REPLICA THEORY AND THE GEOMETRY OF SYMMETRY BREAKING

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Abstract. In the study of Replica Symmetry Breaking (RSB), one is led to consider functions f of *pseudomatrices*, i.e. of matrices of order α, with α a real, rather than an integer, number. We propose a mathematically rigorous definition of pseudomatrices, and show that from this it follows that the minimum of f over the space of pseudomatrices $Q(x,y)$ which depend only on $(x-y)$ is also a critical point for f; this corresponds to a property which is usually assumed without proof in the study of RSB. We also find that generic bifurcations from such a minimum lead to minima corresponding to periodic quasimatrices. These results are obtained through use of Michel's theory of symmetry breaking.
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1. Introduction.

In the study of replica symmetry breaking [1], one is led to look for minima of a function $f(Q)$ of a *pseudomatrix* Q; in this physical situation, f is typically an energy functional depending on a partition function $\mathcal{Z}(Q)$, defined on pseudomatrices.

These functions on pseudomatrices are usually studied in the physical literature starting from corresponding functions on $N\times N$ matrices $M=\{M_{ij}\}$ (with $M_{ii}=0$) via a precise albeit not so mathematically well defined procedure involving several steps. Such steps (to be taken in a precise given order) include in particular a limit for $N\to\infty$, analytical continuation of N to a continuous variable – so to consider derivatives with respect to N of the partition function $Z_N(M)$, defined on $N\times N$ matrices – and finally the limit $N\to\infty$ (see [1,2] for more detail).

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This seemingly self-contradictory extension of the usual concept of functions of a matrix can be cast into a coherent procedure, and provides significant results in the study of disordered systems [2].

In this note, we propose a mathematically coherent way to define pseudomatrices Q; this also has the advantage that we can use the "Symmetric Criticality Principle" of Palais [3] in the search for minima of $f(Q)$. Actually, we will be able to describe generically the location of minima of $f(Q)$, which will be in a set of symmetric psudomatrices, and of symmetry-breaking solutions.

The present paper is written in a quite mathematical style; this is quite unavoidable, as its goal is to provide mathematical foundations (a posteriori) to the work of physicists active in the field. We hope the physicists readers will bear with such a style.

2. Matrices and line bundles.

A real line bundle (RL bundle) is a fiber bundle with fiber $F \simeq \mathbf{R}$; a complex line bundle, or line bundle *tout court*, is a bundle with fiber $F \simeq \mathbf{C}$. In the following we will consider RL bundles, for which it will be understood without further mentioning that $F = \mathbf{R}^1$; notice that if we had to consider complex (pseudo)matrices, our considerations would have to be cast in terms of complex line bundles, which would be immediate.

A real $N \times N$ matrix $M = \{M_{ij}\,;\, i,j = 1, ..., N\}$ can be seen as a section μ of the RL bundle (E_N, π_N, B_N) with base space $B_N = \mathbf{Z}_N \times \mathbf{Z}_N$. Points $(i,j) \in \mathbf{Z}_N \times \mathbf{Z}_N = B_N$ corresponds to sites of the matrix, and the values $\mu(i,j)$ of the section μ at the points $(i,j) \in B_N$ correspond to the entries of the matrix; that is, $\mu(i,j) \equiv M_{ij}$. The projection $\pi_N : E_N \to B_N$ assigns to the entry M_{ij} its indices: $\pi_N(M_{ij}) = (i,j)$.

In this setting, taking the limit $N \to \infty$ corresponds then to consider the RL bundle (E_*, π_*, B_*) with base space $B_* = \mathbf{Z} \times \mathbf{Z}$ and projection $\pi_* : E_* \to B_*$ as before.

Notice that these bundles are trivial ones; notice also that E_N, E_* can be seen both as vector bundles and as principal fiber bundles, since $\mathbf{R}$ is both a vector space and a Lie group.

We come now to a simple but useful remark: the space $\mathbf{Z} \times \mathbf{Z} = B_*$ has a natural immersion in the space $\mathbf{R} \times \mathbf{R} := B_0$; correspondingly, the RL bundle $(E_*, \pi_*, B_*) = B_* \times \mathbf{R}$ is naturally immersed in the RL bundle $(E_0, \pi_0, B_0) = B_0 \times \mathbf{R}$. Thus, sections $\sigma : B_0 \to E_0$ of this bundle are the natural generalization of (infinite) matrices.

It is maybe worth stressing that the immersion of the base space of the bundle into a continuous space can be performed also without taking the limit $N \to \infty$. In this case the natural immersion is $\mathbf{Z}_N \subset I_N \simeq I$, where I_N is the real line interval $I_N = (0, N]$, $I = I_1$, and $\simeq$ denotes isomorphism. In this way, any E_N with finite N is naturally immersed into the RL bundle (E_I, π_I, B_I), where $B_I = I \times I$.

Up to now, we have not considered any special condition (such as $M_{ii} = 0$) on matrices, nor we have considered base space automorphisms, of relevance in the following; considering these will be our next step.

Before proceeding to this, we would like to call the attention of the reader to an important fact: the present interpretation of a matrix as a section of the pertinent bundle requires that the two spaces (we denote them generically by S) entering in the product to give the base spaces of the bundles we consider here, $B = S \times S$, should be seen as *two copies of the same space* (mirrors of each other in the Web jargon), as it will be discussed in a moment.

In the algebraic language [4], we should consider a "doubling functor" δ which assigns to a space S a pair of copies of the same space, $\delta(S) = (S \times S) \equiv S_\delta$; and to morphisms $\rho : S \to S$ of a space S a pair of copies of the same morphism, each acting on a copy of the space. Thus, $\delta(\rho) = (\rho \times \rho) \equiv \rho_\delta$, with $\rho_\delta : S_\delta \to S_\delta$ acting as $\rho_\delta(S \times S) = \rho(S) \times \rho(S)$.

3. Base space morphisms.

As mentioned above, the interpretation of matrices as sections of a bundle requires that the two spaces whose product is the base of the bundle should be seen as mirror copies of the same space. This means in particular that if we consider morphisms $\theta : B \to B$ of the base space $B = S \times S$, they are acceptable only if they are of the form $\theta = \rho \times \rho$, with $\rho : S \to S$, i.e. if $\theta = \delta(\rho)$ for δ the doubling functor and ρ some morphism of S.

We will denote by $\mathcal{M}(\alpha)$ the set of morphisms of α; thus the above remark means that we should not consider the whole $\mathcal{M}(B)$, but only $\delta\left[\mathcal{M}(S)\right] \subset \mathcal{M}(B)$.

In the case of $N \times N$ matrices, $S = \mathbf{Z}_N$, this means that if $\rho : \mathbf{Z}_N \to \mathbf{Z}_N$ is a permutation, we should apply it to both the row and the column indices (which is indeed a quite natural requirement).

Let us now consider in more detail $\mathcal{M}(S)$ for the various kind of S we have considered so far, i.e. $S = \mathbf{Z}_N, \mathbf{Z}, I, \mathbf{R}$; and let us wonder if all elements of $\mathcal{M}(S)$ are acceptable. In the case of matrices, it is clear that only permutations are acceptable, while a noninvertible map $\rho : \mathbf{Z}_N \to \mathbf{Z}_N$ should not be considered as legitimate. For later extension, we characterize the allowed transformations (permutations) as *invertible* and *volume preserving*. We denote the group of invertible applications on S as $\mathcal{M}_0(S) \subset \mathcal{M}(S)$.

Summarizing our discussion so far, an $N \times N$ matrix should be seen as a section $\mu : B_N \to E_N$ of the fiber bundle E_N, on which operates the group $\delta\left[\mathcal{M}_0(\mathbf{Z}_N)\right] \subset \delta(B_N)$ of base space morphisms.

The generalization of this to the bundle E_* is immediate, as now it suffices to consider as $\mathcal{M}_0(\mathbf{Z})$ the group of invertible applications $\rho : \mathbf{Z} \to \mathbf{Z}$.

The situation is much less clear when we pass to a continuous base space (i.e. for $S = I$ or $S = \mathbf{R}$); in this case the group of invertible applications is still well defined, but very large. Moreover, invertible applications can be not volume-preserving, and/or not continuous (and even nowhere continuous, as for the $\rho : \mathbf{R} \to \mathbf{R}$ defined as $\rho(x) = x$ for x rational, and $\rho(x) = 1/x$ for x irrational) .

Two natural classes of applications to consider are indeed (invertible) volume-preserving applications, and (invertible) continuous ones; or the intersection of the two. We choose – in view of physical motivation – to consider the latter case.

Definition 1. Given a space S, the group of *proper morphisms* of S, denoted by $\mathcal{M}_0(S)$, is the group of invertible, volume-preserving, continuous applications $\rho : S \to S$.

For S a discrete space, any map will be considered as continuous. The following definitions formalize the discussion conducted so far.

Definition 2. Given a space S, the associated (real) *pseudomatrix bundle* E_S is the trivial (real) line bundle $E_S = \delta(S) \times \mathbf{R}$, with projection $\pi_S : E_S \to B_S = \delta(S)$.

Definition 3. Given a space S, a (real) pseudomatrix over S is a section $\mu : B_S \to E_S$ of the associated (real) pseudomatrix bundle.

In the standard matrix case, two matrices which can be identified by a simultaneous permutation of rows and column indices can be considered as equivalent; correspondingly,

Definition 4. Given a space S, two pseudomatrices over S are equivalent if they are in the same equivalence class under $\delta[\mathcal{M}_0(S)]$; i.e. two pseudomatrices μ_1, μ_2 are equivalent if exists a $\theta \in \delta[\mathcal{M}_0(S)]$ such that, for all $z \in B_S = \delta(S)$, they satisfy $\mu_1(z) = \mu_2(\theta z)$.

4. Boundary conditions on pseudomatrices.

As mentioned in the introduction, in replica symmetry breaking (RSB) one considers indeed (the generalization of) matrices with diagonal element equal to zero, $M_{ii} = 0$. In the present language, this means that we should impose the

Condition A. $\mu(x, x) = 0$ for any $x \in S$.

It should be stressed that the set of pseudomatrices satisfying condition A is invariant under $\delta[\mathcal{M}_0(S)]$, and actually under the whole $\delta[\mathcal{M}(S)]$. Thus, condition A is compatible with the natural equivalence relation of definition 4.

In RSB, one does also sometimes consider "periodic matrices"; this means matrices with indices "modulo N" (i.e. $j + N \approx j$). In this case, the continuous base space generalization should be $S = S^1$ rather than $S = I$; the corresponding pseudomatrix bundle will be denoted as $E_c = B_c \times \mathbf{R}$ (notice that now a nontrivial structure of the bundle would be in general possible), with $B_c = \delta(S^1)$; the subscript "c" stands for "circle".

Notice that while $\mathcal{M}_0(I) = \{e\}$, in this case we have to deal with $\mathcal{M}_0(S^1) = \mathbf{R}$, corresponding to rotations of Σ^1 through an angle $\theta \in \mathbf{R}$; when we take into account equivalence of rotations through θ and through $\theta + 2\pi$, we have $\mathcal{M}_0(S^1) = S^1 = \mathbf{R}/\mathbf{Z}$.

5. Critical points of functions over pseudomatrices and minimal RSB.

Having obtained a proper mathematical setting for pseudomatrices, we can pass to consider function on pseudomatrices, and in particular the type of minimization problems encountered in RSB.

Given a space S, we will denote by $\mathcal{Q}(S)$ the set of pseudomatrices over S, i.e. of sections of the RL bundle over $\delta(S)$. With this notation, we have to consider critical points of smooth functions $f : \mathcal{Q}(S) \to \mathbf{R}$.

In particular, we would like to consider functions which are compatible with the natural equivalence relation considered in definition 4, i.e. which take equal values on equivalent pseudomatrices; hence f should be invariant under $G = \delta[\mathcal{M}_0(S)]$. We will refer to such an f as the invariant energy functional, and to G as the symmetry (invariance) group.

In view of the physical problem one has to consider, it is natural to introduce a *finite energy condition*, i.e. to consider only the sections which lead to a finite value of the energy functional f; once we define a scalar product in $\mathcal{Q}$, this finite energy condition leads naturally to consider a Sobolev space of sections in $\mathcal{Q}$. From now on, we assume this has been selected and we have restricted to it, and just denote it by $\mathcal{Q}$.

Notice that it is easy to introduce a (fibered) scalar product in $\mathcal{Q}$, defined e.g. as

$$\langle \mu, \nu \rangle \ = \ \begin{cases} \sum_{x \in B} \left(\mu(x) \, \nu(x) \right) & \text{for } B \text{ discrete} \\ \int_B \mu(x) \, \nu(x) \, \mathrm{d}x & \text{for } B \text{ continuous.} \end{cases}$$

A Sobolev product would be defined in the same way, involving also derivatives (or finite differences) of sections; we will not dwell any further on these technical points.

If we make abstraction from the RSB setting, we have a pretty classical problem: namely, look for extremal points of a functional f defined on a space $\mathcal{Q}$ of sections of a fiber bundle E, and invariant under a group G which acts on $\mathcal{Q}$.

The work of previous sections providing a sound mathematical setting for the "objects Q"permits therefore to apply the known results available in the literature for this classical problem, even to the problem of RSB. We will now briefly mention which ones of these classical results are specially useful to deal with RSB, referring to literature for detail.

The key result in this field is Michel's theorem [5] (see also [6]), dealing with a potential V over a smooth finite dimensional manifold M and invariant under the action of a compact Lie group G (as for the $SU(3)$ theory of strong interactions, which provided motivation for Michel's work). The proper statement of Michel's theorem would require to introduce the notion of *(isotropy) stratification of a G-manifold*; here we will just give a rough idea of the final result, referring to [6,7,8] for detail.

If $V : M \to \mathbf{R}$ is G-invariant, it can be thought as a function Φ on the orbit space $\Omega = M/G$. A point $\omega_0 \in \Omega$ – i.e. a G-orbit in M – has an isotropy type $[G_0]$ corresponding to the conjugacy class of isotropy subgroups $G_x \subseteq G$ for points x on the G-orbit; orbits having the same isotropy type are said to form a *stratum*.

Michel's theorem asserts that if ω_0 is isolated in its stratum, then ω_0 is critical for any Φ, i.e. $x \in \omega_0 \subset M$ is a critical point for **any** G-invariant potential V.

The simplest example is given by $G = \mathbf{Z}_2$ acting on $M = \mathbf{R}$ by $g(x) = -x$; here there is a stratum made of only one orbit ω_0 (the origin), which is indeed a critical point for any even potential. A less trivial example is provided by $G = SU(3)$ acting via the adjoint action on its Lie algebra $\mathcal{G} \simeq \mathbf{R}^8$, or more precisely on the unit sphere in this, $\mathcal{G}_0 \simeq S^7$. Here $M/G \simeq \mathbf{R}^2$, $\mathcal{G}_0/G = S^1$, and critical orbits correspond to physical particles in the $SU(3)$ octet [9].

A simple result which, although obtained indipendently, can be seen as a corollary of Michel's theorem [6,10] and which has quite far-reaching applications is the "Equivariant Branching Lemma" (EBL) of Cicogna and Vanderbauwhede [11] (extended by Golubitsky and Stewart [12] to Hopf bifurcations), which together with the "Reduction Lemma" of Golubitsky and Stewart [12] is a cornerstone of equivariant bifurcation theory [13,8], i.e. in the mathematical theory of phase transitions.

Roughly speaking, this ensures that if a subgroup $H \subseteq G$ has a one-dimensional fixed space, then when a maximally symmetric solution (invariant under the whole G) looses stability, the existence of a stable symmetry breaking solution x_0 with symmetry $H \subseteq G$ is generically guaranteed.

Michel's theorem can be extended to the present setting, i.e. G-invariant functionals on a (Sobolev) space of sections of a fiber bundle [14]; this extension does essentially amount to rephrase results by Palais, known as the "Symmetric Criticality Principle" (SCP) [3]. The extension carries over to the bifurcation setting and to the EBL [15], i.e. to the symmetry breaking case, and can therefore be applied to our problem of RSB.

Referring again to the literature for detail [3], let us see what is the content of the SCP in the present notation. Given the functional $f : \mathcal{Q} \to \mathbf{R}$, invariant under G, we can consider the restriction f_H of f to the space $\mathcal{Q}_H$ of H-invariant sections, where $H \subseteq G$. Suppose now that μ is a critical point for f: then, the SCP guarantees that μ is also a critical point for the unrestricted functional f.

We stress that this follows from the fact we managed to set our problem so to satisfy the conditions of Palais' theorem (in particular we have a Sobolev space of sections of a bundle, and the G-action is linearizable around any symmetric point [3,14,15]), but the SCP is *not* valid, contrary to what one could think at first, in more general situations; partial extensions are however possible [3,15].

Making use of this, it is easy to see that an extension of the EBL [15] ensures that if $H \subset G$ is a maximal isotropy subgroup of G (this very notion can be quite delicate for infinite groups, and in more general cases one has either to rely heavily on geometry, or limit to a dense subset of the space of sections, see [16] for details) and the fully symmetric solution looses stability, then generically there is a stable solution, i.e. a minimum of f, given by some $\mu \in \mathcal{Q}_H$.

6. Application to RSB.

Let us now apply the above mentioned results to the problem of RSB. In the case $S = \mathbf{R}$ or $S = S^1$, the group $G = \mathcal{M}_0(S)$ is just $G = \mathbf{R}$, acting as translations. The G-invariant functionals $f : \mathcal{Q} \to \mathbf{R}$ are therefore invariant under translations; G acts explicitely as $g_a[\mu(x,y)] = \mu(g_a x, g_a y) \equiv \mu(x+a, y+a)$. It is clear from this that any $\mu \in \mathcal{Q}_G$ can be written as $\mu(x,y) = \widetilde{\mu}(x-y)$; the SCP guarantees that we can consider f_G, the restriction of f to G-invariant sections.

Considering pseudomatrices of this form is indeed what is usually done in the study of RSB [1,2]. We have therefore provided a sound mathematical justification for the current method of analysis for RSB. We summarize this as the

Proposition 1. *The pseudomatrix which minimizes f_G is also a critical point for f.*

As already mentioned, this is nothing else than a corollary of the SCP of Palais; it should be stressed that the SCP ensures only that μ_0 is a critical point for f, but not necessarily a minimum. The extension of the EBL mentioned above [15] gives the following corollary when f depends on a real parameter λ belonging to an interval $\Lambda \subseteq \mathbf{R}$:

Proposition 2. *Suppose $f = f^\lambda : \mathcal{Q} \to \mathbf{R}$ depends smoothly on a real parameter λ, and let $f_G = f_G^\lambda$ be the restriction of f^λ to $\mathcal{Q}_G \subset \mathcal{Q}$. Let $\mu_0 \equiv \mu_0^\lambda \in \mathcal{Q}_G$ be the minimum of f_G^λ, and suppose μ_G^λ is a minimum of f^λ for $\lambda < \lambda_0$, and looses stability at $\lambda = \lambda_0$. Then there is an $\varepsilon_* > 0$ such that for each maximal isotropy subgroup $H \subseteq G$ there is a stable critical point $\mu_{[H]}^\lambda \in \mathcal{Q}_H$, for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$.*

Notice that if $\mu_0 = 0$, this shows that when μ_0 becomes unstable we have a bifurcation of a branch of fully symmetric (i.e. G-invariant) minima.

Notice also that these results are formulated for general G; indeed, they would apply for any G-action on $\mathcal{Q}$ satisfying the conditions required for the SCP to hold.

In our concrete case, we have $G = \mathbf{R}$ acting as simultaneous translations in x and y; and thus the maximal isotropy subgroups of G for this action on $\mathcal{Q}$ are isomorphic to $\mathbf{Z}$. The sections over $\delta(\mathbf{R})$ invariant under $\mathbf{Z}$, i.e. such that $\mu(x,y) = \mu(x+\alpha, y+\alpha)$ with α a fixed real number, corresponds to "periodic pseudomatrices" (see sect.4), and can be thought as sections over $\delta(\mathbf{R}/\mathbf{Z}) = \delta(S^1)$, so that to a periodic $\mu \in \mathcal{Q}$ we can associate a section $\widetilde{\mu}_1(\varphi, \vartheta)$ of a RL bundle over the torus $\mathbf{T}^2$.

This means that if $\mu_0(x,y) = \widetilde{\mu}_0(x-y)$ looses stability, the symmetry breaking solution will be generically given by a periodic pseudomatrix $\mu_1(x,y)$, i.e. a pseudomatrix for which $\mu_1(x,y) = \mu_1(x+\alpha, y+\alpha)$, as discussed above.

7. Extensions, and conclusions.

We would now like to briefly present some remarks concerning possible extensions of the present work, and then summarize this short note.

First of all, it should be mentioned that fiber bundles corresponding to the product of a continuous and a discrete space are also of use in Connes' *Non-Commutative Geometry* [17], and in the approach to Field Theory of fundamental interactions based on this [18]; although in that case the role of teh two spaces is interchanged with respect to the present setting (i.e. one has a continuous base and a discrete fiber), Connes' work shows that we can have a highly nontrivial connection on a discrete space. An approach based on Non-Commutative Geometry has also been used to address problems arising in Condensed Matter Physics, where the discrete structure is by all means essential for the Physics of the system[19].

In this respect, we mention that bundles with discrete base space – as those we are considering – can be equipped with the discrete equivalent of a connection, i.e. with a set of maps $T_{\alpha\beta} : F_\beta \to F_\alpha$, where F_x is the fiber over x, and α, β are points in the base space.

If, as in the present note, we only want to consider trivial bundles, we can take $T_{\alpha\beta}$ to be the identiy for any pair of points in the discrete base space B; notice that when we are embedding B in the contractible space $\mathbf{R}$, it is entirely natural to consider maps such that $T_{xy} = T_{xz} T_{zy}$ (no sum on z) for all x, y and any z, and for the open chain $B = \mathbf{Z}$ these are equivalent to choosing $T_{xy} = I$.

On the other side, when we consider periodic matrices and pseudomatrices, so that B is a periodic chain – and is finally embedded in S^1 – maps satisfying the transfer property $T_{xy} = T_{xz} T_{zy}$ can however have a nontrivial holoedry; in this case it is natural to embed the bundle over B into a nontrivial RL bundle over S^1, so that one is led to considering nontrivial bundle structures.

In the present note we have implicitely assumed that T_{ij} is the identity for all (i, j) – i.e. just consider trivial RL bundle – as the Physics we are interested in does not require more complicate structures. However, it would be interesting to expand the present approach, especially in the case of complex pseudomatrices and complex line bundles, to consider nontrivial connections; in physical terms this would correspond to introducing a "geometric frustration" [20] in the system.

As already stressed in section 6, the theorems given there apply not only for the case at hand – i.e. $G = \mathbf{R}$ acting as translations – but also for any G-action on $\mathcal{Q}$ satisfying the conditions required for the SCP to hold (discussed in [3] and e.g. in [15]).

It should be stressed that in this general case the theorem does not affirm that other bifurcations – i.e. direct breaking to smaller isotropy subgroups – are not possible: indeed, this can be possible depending on the stratified geometry of $\mathcal{Q}/G$; on the other side, it affirms that certain solutions will however be present and generic. In many cases, these will indeed be the only ones. A discussion of these points would require to introduce a substantial additional amount of notions from Michel's theory of symmetry breaking and from Geometry, and is thus not given here.

Finally, let us briefly summarize our discussion. In the present note, we have defined (real) pseudomatrices in a mathematically precise way, i.e. as sections of certain (real) line bundles. We have then considered the situation encountered in studying Replica Symmetry Breaking, i.e. the minimization of a functional f (free energy) which depends on a partition function $\mathcal{Z}(Q)$ defined on pseudomatrices; by imposing a finite energy condition, whose precise form depends on f, we were led to selecting a Sobolev space of pseudomatrices $\mathcal{Q}$. Such an abstract setting permits to utilize at once Michel's geometric theory of symmetry breaking and the Symmetric Criticality Principle of Palais.

A straightforward application of these powerful tools gave our main results, and confirmed that – as assumed in the physical literature – the minima of f correspond to symmetric pseudomatrices, i.e. to pseudomatrices $\mu(x, y)$ which depend only on $(x - y)$, i.e. $\mu(x, y) = \widetilde{\mu}(x - y)$. We were also able to predict that generically a phase transition will lead to new minima corresponding to periodic pseudomatrices i.e. to pseudomatrices satisfying $\mu(x, y) = \mu(x + \alpha, y + \alpha)$ for some real number α.

These results are neither surprising nor new from the point of view of the Physics of disordered systems, but provide a solid mathematical foundation to physicists' work.

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