ON THE VIRIAL THEOREM FOR THE RELATIVISTIC OPERATOR OF BROWN AND RAVENHALL, AND THE ABSENCE OF EMBEDDED EIGENVALUES

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ABSTRACT. A virial theorem is established for the operator proposed by Brown and Ravenhall as a model for relativistic one-electron atoms. As a consequence, it is proved that the operator has no eigenvalues greater than $\max(mc^2, 2\alpha Z - \frac{1}{2})$, where α is the fine structure constant, for all values of the nuclear charge Z below the critical value Z_c: in particular there are no eigenvalues embedded in the essential spectrum when $Z \leq 3/4\alpha$. Implications for the operators in the partial wave decomposition are also described.

1. INTRODUCTION

The formal operator proposed by Brown and Ravenhall in [1] to include relativistic effects in the description of an electron in the field of its nucleus is of the form

$$\mathbf{B} := \Lambda_+ \left(D_0 - \frac{e^2 Z}{|\cdot|} \right) \Lambda_+. \tag{1.1}$$

In (1.1), the notation is as follows (see [3]):

- D_0 is the free Dirac operator

$$D_0 = c\boldsymbol{\alpha} \cdot \frac{\hbar}{i} \nabla + mc^2 \beta \equiv \sum_{j=1}^{3} c \frac{\hbar}{i} \alpha_j \frac{\partial}{\partial x_j} + mc^2 \beta,$$

  where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β are the Dirac matrices given by

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix}$$

  with $\mathbf{0_2}, \mathbf{1_2}$ the zero and unit 2×2 matrices respectively and σ_j the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Λ_+ denotes the projection of $\mathrm{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ onto the positive spectral subspace of D_0, that is $\chi_{(0,\infty)}(D_0)$, where $\chi_{(0,\infty)}$ is the characteristic function of $(0, \infty)$. If we set

$$\widehat{f}(\mathbf{p}) \equiv \mathcal{F}(f)(\mathbf{p}) = \left(\frac{1}{2\pi\hbar} \right)^{3/2} \int_{\mathbb{R}^3} e^{-i\, \mathbf{x}\cdot\mathbf{p}/\hbar} f(\mathbf{x})\, \mathrm{d}\,\mathbf{x}$$

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for the Fourier transform of f, then it follows that

$$(\Lambda_+ f)^\wedge(\mathbf{p}) = \Lambda_+(\mathbf{p})\widehat{f}(\mathbf{p}),$$

where

$$\Lambda_+(\mathbf{p}) = \frac{1}{2} + \frac{c\boldsymbol{\alpha}\cdot\mathbf{p} + mc^2\beta}{2\mathbf{e}(p)}, \quad \mathbf{e}(p) = \sqrt{c^2p^2 + m^2c^4} \tag{1.2}$$

with $p = |\mathbf{p}|$.

- $2\pi\hbar$ is Planck's constant, c the velocity of light, m the electron mass, $-e$ the electron charge, and Z the nuclear charge.

The underlying Hilbert space in which $\mathbf{B}$ acts is

$$\mathcal{H} = \Lambda_+(\mathrm{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^4). \tag{1.3}$$

The Fourier transform of any spinor ψ in the positive spectral subspace of D_0 can be written

$$\widehat{\psi}(\mathbf{p}) = \frac{1}{\mathbf{n}(p)} \begin{pmatrix} [\mathbf{e}(p) + \mathbf{e}(0)]\ u(\mathbf{p}) \\ c(\mathbf{p}\cdot\boldsymbol{\sigma})\ u(\mathbf{p}) \end{pmatrix}, \tag{1.4}$$

where $u \in \mathrm{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, a Pauli spinor, and $\mathbf{n}(p) = [2\mathbf{e}(p)(\mathbf{e}(p) + \mathbf{e}(0))]^{1/2}$. Conversely any Dirac spinor of the form (1.4) is in the image of $\mathcal{H}$ under the Fourier transform. We than have formally that, if v is the Pauli spinor related by (1.4) to a Dirac spinor φ,

$$(\varphi, \mathbf{B}\psi) = \boldsymbol{\beta}[v, u] := \mathbf{e}[v, u] - \gamma\mathbf{k}[v, u], \tag{1.5}$$

where $(\cdot\,,\cdot)$ is the inner-product on $(\mathrm{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^4)$ (being linear in the second argument),

$$\mathbf{e}[v, u] = \int_{\mathbb{R}^3} \mathbf{e}(p)\ v(\mathbf{p})^*\ u(\mathbf{p})\,\mathrm{d}\,\mathbf{p}, \tag{1.6}$$

$$\mathbf{k}[v, u] = \iint_{\mathbb{R}^3\times\mathbb{R}^3} v(\mathbf{p}')^*\mathbf{K}(\mathbf{p}', \mathbf{p})u(\mathbf{p})\,\mathrm{d}\,\mathbf{p}'\,\mathrm{d}\,\mathbf{p}, \tag{1.7}$$

and

$$\gamma = \frac{\alpha c Z}{2\pi^2}. \tag{1.8}$$

In (1.6)-(1.8), $*$ denotes the Hermitian conjugate, $\alpha = e^2/(\hbar c)$ is Sommerfeld's fine structure constant and the kernel $\mathbf{K}$ in (1.7) is the 2×2 matrix-valued function

$$\mathbf{K}(\mathbf{p}', \mathbf{p}) = \frac{[\mathbf{e}(p') + \mathbf{e}(0)][\mathbf{e}(p) + \mathbf{e}(0)]\mathbf{1_2} + c^2(\mathbf{p}'\cdot\boldsymbol{\sigma})(\mathbf{p}\cdot\boldsymbol{\sigma})}{\mathbf{n}(p')\,|\mathbf{p} - \mathbf{p}'|^2\,\mathbf{n}(p)}. \tag{1.9}$$

The form $\mathbf{e}[\cdot]$ with domain $\mathrm{L}^2(\mathbb{R}^3; \sqrt{1+p^2}\,\mathrm{d}\,\mathbf{p}) \otimes \mathbb{C}^2$ is closed and non-negative in $\mathrm{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^2$. In [2] and [7] it was proved that for $u \in \mathrm{L}^2(\mathbb{R}^3; \sqrt{1+p^2}\,\mathrm{d}\,\mathbf{p}) \otimes \mathbb{C}^2$ and $Z \leq Z_c = 2/[(\frac{\pi}{2} + \frac{2}{\pi})\alpha]$,

$$\gamma\mathbf{k}[u, u] \leq \frac{Z}{Z_c}\mathbf{e}[u, u]. \tag{1.10}$$

It had earlier been established in [3] that $\boldsymbol{\beta}$ is bounded below if and only if $Z \leq Z_c$, confirming a prediction of Hardenkopf and Sucher [4] based on numerical considerations.

The strict positivity of $\boldsymbol{\beta}$ is proved in both [2] [7], a positive lower bound being exhibited in [7] even for $Z = Z_c$. If $Z < Z_c$, it follows from (1.5) and (1.10) that $\boldsymbol{\beta}$ is a closed positive form with domain $\mathcal{D}(\boldsymbol{\beta}) = \mathrm{L}^2(\mathbb{R}^3; \sqrt{1+p^2}\,\mathrm{d}\,\mathbf{p}) \otimes \mathbb{C}^2$.There is therefore defined a positive self-adjoint operator $\mathbf{b}$ in $\mathrm{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ satisfying

$$(v, \mathbf{b}u) = \boldsymbol{\beta}[v, u], \quad u \in \mathcal{D}(\mathbf{b}),\ v \in \mathcal{D}(\boldsymbol{\beta}), \tag{1.11}$$

where the domain $\mathcal{D}(\mathbf{b})$ of $\mathbf{b}$ is dense in its "form domain" $\mathcal{D}(\boldsymbol{\beta})$. We shall follow [3] and refer to this operator $\mathbf{b}$ as the Brown-Ravenhall operator, it being associated to the original operator $\mathbf{B}$ via (1.5).

If $Z = Z_c$, the operator $\mathbf{b}$ is defined as the Friedrichs operator associated with the closure of the form $\boldsymbol{\beta}$ restricted to rapidly decreasing Pauli spinors. The description of the operator $\mathbf{b}$ in this case is studied in [8]. We assume in this paper that $Z < Z_c$, so that $\mathbf{b}$ is defined in (1.11). In this case, it follows from [3, Theorem 2] that $\sigma_{\mathrm{ess}}(\mathbf{b}) = [mc^2, \infty)$ and $\sigma_{\mathrm{sc}}(\mathbf{b}) = \varnothing$, where σ_{ess} and σ_{sc} are respectively the essential and singular continuous spectra. We shall establish a virial theorem for $\mathbf{b}$ which will imply that for $Z \leq Z'_c = \frac{3}{4\alpha}$ there are no eigenvalues embedded in $[mc^2, \infty)$, and hence the spectrum in $[mc^2, \infty)$ is absolutely continuous.

The number Z'_c has a spectral significance in [8] concerning the self-adjointness of operators $b_{l,s}$ in the partial wave decomposition of $\mathbf{B}$ and $\mathbf{b}$: in $b_{l,s}$, l denotes the angular momentum chanel and s the spin, and in [8] the operators are given the domain $\mathrm{L}^2(0, \infty; [\sqrt{1+p^2}]\,\mathrm{d}\,p)$. Tix proves that for $(l,s) \neq (0, 1/2)$ or $(1, -1/2)$, the operators $b_{l,s}$ are all self-adjoint for $Z < Z_c$, but $b_{0,1/2}$ and $b_{1,-1/2}$ are self-adjoint if $Z < Z'_c$, essentially self-adjoint when $Z = Z'_c$ and are symmetric with a one-parameter family of self-adjoint extensions when $Z'_c < Z \leq Z_c$. We also prove that for all $(l,s) \neq (0,1/2)$ or $(1,-1/2)$, the $b_{l,s}$ have no eigenvalues in $[mc^2, \infty)$ for $Z < Z_c$, but only for $Z \leq Z'_c$ in the case $(l,s) = (1, -1/2)$. The operator $b_{0,1/2}$ (its Friedrichs extension when $Z \geq Z'_c$) has no eigenvalues in $[mc^2, \infty)$ for the whole range $Z < Z_c$.

2. The Virial Theorem

We begin by proving an abstract virial theorem.

Lemma 2.1. *Let $U(a)$, $a \in \mathbb{R}_+$, be a one parameter family of unitary operators on a Hilbert space $\mathcal{H}$ which converges strongly to the identity as $a \to 1$. Let T be a self-adjoint operator in $\mathcal{H}$ and $T_a = f(a)\, U(a) T U(a)^{-1}$, where $f(1) = 1$ and $f'(1)$ exists. If $\varphi \in \mathcal{D}(T) \cap \mathcal{D}(T_a)$ is an eigenvector of T corresponding to an eigenvalue λ then*

$$\lim_{a \to 1} \left(\varphi_a, \left[\frac{T_a - T}{a - 1} \right] \varphi \right) = \lambda f'(1) \left\| \varphi \right\|^2, \tag{2.1}$$

where $\varphi_a = U(a)\varphi$.

Proof. ¿From $T\varphi = \lambda\varphi$ we have $T_a \varphi_a = \lambda f(a) \varphi_a$. Hence

$$(\varphi_a, T\varphi) = \lambda(\varphi_a, \varphi)$$

and

$$(T_a\varphi_a,\varphi)=\lambda f(a)(\varphi_a,\varphi).$$

Consequently

$$\left(\varphi_a,\left[\frac{T_a-T}{a-1}\right]\varphi\right)=\lambda\left[\frac{f(a)-1}{a-1}\right](\varphi_a,\varphi)$$

and the result follows as allowing $a\to 1$. $\square$

Our main results are the following Theorem and Corollary which result from the application of Lemma 2.1 to the Brown-Ravenhall operator $\mathbf{b}$.

Theorem 2.2. *Let $Z<Z_c$ and let $\mathbf{b}$ be defined by (1.11) in $\mathrm{L}^2(\mathbb{R}^3)\otimes\mathbb{C}^2$ with form domain $\mathcal{D}(\boldsymbol{\beta})=\mathrm{L}^2(\mathbb{R}^3;\sqrt{1+p^2}\,\mathrm{d}\,\mathbf{p})\otimes\mathbb{C}^2$. If λ is an eigenvalue of $\mathbf{b}$ with eigenvector φ, then*

$$\begin{aligned}\lambda\|\varphi\|^2=&\int_{\mathbb{R}^3}\frac{\mathbf{e}(0)^2}{\mathbf{e}(p)}|\varphi(\mathbf{p})|^2\,\mathrm{d}\,\mathbf{p}\\&-\frac{\gamma}{2}\mathbf{e}(0)\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3}\varphi^*(\mathbf{p}')\mathbf{K}^1(\mathbf{p}',\mathbf{p})\varphi(\mathbf{p})\left[\frac{1}{\mathbf{e}(p)}-\frac{\mathbf{e}(0)}{\mathbf{e}(p)^2}+\frac{1}{\mathbf{e}(p')}-\frac{\mathbf{e}(0)}{\mathbf{e}(p')^2}\right]\mathrm{d}\,\mathbf{p}'\,\mathrm{d}\,\mathbf{p}\\&+\frac{\gamma}{2}\mathbf{e}(0)\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3}\varphi^*(\mathbf{p}')\mathbf{K}^2(\mathbf{p}',\mathbf{p})\varphi(\mathbf{p})\left[\frac{1}{\mathbf{e}(p)}+\frac{\mathbf{e}(0)}{\mathbf{e}(p)^2}+\frac{1}{\mathbf{e}(p')}+\frac{\mathbf{e}(0)}{\mathbf{e}(p')^2}\right]\mathrm{d}\,\mathbf{p}'\,\mathrm{d}\,\mathbf{p},\end{aligned}\tag{2.2}$$

where

$$\mathbf{K}^1(\mathbf{p}',\mathbf{p})=\frac{[\mathbf{e}(p')+\mathbf{e}(0)][\mathbf{e}(p)+\mathbf{e}(0)]\mathbf{1_2}}{\mathbf{n}(p')\,|\mathbf{p}-\mathbf{p}'|^2\,\mathbf{n}(p)},\tag{2.3}$$

$$\mathbf{K}^2(\mathbf{p}',\mathbf{p})=\frac{c^2(\mathbf{p}'\cdot\boldsymbol{\sigma})(\mathbf{p}\cdot\boldsymbol{\sigma})}{\mathbf{n}(p')\,|\mathbf{p}-\mathbf{p}'|^2\,\mathbf{n}(p)}.\tag{2.4}$$

Corollary 2.3. *Let $Z<Z_c$ and $\mathbf{b}\varphi=\lambda\varphi$. Then*

$$\begin{aligned}\left(\frac{\lambda}{\mathbf{e}(0)}-1\right)&\int_{\mathbb{R}^3}|\varphi(\mathbf{p})|^2\left\{1-\frac{\mathbf{e}(0)}{\mathbf{e}(p)}+\frac{\mathbf{e}(0)^2}{\mathbf{e}(p)^2}\right\}\mathrm{d}\,\mathbf{p}\\&=\gamma\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3}\varphi^*(\mathbf{p}')\mathbf{K}^2(\mathbf{p}',\mathbf{p})\varphi(\mathbf{p})\left[\frac{1}{\mathbf{e}(p')}+\frac{1}{\mathbf{e}(p)}\right]\mathrm{d}\,\mathbf{p}'\,\mathrm{d}\,\mathbf{p}\\&\qquad-\int_{\mathbb{R}^3}|\varphi(\mathbf{p})|^2\,\frac{[\mathbf{e}(p)-\mathbf{e}(0)][2\mathbf{e}(p)-\mathbf{e}(0)]}{\mathbf{e}(p)^2}\,\mathrm{d}\,\mathbf{p}.\end{aligned}\tag{2.5}$$

Before proving Theorem 2.2 and Corollary 2.3, we need the following lemma. We shall hereafter in this section write $\mathbf{b}_m$, $\mathbf{e}_m$, $\mathbf{K}_m$, to indicate the dependence on m.

Lemma 2.4. *For any $m\in\mathbb{R}_+$, $\mathbf{b}_m$ and $\mathbf{b}_0$ have the same domain and $\mathbf{b}_m-\mathbf{b}_0$ can be extended to a bounded operator on $\mathrm{L}^2(\mathbb{R}^3)\otimes\mathbb{C}^2$.*

Proof. The result is essentially proved in [8, Theorem 1] but we give a short direct proof for completeness. The following estimates are readily verified (cf [8]):

$$0 \le \mathbf{e}_m(p) - \mathbf{e}_0(p) \le mc^2,$$

$$\begin{aligned} |\mathbf{K}_m(\mathbf{p}',\mathbf{p}) - \mathbf{K}_0(\mathbf{p}',\mathbf{p})| &\le \frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\left\{\frac{m}{2\mathbf{e}_m(p)} + \frac{m}{2\mathbf{e}_m(p')} + \frac{m^2}{4\mathbf{e}_m(p)\mathbf{e}_m(p')}\right\} \\ &\le \frac{k(m)}{|\mathbf{p}-\mathbf{p}'|^2}\left\{\frac{1}{\mathbf{e}_m(p)} + \frac{1}{\mathbf{e}_m(p')}\right\}, \end{aligned}$$

where $k(m)$ is a constant depending on m. Hence, for $\varphi \in \mathcal{D}(\boldsymbol{\beta}_m) = \mathcal{D}(\boldsymbol{\beta}_0)$,

$$\begin{aligned} |\boldsymbol{\beta}_m[\varphi] - \boldsymbol{\beta}_0[\varphi]| &\le mc^2 \, \|\varphi\|^2 \\ &\quad + \gamma k(m) \iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\left\{\frac{1}{\mathbf{e}_m(p)} + \frac{1}{\mathbf{e}_m(p')}\right\} |\varphi(\mathbf{p})| \, |\varphi(\mathbf{p}')| \, \mathrm{d}\,\mathbf{p}' \, \mathrm{d}\,\mathbf{p}. \end{aligned}$$

By the Cauchy-Schwarz inequality, the last integral is no greater than

$$\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} |\varphi(\mathbf{p})|^2 \frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\left\{\frac{1}{\mathbf{e}_m(p)} + \frac{1}{\mathbf{e}_m(p')}\right\} \frac{h(\mathbf{p}')}{h(\mathbf{p})} \, \mathrm{d}\,\mathbf{p}' \, \mathrm{d}\,\mathbf{p},$$

where we choose $h(\mathbf{p}) = |\mathbf{p}|^{-\alpha}$, $1 < \alpha < 2$. Then (see [6, p124])

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{p}-\mathbf{p}'|^2} \frac{h(\mathbf{p}')}{h(\mathbf{p})\mathbf{e}_m(p)} \, \mathrm{d}\,\mathbf{p}' &= \frac{|\mathbf{p}|^\alpha}{\mathbf{e}_m(p)} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{p}-\mathbf{p}'|^2} \frac{1}{|\mathbf{p}'|^\alpha} \, \mathrm{d}\,\mathbf{p}' \\ &\le \frac{|\mathbf{p}|^{\alpha-1}}{c} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{p}-\mathbf{p}'|^2} \frac{1}{|\mathbf{p}'|^\alpha} \, \mathrm{d}\,\mathbf{p}' = O(1) \quad \text{if} \quad 1 < \alpha < 3, \end{aligned}$$

$$\int_{\mathbb{R}^3} \frac{1}{|\mathbf{p}-\mathbf{p}'|^2} \frac{h(\mathbf{p}')}{h(\mathbf{p})\mathbf{e}_m(p')} \, \mathrm{d}\,\mathbf{p}' \le \frac{|\mathbf{p}|^\alpha}{c} \int_{\mathbb{R}^3} \frac{1}{|\mathbf{p}-\mathbf{p}'|^2} \frac{1}{|\mathbf{p}'|^{\alpha+1}} \, \mathrm{d}\,\mathbf{p}' = O(1) \quad \text{if} \quad 0 < \alpha < 2.$$

Hence,

$$|\boldsymbol{\beta}_m[\varphi] - \boldsymbol{\beta}_0[\varphi]| \le k(m) \, \|\varphi\|^2$$

and $(\boldsymbol{\beta}_m - \boldsymbol{\beta}_0)[\cdot\,,\cdot]$ can be extended to a bounded sesquilinear form on $\left\{\mathrm{L}^2(\mathbb{R}^3)\otimes\mathbb{C}^2\right\} \times \left\{\mathrm{L}^2(\mathbb{R}^3)\otimes\mathbb{C}^2\right\}$. It follows that $\mathbf{b}_m$ and $\mathbf{b}_0$ have the same domain and this in turn implies the rest of the lemma. □

Proof of Theorem 2.2. We apply Lemma 2.1 to $T = \mathbf{b}_m$, $f(a) = a$ and $U(a)$ defined by $U(a)\varphi(\mathbf{p}) = a^{-3/2}\varphi(\mathbf{p}/a) =: \varphi_a(\mathbf{p})$. Then $U(a) \to \mathrm{I}$ strongly in $\mathrm{L}^2(\mathbb{R}^3)\otimes\mathbb{C}^2$ and

$$T_a := aU(a)\mathbf{b}_m U(a)^{-1} = \mathbf{b}_{ma}.$$

By Lemma 2.4, an eigenvector φ of $\mathbf{b}_m$, lies in $\mathcal{D}(\mathbf{b}_{ma})$ for any a. Thus, it remains to evaluate the limit on the left-hand side of (2.1). We have

$$\left(\varphi_a, \left[\frac{\mathbf{b}_{ma} - \mathbf{b}_m}{a-1}\right]\varphi\right) = \int_{\mathbb{R}^3} \left[\frac{\mathbf{e}_{ma}(p) - \mathbf{e}_m(p)}{a-1}\right] |\varphi(\mathbf{p})|^2 \, \mathrm{d}\,\mathbf{p}$$
$$- \gamma \iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \varphi_a(\mathbf{p}')^* \left[\frac{\mathbf{K}_{ma}(\mathbf{p}',\mathbf{p}) - \mathbf{K}_m(\mathbf{p}',\mathbf{p})}{a-1}\right]\varphi(\mathbf{p})\, \mathrm{d}\,\mathbf{p}'\, \mathrm{d}\,\mathbf{p} =: I_1 - \gamma I_2, \quad (2.6)$$

and (2.2) will follow if we can justify taking the limit as $a \to 1$ under the integral signs on the right-hand side of (2.6).

In I_1 the integrand is majorised by $|\varphi(\mathbf{p})|^2$ and so the dominated convergence theorem applies. We write I_2 as

$$\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} [\varphi_a(\mathbf{p}') - \varphi(\mathbf{p}')]^* \left[\frac{\mathbf{K}_{ma}(\mathbf{p}',\mathbf{p}) - \mathbf{K}_m(\mathbf{p}',\mathbf{p})}{a-1}\right]\varphi(\mathbf{p})\, \mathrm{d}\,\mathbf{p}'\, \mathrm{d}\,\mathbf{p}$$

$$+ \iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \varphi(\mathbf{p}')^* \left[\frac{\mathbf{K}_{ma}(\mathbf{p}',\mathbf{p}) - \mathbf{K}_m(\mathbf{p}',\mathbf{p})}{a-1}\right]\varphi(\mathbf{p})\, \mathrm{d}\,\mathbf{p}'\, \mathrm{d}\,\mathbf{p} =: I_3 + I_4,$$

and use the readily verified estimate

$$\left|\frac{\mathbf{K}_{ma}(\mathbf{p}',\mathbf{p}) - \mathbf{K}_m(\mathbf{p}',\mathbf{p})}{a-1}\right| \le k_a / |\mathbf{p} - \mathbf{p}'|^2,$$

where $k_a \to 1$ as $a \to 1$. Thus I_3 bounded by

$$\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} |\varphi_a(\mathbf{p}') - \varphi(\mathbf{p}')|\; |\varphi(\mathbf{p})| \frac{\mathrm{d}\,\mathbf{p}'\,\mathrm{d}\,\mathbf{p}}{|\mathbf{p} - \mathbf{p}'|^2}$$

and, on using the Parseval identity and $\mathcal{F}\big(\frac{1}{|\cdot|}\big) = \sqrt{\frac{2}{\pi}}\frac{1}{|\cdot|^2}$ (with $\mathcal{F}$ now the standard Fourier transform with $\hbar = 1$), this is equal to

$$2\pi^2 \int_{\mathbb{R}^3} u(\mathbf{x}) v(\mathbf{x}) \frac{\mathrm{d}\,\mathbf{x}}{|\mathbf{x}|},$$

where $\widehat{u}(\mathbf{p}) = |\varphi_a(\mathbf{p}) - \varphi(\mathbf{p})|$ and $\widehat{v}(\mathbf{p}) = |\varphi(\mathbf{p})|$,

$$\le \pi^3 \left(\int_{\mathbb{R}^3} |\varphi_a(\mathbf{p}) - \varphi(\mathbf{p})|^2 \; |\mathbf{p}|\, \mathrm{d}\,\mathbf{p}\right)^{1/2} \left(\int_{\mathbb{R}^3} |\varphi(\mathbf{p})|^2 \; |\mathbf{p}|\, \mathrm{d}\,\mathbf{p}\right)^{1/2}$$

by the Cauchy-Schwarz inequality and Kato's inequality [5, p. 307]. Thus $I_3 \to 0$ as $a \to 1$ since $\varphi \in \mathrm{L}^2(\mathbb{R}^3; \sqrt{1+p^2}\, \mathrm{d}\,\mathbf{p}) \otimes \mathbb{C}^2$. In I_4 the integrand is majorised by the function $\frac{1}{|\mathbf{p}-\mathbf{p}'|^2} |\varphi(\mathbf{p}')|\; |\varphi(\mathbf{p})|$ which is integrable by Kato's inequality and hence the dominated convergence theorem applies. $\square$

Proof of Corollary 2.3. ¿From (2.2) we have

$$\begin{aligned}\lambda\,\|\varphi\|^2 = \mathbf{e}(0)^2 \int_{\mathbb{R}^3} |\varphi(\mathbf{p})|^2 \frac{\mathrm{d}\,\mathbf{p}}{\mathbf{e}(p)} \\ - \gamma\mathbf{e}(0)\,\mathrm{Re}\Bigg[\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \varphi^*(\mathbf{p}')\mathbf{K}(\mathbf{p}',\mathbf{p})\varphi(\mathbf{p})\left\{\frac{1}{\mathbf{e}(p)} - \frac{\mathbf{e}(0)}{\mathbf{e}(p)^2}\right\}\mathrm{d}\,\mathbf{p}'\,\mathrm{d}\,\mathbf{p}\Bigg] \\ + \gamma\mathbf{e}(0)\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \varphi^*(\mathbf{p}')\mathbf{K}^2(\mathbf{p}',\mathbf{p})\varphi(\mathbf{p})\left\{\frac{1}{\mathbf{e}(p)} + \frac{1}{\mathbf{e}(p')}\right\}\mathrm{d}\,\mathbf{p}'\,\mathrm{d}\,\mathbf{p}. \quad (2.7)\end{aligned}$$

Also, for all $\psi \in \mathcal{D}(\boldsymbol{\beta})$,

$$\begin{aligned}\int_{\mathbb{R}^3} \mathbf{e}(p)\psi^*(\mathbf{p})\varphi(\mathbf{p})\,\mathrm{d}\,\mathbf{p} - \gamma \iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \psi^*(\mathbf{p}')\mathbf{K}(\mathbf{p}',\mathbf{p})\varphi(\mathbf{p})\,\mathrm{d}\,\mathbf{p}'\,\mathrm{d}\,\mathbf{p} \\ = \lambda \int_{\mathbb{R}^3} \psi^*(\mathbf{p})\varphi(\mathbf{p})\,\mathrm{d}\,\mathbf{p}. \quad (2.8)\end{aligned}$$

We choose $\psi(\mathbf{p}) = \left[\frac{1}{\mathbf{e}(p)} - \frac{\mathbf{e}(0)}{\mathbf{e}(p)^2}\right]\varphi(\mathbf{p})$ in (2.8): clearly $\psi \in \mathcal{D}(\boldsymbol{\beta})$. Then

$$\begin{aligned}\gamma \iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \varphi^*(\mathbf{p}')\mathbf{K}(\mathbf{p}',\mathbf{p})\varphi(\mathbf{p})\left\{\frac{1}{\mathbf{e}(p)} - \frac{\mathbf{e}(0)}{\mathbf{e}(p)^2}\right\}\mathrm{d}\,\mathbf{p}'\,\mathrm{d}\,\mathbf{p} \\ = \int_{\mathbb{R}^3} [\mathbf{e}(p) - \lambda]\left\{\frac{1}{\mathbf{e}(p)} - \frac{\mathbf{e}(0)}{\mathbf{e}(p)^2}\right\}|\varphi(\mathbf{p})|^2\,\mathrm{d}\,\mathbf{p}. \quad (2.9)\end{aligned}$$

On substituting (2.9) in (2.7),

$$\begin{aligned}\lambda\,\|\varphi\|^2 = \mathbf{e}(0)^2 \int_{\mathbb{R}^3} |\varphi(\mathbf{p})|^2 \frac{\mathrm{d}\,\mathbf{p}}{\mathbf{e}(p)} - \mathbf{e}(0)\int_{\mathbb{R}^3} [\mathbf{e}(p) - \lambda]\left\{\frac{1}{\mathbf{e}(p)} - \frac{\mathbf{e}(0)}{\mathbf{e}(p)^2}\right\}|\varphi(\mathbf{p})|^2\,\mathrm{d}\,\mathbf{p} \\ + \gamma\mathbf{e}(0)\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \varphi^*(\mathbf{p}')\mathbf{K}^2(\mathbf{p}',\mathbf{p})\varphi(\mathbf{p})\left\{\frac{1}{\mathbf{e}(p)} + \frac{1}{\mathbf{e}(p')}\right\}\mathrm{d}\,\mathbf{p}'\,\mathrm{d}\,\mathbf{p},\end{aligned}$$

whence

$$\begin{aligned}\lambda \int_{\mathbb{R}^3} |\varphi(\mathbf{p})|^2 \left\{1 - \frac{\mathbf{e}(0)}{\mathbf{e}(p)} + \frac{\mathbf{e}(0)^2}{\mathbf{e}(p)^2}\right\}\mathrm{d}\,\mathbf{p} = \mathbf{e}(0)\int_{\mathbb{R}^3} |\varphi(\mathbf{p})|^2 \left\{\frac{\mathbf{e}(0)}{\mathbf{e}(p)} - 1 + \frac{\mathbf{e}(0)}{\mathbf{e}(p)}\right\}\mathrm{d}\,\mathbf{p} \\ + \gamma\mathbf{e}(0)\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \varphi^*(\mathbf{p}')\mathbf{K}^2(\mathbf{p}',\mathbf{p})\varphi(\mathbf{p})\left[\frac{1}{\mathbf{e}(p')} + \frac{1}{\mathbf{e}(p)}\right]\mathrm{d}\,\mathbf{p}'\,\mathrm{d}\,\mathbf{p},\end{aligned}$$

and

$$(\lambda - \mathbf{e}(0)) \int_{\mathbb{R}^3} |\varphi(\mathbf{p})|^2 \left\{1 - \frac{\mathbf{e}(0)}{\mathbf{e}(p)} + \frac{\mathbf{e}(0)^2}{\mathbf{e}(p)^2}\right\} \mathrm{d}\,\mathbf{p}$$
$$= \mathbf{e}(0) \int_{\mathbb{R}^3} |\varphi(\mathbf{p})|^2 \frac{[\mathbf{e}(p) - \mathbf{e}(0)][\mathbf{e}(0) - 2\mathbf{e}(p)]}{\mathbf{e}(p)^2} \mathrm{d}\,\mathbf{p}$$
$$+ \gamma \mathbf{e}(0) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi^*(\mathbf{p}')\mathbf{K}^2(\mathbf{p}', \mathbf{p})\varphi(\mathbf{p}) \left[\frac{1}{\mathbf{e}(p')} + \frac{1}{\mathbf{e}(p)}\right] \mathrm{d}\,\mathbf{p}' \,\mathrm{d}\,\mathbf{p}. \quad (2.10)$$

The corollary is therefore proved. □

3. The absence of embedded eigenvalues

Theorem 3.1. *If $Z < Z_c$, the operator* $\mathbf{b}$ *has no eigenvalues in* $[\max\{1, 2\alpha Z - 1/2\}mc^2, \infty)$. *In particular, if $Z \leq Z_c' = 3/(4\alpha)$, there are no eigenvalues in $[mc^2, \infty)$ and the spectrum of* $\mathbf{b}$ *is absolutely continuous.*

Proof. ¿From (2.5)

$$\left(\frac{\lambda}{\mathbf{e}(0)} - 1\right) \int_{\mathbb{R}^3} |\varphi(\mathbf{p})|^2 \left\{1 - \frac{\mathbf{e}(0)}{\mathbf{e}(p)} + \frac{\mathbf{e}(0)^2}{\mathbf{e}(p)^2}\right\} \mathrm{d}\,\mathbf{p}$$
$$\leq \gamma \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi^*(\mathbf{p}')\mathbf{H}(\mathbf{p}', \mathbf{p})\, |\varphi(\mathbf{p})|\; |\varphi(\mathbf{p})| \,\mathrm{d}\,\mathbf{p}' \,\mathrm{d}\,\mathbf{p}$$
$$- \int_{\mathbb{R}^3} |\varphi(\mathbf{p})|^2 \frac{[\mathbf{e}(p) - \mathbf{e}(0)][2\mathbf{e}(p) - \mathbf{e}(0)]}{\mathbf{e}(p)^2} \mathrm{d}\,\mathbf{p}, \quad (3.1)$$

where

$$\mathbf{H}(\mathbf{p}', \mathbf{p}) = \frac{c^2 pp'}{\mathbf{n}(p')\mathbf{n}(p)\, |\mathbf{p} - \mathbf{p}'|^2} \left[\frac{1}{\mathbf{e}(p')} + \frac{1}{\mathbf{e}(p)}\right].$$

By the Cauchy-Schwarz inequality

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathbf{H}(\mathbf{p}', \mathbf{p})\, |\varphi(\mathbf{p}')|\; |\varphi(\mathbf{p})| \,\mathrm{d}\,\mathbf{p}' \,\mathrm{d}\,\mathbf{p} \leq c^2 \int_{\mathbb{R}^3} \frac{p^2}{\mathbf{n}^2(p)} |\varphi(\mathbf{p})|^2 \,\mathrm{d}\,\mathbf{p} \int_{\mathbb{R}^3} \frac{h(\mathbf{p}')}{h(\mathbf{p})} \mathbf{J}(\mathbf{p}', \mathbf{p}) \,\mathrm{d}\,\mathbf{p}', \quad (3.2)$$

where

$$\mathbf{J}(\mathbf{p}', \mathbf{p}) = \frac{1}{|\mathbf{p} - \mathbf{p}'|^2} \left[\frac{1}{\mathbf{e}(p')} + \frac{1}{\mathbf{e}(p)}\right]$$

and $h(\cdot)$ is an arbitrary positive function. We make the choice $h(\mathbf{p}) = p^{-3/2}$ and use the following (see [6, p 124]):

$$\int_{\mathbb{R}^3} \frac{1}{|\mathbf{p} - \mathbf{p}'|^2} \frac{1}{|\mathbf{p}'|^{3/2}} \,\mathrm{d}\,\mathbf{p}' = 4\pi^2 p^{-1/2},$$

$$\int_{\mathbb{R}^3} \frac{1}{|\mathbf{p}-\mathbf{p}'|^2} \frac{1}{|\mathbf{p}'|^{5/2}} \,\mathrm{d}\,\mathbf{p}' = 4\pi^2 p^{-3/2}.$$

On substituting in (3.2), we have

$$\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \mathbf{H}(\mathbf{p}',\mathbf{p})\,|\varphi(\mathbf{p}')|\;|\varphi(\mathbf{p})|\,\mathrm{d}\,\mathbf{p}'\,\mathrm{d}\,\mathbf{p} < 4\pi^2c^2 \int_{\mathbb{R}^3} \frac{p^2}{\mathbf{n}^2(p)}\left[\frac{p}{\mathbf{e}(p)}+\frac{1}{c}\right]|\varphi(\mathbf{p})|^2\,\mathrm{d}\,\mathbf{p}$$

and hence (3.1) yields

$$\begin{aligned}(\frac{\lambda}{\mathbf{e}(0)}-1)&\int_{\mathbb{R}^3}|\varphi(\mathbf{p})|^2\left\{1-\frac{\mathbf{e}(0)}{\mathbf{e}(p)}+\frac{\mathbf{e}(0)^2}{\mathbf{e}(p)^2}\right\}\mathrm{d}\,\mathbf{p}\\ &<\int_{\mathbb{R}^3}|\varphi(\mathbf{p})|^2\left\{4\pi^2c^2\gamma\frac{p^2}{\mathbf{n}^2(p)}\left[\frac{p}{\mathbf{e}(p)}+\frac{1}{c}\right]-\frac{[\mathbf{e}(p)-\mathbf{e}(0)][2\mathbf{e}(p)-\mathbf{e}(0)]}{\mathbf{e}(p)^2}\right\}\mathrm{d}\,\mathbf{p}.\end{aligned}$$

On replacing $\mathbf{p}$ by $mc\mathbf{p}$ and simplifying, we derive

$$0<\int_{\mathbb{R}^3}|\varphi(mc\mathbf{p})|^2\left(1-\frac{1}{\sqrt{p^2+1}}+\frac{1}{p^2+1}\right)\left(1-\frac{\lambda}{mc^2}+\Phi(p)\right)\mathrm{d}\,\mathbf{p}, \tag{3.3}$$

where

$$\begin{aligned}\Phi(p)&=\frac{p^2(p+\sqrt{p^2+1})}{(\sqrt{p^2+1}+1)(p^2+2-\sqrt{p^2+1})}\left\{2\pi^2\left(\frac{\gamma}{c}\right)-\frac{2\sqrt{p^2+1}-1}{p+\sqrt{p^2+1}}\right\}\\ &\leq\frac{p^2(p+\sqrt{p^2+1})}{(\sqrt{p^2+1}+1)(p^2+2-\sqrt{p^2+1})}\left\{2\pi^2\left(\frac{\gamma}{c}\right)-\frac{3}{4}\right\}\end{aligned}$$

since $\min\limits_{\mathbb{R}_+}\left\{\frac{2\sqrt{p^2+1}-1}{p+\sqrt{p^2+1}}\right\}=\frac{3}{4}$. Thus, if $2\pi^2\left(\frac{\gamma}{c}\right)\equiv\alpha Z\leq\frac{3}{4}$, (3.3) implies that $\lambda<mc^2$. Since $\sup\limits_{\mathbb{R}_+}\left\{\frac{p^2(p+\sqrt{p^2+1})}{(\sqrt{p^2+1}+1)(p^2+2-\sqrt{p^2+1})}\right\}=2$, it follows from (3.3) that, if $2\pi^2\left(\frac{\gamma}{c}\right)\equiv\alpha Z>\frac{3}{4}$, $\lambda<mc^2[1+2\alpha Z-\frac{3}{2}]$. The theorem is therefore proved □

Finally, we analyse the implications of the virial theorem for the operators $b_{l,s}$.

In the partial wave decomposition of $\mathbf{B}$ and $\mathbf{b}$ (see [3]), spinors $\varphi\in \mathrm{L}^2\ (\mathbb{R}^3)\otimes\mathbb{C}^2$ are expanded in terms of spherical spinors $\Omega_{l,n,s}$

$$\varphi(\mathbf{p})=\sum_{(l,n,s)\in I}p^{-1}\varphi_{l,n,s}(p)\Omega_{l,n,s}(\omega) \tag{3.4}$$

say, where $I=\{(l,n,s):l\in\mathcal{N}_0,n=-l-\frac{1}{2},\cdots,l+\frac{1}{2},s=\pm\frac{1}{2},\Omega_{l,n,s}\neq0\}$ and

$$\int_{\mathbb{R}^3}|\varphi(\mathbf{p})|^2\,\mathrm{d}\,\mathbf{p}=\sum_{(l,n,s)\in I}\int_0^\infty|\varphi_{l,n,s}(p)|^2\,\mathrm{d}\,p. \tag{3.5}$$

On substituting in (1.5), we get

$$\boldsymbol{\beta}[\varphi,\psi]=\sum_{(l,n,s)\in I}\boldsymbol{\beta}_{l,s}[\varphi_{l,n,s},\psi_{l,n,s}], \tag{3.6}$$

where

$$\boldsymbol{\beta}_{l,s}[\varphi_{l,n,s},\psi_{l,n,s}] = \int_0^\infty \mathbf{e}(p)\overline{\varphi}_{l,n,s}(p)\psi_{l,n,s}(p)\,\mathrm{d}\,p - \frac{\alpha c Z}{\pi}\int_0^\infty\int_0^\infty \overline{\varphi}_{l,n,s}(p')k_{l,s}(p',p)\psi_{l,n,s}(p)\,\mathrm{d}\,p'\,\mathrm{d}\,p \quad (3.7)$$

and, with Q_l denoting Legendre functions of the second kind,

$$\begin{aligned} k_{l,s}(p',p) &= \frac{[\mathbf{e}(p')+\mathbf{e}(0)]Q_l\big(\frac{1}{2}\big[\frac{p'}{p}+\frac{p}{p'}\big]\big)[\mathbf{e}(p)+\mathbf{e}(0)]}{\mathbf{n}(p')\mathbf{n}(p)} + \frac{c^2p'Q_{l+2s}\big(\frac{1}{2}[\frac{p'}{p}+\frac{p}{p'}]\big)p}{\mathbf{n}(p')\mathbf{n}(p)} \\ &=: k^1_{l,m}(p',p)+k^2_{l,m}(p',p) \end{aligned} \quad (3.8)$$

say. For $Z < Z_c$, the forms with domain $\mathrm{L}^2(0,\infty;\sqrt{1+p^2}\,\mathrm{d}\,p)$ are closed and positive in $\mathrm{L}^2(0,\infty)$, and we shall denote the associated self-adjoint operators by $b_{l,s}$. These operators $b_{l,s}$ coincide with the operators $b_{l,s}$ of Tix in [8] when the later are self-adjoint, but are otherwise their Friedrichs extensions. Note that it follows from [3, Theorem2] that for all values of l, s, $\sigma_{\mathrm{ess}}(b_{ls}) = [mc^2,\infty)$.

Theorem 3.2. *For $(l,s)\neq(1,-\frac{1}{2})$, the operators $b_{l,s}$ have no eigenvalues in $[mc^2,\infty)$ if $Z < Z_c$. If $(l,s)=(1,-\frac{1}{2})$, $b_{l,s}$ has no eigenvalues in $[\max\{1,2\alpha Z-1/2\}mc^2,\infty)$: in particular $b_{1,-1/2}$ has no eigenvalues in $[mc^2,\infty)$ if $Z\le Z_c'=\frac{3}{4\alpha}$.*

Proof. The analogue of (2.5) is

$$\begin{aligned} &\left(\frac{\lambda}{\mathbf{e}(0)}-1\right)\int_0^\infty |\varphi_{l,n,s}(p)|^2\left\{1-\frac{\mathbf{e}(0)}{\mathbf{e}(p)}+\frac{\mathbf{e}(0)^2}{\mathbf{e}(p)^2}\right\}\mathrm{d}\,p \\ &\qquad = 2\pi\gamma\int_0^\infty\int_0^\infty \overline{\varphi}_{l,n,s}(p')\mathbf{k}^2_{l,s}(p',p)\varphi(p)\left[\frac{1}{\mathbf{e}(p')}+\frac{1}{\mathbf{e}(p)}\right]\mathrm{d}\,p'\,\mathrm{d}\,p \\ &\qquad\quad - \int_0^\infty |\varphi_{l,n,s}(p)|^2\,\frac{[\mathbf{e}(p)-\mathbf{e}(0)][2\mathbf{e}(p)-\mathbf{e}(0)]}{\mathbf{e}(p)^2}\,\mathrm{d}\,p. \end{aligned} \quad (3.9)$$

Since $Q_0(t)\ge\cdots\ge Q_l(t)\ge 0$ for all $t>1$, it follows that for all $(l,s)\neq(1,-1/2)$

$$k^2_{l,s}(p',p)\le\frac{c^2p'p}{\mathbf{n}(p')\mathbf{n}(p)}Q_1\left(\frac{1}{2}\left[\frac{p'}{p}+\frac{p}{p'}\right]\right)$$

and hence, on writing φ for $\varphi_{l,n,s}$ in (3.9), we get

$$\left(\frac{\lambda}{\mathbf{e}(0)}-1\right)\int_0^\infty |\varphi(p)|^2\left\{1-\frac{\mathbf{e}(0)}{\mathbf{e}(p)}+\frac{\mathbf{e}(0)^2}{\mathbf{e}(p)^2}\right\}\mathrm{d}\,p$$
$$\leq 2\pi\gamma c^2\int_0^\infty |\varphi(p)|^2\frac{p^2}{\mathbf{n}^2(p)h(p)}\,\mathrm{d}\,p\int_0^\infty h(p')Q_1\left(\frac{1}{2}\left[\frac{p'}{p}+\frac{p}{p'}\right]\right)\left(\frac{1}{\mathbf{e}(p')}+\frac{1}{\mathbf{e}(p)}\right)\mathrm{d}\,p'$$
$$-\int_0^\infty |\varphi(p)|^2\,\frac{[\mathbf{e}(p)-\mathbf{e}(0)][2\mathbf{e}(p)-\mathbf{e}(0)]}{\mathbf{e}(p)^2}\,\mathrm{d}\,p, \quad (3.10)$$

for any positive function $h(\cdot)$, on applying the Cauchy-Schwarz inequality. We choose $h(t)=1/t$, so that in the first term on the right-hand side of (3.10)we have

$$2\pi\gamma c^2\int_0^\infty |\varphi(p)|^2\frac{p^2}{\mathbf{n}^2(p)}\left\{\frac{p}{\mathbf{e}(p)}\int_0^\infty \frac{1}{p'}Q_1\left(\frac{1}{2}\left[\frac{p'}{p}+\frac{p}{p'}\right]\right)\mathrm{d}\,p'\right.$$
$$\left.+\,p\int_0^\infty \frac{1}{p'\mathbf{e}(p')}Q_1\left(\frac{1}{2}\left[\frac{p'}{p}+\frac{p}{p'}\right]\right)\mathrm{d}\,p'\right\}\mathrm{d}\,p$$
$$=2\pi\gamma c^2\int_0^\infty |\varphi(p)|^2\,\frac{p^2}{\mathbf{n}^2(p)}\Big\{I_1+I_2\Big\}\,\mathrm{d}\,p$$

say. Let $g_1(u)=Q_1\left(\frac{1}{2}\left[u+\frac{1}{u}\right]\right)$. Then

$$cI_1\leq\int_0^\infty g_1(u)\frac{\mathrm{d}\,u}{u}=2,$$

(see [2, (3.8)] or [3, § 2.3]), and

$$cI_2\leq\int_0^\infty g_1(u)\frac{\mathrm{d}\,u}{u^2}=\int_0^\infty g_1(u)\,\mathrm{d}\,u.$$

We have (see [2, (3.7)])

$$\int_0^1 g_1(u)\,\mathrm{d}\,u=\frac{1}{2}\int_0^1 u\ln\left(\frac{u+1}{1-u}\right)\mathrm{d}\,u+\frac{1}{2}\int_0^1\frac{1}{u}\ln\left(\frac{u+1}{1-u}\right)\mathrm{d}\,u-\int_0^1\mathrm{d}\,u$$
$$=\frac{1}{2}+\frac{1}{2}\frac{\pi^2}{4}-1=\frac{\pi^2}{8}-\frac{1}{2}$$

and

$$\int_1^\infty g_1(u)\,\mathrm{d}\,u = \frac{1}{2}\int_1^\infty \frac{1}{u}\ln\left(\frac{u+1}{u-1}\right)\mathrm{d}\,u + \int_1^\infty \left\{\frac{u}{2}\ln\left(\frac{u+1}{u-1}\right) - 1\right\}\mathrm{d}\,u$$

$$= \frac{\pi^2}{8} + \left[\frac{1}{4}(u^2-1)\ln\left(\frac{u+1}{u-1}\right) - \frac{u}{2}\right]_1^\infty = \frac{\pi^2}{8} + \frac{1}{2}.$$

Hence $cI_2 \leq \frac{\pi^2}{4}$, and from (3.10)

$$\left(\frac{\lambda}{\mathbf{e}(0)} - 1\right)\int_0^\infty |\varphi(p)|^2 \left\{1 - \frac{\mathbf{e}(0)}{\mathbf{e}(p)} + \frac{\mathbf{e}(0)^2}{\mathbf{e}(p)^2}\right\}\mathrm{d}\,p$$

$$\leq 2\pi\gamma c \int_0^\infty |\varphi(p)|^2 \frac{p^2}{\mathbf{n}^2(p)}\big(\frac{\pi^2}{4} + 2\big)\,\mathrm{d}\,p - \int_0^\infty |\varphi(p)|^2 \frac{[\mathbf{e}(p) - \mathbf{e}(0)][2\mathbf{e}(p) - \mathbf{e}(0)]}{\mathbf{e}(p)^2}\,\mathrm{d}\,p.$$

On replacing p by mcp and simplifying, we obtain

$$0 < \int_0^\infty |\varphi(mc\mathbf{p})|^2 \left(1 - \frac{1}{\sqrt{p^2+1}} + \frac{1}{p^2+1}\right)\left(1 - \frac{\lambda}{mc^2} + \Psi(p)\right)\mathrm{d}\,p,$$

where

$$\Psi(p) = \frac{p^2(\sqrt{p^2+1})}{(\sqrt{p^2+1}+1)(p^2+2-\sqrt{p^2+1})}\left\{\pi\left(\frac{\gamma}{c}\right)\left(\frac{\pi^2}{4}+2\right) - \frac{2\sqrt{p^2+1}-1}{\sqrt{p^2+1}}\right\}$$

$$\leq \frac{p^2\sqrt{p^2+1}}{(\sqrt{p^2+1}+1)(p^2+2-\sqrt{p^2+1})}\left\{\frac{2}{\pi^2+4}\left(\frac{\pi^2}{4}+2\right) - 1\right\} < 0$$

for $2\pi^2\frac{\gamma}{c} \equiv \alpha Z \leq \alpha Z_c$. Hence $\lambda < mc^2$.

For the case $(l, s) = (1, -1/2)$, the Legendre function Q_1 in (3.10) has to be replaced by Q_0, and we have to consider

$$\frac{1}{h(p)}\int_0^\infty h(p')Q_0\left(\frac{1}{2}\left[\frac{p'}{p} + \frac{p}{p'}\right]\right)\left[\frac{1}{\mathbf{e}(p')} + \frac{1}{\mathbf{e}(p)}\right]\mathrm{d}\,p'. \tag{3.11}$$

We now make the choice $h(u) = 1/\sqrt{u}$, and, with $g_0(u) = Q_0\left(\frac{1}{2}\left[u + \frac{1}{u}\right]\right) \equiv \ln\left|\frac{u+1}{u-1}\right|$, (3.11) becomes $J_1 + J_2$ say, where

$$J_1 = p\int_0^\infty \frac{1}{\sqrt{u}\mathbf{e}(pu)}g_0(u)\,\mathrm{d}\,u < \frac{1}{c}\int_0^\infty \frac{g_0(u)}{u^{3/2}}\,\mathrm{d}\,u$$

and

$$J_2 = \frac{p}{\mathbf{e}(p)}\int_0^\infty \frac{1}{\sqrt{u}}g_0(u)\,\mathrm{d}\,u.$$

Since

$$\int_0^1 \frac{g_0(u)}{u^{1/2}}\,\mathrm{d}\,u = \int_1^\infty \frac{g_0(u)}{u^{3/2}}\,\mathrm{d}\,u,$$

$$\int_1^\infty \frac{g_0(u)}{u^{1/2}}\,\mathrm{d}\,u = \int_0^1 \frac{g_0(u)}{u^{3/2}}\,\mathrm{d}\,u$$

we have

$$\int_0^\infty \frac{g_0(u)}{u^{1/2}}\,\mathrm{d}\,u = \int_0^\infty \frac{g_0(u)}{u^{3/2}}\,\mathrm{d}\,u = \int_1^\infty g_0(u)\left(1+\frac{1}{u}\right)\frac{\mathrm{d}\,u}{\sqrt{u}}$$

$$= 4\int_1^\infty (\ln y)\frac{y}{(y^2-1)^{3/2}}\,\mathrm{d}\,y$$

on setting $\frac{u+1}{u-1} = y$,

$$= \int_1^\infty (\ln z)\frac{\mathrm{d}\,z}{(z-1)^{3/2}} = 2\int_1^\infty \frac{1}{z(z-1)^{1/2}}\,\mathrm{d}\,z = 2\pi.$$

Hence, on substituting in (3.10) (with Q_1 replaced by Q_0) we infer that

$$\left(\frac{\lambda}{\mathbf{e}(0)}-1\right)\int_0^\infty |\varphi(p)|^2\left\{1-\frac{\mathbf{e}(0)}{\mathbf{e}(p)}+\frac{\mathbf{e}(0)^2}{\mathbf{e}(p)^2}\right\}\mathrm{d}\,p$$

$$< 2\pi\gamma c\int_0^\infty |\varphi(p)|^2\,\frac{p^2}{\mathbf{n}^2(p)}\left[2\pi\left(\frac{cp}{\mathbf{e}(p)}+1\right)\right]\mathrm{d}\,p - \int_0^\infty |\varphi(p)|^2\,\frac{[\mathbf{e}(p)-\mathbf{e}(0)][2\mathbf{e}(p)-\mathbf{e}(0)]}{\mathbf{e}(p)^2}\,\mathrm{d}\,p.$$

On replacing p by mcp and simplifying, this yields

$$0 < \int_0^\infty |\varphi(mc\mathbf{p})|^2\left(1-\frac{1}{\sqrt{p^2+1}}+\frac{1}{p^2+1}\right)\left(1-\frac{\lambda}{mc^2}+\Theta(p)\right)\mathrm{d}\,p,$$

where

$$\Theta(p) = \frac{p^2(p+\sqrt{p^2+1})}{(\sqrt{p^2+1}+1)(p^2+2-\sqrt{p^2+1})}\left\{2\pi^2\left(\frac{\gamma}{c}\right)-\frac{2\sqrt{p^2+1}-1}{p+\sqrt{p^2+1}}\right\}$$

$$\leq \frac{p^2(\sqrt{p^2+1}+p)}{(\sqrt{p^2+1}+1)(p^2+2-\sqrt{p^2+1})}\left\{2\pi^2\left(\frac{\gamma}{c}\right)-\frac{3}{4}\right\} \leq 0$$

if $2\pi^2\left(\frac{\gamma}{c}\right) = \alpha Z \leq \alpha Z_c' = 3/4$. Hence in this case $\lambda < mc^2$. If $\alpha Z > 3/4$, we have

$$\Theta(p) \leq 2\left(\alpha Z - \frac{3}{4}\right)$$

and $\lambda < mc^2\left(\alpha Z - \frac{1}{2}\right)$. The theorem is therefore proved. □

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