some nondegeneracy and nonresonance conditions insensitive to small perturbations of the Hamilton function. However, to obtain exponential "condensation" of the tori while approaching 0 for $r \geq 2$, it is necessary to impose certain Diophantine hypotheses. In paper [14], such "condensation" was proven for the first time for a particular case of $m$-tori in a neighborhood of an elliptic equilibrium point of a Hamiltonian system with $m$ degrees of freedom. In that paper, the theorem on an exponentially small estimate for the measure of the complement of the union of the invariant tori was also generalized to the case where the collection of the eigenfrequencies is not Diophantine but only “quasi-Diophantine” (Diophantine “up to precision $\delta$”). The property of being “quasi-Diophantine” is structurally stable, i.e., it persists under small perturbations. To obtain an exponentially small estimate in the “quasi-Diophantine” case, one should neglect a neighborhood of the equilibrium point of radius $O(\delta)$.

## 4. Excitation of elliptic normal modes of an analytic family of invariant tori

The proof and even the statement of theorem 1 exploit the advanced technique of normal forms for a Hamiltonian system in a neighborhood of an invariant torus. The nondegeneracy condition (11) and the nonresonance condition (12) involve terms of the normal form (10) for the Hamilton function that are of degree 4 in the “normal coordinates” $z$ (since $2I_j^{\rm all} = z_{j-n}^2 + z_{j-n+m}^2$ for $n+1 \le j \le n+r$). The left-hand side of the exponential estimate (13) involves (via $W(R)$) terms of the normal form for the Hamilton function that are of arbitrarily high degree in the “normal coordinates”.

It turns out that for $n \ge 1$, the excitation of normal modes of invariant tori can be established without the normal form technique and without considering the terms of the Hamilton function of degrees higher than 2 in the “normal coordinates” [7, 15, 16]. For this purpose, one has to change somewhat the statement of the problem and to formulate *new* nondegeneracy and nonresonance conditions for a Hamiltonian system $X_0$ possessing an *analytic* (not merely Whitney-smooth) $n$-parameter family of reducible isotropic invariant $n$-tori with parallel flows. Since for $n \ge 2$ the space of such *partially integrable* systems is of infinite codimension, the existence of invariant tori of dimensions larger than $n$ should be verified in this context not for the system $X_0$ only but *for all its sufficiently small perturbations* as well (perturbed systems are already generic).

Consider a Hamiltonian system governed by a Hamilton function of the form

$$H = H(\varphi, I, z) = P(I) + \tfrac{1}{2}\langle z, B(I)z\rangle + O(|z|^3) \tag{14}$$

with respect to symplectic structure (1), where $\varphi \in \mathbf{T}^n$ ($n \ge 1$), $I \in G \subset \mathbf{R}^n$ ($G$ being a bounded connected domain in $\mathbf{R}^n$), $z$ ranges in a neighborhood of the origin in $\mathbf{R}^{2m}$, and $2m \times 2m$ matrix $B(I)$ is symmetric for each value of $I$. The Hamilton function (14) affords the equations of motion

$$\dot\varphi = \omega(I) + O(|z|^2), \quad \dot I = O(|z|^3), \quad \dot z = \Omega(I)z + O(|z|^2), \tag{15}$$

where $\omega(I) = \partial P(I)/\partial I$ and $\Omega(I) = J_m B(I)$ while $J_m$ is matrix (4). The system in question thus possesses the $n$-parameter family of reducible isotropic invariant $n$-tori $\{I = \mathrm{const}, z = 0\}$, the flow on each torus being parallel with frequency vector $\omega(I)$.

Now suppose that for each $I \in G$, among the eigenvalues of matrix $\Omega(I)$, there are $\nu$ pairs ($0 \le \nu \le m$) of purely imaginary numbers. We wonder whether system (15) has invariant tori of dimensions $n+1, \dots, n+\nu$ near the $2n$-dimensional surface $\{z = 0\}$ and whether sufficiently small perturbations of this system possess invariant tori of dimensions $n, n+1, \dots, n+\nu$.

Fix an arbitrary integer $r$ in the interval $0 \le r \le \nu$. Assume that for each $I \in G$, all the eigenvalues of matrix $\Omega(I)$ are pairwise distinct, and fix $r$ pairs $\pm i\omega_1^N(I), \dots, \pm i\omega_r^N(I)$ of purely imaginary eigenvalues of matrix $\Omega(I)$ that depend on $I$ analytically. Denote the remaining eigenvalues of this matrix by

$$\begin{array}{ll} \pm i\theta_j(I), & 1 \le j \le \nu - r, \\ \pm \eta_j(I), & 1 \le j \le m - \nu - 2s, \\ \pm \alpha_j(I) \pm i\beta_j(I), & 1 \le j \le s, \end{array}$$

where $0 \le s \le \frac{1}{2}(m-\nu)$ while $\theta_j(I)$, $\eta_j(I)$, $\alpha_j(I)$, $\beta_j(I)$ are analytic functions. Introduce the notation

$$\omega^{\mathrm{all}} = \omega^{\mathrm{all}}(I) = (\omega_1, \dots, \omega_n, \omega_1^N, \dots, \omega_r^N),$$
$$\theta^{\mathrm{all}} = \theta^{\mathrm{all}}(I) = (\theta_1, \dots, \theta_{\nu-r}, \beta_1, \dots, \beta_s).$$

The conditions guaranteeing the existence of invariant tori of dimension $n+r$ in system (15) and any of its sufficiently small perturbations can be expressed in terms of the intrinsic frequencies $\omega_i$ of the unperturbed $n$-tori $\{I = \mathrm{const}, z = 0\}$ and the eigenvalues $\pm i\omega_j^N$, $\pm i\theta_j$, $\pm\alpha_j \pm i\beta_j$ of matrices $\Omega(I)$, but those conditions are rather complicated and involve derivatives with respect to variables $I$ of an arbitrarily high order. Let $q \in \mathbf{Z}_+^n$, $x \in \mathbf{R}^n$, $e \in \mathbf{R}^{n+r}$, $\mu \in \mathbf{Z}$, $Q \in \mathbf{Z}$, $Q > 0$, $l \in \mathbf{Z}^{\nu-r+s}$, where $\mathbf{Z}_+^n = \mathbf{Z}^n \cap \mathbf{R}_+^n$ denotes the set of integer vectors of length $n$ with non-negative components. Introduce the notation

$$D^q\omega^{\mathrm{all}} = \frac{\partial^{|q|}\omega^{\mathrm{all}}}{\partial I_1^{q_1} \cdots \partial I_n^{q_n}}, \qquad D^q\theta^{\mathrm{all}} = \frac{\partial^{|q|}\theta^{\mathrm{all}}}{\partial I_1^{q_1} \cdots \partial I_n^{q_n}},$$
$$x^q = x_1^{q_1} \cdots x_n^{q_n},$$
$$\rho(I,Q) = \min_{\|e\|=1} \max_{\mu=0}^{Q} \max_{\|x\|=1} \left| \sum_{|q|=\mu} \langle e, D^q\omega^{\mathrm{all}}(I)\rangle x^q \right|,$$
$$\psi(I,Q,l) = \max_{\mu=0}^{Q} \max_{\|x\|=1} \left| \sum_{|q|=\mu} \langle l, D^q\theta^{\mathrm{all}}(I)\rangle x^q \right|.$$

Recall that the norms $|\cdot|$ and $\|\cdot\|$ are respectively defined by equalities (7) and (8).

THEOREM 2 [7, 16]. *Suppose that for each $I \in G$, all the eigenvalues of matrix $\Omega(I)$ are pairwise distinct and the following nondegeneracy and nonresonance conditions on the couple $(\omega(I), \Omega(I))$ are satisfied:*

*1) Nondegeneracy condition: there exists an integer $Q > 0$ such that $\rho(I,Q) > 0$ for each $I \in G$.*

*2) Nonresonance condition: for any $I \in G$, $l \in \mathbf{Z}^{\nu-r+s}$, $k \in \mathbf{Z}^{n+r}$ such that*

$$1 \le |l| \le 2, \quad 1 \le \|k\| \le \frac{\psi(I,Q,l)}{\rho(I,Q)},$$

*the inequality $\langle k, \omega^{\mathrm{all}}(I)\rangle \ne \langle l, \theta^{\mathrm{all}}(I)\rangle$ holds.*

*Then for any $\sigma > 0$, any sufficiently small Hamiltonian perturbation of system* (15) *possesses, in the $\sigma$-neighborhood of the surface $\{z = 0\}$, a Whitney-smooth $(n+r)$-parameter family of reducible Diophantine isotropic invariant $(n+r)$-tori, the $2(n+r)$-dimensional Hausdorff measure of the union of all these tori being positive (the perturbation smallness needed depends on $\sigma$). The frequencies of an $(n+r)$-torus located near the $n$-torus $\{I = I_0, z = 0\}$ are close to $\omega_1^{\mathrm{all}}(I_0), \dots, \omega_{n+r}^{\mathrm{all}}(I_0)$.*

Note that the nondegeneracy condition in this theorem is equivalent to the following geometric condition $\Psi$: the image of the frequency map $\omega^{\mathrm{all}} : G \to \mathbf{R}^{n+r}$ does not lie in any linear hyperplane passing through the origin [6, 7]. This condition is very weak (for example, among the maps $\omega^{\mathrm{all}}$ satisfying condition $\Psi$, there are maps whose image is a one-dimensional curve in the frequency space $\mathbf{R}^{n+r}$).

Although no one of theorems 1 or 2 is a corollary of the other due to different scenarios of the existence of tori and entirely different nondegeneracy and nonresonance conditions, theorem 2, as a whole, seems to be a weaker statement than theorem 1. Theorem 2 contains no exponential estimates, it is not applicable for $n = 0$, whereas for $n \ge 2$ it describes only systems close to partially integrable ones (integrable on the surface $\{z = 0\}$). On the other hand, theorem 2 can be easily carried over to *reversible* systems (the excitation of elliptic normal modes of invariant tori in reversible systems is considered in, e.g., [7, 17]), in particular, for the case where the manifold $\Sigma$ of the fixed points of the reversing involution is subject to inequality $\dim\Sigma < \operatorname{codim}\Sigma - 1$ (cf. [17]). The proof and even the formulation of the analogue of theorem 1 for reversible systems in the case $\dim\Sigma < \operatorname{codim}\Sigma - 1$ seem to be a very difficult problem.

The proofs of theorems 1 and 2 are given in original works [8, 11] and [7, 16], respectively.

Both the theorems can be carried over to "Hamiltonian systems with discrete time", i.e., exact symplectic diffeomorphisms.

## 5. Bibliographical remarks

In this concluding section, we list, without claiming to be complete, some basic works pertaining to two aspects of the theory under consideration: the *existence* of invariant tori of dimensions $p > n$ in a neighborhood of invariant tori of dimension $n \geq 2$ and *exponential "condensation"* of invariant tori of dimensions $p \geq n$ while approaching a given invariant torus of dimension $n \geq 0$.

Diophantine Lagrangian invariant $(n+m)$-tori in a neighborhood of invariant $n$-tori in Hamiltonian systems with $n+m$ degrees of freedom were first considered by V. I. Arnold [18, 19]. A. D. Bruno [20, 21] studied *analytic* families of Diophantine isotropic invariant tori of dimensions $p \geq n$ (not necessarily reducible) passing through a given invariant $n$-torus in systems with $n+m$ degrees of freedom. Cantor $p$-parameter families of invariant $p$-tori near invariant $n$-tori in systems with $n+m$ degrees of freedom for any $n$, $m$ and $p$ $(2 \leq n \leq p \leq n+m)$ were constructed in works by À. Jorba, J. Villanueva [8, 11], H. W. Broer, G. B. Huitema, and the author [7, 16].

An exponentially small estimate for the measure of the complement of the union of invariant tori was first obtained by A. I. Neĭshtadt [22]. In paper [22], a nearly-integrable system with two degrees of freedom was considered in the presence of the so called proper degeneracy, the measure of the complement of the union of invariant tori turning out to be exponentially small with respect to the perturbation magnitude. In the present survey, we have described the phenomenon of exponential "condensation" of a family of invariant $p$-tori while approaching an invariant $n$-torus in a Hamiltonian system with $n+m$ degrees of freedom. Here exponential smallness of the "slits" between the $p$-tori is understood with respect to the distance from the given $n$-torus. The exponential "condensation" effect was discovered by A. Morbidelli and A. Giorgilli [23] who considered the case $p = n$, $m = 0$. The results of [23] were confirmed numerically in [24]. The case $p = m$, $n = 0$ was examined by A. Delshams and P. Gutiérrez [14] while the case where $p = n$ and $m$ is arbitrary, by À. Jorba and J. Villanueva [25]. Finally, À. Jorba and J. Villanueva have proven exponential "condensation" for any $n$, $m$ and $p$ [8, 11].

Recently, À. Jorba, J. Villanueva and co-authors obtained exponential estimates for the measure of "exceptional sets" or the magnitude of "remainders" of normal forms in some other problems of the KAM theory as well (see, e.g., [8, 25, 26, 27]). Exponentially small effects occur frequently in analytic dynamical systems in rather diverse situations.

The applications of the results discussed in the present survey to some problems of celestial mechanics are considered in, e.g., [8, 18, 19, 25].

I am indebted to V. I. Arnold who introduced me to the whimsical world of Cantor families of invariant tori. I am also very grateful to H. W. Broer and G. B. Huitema for extremely fruitful cooperation during the last several years and to À. Jorba who gave me the preprints of his papers [11, 25, 26, 27] prior to publication and the manuscript of the thesis work of his student J. Villanueva [8].

# References


[1] *Lazutkin V. F.* KAM Theory and Semiclassical Approximations to Eigenfunctions. Berlin, Springer, 1993.

[2] *Chierchia L., Gallavotti G.* Drift and diffusion in phase space // Ann. Institut Henri Poincaré, Physique Théorique, 1994, V. 60, No. 1, p. 1–144.

[3] *Mather J. N., Forni G.* Action minimizing orbits in Hamiltonian systems // In: Transition to Chaos in Classical and Quantum Mechanics, ed. *Graffi S.* (Lect. Notes Math., V. 1589). Berlin, Springer, 1994, p. 92–186.

[4] *MacKay R. S.* Recent progress and outstanding problems in Hamiltonian dynamics // Physica D, 1995, V. 86, No. 1–2, p. 122–133.

[5] *Kozlov V. V.* Symmetries, Topology, and Resonances in Hamiltonian Mechanics. Berlin, Springer, 1996.

[6] *Broer H. W., Huitema G. B., Sevryuk M. B.* Families of quasi-periodic motions in dynamical systems depending on parameters // In: Nonlinear Dynamical Systems and Chaos, eds. *Broer H. W., van Gils S. A., Hoveijn I., Takens F.* (Progress Nonlinear Differ. Equations and Their Appl., V. 19). Basel, Birkhäuser, 1996, p. 171–211.

[7] *Broer H. W., Huitema G. B., Sevryuk M. B.* Quasi-Periodic Motions in Families of Dynamical Systems: Order amidst Chaos (Lect. Notes Math., V. 1645). Berlin, Springer, 1996.

[8] *Villanueva J.* Normal Forms around Lower Dimensional Tori of Hamiltonian Systems. PhD thesis, Universitat Politècnica de Catalunya, Barcelona (Spain), 1997.

[9] *Herman M. R.* Inégalités "a priori" pour des tores lagrangiens invariants par des difféomorphismes symplectiques // Publ. Math. IHES, 1990, V. 70, p. 47–101.

[10] *Morbidelli A., Giorgilli A.* On a connection between KAM and Nekhoroshev's theorems // Physica D, 1995, V. 86, No. 3, p. 514–516.

[11] *Jorba À., Villanueva J.* On the normal behaviour of partially elliptic lower dimensional tori of Hamiltonian systems // Nonlinearity, 1997, V. 10, No. 4, p. 783–822.

[12] *Eliasson L. H.* Perturbations of stable invariant tori for Hamiltonian systems // Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV Ser., 1988, V. 15, No. 1, p. 115–147.

[13] *Pöschel J.* On elliptic lower dimensional tori in Hamiltonian systems // Math. Zeitschr., 1989, V. 202, No. 4, p. 559–608.

[14] *Delshams A., Gutiérrez P.* Estimates on invariant tori near an elliptic equilibrium point of a Hamiltonian system // J. Differ. Equat., 1996, V. 131, No. 2, p. 277–303.

[15] *Sevryuk M. B.* Excitation of normal modes of invariant tori in Hamiltonian systems // Uspekhi Mat. Nauk, 1996, V. 51, No. 5, p. 171 (in Russian).

[16] *Sevryuk M. B.* Excitation of elliptic normal modes of invariant tori in Hamiltonian systems // In: Topics in Singularity Theory. V. I. Arnold's 60th Anniversary Collection, eds. *Khovanskiĭ A. G., Varchenko A. N., Vassiliev V. A.* (Amer. Math. Soc. Transl., Ser. 2, V. 180) (Adv. Math. Sci., V. 34). Providence, AMS, 1997, p. 209–218.

[17] *Sevryuk M. B.* The iteration-approximation decoupling in the reversible KAM theory // Chaos, 1995, V. 5, No. 3, p. 552–565.

[18] *Arnold V. I.* On the classical perturbation theory and the problem of stability of planetary systems // Sov. Math. Dokl., 1962, V. 3, No. 4, p. 1008–1012.

[19] *Arnold V. I.* Small denominators and problems of stability of motion in classical and celestial mechanics // Russian Math. Surveys, 1963, V. 18, No. 6, p. 85–191.

[20] *Bruno A. D.* The sets of analyticity of a normalizing transformation // The USSR Academy of Sciences Institute of Applied Mathematics, preprints 97 and 98, 1974 (in Russian).

[21] *Bruno A. D.* Local Methods in Nonlinear Differential Equations. Berlin, Springer, 1989.

[22] *Neĭshtadt A. I.* Estimates in the Kolmogorov theorem on the persistence of quasi-periodic motions // J. Appl. Math. Mech., 1981, V. 45, No. 6, p. 766–772.

[23] *Morbidelli A., Giorgilli A.* Superexponential stability of KAM tori // J. Stat. Phys., 1995, V. 78, No. 5–6, p. 1607–1617.

[24] *Lega E., Froeschlé C.* Numerical investigations of the structure around an invariant KAM torus using the frequency map analysis // Physica D, 1996, V. 95, No. 2, p. 97–106.

[25] *Jorba À., Villanueva J.* On the persistence of lower dimensional invariant tori under quasiperiodic perturbations // J. Nonlinear Science, 1997, V. 7, No. 5, p. 427–473.

[26] *Jorba À., Simó C.* On quasiperiodic perturbations of elliptic equilibrium points // SIAM J. Math. Anal., 1996, V. 27, No. 6, p. 1704–1737.

[27] *Jorba À., Ramírez-Ros R., Villanueva J.* Effective reducibility of quasiperiodic linear equations close to constant coefficients // SIAM J. Math. Anal., 1997, V. 28, No. 1, p. 178–188.