QUANTUM METHODS FOR INTERACTING PARTICLE SYSTEMS III, STATISTICAL MECHANICS OF ISING MODELS

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Abstract. We discuss the structure of the Hilbert states space of some selected classical 2-state spin models, in the algebraic framework given in [QMIPS I]. we compute the partition function of some selected models in terms of the eigenvalue of the associated Hamiltonian operators. In particular we focus our attention on the ferromagnatic Ising model deriving, in an alternative way, contour expansions which are a straightforward extension in dimension grater than two of the classical high-temperature, low-temperature expansions. We also obtain a different derivation of the Fortuin-Kasteleyn representation for the Ising model involving a contour expansion similar to the previous ones.

1. Introduction

In this paper we describe equilibrium statistical mechanics of a particular class of interacting particle systems (Ising spin systems) making use of the algebraic techniques borrowed from equilibrium quantum statistical mechanics. We refer the reader to [QMIPS I] for a general discussion and reference to other results obtained by a similar approach, and to [QMIPS II] and to [QMIPS IV] for some application to non-equilibrium statistical mechanics.

We consider the $\mathbb{Z}^d$ lattice. As an introduction to the formalism, we consider the Hilbert space of complex square summable function on the single site configuration space $\mathbb{Z}_2^{(x)}$ with respect to the symmetric Bernoulli measure. Namely $\forall x \in \mathbb{Z}^d$

$$\begin{aligned} \mathcal{H}_x \quad &: = span\left\{ |\emptyset\rangle_x , |\mathrm{x}\rangle_x \right\} \cong \mathbb{C}^2 \\ |\emptyset\rangle_x \quad &\equiv \left(\begin{array}{c} 1 \\ 0 \end{array} \right)_x \quad |\mathrm{x}\rangle_x \equiv \left(\begin{array}{c} 0 \\ 1 \end{array} \right)_x \end{aligned}$$

$\mathcal{U}_x = M(2,\mathbb{C})$ is the algebra of linear operators on $\mathcal{H}_x$[1]. We define the *spin* operator

$$\boldsymbol{\sigma}_x \in \mathcal{U}_x : \boldsymbol{\sigma}_x \left\{ \begin{array}{l} |\emptyset\rangle_x = |\mathrm{x}\rangle_x \\ |\mathrm{x}\rangle_x = |\emptyset\rangle_x \end{array} \right.$$

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[1]Here, see [QMIPS I] and [M] for more details, we think of $\mathcal{H}_x$ as spanned by two (orthonormal) vectors labelled by the "empty site" and the "full site" configurations. Consequently any operator acting on the configuration space is lifted to a linear operator acting on $\mathcal{H}_x$ and a probability density on the configuration space becomes a convex combination of the projectors on the subspaces spanned by the basis vectors of $\mathcal{H}_x$.

equivalent to the Pauli matrix $\sigma^{(1)}$

$$\sigma^{(1)} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the *spin flip* operator

$$\mathbf{f}_x \in \mathcal{U}_x : \mathbf{f}_x \begin{cases} |\emptyset\rangle_x = |\emptyset\rangle_x \\ |\mathrm{x}\rangle_x = -|\mathrm{x}\rangle_x \end{cases}$$

equivalent to the Pauli matrix $\sigma^{(3)}$

$$\sigma^{(3)} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Consider also the automorphism $\mathbf{U}_x$ on $\mathcal{U}_x$ such that

$$\mathbf{U}_x \in \mathcal{U}_x : \mathbf{U}_x \boldsymbol{\sigma}_x \mathbf{U}_x = \mathbf{f}_x \qquad \mathbf{U}_x^2 = \mathbf{I}_x$$

Setting $\mathcal{C}_x^{(s)}, \mathcal{C}_x^{(f)} \subset \mathcal{U}_x$ to be the commutative algebras generated by the identity operator $\mathbf{I}_x$ and by respectively $\boldsymbol{\sigma}_x$, $\mathbf{f}_x$, from the definition of $\mathbf{U}_x$ follows that $\mathbf{U}_x \mathcal{C}_x^{(s)} \mathbf{U}_x = \mathcal{C}_x^{(f)}$. In the forthcoming analysis we will denote the algebra of observable with $\mathcal{C}_x$ which will be identified either with $\mathcal{C}_x^{(s)}$ or with $\mathcal{C}_x^{(f)}$ depending on the case.

Given any finite lattice subset Λ, we can extend the previous construction to $\mathbb{Z}_2^{\Lambda} = \bigotimes_{x\in\Lambda} \mathbb{Z}_2^{(x)}$ simply by tensor product. Then we have

$$\begin{aligned} |\alpha\rangle_\Lambda &= \bigotimes_{x\in\alpha} |\mathrm{x}\rangle_x \bigotimes_{x\in\Lambda\setminus\alpha} |\emptyset\rangle_x \\ \mathcal{H}_\Lambda &= span\left\{ |\alpha\rangle_\Lambda : \alpha \subseteq \Lambda \right\} \end{aligned}$$

Moreover $\mathcal{U}_\Lambda = M\left(2^{|\Lambda|}, \mathbb{C}\right)$, $\mathcal{C}_\Lambda$ is the algebra of polynomials in $\boldsymbol{\sigma}_\alpha$ $(\mathbf{f}_\alpha)$ $\forall \alpha \subset \Lambda$ and

$$\begin{aligned} \boldsymbol{\sigma}_\alpha &= \bigotimes_{x\in\alpha} \boldsymbol{\sigma}_x \bigotimes_{x\in\Lambda\setminus\alpha} \mathbf{I}_x \\ \mathbf{f}_\alpha &= \mathbf{U}_\Lambda \boldsymbol{\sigma}_\alpha \mathbf{U}_\Lambda = \sum_{\emptyset\subseteq\gamma\subseteq\Lambda} (-1)^{|\alpha\cap\gamma|} |\gamma\rangle \langle\gamma|_\Lambda \\ \boldsymbol{\sigma}_\emptyset &= \mathbf{f}_\emptyset = \mathbf{I}_\Lambda \end{aligned}$$

where $|\gamma\rangle \langle\gamma|_\Lambda$ denotes the projector on the subspace of $\mathcal{H}_\Lambda$ spanned by $|\gamma\rangle_\Lambda$. For a further extension of the preceding constructions to the configuration space of the whole particle system see [QMIPS I] and reference quoted therein.

The following analysis involves a finite size system whose kinematical environment is $\left(\mathcal{H}_\Lambda, \mathcal{U}_\Lambda, \mathcal{C}_\Lambda, \omega_\Lambda^0\right)$, where

$$\omega_\Lambda^0 (\mathbf{A}_\Lambda) := tr_{\mathcal{H}_\Lambda} (\mathbf{A}_\Lambda) = \sum_{\emptyset\subseteq\gamma\subseteq\Lambda} \langle\gamma| \mathbf{A}_\Lambda |\gamma\rangle_\Lambda$$

or considering the equivalent representation of $\left(\mathcal{H}_\Lambda,\mathcal{U}_\Lambda,\mathcal{C}_\Lambda,\omega^0_\Lambda\right)$ induced by $\mathbf{U}_\Lambda$

$$\begin{aligned}\bar{\mathbf{A}}_\Lambda &= \mathbf{U}_\Lambda\mathbf{A}_\Lambda\mathbf{U}_\Lambda \quad \mathbf{A}_\Lambda\in\mathcal{U}_\Lambda\\ |0\rangle_\Lambda &= \mathbf{U}_\Lambda\,|\emptyset\rangle_\Lambda = \frac{1}{2^{\frac{|\Lambda|}{2}}}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}|\alpha\rangle_\Lambda\\ \omega^0_\Lambda(\mathbf{A}_\Lambda) &= 2^{|\Lambda|}\,\langle 0|\,\bar{\mathbf{A}}_\Lambda\,|0\rangle_\Lambda = 2^{|\Lambda|}\,\langle\emptyset|\,\mathbf{A}_\Lambda\,|\emptyset\rangle_\Lambda\end{aligned}$$

2. Interacting spin systems

We will always consider a Λ-subsystem $\Lambda\subset\mathbb{Z}^d$, $|\Lambda|<\infty$. The interaction among spins is defined by the Hamiltonian operator $\mathbf{H}_\Lambda\in\mathcal{C}_\Lambda$, namely

$$\mathbf{H}_\Lambda := \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} J_\alpha\boldsymbol{\sigma}_\alpha \qquad \{J_\alpha\}_{\alpha\subseteq\Lambda}\in\mathbb{R} \tag{1}$$

where we can assume, for sake of simplicity, translational invariance [QMIPS I], i.e.

$$\tau_x J_\alpha\boldsymbol{\sigma}_\alpha = J_{\alpha+x}\boldsymbol{\sigma}_{\alpha+x} \quad \forall\alpha\subseteq\Lambda$$

even if, in the following, the formalism we will develop is not affected by this assumption and it can be used to describe more general constructions as shown in the examples given in the next section.

Let's consider also

$$\overline{\mathbf{H}}_\Lambda = \mathbf{U}_\Lambda\mathbf{H}_\Lambda\mathbf{U}_\Lambda = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} J_\alpha\mathbf{f}_\alpha$$

which is a diagonal operator on $\mathcal{H}_\Lambda$

$$\begin{aligned}\overline{\mathbf{H}}_\Lambda\,|\gamma\rangle_\Lambda &= \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} J_\alpha\,(-1)^{|\alpha\cap\gamma|}\,|\gamma\rangle_\Lambda = H_\gamma\,|\gamma\rangle_\Lambda\\ H_\gamma &= \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} J_\alpha\,(-1)^{|\alpha\cap\gamma|} \quad \forall\gamma\subseteq\Lambda\\ \overline{\mathbf{H}}_\Lambda &= \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} H_\alpha\,|\gamma\rangle\,\langle\gamma|_\Lambda\end{aligned} \tag{2}$$

We are now able to compute the partition function

$$\begin{aligned}Z^{(d)}_\Lambda(\beta) \quad &: = tr_{\mathcal{H}_\Lambda}\left(e^{\beta\mathbf{H}_\Lambda}\right) = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-\beta H_\alpha}\\ Z^{(d)}_\Lambda(\beta=0) &= 2^{|\Lambda|}\end{aligned}$$

The density operator defining the Gibbs state is

$$\begin{aligned}\bar{\boldsymbol{\rho}}^{(d)}_\Lambda(\beta) \quad &: = \frac{1}{Z^{(d)}_\Lambda(\beta)}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-\beta H_\alpha}\,|\alpha\rangle\,\langle\alpha|_\Lambda\\ \bar{\boldsymbol{\rho}}^{(d)}_\Lambda(\beta=0) &= \mathbf{I}_\Lambda\end{aligned}$$

and the Gibbs average of any cylinder observable $\boldsymbol{\sigma}_\gamma;\ \gamma \subseteq \Lambda$ is

$$\begin{aligned}\omega_\Lambda^\beta(\boldsymbol{\sigma}_\gamma) \quad &: \quad = \omega_\Lambda^0\left(\boldsymbol{\rho}_\Lambda^{(d)}(\beta)\,\boldsymbol{\sigma}_\gamma\right) = tr_{\mathcal{H}_\Lambda}\left(\frac{e^{\beta \mathbf{H}_\Lambda}}{Z_\Lambda^{(d)}(\beta)}\boldsymbol{\sigma}_\gamma\right) = \\ &= \frac{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-\beta H_\alpha}(-1)^{|\gamma\cap\alpha|}}{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-\beta H_\alpha}} = 1 - 2\frac{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-\beta H_\alpha}\delta(|\gamma\cap\alpha| = 2n+1; n\in\mathbb{N})}{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-\beta H_\alpha}}\end{aligned}$$

where $\delta(*)$ denotes the characteristic function of its argument. Note that by the definition of $\mathcal{C}_\Lambda$ any observable is just a linear combination of cylinder operators.

So far we rewrote known quantities making use of this formalism to show its usefulness, we will now compute explicitly the eigenvalue of the Hamiltonian operators of some well known model of interacting spin systems, which appear also in [QMIPS II]. For sake of simplicity, in the following, all the calculations will be carried out assuming periodic boundary conditions unless differently specified.

Example 2.1. Ising model without external source.

$$\mathbf{H}_\Lambda = -\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\delta(d(\alpha)=1)\,\boldsymbol{\sigma}_\alpha \tag{3}$$

where $d(\alpha)$ indicates the diameter of any lattice subset α. From (2) we have

$$\begin{aligned} H_\gamma &= -\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\delta(d(\alpha)=1)(-1)^{|\alpha\cap\gamma|} = \\ &= -\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\delta(d(\alpha)=1)\left[1-2\delta(|\alpha\cap\gamma| = 2n+1; n\in\mathbb{N})\right] = \\ &= -d|\Lambda| + 2|\partial\gamma| \qquad \forall\gamma\subseteq\Lambda\end{aligned} \tag{4}$$

$d|\Lambda|$ indicates the number of dimers (i.e. connected sets made by two lattice points) in Λ, and $|\partial\gamma|$ the number of ways a dimer can intersect, with a single site only, sites belonging to γ. These sites constitutes the inner boundary of γ, so $|\partial\gamma|$ is the cardinality of the hypersurface bounding γ. For example, if $d=2$, then $|\partial\gamma|$ is the length of the contour surrounding γ. Further results on this model will be given in the next section.

Example 2.2. Dobrushin-Gertsik model *(*[Ge]*). The lattice is $\mathbb{Z}^2$.*

$$\mathbf{H}_\Lambda^{DG}(h, J_1, J_2) = -\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\left[J_1\delta(d(\alpha)=1) + J_\alpha + h\delta(d(\alpha)=0)\right]\boldsymbol{\sigma}_\alpha$$

$$J_\alpha = J_2\delta\left(\alpha = \alpha_1\cup\alpha_2 : d(\alpha_1) = d(\alpha_2) = 0\ ;\ dist(\alpha_1,\alpha_2) = \sqrt{2}\right)$$

where $\delta(d(\alpha)=0)$ means α to be a single lattice site and $h, J_1, J_2 \in \mathbb{R}$. Since

$$\left|\left\{\alpha : \alpha = \alpha_1\cup\alpha_2 : d(\alpha_1) = d(\alpha_2) = 0; dist(\alpha_1,\alpha_2) = \sqrt{2}\right\}\right| = 2|\Lambda|$$

$\forall\gamma\subseteq\Lambda$ *we have*

$$
\begin{aligned}
H_{\gamma}^{DG}(h,J_1,J_2) &= (-2|\Lambda|+2|\partial\gamma|)\,J_1+(-2|\Lambda|+2|\not\partial\gamma|)\,J_2+(-|\Lambda|+2|\gamma|)\,h\\
|\not\partial\gamma| &= \sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\delta\left(\alpha=\alpha_1\cup\alpha_2:d(\alpha_1)=d(\alpha_2)=0\ ;\ dist(\alpha_1,\alpha_2)=\sqrt{2}\right)\times\\
&\qquad\times\delta(|\alpha\cap\gamma|=1)
\end{aligned}
$$

Example 2.3. Ising model with second neighbors antiferromagnetic interaction in an external source. *The lattice is* $\mathbb{Z}^d$

$$
\begin{aligned}
\mathbf{H}_{\Lambda}(h,J) &= -\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\left[\delta(d(\alpha)=1)+J_{\alpha}+h\delta(d(\alpha)=0)\right]\boldsymbol{\sigma}_{\alpha}\\
J_{\alpha} &= -J\delta(\alpha=\alpha_1\cup\alpha_2:d(\alpha_1)=d(\alpha_2)=0\ ;\ dist(\alpha_1,\alpha_2)=2)
\end{aligned}
$$

$h,J>0$. *The associated eigenvalues are* $\forall\gamma\subseteq\Lambda$

$$
\begin{aligned}
H_{\gamma}(h,J) &= (-d|\Lambda|+2|\partial\gamma|)(1-J)-h|\Lambda|+2h|\gamma|-2JY_{\gamma}\\
Y_{\gamma} &= \sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\delta(\alpha=\alpha_1\cup\alpha_2:d(\alpha_1)=d(\alpha_2)=0\ ;\ dist(\alpha_1,\alpha_2)=2)\times\\
&\qquad\times\delta(|\alpha\cap\gamma|=1)\left[\delta(\alpha_1\subset\subset\gamma)\,\delta(\alpha_2\cap\gamma=\emptyset)+\delta(\alpha_2\subset\subset\gamma)\,\delta(\alpha_1\cap\gamma=\emptyset)\right]
\end{aligned}
$$

where $\delta(\alpha_i\subset\subset\gamma)=\delta(\alpha_i\subset\gamma)\,\delta(\alpha_i\cap\partial\gamma=\emptyset)\quad i=1,2$. *Therefore* $Y_{\gamma}\leq 2|\partial\gamma|\quad\forall\gamma\subseteq\Lambda$.

These examples show that the computation of the eigenvalues of the Hamiltonian operator of a model is reduced to a combinatorial argument. Nevertheless, even if the explicit form of such eigenvalues is known, it is insufficient for the computation of the partition function. This requires also the knowledge of the multiplicity of such eigenvalues which is often very hard to compute.

3. Ising model

Let's consider first the Hamiltonian operator describing the *nearest neighbor interaction* Ising model in absence of any external source from **Example 2.1**

$$
\begin{aligned}
\mathbf{H}_{\Lambda} &= -\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\delta(d(\alpha)=1)\,\boldsymbol{\sigma}_{\alpha}\\
H_{\gamma} &= -d|\Lambda|+2|\partial\gamma|\qquad\forall\gamma\subseteq\Lambda
\end{aligned}
$$

The partition function now reads

$$
\begin{aligned}
Z_{\Lambda}^{(d)}(\beta) &= \sum_{\emptyset\subseteq\gamma\subseteq\Lambda}e^{\beta(d|\Lambda|-2|\partial\gamma|)}=\\
&= e^{\beta d|\Lambda|}\sum_{0\leq|\partial\gamma|\leq d|\Lambda|}m(|\partial\gamma|)\,e^{-2\beta|\partial\gamma|}
\end{aligned}
\tag{5}
$$

where $m(|\partial\gamma|)$ is the multiplicity of the Hamiltonian operator's eigenvalues, that is the number of lattice's subsets whose *boundary hypersurface* have the same cadinality. The maximum value for $|\partial\gamma|$ is reached considering the *chessboard* set whose associated eigenvalue H_γ has multiplicity 2. The expression for the Gibbs density operator is

$$\begin{aligned}(6)\qquad \bar{\boldsymbol{\rho}}_\Lambda^{(d)}(\beta) &= \frac{1}{Z_\Lambda^{(d)}(\beta)}\sum_{\emptyset\subseteq\gamma\subseteq\Lambda} e^{\beta(d|\Lambda|-2|\partial\gamma|)}\,|\gamma\rangle\langle\gamma|_\Lambda = \\ &= \frac{e^{\beta d|\Lambda|}}{Z_\Lambda^{(d)}(\beta)}\sum_{0\leq|\partial\alpha|\leq d|\Lambda|}\ \sum_{\{\gamma\subseteq\Lambda:|\partial\gamma|=|\partial\alpha|\}} e^{-2\beta|\partial\gamma|}\,|\gamma\rangle\langle\gamma|_\Lambda\end{aligned}$$

and for any cylinder observable we have

$$\begin{aligned}\omega_\Lambda^\beta(\boldsymbol{\sigma}_\gamma) &= \frac{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-2\beta|\partial\alpha|}(-1)^{|\gamma\cap\alpha|}}{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-2\beta|\partial\alpha|}} = \\ &= 1-2\frac{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-2\beta|\partial\alpha|}\delta(|\gamma\cap\alpha|=2n+1;n\in\mathbb{N})}{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-2\beta|\partial\alpha|}}\end{aligned}$$

Adding a constant external magnetic field $h\in\mathbb{R}$ on the whole lattice, from (3) we get

$$\begin{aligned}\mathbf{H}_\Lambda(h) &= -\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\left[\delta(d(\alpha)=1)+h\delta(d(\alpha)=0)\right]\boldsymbol{\sigma}_\alpha \\ H_\gamma(h) &= -\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\left[\delta(d(\alpha)=1)+h\delta(d(\alpha)=0)\right]\times \\ &\qquad\times\left[1-2\delta(|\alpha\cap\gamma|=2n+1;n\in\mathbb{N})\right] \\ &= -d|\Lambda|+2|\partial\gamma|-h|\Lambda|+2h|\gamma| \qquad \forall\gamma\subseteq\Lambda\end{aligned}$$

which leads to the following expression of the partition function and Gibbs density operator

$$\begin{aligned}(7)\qquad Z_\Lambda^{(d)}(\beta,h) &= e^{\beta(d+h)|\Lambda|}\sum_{\emptyset\subseteq\gamma\subseteq\Lambda} e^{-2\beta(|\partial\gamma|+h|\gamma|)} \\ \bar{\boldsymbol{\rho}}_\Lambda^{(d)}(\beta,h) &= \frac{e^{\beta(d+h)|\Lambda|}}{Z_\Lambda^{(d)}(\beta,h)}\sum_{\emptyset\subseteq\gamma\subseteq\Lambda} e^{-2\beta(|\partial\gamma|+h|\gamma|)}\,|\gamma\rangle\langle\gamma|_\Lambda\end{aligned}$$

and also $\forall\boldsymbol{\sigma}_\gamma\in\mathcal{C}_\Lambda;\ \gamma\subseteq\Lambda$

$$\begin{aligned}\omega_\Lambda^{\beta,h}(\boldsymbol{\sigma}_\gamma) &= \frac{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-2\beta(|\partial\alpha|+h|\alpha|)}(-1)^{|\gamma\cap\alpha|}}{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-2\beta(|\partial\alpha|+h|\alpha|)}} = \\ &= 1-2\frac{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-2\beta(|\partial\alpha|+h|\alpha|)}\delta(|\alpha\cap\gamma|=2n+1;n\in\mathbb{N})}{\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-2\beta(|\partial\alpha|+h|\alpha|)}}\end{aligned}$$

Remark 3.1. One-dimensional Ising model. *Consider the lattice $\mathbb{Z}^1$. By simple combinatorial arguments we have $m(|\partial\gamma|)=2\binom{|\Lambda|}{|\partial\gamma|}$ (the reader may convince himself computing, for example, the number of lattice subsets having cardinality of* *their boundary equal 2). From (5) we get*

$$\begin{aligned}
Z_\Lambda^{(1)}(\beta) &= e^{\beta|\Lambda|}\sum_{k=0}^{|\Lambda|}\sum_{\{\emptyset\subseteq\gamma\subseteq\Lambda:|\partial\gamma|=k\}} e^{-2\beta k}\\
&= e^{\beta|\Lambda|}\sum_{0\le|\partial\gamma|\le|\Lambda|} m(|\partial\gamma|)\, e^{-2\beta|\partial\gamma|}\\
&= 2e^{\beta|\Lambda|}\sum_{0\le|\partial\gamma|\le|\Lambda|}\binom{|\Lambda|}{|\partial\gamma|} e^{-2\beta|\partial\gamma|}\\
&= 2e^{\beta|\Lambda|}\left(1+e^{-2\beta}\right)^{|\Lambda|} = 2^{|\Lambda|+1}(\cosh\beta)^{|\Lambda|}
\end{aligned}$$

Adding an external field, since the number of sets in Λ with cardinality $|\gamma|$ are $\binom{|\Lambda|}{|\gamma|}$, from (7) we have

$$\begin{aligned}
Z_\Lambda^{(1)}(\beta,h) &= e^{\beta|\Lambda|(1+h)}\sum_{\emptyset\subseteq\gamma\subseteq\Lambda} e^{-2\beta(|\partial\gamma|+h|\gamma|)} =\\
&= e^{\beta|\Lambda|(1+h)}\left\{\sum_{n=1}^{\frac{|\Lambda|}{2}} 2\binom{|\Lambda|}{2n} e^{-4\beta n}\sum_{k=n}^{|\Lambda|-n} e^{-2\beta hk} + 1 + e^{-2\beta h|\Lambda|}\right\}
\end{aligned}$$

3.1. Geometric duality and percolation mapping. The expression (5) is similar to the *low temperature* (LT) expansion for the partition function of the Ising model obtained by geometric duality arguments [S]. On the other hand the computation of (5) does not involve any argument of phase separation in the configuration space as in the case of the computation of the *high temperature* (HT) expansion of $Z_\Lambda^{(d)}(\beta)$. In the following we will show our approach to give a description of the configuration space of two-state spin systems in terms of the geometric properties of the underlying lattice structure in a more complete way than that given in [GHM]. This will be done deriving the expression for $Z_\Lambda^{(d)}(\beta)$ in respectively the HT, LT and Fortuin-Kasteleyn (FK) representation from one another.

3.1.1. **HT expression of $Z_\Lambda^{(d)}(\beta)$.** Let $\Delta_\Lambda := \{\gamma\subseteq\Lambda : \delta(d(\gamma)=1)\}$ be the set of dimers in Λ, then $|\Delta_\Lambda| = d|\Lambda|$. We define $\mathfrak{P}_{\Delta_\Lambda}$ as the set of collections of elements of $\Delta_\Lambda\cup\emptyset$ and $\forall\Gamma\in\mathfrak{P}_{\Delta_\Lambda}$ $|\Gamma|$ will denote the number of dimers forming Γ. Let's consider also the maps from $\mathfrak{P}_{\Delta_\Lambda}$ to $\mathfrak{P}_\Lambda$ (the set of the parts of Λ [QMIPS I]) given by the symmetric difference and the union of sets $(\triangle,\cup)$ such that

$$\forall\Gamma\in\mathfrak{P}_{\Delta_\Lambda}\quad (\triangle_{\gamma\in\Gamma}\,\gamma),(\cup_{\gamma\in\Gamma}\,\gamma)\in\mathfrak{P}_\Lambda$$

Then

$$\begin{aligned}
Z_\Lambda^{(d)}(\beta) &= \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{\beta\sum_{\emptyset\subseteq\gamma\subseteq\Lambda}\delta(d(\gamma)=1)(-1)^{|\alpha\cap\gamma|}} =\\
&= \sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\prod_{\gamma\in\Delta_\Lambda}\left[\cosh\beta+(-1)^{|\alpha\cap\gamma|}\sinh\beta\right] =\\
&= (\cosh\beta)^{d|\Lambda|}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\prod_{\gamma\in\Delta_\Lambda}\left[1+(-1)^{|\alpha\cap\gamma|}\tanh\beta\right] \qquad (8)
\end{aligned}$$

Since $\forall\alpha,\gamma,\eta\subseteq\Lambda$ $(-1)^{|\alpha\cap\gamma|+|\alpha\cap\eta|}=(-1)^{|\alpha\cap(\gamma\Delta\eta)|}$ and $\delta_{\gamma,\emptyset}=\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\frac{(-1)^{|\alpha\cap\gamma|}}{2^{|\Lambda|}}$ [QMIPS I] than (8) becomes

$$\begin{aligned} Z_\Lambda^{(d)}(\beta) &= (\cosh\beta)^{d|\Lambda|}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\left[1+\sum_{\Gamma\subseteq\Delta_\Lambda}(-1)^{|\alpha\cap\triangle_{\gamma\in\Gamma}\gamma|}(\tanh\beta)^{|\Gamma|}\right]= \\ &= 2^{|\Lambda|}(\cosh\beta)^{d|\Lambda|}\left[1+\sum_{\Gamma\subseteq\Delta_\Lambda}\delta\left(\triangle_{\gamma\in\Gamma}\gamma=\emptyset\right)(\tanh\beta)^{|\Gamma|}\right]= \\ &= 2^{|\Lambda|}e^{d\beta|\Lambda|}\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}\delta\left(\triangle_{\gamma\in\Gamma}\gamma=\emptyset\right)\left(\frac{1-e^{-2\beta}}{2}\right)^{|\Gamma|}\left(\frac{1+e^{-2\beta}}{2}\right)^{d|\Lambda|-|\Gamma|} \end{aligned}$$

Setting $p_{\mathrm{HT}}(\beta)=\left(\frac{1-e^{-2\beta}}{2}\right)$ and $q_{\mathrm{HT}}(\beta)=\left(\frac{1+e^{-2\beta}}{2}\right)$, we can rewrite the last expression above in terms of a Bernoulli sum on those sets in $\mathfrak{P}_{\Delta_\Lambda}$ whose image in $\mathfrak{P}_\Lambda$ under the operation of symmetric difference is the emptyset, that is their image in $\mathfrak{P}_\Lambda$ under the operation of union of sets is a closed loop (HT graphs). Then

$$Z_\Lambda^{(d)}(\beta)=2^{|\Lambda|}e^{d\beta|\Lambda|}\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}\delta\left(\triangle_{\gamma\in\Gamma}\gamma=\emptyset\right)p_{\mathrm{HT}}^{|\Gamma|}(\beta)\,q_{\mathrm{HT}}^{d|\Lambda|-|\Gamma|}(\beta) \tag{9}$$

The Gibbs density operator can now be rewritten as

$$\bar{\boldsymbol{\rho}}_\Lambda^{(d)}(\beta)=\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\frac{\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}(-1)^{|\alpha\cap\triangle_{\gamma\in\Gamma}\gamma|}p_{\mathrm{HT}}^{|\Gamma|}(\beta)\,q_{\mathrm{HT}}^{d|\Lambda|-|\Gamma|}(\beta)}{2^{|\Lambda|}\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}\delta\left(\triangle_{\gamma\in\Gamma}\gamma=\emptyset\right)p_{\mathrm{HT}}^{|\Gamma|}(\beta)\,q_{\mathrm{HT}}^{d|\Lambda|-|\Gamma|}(\beta)}\left|\alpha\right\rangle\left\langle\alpha\right|_\Lambda$$

Then for any cylinder observable $\boldsymbol{\sigma}_\eta\in\mathcal{C}_\Lambda$; $\eta\subseteq\Lambda$

$$\begin{aligned} \omega_\Lambda^\beta(\boldsymbol{\sigma}_\eta) &= \frac{\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}p_{\mathrm{HT}}^{|\Gamma|}(\beta)\,q_{\mathrm{HT}}^{d|\Lambda|-|\Gamma|}(\beta)\,\frac{1}{2^{|\Lambda|}}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}(-1)^{|\alpha\cap\triangle_{\gamma\in\Gamma}\gamma|}(-1)^{|\alpha\cap\eta|}}{\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}\delta\left(\triangle_{\gamma\in\Gamma}\gamma=\emptyset\right)p_{\mathrm{HT}}^{|\Gamma|}(\beta)\,q_{\mathrm{HT}}^{d|\Lambda|-|\Gamma|}(\beta)}= \\ &= \frac{\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}p_{\mathrm{HT}}^{|\Gamma|}(\beta)\,q_{\mathrm{HT}}^{d|\Lambda|-|\Gamma|}(\beta)\,\frac{1}{2^{|\Lambda|}}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}(-1)^{|\alpha\cap(\triangle_{\gamma\in\Gamma}\gamma)\Delta\eta|}}{\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}\delta\left(\triangle_{\gamma\in\Gamma}\gamma=\emptyset\right)p_{\mathrm{HT}}^{|\Gamma|}(\beta)\,q_{\mathrm{HT}}^{d|\Lambda|-|\Gamma|}(\beta)}= \\ &= \frac{\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}\delta\left(\triangle_{\gamma\in\Gamma}\gamma=\eta\right)p_{\mathrm{HT}}^{|\Gamma|}(\beta)\,q_{\mathrm{HT}}^{d|\Lambda|-|\Gamma|}(\beta)}{\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}\delta\left(\triangle_{\gamma\in\Gamma}\gamma=\emptyset\right)p_{\mathrm{HT}}^{|\Gamma|}(\beta)\,q_{\mathrm{HT}}^{d|\Lambda|-|\Gamma|}(\beta)} \end{aligned}$$

Remark 3.2. *Since* $\forall\gamma\in\Delta_\Lambda$, $\forall\alpha\subseteq\Lambda$ $(-1)^{|\alpha\cap\gamma|}=1-2\delta\left(|\alpha\cap\gamma|=2n+1;n\in\mathbb{N}\right)$ *and* $q_{HT}(\beta)-p_{HT}(\beta)=e^{-2\beta}$, *we have*

$$\begin{aligned}
Z^d_\Lambda(\beta) &= \sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\prod_{\gamma\in\Delta_\Lambda}\left[\cosh\beta+(-1)^{|\alpha\cap\gamma|}\sinh\beta\right] \\
&= \sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\prod_{\gamma\in\Delta_\Lambda}\left[\cosh\beta+\sinh\beta-2\delta\left(|\alpha\cap\gamma|=1\right)\sinh\beta\right]= \\
&= e^{d\beta|\Lambda|}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\prod_{\gamma\in\Delta_\Lambda}\left[1-\left(1-e^{-2\beta}\right)\delta\left(|\alpha\cap\gamma|=1\right)\right]= \\
&= e^{d\beta|\Lambda|}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}e^{-2\beta|\partial\alpha|}= \\
&= e^{\beta d|\Lambda|}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\left(q_{HT}(\beta)-p_{HT}(\beta)\right)^{|\partial\alpha|}\left(p_{HT}(\beta)+q_{HT}(\beta)\right)^{d|\Lambda|-|\partial\alpha|}
\end{aligned}$$

which is exactly (5).

Remark 3.3. *Consider the* $\mathbb{Z}^2$ *lattice. Choose in the computation of (9) a finite cube* $\Lambda:|\Lambda|=L^2$ *with free b.c., and in the computation of (5) a finite cube* Λ' : $|\Lambda'|=(L-1)^2$ *with + b.c. Moreover, if we set* $\sinh 2\beta\sinh 2\beta'=1$, *from the two expressions for the partition function given before we get* $\frac{Z^{(2),free}_\Lambda(\beta)}{2^{|\Lambda|}}=Z^{(2),+}_{\Lambda'}(\beta')$. *This implies that the set of HT graphs in* Λ *coincides with the set of LT graphs in* Λ'. *This argument goes under the name of self-duality* [S].

3.1.2. **FK expression for** $Z^{(d)}_\Lambda(\beta)$. Fortuin-Kasteleyn (FK) representation [FK] allow the description of two-state systems statistical mechanics in terms of *bond percolation processes.* Here we will derive it in a different way. Consider the graph (Λ,Δ_Λ), from (4) and (5) follows

$$\begin{aligned}
e^{-2\beta|\partial\alpha|} &= \sum_{\emptyset\subseteq\alpha\subseteq\Lambda}e^{-2\beta\sum_{\gamma\in\Delta_\Lambda}\delta(|\alpha\cap\gamma|=1)}= \\
e^{-2\beta|\partial\alpha|} &= e^{-2\beta\sum_{\gamma\in\Delta_\Lambda\setminus\bar\gamma}\delta(|\alpha\cap\gamma|=1)}\left[\left(1-\delta\left(|\alpha\cap\bar\gamma|=1\right)\right)+e^{-2\beta}\delta\left(|\alpha\cap\bar\gamma|=1\right)\right]\quad\forall\alpha\subseteq\Lambda
\end{aligned}$$

where $e^{-2\beta\sum_{\gamma\in\Delta_\Lambda\setminus\bar\gamma}\delta(|\alpha\cap\gamma|=1)}$ is $e^{-2\beta|\partial\alpha|}$ calculated on the graph $(\Lambda,\Delta_\Lambda\setminus\bar\gamma)$. Setting

$$\left\{\begin{array}{c} p_{\mathrm{FK}}(\beta)=2p_{\mathrm{HT}}(\beta)=\left(1-e^{-2\beta}\right)\\ q_{\mathrm{FK}}(\beta)=q_{\mathrm{HT}}(\beta)-p_{\mathrm{HT}}(\beta)=e^{-2\beta}\end{array}\right.\tag{10}$$

we obtain

$$e^{-2\beta|\partial\alpha|}=e^{-2\beta\sum_{\gamma\in\Delta_\Lambda\setminus\bar\gamma}\delta(|\alpha\cap\gamma|=1)}\left[\left(1-\delta\left(|\alpha\cap\bar\gamma|=1\right)\right)p_{\mathrm{FK}}(\beta)+q_{\mathrm{FK}}(\beta)\right]$$

Iterating such expression until all the dimers have been deleted, since

$$\left(1-\delta\left(|\alpha\cap\eta|=1\right)\right)=\frac{\left(1+(-1)^{|\alpha\cap\gamma|}\right)}{2}\quad\forall\gamma\in\Delta_\Lambda;\ \forall\alpha\subseteq\Lambda$$

we have

$$e^{-2\beta|\partial\alpha|}=\prod_{\gamma\in\Delta_\Lambda}\left[\frac{\left(1+(-1)^{|\alpha\cap\gamma|}\right)}{2}p_{\mathrm{FK}}(\beta)+q_{\mathrm{FK}}(\beta)\right] \tag{11}$$

Defining $\mathfrak{P}_\Gamma\subseteq\mathfrak{P}_{\Delta_\Lambda}$ as the set of collection of dimers in Γ and restricting $\triangle,\cup:\mathfrak{P}_{\Delta_\Lambda}\to\mathfrak{P}_\Lambda$ to $\mathfrak{P}_\Gamma$, such that $\forall\Gamma\in\mathfrak{P}_{\Delta_\Lambda},\forall c\in\mathfrak{P}_\Gamma\quad(\triangle_{\gamma\in c}\,\gamma)\in\mathfrak{P}_\Lambda$, we get

$$e^{-2\beta|\partial\alpha|}=q_{\mathrm{FK}}^{d|\Lambda|}+\sum_{\Gamma\subseteq\Delta_\Lambda}\frac{p_{\mathrm{FK}}^{|\Gamma|}(\beta)\,q_{\mathrm{FK}}^{d|\Lambda|-|\Gamma|}(\beta)}{2^{|\Gamma|}}\left(1+\sum_{c\subseteq\Gamma}(-1)^{|\alpha\cap\triangle_{\gamma\in c}\,\gamma|}\right)$$

where $c\in\mathfrak{P}_\Gamma$. Then

$$\begin{aligned}Z_\Lambda^{(d)}(\beta)&=e^{d\beta|\Lambda|}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}\frac{p_{\mathrm{FK}}^{|\Gamma|}(\beta)\,q_{\mathrm{FK}}^{d|\Lambda|-|\Gamma|}(\beta)}{2^{|\Gamma|}}\left(1+\sum_{\emptyset\neq c\subseteq\Gamma}(-1)^{|\alpha\cap\triangle_{\gamma\in c}\,\gamma|}\right)\\&=2^{|\Lambda|}e^{d\beta|\Lambda|}\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}p_{\mathrm{FK}}^{|\Gamma|}(\beta)\,q_{\mathrm{FK}}^{d|\Lambda|-|\Gamma|}(\beta)\,2^{-|\Gamma|}\left[1+\sum_{\emptyset\neq c\subseteq\Gamma}\delta\left(\triangle_{\gamma\in c}\,\gamma=\emptyset\right)\right]\end{aligned} \tag{12}$$

which is, in a similar fashion to (9), a sum with Bernoulli weights on the elements of $\mathfrak{P}_{\Delta_\Lambda}$ whose terms involve closed loops in $\mathfrak{P}_\Lambda$.

Remark 3.4. *To obtain the full equivalence of (12) with the FK representation of the model we need to give some more definitions.* $\forall\Gamma\in\mathfrak{P}_{\Delta_\Lambda}$ *two lattice sites* x,y *are said to be connected in* Γ *if either* $x,y\in\gamma:\gamma\in\Gamma$ *or* $\{x,y\}=\triangle_{\gamma\in c}\,\gamma$ *for some* $c\in\mathfrak{P}_\Gamma$. *We will denote by* $\mathfrak{C}_\Gamma$ *the set of connected components of* Γ *and by* C_Γ *each connected component of* Γ *(*Γ*-clusters). The coefficient* 2^{N_Γ} *appearing in* [FK], *where* N_Γ *is the number of cluster in* Γ, *is then just* $2^{|\Lambda|-|\Gamma|}\left[1+\sum_{\emptyset\neq c\subseteq\Gamma}\delta\left(\triangle_{\gamma\in c}\,\gamma=\emptyset\right)\right]$. *This can be easily checked computing (12) for simple graphs in both ways.*

Since

$$\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}p_{\mathrm{FK}}^{|\Gamma|}(\beta)\,q_{\mathrm{FK}}^{d|\Lambda|-|\Gamma|}(\beta)=\sum_{|\Gamma|=0}^{d|\Lambda|}\binom{d\,|\Lambda|}{|\Gamma|}p_{\mathrm{FK}}^{|\Gamma|}(\beta)\,q_{\mathrm{FK}}^{d|\Lambda|-|\Gamma|}(\beta)=1$$

we can build on Δ_Λ a *bond percolation process* with *bond probability* $p_{\mathrm{FK}}(\beta)$ defining the expectation

$$M_{\beta,\Lambda}^{\mathrm{FK}}[f]:=\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset}p_{\mathrm{FK}}^{|\Gamma|}(\beta)\,q_{\mathrm{FK}}^{d|\Lambda|-|\Gamma|}(\beta)\,f_\Gamma$$

for any random variable $f:\Delta_\Lambda\to\mathbb{R}$.. Thus

$$Z_\Lambda^{(d)}(\beta)=e^{d\beta|\Lambda|}M_{\beta,\Lambda}^{\mathrm{FK}}\left[2^{|\Lambda|-|\Gamma|}\left[1+\sum_{\emptyset\neq c\subseteq\Gamma}\delta\left(\triangle_{\gamma\in c}\,\gamma=\emptyset\right)\right]\right]$$

(6) can be replaced by

$$\bar{\boldsymbol{\rho}}_{\Lambda}^{(d)}(\beta) = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} \frac{\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset} p_{\mathrm{FK}}^{|\Gamma|}(\beta)\, q_{\mathrm{FK}}^{d|\Lambda|-|\Gamma|}(\beta)\, 2^{-|\Gamma|}\left[1+\sum_{\emptyset\neq c\subseteq\Gamma}(-1)^{|\alpha\cap\Delta_{\gamma\in c}\,\gamma|}\right]}{2^{|\Lambda|}\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset} p_{\mathrm{FK}}^{|\Gamma|}(\beta)\, q_{\mathrm{FK}}^{d|\Lambda|-|\Gamma|}(\beta)\, 2^{-|\Gamma|}\left[1+\sum_{\emptyset\neq c\subseteq\Gamma}\delta\left(\triangle_{\gamma\in c}\ \gamma=\emptyset\right)\right]} \left|\alpha\right\rangle\left\langle\alpha\right|_{\Lambda} =$$

$$= \sum_{\emptyset\subseteq\alpha\subseteq\Lambda} \frac{M_{\beta,\Lambda}^{\mathrm{FK}}\left[2^{-|\Gamma|}\left[1+\sum_{\emptyset\neq c\subseteq\Gamma}(-1)^{|\alpha\cap\Delta_{\gamma\in c}\,\gamma|}\right]\right]}{M_{\beta,\Lambda}^{\mathrm{FK}}\left[2^{|\Lambda|-|\Gamma|}\left[1+\sum_{\emptyset\neq c\subseteq\Gamma}\delta\left(\triangle_{\gamma\in c}\ \gamma=\emptyset\right)\right]\right]} \left|\alpha\right\rangle\left\langle\alpha\right|_{\Lambda}$$

and $\forall\boldsymbol{\sigma}_\eta\in\mathcal{C}_\Lambda;\ \eta\subseteq\Lambda$ we get

$$\omega_{\Lambda}^{\beta}(\boldsymbol{\sigma}_\eta) = \frac{\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset} p_{\mathrm{FK}}^{|\Gamma|}(\beta)\, q_{\mathrm{FK}}^{d|\Lambda|-|\Gamma|}(\beta)\, 2^{-|\Gamma|}\frac{1}{2^{|\Lambda|}}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\left[1+\sum_{\emptyset\neq c\subseteq\Gamma}(-1)^{|\alpha\cap\Delta_{\gamma\in c}\,\gamma|}\right](-1)^{|\alpha\cap\eta|}}{\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset} p_{\mathrm{FK}}^{|\Gamma|}(\beta)\, q_{\mathrm{FK}}^{d|\Lambda|-|\Gamma|}(\beta)\, 2^{-|\Gamma|}\left[1+\sum_{\emptyset\neq c\subseteq\Gamma}\delta\left(\triangle_{\gamma\in c}\ \gamma=\emptyset\right)\right]} =$$

$$= \frac{\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset} p_{\mathrm{FK}}^{|\Gamma|}(\beta)\, q_{\mathrm{FK}}^{d|\Lambda|-|\Gamma|}(\beta)\, 2^{-|\Gamma|}\left[\delta_{\eta,\emptyset}+\sum_{\emptyset\neq c\subseteq\Gamma}\delta\left(\triangle_{\gamma\in c}\ \gamma=\eta\right)\right]}{\sum_{\Gamma\subseteq\Delta_\Lambda\cup\emptyset} p_{\mathrm{FK}}^{|\Gamma|}(\beta)\, q_{\mathrm{FK}}^{d|\Lambda|-|\Gamma|}(\beta)\, 2^{-|\Gamma|}\left[1+\sum_{\emptyset\neq c\subseteq\Gamma}\delta\left(\triangle_{\gamma\in c}\ \gamma=\emptyset\right)\right]} =$$

$$= \frac{M_{\beta,\Lambda}^{\mathrm{FK}}\left[2^{|\Lambda|-|\Gamma|}\left[\delta_{\eta,\emptyset}+\sum_{\emptyset\neq c\subseteq\Gamma}\delta\left(\triangle_{\gamma\in c}\ \gamma=\eta\right)\right]\right]}{M_{\beta,\Lambda}^{\mathrm{FK}}\left[2^{|\Lambda|-|\Gamma|}\left[1+\sum_{\emptyset\neq c\subseteq\Gamma}\delta\left(\triangle_{\gamma\in c}\ \gamma=\emptyset\right)\right]\right]}$$

Remark 3.5. *We note that a similar description of the model in terms of a bond percolation process could also be derived by the HT representation considering as bond probability $p_{HT}(\beta)$ instead of $p_{FK}(\beta)$. Therefore*

$$Z_{\Lambda}^{(d)}(\beta) = 2^{|\Lambda|}e^{d\beta|\Lambda|}M_{\beta,\Lambda}^{HT}\left[\delta\left(\triangle_{\gamma\in\Gamma}\ \gamma=\emptyset\right)\right]$$

$$\bar{\boldsymbol{\rho}}_{\Lambda}^{(d)}(\beta) = \sum_{\emptyset\subseteq\alpha\subseteq\Lambda}\frac{M_{\beta,\Lambda}^{HT}\left[(-1)^{|\alpha\cap\Delta_{\gamma\in\Gamma}\,\gamma|}\right]}{2^{|\Lambda|}M_{\beta,\Lambda}^{HT}\left[\delta\left(\triangle_{\gamma\in\Gamma}\ \gamma=\emptyset\right)\right]}\left|\alpha\right\rangle\left\langle\alpha\right|_{\Lambda}$$

$$\omega_{\Lambda}^{\beta}(\boldsymbol{\sigma}_\eta) = \frac{M_{\beta,\Lambda}^{HT}\left[\delta\left(\triangle_{\gamma\in\Gamma}\ \gamma=\eta\right)\right]}{M_{\beta,\Lambda}^{HT}\left[\delta\left(\triangle_{\gamma\in\Gamma}\ \gamma=\emptyset\right)\right]}\qquad\forall\boldsymbol{\sigma}_\eta\in\mathcal{C}_\Lambda;\ \eta\subseteq\Lambda$$

Substituting (10) in (11) we have

$$\prod_{\gamma\in\Delta_\Lambda}\left[\left(1+(-1)^{|\alpha\cap\gamma|}\right)p_{HT}(\beta)+q_{HT}(\beta)-p_{HT}(\beta)\right]=\prod_{\gamma\in\Delta_\Lambda}\left[(-1)^{|\alpha\cap\gamma|}p_{HT}(\beta)+q_{HT}(\beta)\right]$$

multiplying by $e^{d\beta|\Lambda|}$ and summing over all $\emptyset\subseteq\alpha\subseteq\Lambda$ we get (9).

The discussion carried on in this section can be summarized by the following

Theorem 3.1. *If Λ is any finite subset of $\mathbb{Z}^d$, assuming periodic b.c., we can derive HT, LT and FK representation for the statistical mechanics of the Ising model on $\mathbb{Z}_2^\Lambda$, in absence of external magnetic fields, one from the other.*

3.1.3. *Proof.* From **Remark 3.2** (9) $\Longrightarrow$ (5); (5) $\Longrightarrow$ (12) follows from (11). Finally from **Remark 3.5** (12) $\Longrightarrow$ (9). ■

So far we have considered only homogeneous two-body interactions, but we can extend the preceding analysis to multibody and inhomogeneous potentials. A straightforward extension is given by the inhomogeneous ferromagnetic Ising model defined trough the Hamiltonian operator

$$\begin{aligned} \mathbf{H}_{\Lambda}(J) \quad &: \quad = -\sum_{\gamma\in\Delta_{\Lambda}} J_{\gamma}\boldsymbol{\sigma}_{\gamma} \qquad \{J_{\gamma}\}_{\gamma\in\Delta_{\Lambda}} \\ H_{\alpha} \quad &= \quad -\sum_{\gamma\in\Delta_{\Lambda}} J_{\gamma} + 2\sum_{\gamma\in\Delta_{\Lambda}} J_{\gamma}\delta\left(|\alpha\cap\gamma| = 1\right) \quad \forall\alpha\subseteq\Lambda \end{aligned}$$

The partition function, the Gibbs density operator and the expectation value of any cylinder observable over the Gibbs state for this model are given by simply substituting $e^{d\beta|\Lambda|}$ with $e^{\beta\sum_{\gamma\in\Delta_{\Lambda}}J_{\gamma}}$, $e^{-2\beta|\partial\alpha|}$ with $\prod_{\{\gamma\in\Delta_{\Lambda}:|\alpha\cap\gamma|=1\}} e^{-2\beta J_{\gamma}}$, $p_{\mathrm{HT}}^{|\Gamma|}(\beta)$ with $\prod_{\gamma\in\Gamma}\left(\frac{1-e^{-2\beta J_{\gamma}}}{2}\right)$ and $p_{\mathrm{FK}}^{|\Gamma|}(\beta)$ with $\prod_{\gamma\in\Gamma}\left(1-e^{-2\beta J_{\gamma}}\right)$ respectively in the LT, HT and FK expressions for $Z_{\Lambda}^{(d)}(\beta)$, $\boldsymbol{\rho}_{\Lambda}(\beta)$ and $\omega_{\Lambda}^{\beta}(\boldsymbol{\sigma}_{\alpha})$ $\forall\boldsymbol{\sigma}_{\alpha}\in\mathcal{C}_{\Lambda};\alpha\subseteq\Lambda$ already computed for the Ising model. As a further example of the applicability of this framework we compute FK and HT expansions for the partition function of the Ising model in a constant magnetic field (7). We begin introducing the *ghost spin (site)* [Gr] [FK] [S] $\{0\}$ interacting with all $x\in\Lambda$ trough the Hamiltonian operator

$$\mathbf{H}_{\Lambda}^{0}(h) := -h\sum_{x\in\Lambda}\boldsymbol{\sigma}_{\{x\}\cup\{0\}}$$

Then we define the graph $G_0 := \left(\Lambda\cup\{0\},\Delta_{\Lambda}^{0}\right)$ where $\Delta_{\Lambda}^{0} := \Delta_{\Lambda}\cup\{\{0,x\}\,;\ \forall x\in\Lambda\}$ and $\Delta_{\Lambda}^{0}\ni\gamma_0 := \gamma\cup\{0,x\}\,;\ \forall x\in\Lambda,\ \gamma\in\Delta_{\Lambda}$. Therefore any collection of dimers in Δ_{Λ}^{0} is now given by $\Gamma_0 = \Gamma\cup\{\{0,x\}\,;\ \forall x\in\eta\}\ \forall\Gamma\in\Delta_{\Lambda},\ \eta\subseteq\Lambda$ with $|\Gamma_0| = |\Gamma|+|\eta|$. In the same fashion to $M_{\beta,\Lambda}^{\mathrm{FK(HT)}}$ we define $M_{\beta,h,\Lambda}^{\mathrm{FK(HT)}}$ as the expectation with respect to the *bond percolation* process defined on Δ_{Λ}^{0}. Since

$$\begin{aligned} e^{-2\beta h|\alpha|} \quad &= \quad e^{-2\beta h\sum_{x\in\Lambda}\delta(x\in\alpha)} = \prod_{x\in\Lambda}\left[\frac{1+(-1)^{|\alpha\cap\{x\}|}}{2}p_{\mathrm{FK}}(\beta h) + q_{\mathrm{FK}}(\beta h)\right] = \\ &= \quad \sum_{\emptyset\subseteq\eta\subseteq\Lambda} p_{\mathrm{FK}}^{|\eta|}(\beta h)\, q_{\mathrm{FK}}^{|\Lambda|-|\eta|}(\beta h)\sum_{\emptyset\subseteq\eta_1\subseteq\eta}(-1)^{|\alpha\cap\eta_1|} \end{aligned}$$

from (12) we have

$$\begin{aligned}
Z_\Lambda^{(d)}(\beta,h) &= 2^{|\Lambda|}e^{\beta|\Lambda|(d+h)}\frac{1}{2^{|\Lambda|}}\sum_{\emptyset\subseteq\alpha\subseteq\Lambda} e^{-2\beta|\partial\alpha|-2\beta h|\alpha|} = \\
&= 2^{|\Lambda|}e^{\beta|\Lambda|(d+h)}M^{\mathrm{FK}}_{\beta,\Lambda}\left[\sum_{\emptyset\subseteq\eta\subseteq\Lambda} p_{\mathrm{FK}}^{|\eta|}(\beta h)\, q_{\mathrm{FK}}^{|\Lambda|-|\eta|}(\beta h)\, 2^{-(|\Gamma|+|\eta|)}\times\right.\\
&\left.\times\left(\sum_{\emptyset\subseteq\eta_1\subseteq\eta}\delta_{\eta_1,\emptyset}+\sum_{\emptyset\neq c\subseteq\Gamma}\sum_{\emptyset\subseteq\eta_1\subseteq\eta}\delta\left(\triangle_{\gamma\in c}\ \gamma=\eta\right)\right)\right]\\
&= e^{\beta|\Lambda|(d+h)}M^{\mathrm{FK}}_{\beta,h,\Lambda}\left[2^{|\Lambda|-|\Gamma_0|}\left(1+\sum_{\emptyset\neq c_0\subseteq\Gamma_0}\delta\left(\triangle_{\gamma_0\in c_0}\ \gamma_0=\emptyset\right)\right)\right]
\end{aligned}$$

In the same way we can compute the HT expansion for (7)

$$\begin{aligned}
e^{\beta h\sum_{x\in\Lambda}(-1)^{|\alpha\cap\{x\}|}} &= \prod_{x\in\Lambda}\left[\cosh(\beta h)+(-1)^{|\alpha\cap\{x\}|}\sinh(\beta h)\right] = \\
&= e^{\beta h|\Lambda|}\sum_{\emptyset\subseteq\eta\subseteq\Lambda} p_{\mathrm{HT}}^{|\eta|}(\beta h)\, q_{\mathrm{HT}}^{|\Lambda|-|\eta|}(\beta h)\,(-1)^{|\alpha\cap\eta|}
\end{aligned}$$

thus from (9)

$$\begin{aligned}
Z_\Lambda^{(d)}(\beta,h) &= 2^{|\Lambda|}e^{\beta|\Lambda|(d+h)}M^{\mathrm{FK}}_{\beta,\Lambda}\left[\sum_{\emptyset\subseteq\eta\subseteq\Lambda} p_{\mathrm{HT}}^{|\eta|}(\beta h)\, q_{\mathrm{HT}}^{|\Lambda|-|\eta|}(\beta h)\,\delta\left(\triangle_{\gamma\in\Gamma}\ \gamma=\eta\right)\right] = \\
&= 2^{|\Lambda|}e^{\beta|\Lambda|(d+h)}M^{\mathrm{FK}}_{\beta,\Lambda}\left[p_{\mathrm{HT}}^{|\triangle_{\gamma\in\Gamma}\ \gamma|}(\beta h)\, q_{\mathrm{HT}}^{|\Lambda|-|\triangle_{\gamma\in\Gamma}\ \gamma|}(\beta h)\right] = \\
&= 2^{|\Lambda|}e^{\beta|\Lambda|(d+h)}M^{\mathrm{FK}}_{\beta,h,\Lambda}\left[\delta\left(\triangle_{\gamma_0\in\Gamma_0}\ \gamma_0=\emptyset\right)\right]
\end{aligned}$$

4. Conclusions

Through all this work we have assumed the lattice to be $\mathbb{Z}^d$, but it is clear, from the analysis carried on up to now, that this choice affects only combinatorial calculations. Thus similar computations can be performed considering other kinds of lattices, by simply modifying the combinatorics. It is also clear that this framework can be extended, with some modification, to higher spin systems.

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