The spectrum of weakly coupled map lattices

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Abstract. We consider weakly coupled analytic expanding circle maps on the lattice $\mathbb{Z}^d$ (for $d \geq 1$), with small coupling strength ϵ and coupling between two sites decaying exponentially with the distance. We study the spectrum of the associated (Perron-Frobenius) transfer operators. We give a Fréchet space on which the operator associated to the full system has a simple eigenvalue at 1 (corresponding to the SRB measure μ_ϵ previously obtained by Bricmont–Kupiainen [BK1]) and the rest of the spectrum, except maybe for continuous spectrum, is inside a disc of radius smaller than one. For $d = 1$ we also construct Banach spaces of densities with respect to μ_ϵ on which perturbation theory, applied to the difference of fixed high iterates of the normalised coupled and uncoupled transfer operators, yields localisation of the full spectrum of the coupled operator (i.e., the first spectral gap and beyond). As a side-effect, we show that the whole spectra of the truncated coupled transfer operators (on bounded analytic functions) are $\mathcal{O}(\epsilon)$-close to the truncated uncoupled spectra, uniformly in the spatial size. Our method uses polymer expansions and also gives the exponential decay of time-correlations for a larger class of observables than those considered in [BK1].

Introduction

In the seventies and eighties, ideas from statistical mechanics have been used to describe the statistical properties of finite-dimensional chaotic dynamical systems. In particular, uniformly hyperbolic or expanding smooth, discrete- or continuous-time, dynamical systems on compact finite-dimensional manifolds M have been showed to admit finitely many (with uniqueness in transitive cases) Sinai-Ruelle-Bowen (SRB) invariant probability measures μ, which are ergodic, and topologically *mixing* for some

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iterate of the dynamics. (One characteristic property of the SRB measure of a map f is that it is obtained as a weak-limit of Birkhoff sums $n^{-1}\sum_{k=0}^{n-1}\delta_{f^k(x)}$ for a set of initial conditions x of positive *Lebesgue* measure m. When the SRB measure is unique, for any continuous function φ the averages $\int \varphi \circ f^n \, dm$ converge to $\int \varphi \, d\mu$ as $n \to \infty$.)

In fact, mixing has been shown to occur at an exponential speed for smooth (Hölder) observables. This means that there exists $\tau < 1$ such that for any two Hölder functions $\varphi, \psi : M \to \mathbb{C}$ the so-called *correlation function* $C_{\varphi,\psi}(n)$ decays exponentially with rate τ, i.e., there is a constant $K(\varphi,\psi) > 0$ so that for all $n \geq 1$

$$|C_{\varphi,\psi}(n)| = \left| \int \varphi \circ f^n \, \psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right| \leq K(\varphi,\psi)\, \tau^n . \tag{0.1}$$

More recently, these results have been extended to some nonuniformly hyperbolic dynamical systems on finite-dimensional manifolds. We refer to the brief and excellent review of Young [Y] for an introduction and recent references.

One way to study decay of correlations is to associate to the dynamics a Perron-Frobenius type *transfer operator* P acting on a suitable Banach space. In expanding situations, the operator acts on densities of absolutely continuous measures like f acts on measures, i.e., $(P\varphi)m = f_*(\varphi\, m)$, and one shows that the operator has a smooth fixed point (the density of the SRB measure), and that the rest of the spectrum is inside a disk of strictly smaller radius. It is this *spectral gap* for the transfer operator which yields exponential decay of correlations. Sometimes, part of the spectrum of the operator can be described beyond the first gap: one shows that the spectrum consists of eigenvalues of finite multiplicity, at least in some peripheral annulus (for the case of expanding analytic maps the operator acting on bounded analytic functions is compact, but in more general cases essential spectrum is present). These eigenvalues (the Ruelle *resonances*) are in bijection with poles of the Fourier transform of the correlation function, and also with poles of suitable dynamical zeta functions or dynamically defined generalised Fredholm determinants (see e.g. the survey [B]).

Coupled map lattices (CMLs) were first introduced by Kaneko in 1983. For an overview, we refer to [K] from which we adapt a definition: CMLs are dynamical systems involving interactions (the couplings) between continuous state elements (e.g. points in a manifold) evolving in discrete time (the maps) and distributed on a discrete space (the finite or infinite lattice). Existence of mixing SRB-type measures for such systems is sometimes referred to as *space-time chaos* (see e.g. [Bu] for a recent review of rigorous and numerical works).

Mathematical studies of CMLs have mainly been successful when the coupling is weak and fastly decaying (see however [J] for recent results on globally coupled maps), and were initiated in 1988 by Bunimovich and Sinai [BS], who considered the case where the single spin map is one-dimensional and locally expanding, using Markov partitions to study the problem of the existence of a mixing invariant measure (for the infinite lattice system) with absolutely continuous densities of finite marginals: the SRB measure. Pesin and Sinai [PS] then generalised this setting to the case where local maps are hyperbolic (Anosov) instead of expanding. Another model, that of weakly coupled, piecewise monotone and piecewise expanding, but non-Markov, interval maps, was analysed by Keller and Künzle [KK], who worked with functions of bounded variation and obtained existence of the SRB measure, but no mixing properties, for the infinite lattice system. Volevich [V] considered the case of hyperbolic local maps and proved existence of the SRB measure using statistical mechanics techniques. After that, Bricmont and Kupianen [BK1], considering analytic expanding circle maps with weak coupling (just like in the present paper), used so-called *polymer (or cluster) expansion* techniques from statistical mechanics to prove existence of the SRB measure, and exponential decay of correlation for locally supported analytic test functions. (The analogue of the constants $K(\varphi, \psi)$ in (0.1) there grows exponentially with the size of the "support" of φ, ψ.) Extending results of Dobrushin–Martirosyan and constructing so-called partial high-temperature expansions, Bricmont–Kupiainen then obtained the same results [BK2] under weaker assumptions (uniform expansion but finite differentiability). Hyperbolic local maps were more recently considered by Jiang [Ji] and Jiang–Pesin [JP], who obtained existence of equilibrium states for suitable potentials, with exponential decay of time- and space-correlations for locally supported observables (again with constants depending on the support of the test function).

To our knowledge, and despite misleading terminology, uniqueness of SRB states for CMLs has only been obtained rigorously up to now in a weak sense, namely restricting to spaces of measures whose finite marginals satisfy some regularity condition (see [BK2, Def. 2, p. 715] for a specific example).

The approach used in [BK1] consists in associating transfer operators to finite truncations of the system, and showing via polymer expansions that they have a spectral gap which is uniform in the size of the system. Due perhaps in part to the terminology, polymer expansion techniques sometimes seem esoteric. The main initial idea in our setting, however, is the most natural possible (the difficulties arise later, in the combinatorics, and when obtaining the bounds necessary to play with infinite sums): since the single-spin transfer operator P can be decomposed as $P = Q + PR$, where Q is the rank-one operator associated to the fixed point, and PR has spectral radius strictly smaller than 1, we may write the uncoupled operator on a finite $\Lambda \subset \mathbb{Z}^d$ as

$$P_{0,\Lambda} = \bigotimes_{a\in\Lambda}(Q_a + P_a R_a) = \sum_{T\subset\Lambda}\Big(\bigotimes_{b\in\Lambda\setminus T} Q_b\Big)\otimes\Big(\bigotimes_{a\in T} P_a R_a\Big)\,. \tag{0.2}$$

Now, the coupled dynamics F_ϵ can be written in many models as $F_\epsilon = \Phi_\epsilon \circ F_0$, where F_0 is the uncoupled dynamics and Φ_ϵ is the interaction, i.e., a change of coordinates whose truncation to a finite box $\Lambda \subset \mathbb{Z}^d$ can have an expansion of the form

$$\Phi_{\epsilon,\Lambda} = \sum_{\substack{\text{sets } \mathcal{S} \text{ of disjoint} \\ S\subset\Lambda}} \Big(\bigotimes_{S\in\mathcal{S}} \delta_S \otimes \mathrm{Id}_{\Lambda\setminus\cup_{S\in\mathcal{S}}S}\Big)\,, \tag{0.3}$$

(see Proposition 2 in [BK1, p. 383]), where the operator norms of the δ_S satisfy bounds involving both the coupling strength ϵ and the coupling decay (see (2.36) below for an explicit example). Combining (0.2) and (0.3), we find the following expansion for the nth iterate of the truncated coupled operator $P^n_{\epsilon,\Lambda}$:

$$\sum_{\substack{\mathcal{T}_k \\ \text{sets of time bonds } T}} \sum_{\substack{\mathcal{S}_k \\ \text{sets of spatial sets } S}} \prod_{k=1}^{n} \Big[\big(\bigotimes_{S\in\mathcal{S}_k} \delta_S \otimes \mathrm{Id}_{\Lambda\setminus\cup_{S\in\mathcal{S}_k} S} \big) \otimes \bigotimes_{T\in\mathcal{T}_k} (Q_{\Lambda\setminus T} \otimes P_T R_T) \Big] . \tag{0.4}$$

Finally, we express (0.4) as a sum over sets of disjoint *polymers,* polymers being connected finite subsets of time-bonds T and spatial subsets S, living at various time levels k in $\mathbb{Z}^{d+1}$. (See Proposition B in Section 2 for such an expansion.)

We now describe the motivations leading to the present work. With the exception of Keller–Künzle [KK] (who did not obtain a spectral gap for the infinite system), the above mentioned authors did not study the transfer operator associated to the full coupled system (acting on measures like $(F_\epsilon)_*$). Note that when the single-site operator P is compact, the spectrum of the truncated uncoupled operator $P_{0,\Lambda} = \otimes_{a\in\Lambda} P$ (for finite $\Lambda \subset \mathbb{Z}^d$) acting (compactly) on the corresponding topological tensor product, is obtained by taking all finite products of eigenvalues of P, with multiplicity growing with the cardinality of Λ, except for the eigenvalue 1 which remains simple. One thus expects that the full uncoupled spectrum should consist of a simple eigenvalue at 1, together with eigenvalues of infinite multiplicity given by products of eigenvalues of P (accumulating only at 0). (This situation, where essential spectrum occurs immediately after the first gap, is different from the usual finite-dimensional dynamics picture.) The best one can hope for the coupled operator is thus to find a Banach space on which, for small enough coupling strength ϵ, there is a simple eigenvalue at 1 and the rest of the spectrum is located $\mathcal{O}(\epsilon)$-close to the uncoupled spectrum, also beyond the first gap.

We have carried out the above program for $d = 1$, and we obtained partial results for general $d \geq 1$. We work with the assumptions of Bricmont and Kupiainen in [BK1], and our approach is based on their polymer expansion. We have attempted to make the present paper as self-contained as possible, by isolating the two main statements we need from [BK1] as Proposition A and Proposition B in Section 2 below, by giving references whenever we apply "standard" polymer techniques, and by mentioning, in Lemma 1.1, a simple but crucial bound for completions of tensor products of analytic functions which is implicitly used in [BK1].

The paper is organised as follows. In Section 1, we define the setup, recall some results on finite-dimensional analytic expanding maps, and prove Lemma 1.1 on completions of tensor products of analytic functions. In Section 2.A, we introduce an abstract Fréchet space $\mathcal{F}$, which contains in particular representants of all those complex Borel measures on the full space whose finite marginals have densities with respect to Lebesgue admitting bounded analytic extensions to polyannuli of fixed polyradii. In Lemma 2.4 (which is proved in Section 2.B by using ideas from [BK1]), we obtain a key uniform exponential bound for semi-norms of iterates of the truncated coupled operators. Besides enabling us to recover existence of the SRB measure (Theorem 2.6 (1), see also Corollary 2.8 (1)) for small coupling (note that our approach does not need the final thermodynamic formalism argument used in [BK1, Section 5] to perform the spatial limit), this bound yields exponential decay of operational time-correlations (Corollary 2.8(2)). Note that we are not concerned directly with spatial mixing in this paper. Finally, Lemma 2.4 allows us to show that, except maybe for continuous spectrum, the spectrum of the coupled operator $\mathcal{P}_\epsilon$ on $\mathcal{F}$ consists of the simple eigenvalue 1 and a subset of a disc of radius $\kappa < 1$, with κ close to the original spectral gap if ϵ is small enough.

In Section 3.A, assuming that $d = 1$, we introduce new Banach space $\mathcal{B}$ of (bounded) functions on the infinite space, which are limits of Cauchy sequences of multidimensional Laurent expansions whose coefficients a_k satisfy decay conditions depending both on the size and on the distance to the origin of the support of the multi-index k (see (3.4)). Working with *normalised* transfer operators $\mathcal{L}_\epsilon$ (for $\epsilon \geq 0$) such that $(\mathcal{L}_\epsilon \varphi)\, \mu_\epsilon = (F_\epsilon)_*(\varphi\, \mu_\epsilon)$, with μ_ϵ the SRB measure from Section 2, we are able to show that the difference $\mathcal{L}_\epsilon^n - \mathcal{L}_0^n$ is small in operator norm for all large enough n (this is done in Lemma 3.3, which is proved in Section 3.B, using polymer expansions together with the key Lemma 3.2). After proving that the spectrum of the uncoupled transfer operator on $\mathcal{B}$ is as described above (Theorem 3.3 (3)), we deduce from our perturbative lemma the desired localisation of the spectrum of $\mathcal{L}_\epsilon$ in Theorem 3.3 (4). (Applicability of perturbation theory does not contradict the fact that the coupled and uncoupled SRB measures are in general mutually singular, since $\mathcal{L}_\epsilon$ acts on densities with respect to μ_ϵ for each $\epsilon \geq 0$.) Since the finite Λ versions of the Banach space are just bounded analytic functions in the polyannulus (with our Banach norm equivalent to supremum, but with constants growing exponentially with $|\Lambda|$), we prove stability of the spectrum of the compact truncated operators $L_{\epsilon,\Lambda}$ and $P_{\epsilon,\Lambda}$ as a side-effect (Theorem 3.3(1)). Finally, exponential decay of time correlations (Corollary 3.5) is obtained for observables in $\mathcal{B}$. The bounds in Section 3 often use results and techniques described in Section 2.

While we were finishing writing the present paper, results of T. Fischer and H.H. Rugh on weakly coupled expanding analytic circle maps [FR] were brought to our attention. Combining the simpler polymer expansion introduced by Maes and van Moffaert in [MM] together with a new kernel representation of the truncated Perron-Frobenius operators (which avoids the exponential growth for the topological tensor product embeddings in Lemma 1.1 below), they construct a Banach space of observables on which the (unnormalised) transfer operators associated to the full coupled systems have a uniform gap. (Their Banach space is the subset of those vectors in our Fréchet space of Section 2 which have exponentially bounded growth of semi-norms $\|\varphi\|_\Lambda$, for a well-chosen rate.) Their analysis, which (just as ours) relies heavily on the analytic properties of the local dynamics, works for any spatial dimension $d \geq 1$. However, they do not study the spectrum of the operator beyond the first gap.

1. Weakly coupled analytic circle maps: definitions and preliminaries

Let $\gamma > 1$, and $f : S^1 \to S^1$ be a real analytic locally γ-expanding circle map (i.e., $\|D_x f\, v\| \geq \gamma \|v\|$ for all $x \in S^1$ and all $v \in T_x S^1$). Assume to fix ideas that f is orientation preserving. It is well-known (see, e.g., [KS]) that such a map f admits a unique invariant Borel probability measure μ_0 on S^1 which is absolutely continuous with respect to Lebesgue measure m, with positive and analytic density h.

For $U \subset \mathbb{C}$ a bounded connected open set, we write $\mathcal{H}(U)$ for the Banach space of functions analytic in U with a bounded extension to $\overline{U}$, endowed with the supremum norm. For $0 < \alpha < 1$ the open annulus $\{z \in \mathbb{C} \mid 1 - \alpha < |z| < 1/(1-\alpha)\}$ is denoted by $D(\alpha)$. For $n \geq 1$, we write $D(\alpha)^n = D(\alpha) \times \cdots \times D(\alpha)$ for the corresponding polyannulus in $\mathbb{C}^n$.

Let $0 < \alpha_1 < \alpha_0 < 1$ be fixed small enough such that each of the (finitely many) local inverse branches of f extends to $D(\alpha_0)$ and maps the closure of $D(\alpha_0)$ into $D(\alpha_1)$. From now on, we fix such $0 < \alpha_1 < \alpha < \alpha_0 < 1$ (in Proposition A we will modify them slightly in order to take the coupling into account), and use the notation

$$D = D(\alpha)\,, \tag{1.1}$$

and continue to write f for the complex extensions of f. In fact, the density of μ_0 is the restriction to S^1 of the unique fixed point (denoted also by h) of the transfer operator $P : \mathcal{H}(D) \to \mathcal{H}(D)$ defined by

$$(P\varphi)(x) = \sum_{\substack{y \in D \\ f(y) = x}} \frac{\varphi(y)}{Df(y)}\,. \tag{1.2}$$

Note that

$$\int_{S^1} \psi \circ f \cdot \varphi \, dm = \int_{S^1} \psi \cdot P(\varphi)\, dm \tag{1.3}$$

for all $\psi, \varphi \in C(S^1)$, where $C(S^1)$ is the set of continuous (complex valued) functions on S^1. Thus the dual of the restriction of P to functions on the circle preserves Lesbesgue measure.

For further use, we recall some properties of P (see [Ru1], [Ma]): it is a compact operator with spectral radius equal to 1, a simple eigenvalue at 1 (for the eigenfunction h), and no other eigenvalues of modulus 1. In particular, the nonzero eigenvalues of P form an at most countable set

$$\Sigma = \mathrm{sp}\,(P) = \{\rho_0 = 1\} \cup \{\rho_i \mid i \geq 1\}\,. \tag{1.4}$$

We set

$$\kappa_1 = \sup\{|z| \mid z \in \Sigma\,, z \neq 1\} < 1\,. \tag{1.5}$$

The infinite-dimensional spaces on which our dynamics takes place are the spaces

$$\mathcal{X} = \prod_{\mathbb{Z}^d} S^1 \,, \mathcal{D} = \prod_{\mathbb{Z}^d} D \,, \tag{1.6}$$

(where $d \geq 1$), endowed with the product topology and the Borel σ-algebra inherited from S^1, respectively D. The *uncoupled lattice map* $F : \mathcal{X} \to \mathcal{X}$ is defined by

$$F = \underset{\mathbb{Z}^d}{\otimes} f \,, \quad \text{ i.e., } F(x)_i = f(x_i) \,. \tag{1.7}$$

For any nonempty $\Lambda \subset \mathbb{Z}^d$, we write $|\Lambda|$ for the cardinality of Λ, and we set

$$F_\Lambda = \underset{\Lambda}{\otimes} f \,, \quad \mathcal{X}_\Lambda = \prod_\Lambda S^1 \,, \quad \text{and} \quad \mathcal{D}_\Lambda = \prod_\Lambda D \,. \tag{1.8}$$

Each of the finitely many inverse branches of F_Λ extends to $\mathcal{D}_\Lambda$.

The *coupling map* $\Phi_\epsilon : \mathcal{X} \to \mathcal{X}$ is taken as in [BK1]:

$$(\Phi_\epsilon(x))_i = x_i \, \exp\left(2\pi i \epsilon \sum_{j \in \mathbb{Z}^d} g_{|i-j|}(x_i, x_j) \right) , \tag{1.9}$$

where $\epsilon > 0$ is a small coupling parameter, each $g_\ell : D \times D \to \mathbb{C}$ is analytic and bounded on $D \times D$ such that its restriction to $S^1 \times S^1$ is real valued, and there are $C > 0$ and $\lambda > 0$ so that

$$|g_\ell(u,v)| \leq C e^{-\lambda \ell} \,, \forall \ell \geq 0 \,, \forall u, v \in D \,. \tag{1.10}$$

Finally, the *coupled lattice map* $F_\epsilon : \mathcal{X} \to \mathcal{X}$ is defined by

$$F_\epsilon = \Phi_\epsilon \circ F \,. \tag{1.11}$$

Note that $F_0 = F$. By definition, an F_ϵ-invariant Borel probability measure μ on $\mathcal{X}$ is called an *SRB measure* if all its marginals on the finite tori $\mathcal{X}_\Lambda$ are absolutely continuous with respect to Lebesgue measure. (See also Corollary 2.8 (1) below.)

We end with a useful preliminary result on topological tensor products of spaces of analytic functions, which is certainly classical, but for which we found no reference (the computation was pointed out to us by H.H. Rugh). If $\mathcal{B}$ is a Banach space, we write $\mathcal{B} \hat{\otimes} \mathcal{B}$ for the norm completion of the topological tensor product, see e.g. [Tr].

Lemma 1.1. *For any $0 < \alpha' < \alpha$ and any $n \geq 1$, we have*

$$\mathcal{H}(D(\alpha)^n) \subset \hat{\otimes}_{i=1}^n \mathcal{H}(D(\alpha')) \,. \tag{1.12}$$

The $\mathcal{H}(D(\alpha)^n)$ and $\hat{\otimes}_{i=1}^n \mathcal{H}(D(\alpha'))$ norms of $\varphi \in \mathcal{H}(D(\alpha)^n)$ are related by

$$\|\varphi\|_{\hat{\otimes}_{i=1}^n \mathcal{H}(D(\alpha'))} \leq \left(2 \frac{1-\alpha'}{\alpha - \alpha'} \right)^n \|\varphi\|_{\mathcal{H}(D(\alpha)^n)} . \tag{1.13}$$

Note that the inclusion

$$\hat{\otimes}_{i=1}^n \mathcal{H}(D(\alpha')) \subset \mathcal{H}(D(\alpha')^n) \tag{1.14}$$

with $\|\varphi\|_{\mathcal{H}(D(\alpha')^n)} \le \|\varphi\|_{\hat{\otimes}_{i=1}^n \mathcal{H}(D(\alpha'))}$ is clear.

Proof of Lemma 1.1. We may assume that $\|\varphi\|_{\mathcal{H}(\mathcal{D}(\alpha)^n)} = 1$. Writing $\Gamma_\alpha^- = \{w \in \mathbb{C} \mid |w| = 1-\alpha\}$, $\Gamma_\alpha^+ = \{w \in \mathbb{C} \mid |w| = (1-\alpha)^{-1}\}$, and $\Gamma_\alpha = \Gamma_\alpha^- \cup \Gamma_\alpha^+$ for the boundary of the annulus $D(\alpha)$, the Cauchy formula gives the following tensor product decomposition (the Kronecker delta is denoted by $\delta(u,v)$) for φ:

$$\begin{aligned}\varphi(z_1,\dots,z_n) &= \oint_{\Gamma_\alpha} \frac{dw_1}{2i\pi} \cdots \oint_{\Gamma_\alpha} \frac{dw_n}{2i\pi} \frac{\varphi(w_1,\dots,w_n)}{(w_1-z_1)\cdots(w_n-z_n)} \\ &= \sum_{s_i \in \{\pm\}} (\prod_i s_i) \sum_{k_1=0}^\infty \cdots \sum_{k_n=0}^\infty z_1^{s_1 k_1} \cdots z_n^{s_n k_n} \oint_{\Gamma_\alpha^{s_1}} \frac{dw_1}{2i\pi} \cdots \oint_{\Gamma_\alpha^{s_n}} \frac{dw_n}{2i\pi} \\ &\qquad \frac{1}{z_1^{\delta(s_1,-)} \cdots z_n^{\delta(s_n,-)} w_1^{\delta(s_1,+)} \cdots w_n^{\delta(s_n,+)}} \frac{\varphi(w_1,\dots,w_n)}{w_1^{s_1 k_1} \cdots w_n^{s_n k_n}} .\end{aligned} \tag{1.15}$$

Each

$$\begin{aligned}\psi(z_1,\dots,z_n) = z_1^{s_1 k_1} \cdots z_n^{s_n k_n} \oint_{\Gamma_\alpha^{s_1}} \frac{dw_1}{2i\pi} \cdots \oint_{\Gamma_\alpha^{s_n}} \frac{dw_n}{2i\pi} \\ \frac{1}{z_1^{\delta(s_1,-)} \cdots z_n^{\delta(s_n,-)} w_1^{\delta(s_1,+)} \cdots w_n^{\delta(s_n,+)}} \frac{\varphi(w_1,\dots,w_n)}{w_1^{s_1 k_1} \cdots w_n^{s_n k_n}}\end{aligned} \tag{1.16}$$

is in $\otimes_i \mathcal{H}(D(\alpha'))$, with supremum bounded by $((1-\alpha)/(1-\alpha'))^{\sum_i k_i}$. We thus find

$$\|\varphi\|_{\hat{\otimes}_{i=1}^n \mathcal{H}(D(\alpha'))} \le 2^n \sum_{k_1=0}^\infty \cdots \sum_{k_n=0}^\infty \left(\frac{1-\alpha}{1-\alpha'}\right)^{\sum_{i=1}^n k_i} \le \left(\frac{2}{1-\frac{1-\alpha}{1-\alpha'}}\right)^n , \tag{1.17}$$

as claimed. □

Remark 1.2. As an easy consequence of Lemma 1.1, we find that if Q_i, $i = 1,\dots,n$, are bounded operators on $\mathcal{H}(D(\alpha'))$, then for all $\varphi \in \mathcal{H}(D(\alpha)^n)$

$$\|(\otimes_{i=1}^n Q_i)\,\varphi\|_{\mathcal{H}(D(\alpha')^n)} \le \left(2\frac{1-\alpha'}{\alpha-\alpha'}\right)^n (\prod_{i=1}^n \|Q_i\|_{\mathcal{H}(D(\alpha'))})\,\|\varphi\|_{\mathcal{H}(D(\alpha)^n)} . \tag{1.18}$$

If we make the stronger assumption that the $Q_i' : \mathcal{H}(D(\alpha')) \to \mathcal{H}(D(\alpha))$, $i = 1,\dots,n$, are bounded operators for some $0 < \alpha' < \alpha$, with operator norm $\|Q_i'\|$, then for all $0 < \alpha' < \alpha'' \le \alpha$ and all $\varphi \in \mathcal{H}(D(\alpha)^n)$

$$\|(\otimes_{i=1}^n Q_i')\,\varphi\|_{\mathcal{H}(D(\alpha)^n)} \le \left(2\frac{1-\alpha'}{\alpha''-\alpha'}\right)^n (\prod_{i=1}^n \|Q_i'\|)\,\|\varphi\|_{\mathcal{H}(D(\alpha'')^n)} . \tag{1.19}$$

Remark 1.3. Modulo technical complications (related in particular to our use of Laurent coefficients in Section 3), our results should apply to real analytic locally γ-expanding transformations ($\gamma > 1$) of compact connected real analytic Riemann manifolds.

2. Spectrum on a Fréchet space of observables

2.A Preliminaries and statements.

In this section we describe the spectrum of the coupled transfer operator $\mathcal{P}_\epsilon$ associated to F_ϵ on a Fréchet space $\mathcal{F}$ (see e.g. [Yo] for the spectral theory of linear operators on complex linear topological spaces, in particular Fréchet spaces). To define $\mathcal{F}$, we need more notation. For any finite nonempty $\Lambda \subset \mathbb{Z}^d$ we write $\mathcal{H}^\Lambda$ for $\mathcal{H}(\mathcal{D}_\Lambda)$ (the polyannulus $\mathcal{D}_\Lambda = \mathcal{D}_\Lambda(\alpha)$ was defined in (1.8)). By definition, $\mathcal{H}^\emptyset$ is identified with $\mathbb{C}$. Let m denote normalised Lebesgue measure on S^1. For any finite non-empty $\Lambda \subset \mathbb{Z}^d$ write m^Λ for the probability measure $\otimes_{a\in\Lambda} m$, and l_Λ for the continuous operators defined on each $\mathcal{H}^{\Lambda'}$ (which Λ' is being considered will be clear from the context) with $\Lambda \subset \Lambda'$ finite by

$$l_\Lambda : \mathcal{H}^{\Lambda'} \to \mathcal{H}^{\Lambda'\setminus\Lambda}\,, \qquad (l_\Lambda(\varphi))|\mathcal{X}_{\Lambda'\setminus\Lambda} = \int_{\mathcal{X}_\Lambda} \varphi \, dm^\Lambda\,. \tag{2.1}$$

For any non-empty $\Lambda \subset \Lambda' \subseteq \mathbb{Z}^d$, we denote by $\pi_\Lambda^{\Lambda'}$ the canonical projection

$$\pi_\Lambda^{\Lambda'} : \mathcal{D}_{\Lambda'} \to \mathcal{D}_\Lambda\,, \tag{2.2}$$

writing also $\pi_\Lambda^{\Lambda'}$ for the restriction $\pi_\Lambda^{\Lambda'}|_{\mathcal{X}_{\Lambda'}} : \mathcal{X}_{\Lambda'} \to \mathcal{X}_\Lambda$. (If $\Lambda' = \mathbb{Z}^d$ we just write π_Λ.)

Definition 2.1. Let $\mathcal{F}$ be the space of families

$$\varphi = (\varphi_\Lambda \in \mathcal{H}^\Lambda \mid \Lambda = \emptyset \text{ or } \Lambda = [-p,p]^d \subset \mathbb{Z}^d\,,\, p \in \mathbb{Z}^+)\,, \tag{2.3}$$

such that for any finite square boxes $\Lambda \subset \Lambda'$ in $\mathbb{Z}^d$, we have

$$\varphi_\Lambda|\mathcal{X}_\Lambda = l_{\Lambda'\setminus\Lambda}(\varphi_{\Lambda'})|\mathcal{X}_\Lambda = \int_{\mathcal{X}_{\Lambda'\setminus\Lambda}} \varphi_{\Lambda'} \, dm^{\Lambda'\setminus\Lambda}\,. \tag{2.4}$$

The topology of $\mathcal{F}$ is generated by the (countable) family of semi-norms

$$\|\varphi\|_\Lambda = \|\varphi_\Lambda\|_{\mathcal{H}^\Lambda} = \sup_{\overline{\mathcal{D}}_\Lambda} |\varphi_\Lambda|\,, \tag{2.5}$$

indexed by all finite square boxes $\Lambda \subset \mathbb{Z}^d$, together with the empty set $\Lambda = \emptyset$. An element φ of $\mathcal{F}$ is called *locally supported* if there exists a finite Λ such that $\varphi_{\Lambda'} = \varphi_\Lambda \circ \pi_\Lambda^{\Lambda'}$, for all finite boxes $\Lambda' \supset \Lambda$, and the smallest such Λ is the *support* of φ. (We then frequently write φ_Λ instead of φ, slightly abusing notation.)

We now verify that $\mathcal{F}$ is a Fréchet space: Consider a Cauchy sequence $\nu^{(l)} \in \mathcal{F}$, i.e., for any finite $\Lambda \subset \mathbb{Z}^d$ and any $\zeta > 0$, there is $N_0(\Lambda) \in \mathbb{N}$ so that for all $l, k \geq N_0$ we have $\|\varphi^{(l)} - \varphi^{(k)}\|_\Lambda < \zeta$. Since $\mathcal{H}^\Lambda$ is a Banach space we find that $\varphi_\Lambda^{(l)} \to \varphi_\Lambda \in \mathcal{H}^\Lambda$.

Since, for all l and all finite $\Lambda \subset \Lambda'$, we have $\varphi^{(l)}_\Lambda|_{\mathcal{X}_\Lambda} = \int_{\mathcal{X}_{\Lambda'\setminus\Lambda}} \varphi^{(l)}_{\Lambda'}|_{\mathcal{X}_{\Lambda'}}\, dm^{\Lambda'\setminus\Lambda}$, we get (2.4) by passing to the limit.

Let $\mathcal{M}$ be the space of (finite) complex Borel measures on $\mathcal{X}$ (see, e.g., [DS] or [R] for definitions). If $\nu \in \mathcal{M}$, we write $\nu_\Lambda = (\pi_\Lambda)_* \nu$ for its marginal on $\mathcal{X}_\Lambda$ for any $\Lambda \subset \mathbb{Z}^d$, and $\nu_\emptyset$ for its mass $\nu_\emptyset = \nu(\mathcal{X}) \in \mathbb{C}$. A complex Borel measure ν on $\mathcal{X}$ can be represented by the family ν_Λ of its marginals on $\mathcal{X}_\Lambda$ for all finite square boxes $\Lambda \subset \mathbb{Z}^d$, by definition of the product topology. By a slight abuse of notation, we write $\mathcal{F} \cap \mathcal{M}$ for the vector space of those $\nu \in \mathcal{M}$ for which each ν_Λ, with Λ finite, is absolutely continuous with respect to Lebesgue measure m^Λ on $\mathcal{X}_\Lambda$, with an analytic density which can be extended to a function $\varphi_\Lambda \in \mathcal{H}^\Lambda$ (condition (2.4) is automatically satisfied). Any locally supported $\varphi \in \mathcal{F}$ is in $\mathcal{F} \cap \mathcal{M}$, but the space $\mathcal{F} \cap \mathcal{M}$ is not complete (see Remark 2.2 below). Note that $\varphi \in \mathcal{F}$ defines an element of $\mathcal{M}$ if and only if there is a constant $C > 0$ with $\int_{\mathcal{X}_\Lambda} |\varphi_\Lambda|\, dm^\Lambda \le C$ for all finite Λ. (Use the Riesz representation theorem which says that complex measures are the dual space to continuous functions on $\mathcal{X}$.)

Remark 2.2. The Fréchet space $\mathcal{F}$ is quite large and in particular not a subset of $\mathcal{M}$. Assuming $d = 1$ for simplicity, here is an example of a Cauchy sequence in $\mathcal{F}$:

$$\varphi^{(\ell)}_\emptyset = 1\,, \quad \varphi^{(\ell)}_{[-p,p]}(x_{-p}, \ldots, x_p) = \prod_{j=-\min(\ell,p)}^{\min(\ell,p)} (2\sin(\frac{\log x_j}{2i\pi}) + 1)\,. \tag{2.6}$$

Each $\varphi^{(\ell)}$ corresponds to the marginal of a (finite) signed measure on $\mathcal{X}$, with total variation $\int_{\mathcal{X}_{[-\ell,\ell]}} |\varphi^{(\ell)}|\, dm^{[-\ell,\ell]} \ge (3/2)^{2\ell}$. The limit of the sequence is not associated to a finite signed measure on $\mathcal{X}$. Since the total variation does not seem to be easily controllable (see in particular Lemma 2.4), we work with the abstract space $\mathcal{F}$.

For small $\epsilon \ge 0$, we will next define $\mathcal{P}_\epsilon$ on $\mathcal{F}$ such that its restriction to $\mathcal{F} \cap \mathcal{M}$ is the usual Perron-Frobenius-Ruelle transfer operator of F_ϵ, i.e., for $\mu \in \mathcal{F} \cap \mathcal{M}$ and each Borel set $E \subset \mathcal{X}$ we will have

$$(\mathcal{P}_\epsilon \mu)(E) = ((F_\epsilon)_* \mu)(E) = \mu(F_\epsilon^{-1} E)\,. \tag{2.7}$$

The uncoupled operator

We define the uncoupled transfer operator $\mathcal{P}_0$ on $\mathcal{F}$ by setting $(\mathcal{P}_0 \varphi)_\Lambda = P_\Lambda(\varphi_\Lambda)$ for each finite box $\Lambda \subset \mathbb{Z}^d$, where

$$P_\Lambda \varphi_\Lambda(x) = \sum_{\substack{y \in \mathcal{D}_\Lambda \\ F_\Lambda(y) = x}} \varphi_\Lambda(y) \prod_{i \in \Lambda} \frac{1}{Df(y_i)}\,. \tag{2.8}$$

It is easy to see that $\mathcal{P}_0 \varphi \in \mathcal{F}$ (i.e., satisfies (2.4)) if $\varphi \in \mathcal{F}$. To check that the definition of $\mathcal{P}_0$ is compatible with (2.7), let $\varphi = \nu \in \mathcal{F} \cap \mathcal{M}$ ($\nu_\Lambda = \varphi_\Lambda\, m^\Lambda$ on $\mathcal{X}_\Lambda$, with density $\varphi_\Lambda \in \mathcal{H}^\Lambda$). We need to check that $((F_0)_* \nu)_\Lambda = (\mathcal{P}_0 \varphi)_\Lambda m^\Lambda$ for each finite box $\Lambda \subset \mathbb{Z}^d$

which will also prove that $\mathcal{P}_0 : \mathcal{M} \cap \mathcal{F} \to \mathcal{M} \cap \mathcal{F}$. This is clear since for any $\psi \in C(\mathcal{X}_\Lambda)$ we have

$$\begin{aligned}\int_{\mathcal{X}_\Lambda} \psi \, d(F_* \nu)_\Lambda &= \int_{\mathcal{X}} \psi \circ F \, d\nu \\ &= \int_{\mathcal{X}_\Lambda} \psi \circ F \, d\nu_\Lambda = \int_{\mathcal{X}_\Lambda} \psi \cdot P_\Lambda(\varphi_\Lambda) \, dm^\Lambda \, . \end{aligned} \tag{2.9}$$

Proposition 2.3 (Uncoupled spectrum).

(1) *The operator $\mathcal{P}_0 : \mathcal{F} \to \mathcal{F}$ is bounded.*

(2) *The nonzero spectrum of $\mathcal{P}_0 : \mathcal{F} \to \mathcal{F}$ consists of the set*

$$\Sigma_\infty = \{1\} \cup \{\rho = \prod_{\ell=1}^{m} \rho_{i_\ell} \mid m \geq 1 \, , 1 \neq \rho_\ell \in \Sigma\} \, , \tag{2.10}$$

with Σ as in (1.4). Furthermore, 1 is a simple eigenvalue, and the other elements of Σ_∞ are (isolated) eigenvalues of infinite multiplicity.

For $\Lambda \subset \mathbb{Z}^d$ we write $h_\Lambda = \otimes_{a \in \Lambda} h$, where h is the density for the one-dimensional SRB-measure with respect to normalised Lebesgue measure introduced in Section 1.

Proof of Proposition 2.3. (1) Our choice of $0 < \alpha_1 < \alpha < \alpha_0$ implies that the single-site operator $P : \mathcal{H}(D(\alpha_1)) \to \mathcal{H}(D(\alpha))$ is bounded. Fix a finite $\Lambda \subset \mathbb{Z}^d$. Since $P_\Lambda = \otimes_\Lambda P$ we see by applying Remark 1.2 that there is a constant $C > 0$ so that

$$\|P_\Lambda \varphi_\Lambda\|_{\mathcal{H}^\Lambda} \leq C^{|\Lambda|} \|\varphi_\Lambda\|_{\mathcal{H}^\Lambda} \, , \tag{2.11}$$

for all $\varphi_\Lambda \in \mathcal{H}^\Lambda$. This proves that the operator $\mathcal{P}_0 : \mathcal{F} \to \mathcal{F}$ is continuous.

(2) For any finite $\Lambda \subset \mathbb{Z}^d$, the operator P_Λ acting on $\mathcal{H}^\Lambda$ is compact (this follows e.g. from Montel's theorem). One can show (for example by using the associated Fredholm determinants, see [Ru1, Ru2]) that the nonzero spectrum Σ_Λ of P_Λ consists of the simple eigenvalue 1, together with isolated eigenvalues of finite multiplicity at all points

$$\rho = \prod_{\ell=1}^{|\Lambda|} \rho_{i_\ell} \tag{2.12}$$

with the $\rho_{i_\ell} \in \Sigma$, at least one of them different from 1. It follows in particular that the nonzero spectrum of P_Λ is contained in the spectrum of $P_{\Lambda'}$ whenever $|\Lambda| \leq |\Lambda'|$. Note that, although the multiplicity of any given eigenvalue $\rho \neq 1$ of the form (2.12) increases with $|\Lambda|$, there is for any fixed $\kappa > 0$ some $\ell(\kappa) \geq 1$ so that the sets

$$\Sigma_{\Lambda, > \kappa} = \{z \in \Sigma_\Lambda \mid |z| > \kappa\} \tag{2.13}$$

are identical for all finite Λ with $|\Lambda| > \ell(\kappa)$.

Clearly, 1 is an eigenvalue for $\mathcal{P}_0$ (with eigenfunction $\otimes_{\mathbb{Z}^d} h \in \mathcal{F} \cap \mathcal{M}$). We next show that any ρ of the form (2.12) is an eigenvalue of infinite multiplicity of $\mathcal{P}_0$. Let $\psi_\ell \in \mathcal{H}(D)$, $\ell = 1, \dots, |\Lambda|$, be a fixed family of eigenfunctions corresponding to the eigenvalues ρ_{i_ℓ} appearing in the decomposition (2.12) of ρ. Then it is easy to check that for any numbering j_ℓ of the points in Λ, the function ψ_Λ defined by

$$\psi_\Lambda(x) = \psi_1(x_{j_1}) \otimes \cdots \otimes \psi_{|\Lambda|}(x_{j_{|\Lambda|}}) \tag{2.14}$$

satisfies $P_\Lambda \psi_\Lambda = \rho \psi_\Lambda$. If ψ is the locally supported element of $\mathcal{F}$ associated to ψ_Λ, we have $\mathcal{P}_0 \psi = \rho \psi$, and we may generate an infinite dimensional eigenspace for $\mathcal{P}_0$ corresponding to the eigenvalue ρ by acting on ψ with spatial translations of $\mathbb{Z}^d$. Generalised eigenfunctions (that is, $(\rho - \mathcal{P}_0)^k \varphi = 0$ for some $k \geq 2$) can be treated similarly.

To prove that 1 is a simple eigenvalue and that $\mathcal{P}_0$ does not have anything else in its spectrum, we shall need the following consequence of the above-mentioned spectral properties of P_Λ. For any $\kappa > 0$ such that Σ_∞ contains no element of modulus κ, and for any finite $\Lambda \subset \mathbb{Z}^d$, writing $R_{\Lambda, <\kappa}$ for the spectral projection corresponding to $\{z \in \mathrm{sp}\,(P_\Lambda) \mid |z| < \kappa\}$, the spectral radius formula says that there is $C(\Lambda) \geq 1$ so that

$$\| P_\Lambda^n R_{\Lambda, <\kappa} \varphi_\Lambda \|_{\mathcal{H}^\Lambda} \leq C(\Lambda)\, \kappa^n \, \|\varphi_\Lambda\|_{\mathcal{H}^\Lambda} \,, \tag{2.15}$$

for all $n \geq 0$ and all $\varphi_\Lambda \in \mathcal{H}^\Lambda$. Write $Q_{\Lambda, >\kappa} = \mathrm{Id} - R_{\Lambda, <\kappa}$ for the complementary (finite rank) projection, and define continuous operators $\mathcal{R}_{<\kappa}$ and $\mathcal{Q}_{>\kappa}$ on $\mathcal{F}$ by $(\mathcal{R}_{<\kappa} \varphi)_\Lambda = R_{\Lambda, <\kappa} \varphi_\Lambda$ and $\mathcal{Q}_{>\kappa} = \mathrm{Id} - \mathcal{R}_{<\kappa}$ or equivalently $(\mathcal{Q}_{>\kappa} \varphi)_\Lambda = Q_{\Lambda, >\kappa} \varphi_\Lambda$. We have $\mathcal{R}_{<\kappa}^2 = \mathcal{R}_{<\kappa}$ (since $R_{\Lambda, <\kappa}^2 = R_{\Lambda, <\kappa}$ for all finite Λ) and $\mathcal{Q}_{>\kappa}^2 = \mathcal{Q}_{>\kappa}$. Also $\mathcal{R}_{<\kappa} \mathcal{Q}_{>\kappa} = \mathcal{Q}_{>\kappa} \mathcal{R}_{<\kappa} = 0$. (We check similarly that $\mathcal{P}_0 \mathcal{Q}_{>\kappa} = \mathcal{Q}_{>\kappa} \mathcal{P}_0$ and thus $\mathcal{P}_0 \mathcal{R}_{<\kappa} = \mathcal{R}_{<\kappa} \mathcal{P}_0$.) It follows that the spectrum of $\mathcal{P}_0$ on $\mathcal{F}$ is a subset of the union of the spectra of $\mathcal{P}_0 \mathcal{Q}_{>\kappa}$ and $\mathcal{P}_0 \mathcal{R}_{<\kappa}$. Indeed, for z in the intersection of the resolvent sets of $\mathcal{P}_0 \mathcal{Q}_{>\kappa}$ and $\mathcal{P}_0 \mathcal{R}_{<\kappa}$

$$(z - \mathcal{P}_0)^{-1} = (z - \mathcal{P}_0 \mathcal{R}_{<\kappa})^{-1} \mathcal{R}_{<\kappa} + (z - \mathcal{P}_0 \mathcal{Q}_{>\kappa})^{-1} \mathcal{Q}_{>\kappa} \,. \tag{2.16}$$

We shall prove that the spectrum of $\mathcal{P}_0 \mathcal{R}_{<\kappa}$ is a subset of the disc of radius κ and that the spectrum of $\mathcal{P}_0 \mathcal{Q}_{>\kappa}$ is composed of the simple eigenvalue 1 together with eigenvalues of infinite multiplicities in the set $\Sigma_{\infty, >\kappa} \setminus \{1\}$ where $\Sigma_{\infty, >\kappa} = \{z \in \Sigma_\infty \mid |z| > \kappa\}$. Since $\kappa > 0$ is arbitrarily small, this will prove the second claim of Proposition 2.3.

To show that the spectrum of $\mathcal{P}_0 \mathcal{R}_{<\kappa}$ is contained in the disc of radius κ, we find for any $|z| > \kappa$, using (2.15) for each finite Λ, a convergent Neumann series for the Λ-semi-norm of the resolvent:

$$\begin{aligned} \|(z - \mathcal{P}_0 \mathcal{R}_{<\kappa})^{-1} \varphi\|_\Lambda &\leq \frac{1}{|z|} \sum_{j=0}^\infty \frac{\|P_\Lambda^j R_{\Lambda, <\kappa}\|_\Lambda}{|z|^j} \|\varphi\|_\Lambda \\ &\leq \frac{1}{|z|} \sum_{j=0}^\infty C(\Lambda) \left| \frac{\kappa}{z} \right|^j \|\varphi\|_\Lambda \\ &\leq \frac{C(\Lambda)}{|z| - \kappa} \|\varphi\|_\Lambda \,. \end{aligned} \tag{2.17}$$

We now describe the spectrum of $\mathcal{P}_0\mathcal{Q}_{>\kappa}$. For each fixed finite $\Lambda \subset \mathbb{Z}^d$, the operator $P_\Lambda Q_{\Lambda,>\kappa}$ is a finite rank operator with spectrum in $\Sigma_{\Lambda,>\kappa}$. Therefore, for $z \notin \Sigma_{\Lambda,>\kappa}$ the resolvent $(z - P_\Lambda Q_{\Lambda,>\kappa})^{-1}$ is a bounded operator on $\mathcal{H}^\Lambda$, with norm say $C_z(\Lambda)$. Since $(\mathcal{P}\mathcal{Q}_{>\kappa}\varphi)_\Lambda = P_\Lambda Q_{\Lambda,>\kappa}\varphi_\Lambda$, we have for any $z \notin \Sigma_{\infty,>\kappa}$

$$\|(z - \mathcal{P}\mathcal{Q}_{>\kappa})^{-1}\varphi\|_\Lambda \le C_z(\Lambda)\|\varphi\|_\Lambda\,, \tag{2.18}$$

proving that $(z - \mathcal{P}\mathcal{Q}_{>\kappa})^{-1}$ is a continuous operator on $\mathcal{F}$. Simplicity of the eigenvalue 1 follows from the fact that for $\kappa_1 < \kappa < 1$ we have $Q_{\Lambda,>\kappa}\varphi_\Lambda = h_\Lambda \int \varphi_\Lambda\, dm^\Lambda$, so that $(\mathcal{Q}_{>\kappa}\varphi)_\Lambda = h_\Lambda \cdot \int \varphi_\Lambda\, dm^\Lambda$, showing that $Q_{>\kappa}$ is rank one. □

The coupled operator

We now define $\mathcal{P}_\epsilon : \mathrm{Dom}(\mathcal{P}_\epsilon) \to \mathcal{F}$, where the domain of $\mathcal{P}_\epsilon$ is the dense vector subspace of locally supported elements of $\mathcal{F}$. For any finite $\Lambda \subset \mathbb{Z}^d$, we define a cutoff (of open boundary condition type) of F_ϵ on $\mathcal{X}_\Lambda$:

$$F_{\epsilon,\Lambda} : \mathcal{X}_\Lambda \to \mathcal{X}_\Lambda\,, \qquad F_{\epsilon,\Lambda} = \Phi_{\epsilon,\Lambda} \circ F_\Lambda\,, \tag{2.19}$$

where $\Phi_{\epsilon,\Lambda}$ is defined by (1.9) such that the sum is only over $j \in \Lambda$. Each map $F_{\epsilon,\Lambda}$ is analytic. Let $P_{\epsilon,\Lambda}$ be the transfer operator associated to $F_{\epsilon,\Lambda}$, which we view first as acting on functions defined on $\mathcal{X}_\Lambda$:

$$(P_{\epsilon,\Lambda}\varphi_\Lambda)(x) = \sum_{\substack{y\in\mathcal{X}_\Lambda \\ F_{\epsilon,\Lambda}(y)=x}} \frac{\varphi_\Lambda(y)}{\mathrm{Det}DF_{\epsilon,\Lambda}(y)}\,. \tag{2.20}$$

Clearly, l_Λ is an eigenfunctional of the dual of $P_{\epsilon,\Lambda}$ acting on Radon measures on $\mathcal{X}_\Lambda$.

We shall need the following result of Bricmont and Kupiainen (which implicitly uses (1.19)):

Proposition A [BK1, Proposition 2, Remark p. 384]. *If $0 < \alpha_0 < 1$ is such that each local inverse branch of f maps the closure of $D(\alpha_0)$ into its interior, then for any $0 < \alpha < \alpha_0$ there is $\epsilon_0 > 0$ (which depends on λ as defined in (1.10)), so that if $0 \le \epsilon < \epsilon_0$ there is $0 < \alpha' < \alpha$ such that for any finite $\Lambda \subset \mathbb{Z}^d$, the operator $P_{\epsilon,\Lambda}$ may be extended to a bounded operator on $\mathcal{H}^\Lambda = \mathcal{H}(\mathcal{D}_\Lambda(\alpha))$, and, in fact, to a bounded operator from $\mathcal{H}(\mathcal{D}_\Lambda(\alpha'))$ to $\mathcal{H}(\mathcal{D}_\Lambda(\alpha))$.*

We assume in the sequel that $0 \le \epsilon < \epsilon_0$ and $0 < \alpha' < \alpha$ are as in the above proposition (strenghtening the condition introduced before (1.1)). We now state our main bound, which will be proved in Subsection 2.B using cluster expansions.

Lemma 2.4 (Main Fréchet bounds). *Let $0 < \alpha' < \alpha$ and $0 \le \epsilon < \epsilon_0$ be as in Proposition A. For a finite square box $\Lambda' \subset \mathbb{Z}^d$ let $P_{\epsilon,\Lambda'}$ be the operator associated to the cutoff map $F_{\epsilon,\Lambda'}$.*

(1) *There is $0 < \epsilon_1 < \epsilon_0$ such that if $0 \le \epsilon < \epsilon_1$ then for all finite boxes Λ', $P_{\epsilon,\Lambda'}$ acting on $\mathcal{H}^{\Lambda'}$ has a simple fixed function $\Omega_{\epsilon,\Lambda'}$ whose restriction to $\mathcal{X}_{\Lambda'}$ is positive. (We normalise $\Omega_{\epsilon,\Lambda'}$ so that $l_{\Lambda'}\Omega_{\epsilon,\Lambda'} = 1$.)*

(2) *Let $R_{\epsilon,\Lambda'} = Id - Q_{\epsilon,\Lambda'}$, where $Q_{\epsilon,\Lambda'}\varphi = \Omega_{\epsilon,\Lambda'} l_{\Lambda'}\varphi$ is the projection onto the one-dimensional eigenspace of $P_{\epsilon,\Lambda'}$ for the eigenvalue 1. For any $\kappa > \kappa_1$ (defined by (1.5)) there are $C > 0$ and $0 < \epsilon_2 < \epsilon_1$ such that for all $0 \le \epsilon < \epsilon_2$, all $\varphi \in \mathcal{F}$, all $n \ge 0$, and all finite boxes $\Lambda \subset \Lambda' \subset \mathbb{Z}^d$*

$$\|P^n_{\epsilon,\Lambda'} R_{\epsilon,\Lambda'} \varphi_{\Lambda'}\|_\Lambda \le \kappa^n e^{C|\Lambda|} \|\varphi\|_{\Lambda'} , \tag{2.21}$$

(where $\|\psi_{\Lambda'}\|_\Lambda$ is defined to be $\|l_{\Lambda'\setminus\Lambda}\psi_{\Lambda'}\|_\Lambda$); also, there exists $\tilde\lambda > 0$ such that for all finite boxes $\Lambda \subset \Lambda' \subset \Lambda'' \subset \mathbb{Z}^d$, writing $d(A, B)$ for the distance between two subsets of $\mathbb{Z}^d$,

$$\|((P^n_{\epsilon,\Lambda'} R_{\epsilon,\Lambda'} \otimes l_{\Lambda''\setminus\Lambda'}) - P^n_{\epsilon,\Lambda''} R_{\epsilon,\Lambda''})\varphi_{\Lambda''}\|_\Lambda \le e^{-\tilde\lambda d(\Lambda,\mathbb{Z}^d\setminus\Lambda')} e^{C|\Lambda|} \kappa^n \|\varphi\|_{\Lambda''} . \tag{2.22}$$

Remark 2.5. It is important that the exponential factor in Lemma 2.4 (2) is $e^{C|\Lambda|}$ and not $e^{C|\Lambda'|}$ or $e^{C|\Lambda''|}$ as in [BK1, (43) and (47) p. 388]. Lemma 2.4 in some sense combines both Propositions 5 and 6 in [BK1]. Note that in Lemma 2.4 ϵ_1 and ϵ_2 depend on λ (similarly as in Proposition A).

We may now state and prove (using Lemma 2.4) the main result of Section 2:

Theorem 2.6 (Coupled operator on Fréchet space: first gap). *Let $0 < \alpha' < \alpha$ be as in Proposition A, $\kappa > \kappa_1$, and $0 \le \epsilon < \epsilon_2$ be as in Lemma 2.4. For each finite box $\Lambda' \subset \mathbb{Z}^d$ let $P_{\epsilon,\Lambda'}$ be the operator associated to the cutoff map $F_{\epsilon,\Lambda'}$.*

(1) *There is $h_\epsilon \in \mathcal{F}\cap\mathcal{M}$ such that each $(h_\epsilon)_\Lambda$ restricted to $\mathcal{X}_\Lambda$ is positive with integral 1, and such that the corresponding Borel measure μ_ϵ on $\mathcal{X}$ is an F_ϵ-invariant probability measure. The measure μ_ϵ is thus an SRB measure for $F_\epsilon : \mathcal{X} \to \mathcal{X}$.*

(2) *For any locally supported $\varphi \in \mathcal{F}$ and each finite box Λ, the limit*

$$(\mathcal{P}_\epsilon\varphi)_\Lambda|_{\mathcal{X}_\Lambda} = \lim_{\Lambda'\to\mathbb{Z}^d} \int_{\mathcal{X}_{\Lambda'\setminus\Lambda}} (P_{\epsilon,\Lambda'}\varphi_{\Lambda'})|_{\mathcal{X}'_\Lambda}\, dm^{\Lambda'\setminus\Lambda} \tag{2.23}$$

(where the $\Lambda' \to \mathbb{Z}^d$ are finite square boxes) exists and extends to an element of $\mathcal{H}^\Lambda$. The coupled transfer operator $\mathcal{P}_\epsilon$ defined by (2.23) on $Dom(\mathcal{P}_\epsilon) \subset \mathcal{F}$, the vector space of locally supported $\varphi \in \mathcal{F}$, is consistent with (2.7). If Φ_ϵ is a finite range coupling, that is, if there exists $M < \infty$ such that $g_l = 0$ for $l \ge M$ (see (1.9)), then $\mathcal{P}_\epsilon$ extends to a continuous operator on $\mathcal{F}$.

(3) *The spectrum of the linear operator $\mathcal{P}_\epsilon$ (with domain locally supported functions in $\mathcal{F}$) outside of the disc of radius κ consists of the simple eigenvalue at 1, and maybe continuous spectrum (i.e., no other eigenvalues and no residual spectrum).*

Remark 2.7. The coupled operator $\mathcal{P}_\epsilon$ does not seem to be closable (see e.g. [Yo] for definitions) in general. Also, it is not clear in general that the spectrum of $\mathcal{P}_\epsilon$ on $\mathcal{F}$ is contained in the unit disc (or in any other finite disc). Note finally that we do not make claims about possible continuous spectrum of $\mathcal{P}_\epsilon$. In fact, even if we were able to show that the spectrum of $\mathcal{P}_\epsilon$ is composed of the simple eigenvalue 1 together with a subset of a disc of radius $\kappa < 1$, this would not be a priori helpful to obtain stronger statements e.g. on decay of correlations. Indeed, the existence of a Neumann series for an operator only makes sense in general in the context of Banach spaces. (The uncoupled situation of Proposition 2.3 was somewhat different, since we could control the semi-norm of the image by the *same* semi-norm for the pre-image, see e.g. (2.15).)

Proof of Theorem 2.6. For finite $\Lambda \subset \mathbb{Z}^d$, we use the notations F_Λ, P_Λ, Q_Λ, R_Λ, and Ω_Λ from the statement of Lemma 2.4, omitting the reference to ϵ.

(1) Using the case $n = 0$ of (2.22), for all finite $\Lambda \subset \mathbb{Z}^d$ and each locally supported $\varphi \in \mathcal{F}$, we see that both limits $\lim_{\Lambda' \to \mathbb{Z}^d} (R_{\Lambda'}\varphi)_\Lambda$ and $\lim_{\Lambda' \to \mathbb{Z}^d} (Q_{\Lambda'}\varphi)_\Lambda$ exist. The notation $R_{\Lambda'}\varphi$ is a shorthand for $R_{\Lambda'}\varphi_{\Lambda'} = R_{\Lambda'}(\varphi_{\Lambda_\varphi} \circ \pi^{\Lambda'}_{\Lambda_\varphi})$, with Λ_φ the support of φ and $\Lambda_\varphi \subset \Lambda'$. We may therefore, on the one hand, define a linear operator $\mathcal{R}_\epsilon$, whose domain is the locally supported elements of $\mathcal{F}$, by

$$(\mathcal{R}_\epsilon \varphi)_\Lambda = \lim_{\Lambda' \to \mathbb{Z}^d} (R_{\Lambda'}\varphi)_\Lambda \,, \tag{2.24}$$

(one easily sees that (2.4) is satisfied for $\mathcal{R}_\epsilon\varphi$) and, on the other, define $h_\epsilon \in \mathcal{F}$ (again, (2.4) is easily checked) by

$$(h_\epsilon)_\Lambda = \lim_{\Lambda' \to \mathbb{Z}^d} (Q_{\Lambda'}1)_\Lambda = \lim_{\Lambda' \to \mathbb{Z}^d} (\Omega_{\Lambda'})_\Lambda \,. \tag{2.25}$$

(Note that $(\Omega_{\Lambda'})_\Lambda$ does not coincide with Ω_Λ in general.) Clearly, the restriction of $(h_\epsilon)_\Lambda$ to $\mathcal{X}_\Lambda$ is a positive function with integral 1 with respect to m^Λ. Thus, h_ϵ defines a Borel probability measure μ_ϵ on $\mathcal{X}$, the densities of the Λ-marginals of which are in $\mathcal{H}^\Lambda$ by construction.

It is easy to check that the probability measures $\Omega_\Lambda m^{\mathbb{Z}^d}$ converge weakly to μ_ϵ as $\Lambda \to \mathbb{Z}^d$. Since $\psi \circ F_\Lambda \circ \pi_\Lambda$ converges to $\psi \circ F_\epsilon$ as $\Lambda \to \mathbb{Z}^d$ in the supremum norm, for any continuous locally supported function ψ on $\mathcal{X}$, we find

$$\begin{aligned}\int_{\mathcal{X}} \psi \circ F_\epsilon d\mu_\epsilon &= \lim_{\Lambda \to \mathbb{Z}^d} \lim_{\Lambda' \to \mathbb{Z}^d} \int \psi \circ F_\Lambda \Omega_{\Lambda'} \, dm^{\Lambda'} \\ &= \lim_{\Lambda \to \mathbb{Z}^d} \int \psi \, P_\Lambda(\Omega_\Lambda) \, dm^\Lambda \\ &= \lim_{\Lambda \to \mathbb{Z}^d} \int \psi \, \Omega_\Lambda \, dm^\Lambda = \int \psi \, d\mu_\epsilon \,.\end{aligned} \tag{2.26}$$

(The double limit can be replaced by a limit along the diagonal sequence.) Since locally supported functions are dense in $C(\mathcal{X})$, μ_ϵ is F_ϵ-invariant and therefore an SRB measure.

(2) It follows from (2.22) for $n = 1$ that the limit $\lim_{\Lambda \to \mathbb{Z}^d} P_\Lambda R_\Lambda \varphi$ exists in $\mathcal{F}$ for any locally supported φ in $\mathcal{F}$. Decomposing $P_\Lambda = P_\Lambda R_\Lambda + \Omega_\Lambda l_\Lambda$, and using the results from the first part of the proof, we see that the limit (2.23) exists as claimed for any locally supported function, defining $\mathcal{P}_\epsilon$. Property (2.7) is easily verified.

Assume now that the coupling is finite range. Then, for all finite $\Lambda \subset \mathbb{Z}^d$, all $\psi \in C(\mathcal{X}_\Lambda)$, and all $\nu \in \mathcal{F} \cap \mathcal{M}$

$$\begin{aligned}\int_{\mathcal{X}_\Lambda} \psi \, d((F_\epsilon)_* \nu)_\Lambda &= \int_{\mathcal{X}} \psi \circ F_\epsilon \, d\nu = \int_{\mathcal{X}} \psi \circ F_{\epsilon,\Lambda_1} \, d\nu \\ &= \int_{\mathcal{X}_{\Lambda_1}} \psi \circ F_{\epsilon,\Lambda_1} d\nu_{\Lambda_1} = \int_{\mathcal{X}_{\Lambda_1}} \psi d((F_{\epsilon,\Lambda_1})_* \nu_{\Lambda_1}) ,\end{aligned} \tag{2.27}$$

where $\Lambda_1 = \{a \in \mathbb{Z}^d \mid d(\{a\}, \Lambda) < M\}$. Thus, using (2.7) and the definition of P_{ϵ,Λ_1}, we see that for any locally supported φ and any finite square box Λ

$$(\mathcal{P}_\epsilon \varphi)_\Lambda = \int_{\mathcal{X}_{\Lambda_1 \setminus \Lambda}} P_{\epsilon,\Lambda_1} \varphi_{\Lambda_1} dm^{\Lambda_1 \setminus \Lambda} \le C(\Lambda_1, \epsilon) \|\varphi\|_{\Lambda_1} . \tag{2.28}$$

Since locally supported elements are dense in $\mathcal{F}$, this proves the claimed continuity of $\mathcal{P}_\epsilon$.

(3) The operator $\mathcal{Q}_\epsilon$ defined on locally supported elements of $\mathcal{F}$ by $\mathcal{Q}_\epsilon = \mathrm{Id} - \mathcal{R}_\epsilon$, or equivalently $(\mathcal{Q}_\epsilon \varphi)_\Lambda = \lim_{\Lambda' \to \mathbb{Z}^d} (Q_{\Lambda'} \varphi)_\Lambda$, satisfies $\mathcal{Q}_\epsilon \varphi = h_\epsilon \cdot \varphi_\emptyset$. Therefore, both $\mathcal{Q}_\epsilon$ and $\mathcal{R}_\epsilon$ can be extended to continuous operators on the whole space $\mathcal{F}$, since for any finite square box Λ we have $\|\mathcal{Q}_\epsilon \varphi\|_\Lambda \le \|\varphi\|_\emptyset \cdot \|h_\epsilon\|_\Lambda$. Moreover, we set $\mathcal{P}_\epsilon h_\epsilon$ to be h_ϵ (according to (2.7)), so that $\mathcal{P}_\epsilon \mathcal{Q}_\epsilon = \mathcal{Q}_\epsilon = \mathcal{Q}_\epsilon \mathcal{P}_\epsilon$, where the last equality is valid for locally supported φ. We define $\mathcal{P}_\epsilon$ of $\mathcal{R}_\epsilon \varphi$ for locally supported φ by linearity, i.e., $\mathcal{P}_\epsilon \mathcal{R}_\epsilon = \mathcal{P}_\epsilon - \mathcal{Q}_\epsilon$. Clearly, $\lim_{\Lambda' \to \mathbb{Z}^d} (P_{\Lambda'} R_{\Lambda'} \varphi)_\Lambda = (\mathcal{P}_\epsilon \mathcal{R}_\epsilon \varphi)_\Lambda$ for all locally supported φ and all finite square $\Lambda \subset \mathbb{Z}^d$.

Clearly $\mathcal{Q}_\epsilon^2 = \mathcal{Q}_\epsilon$, thus $\mathcal{R}_\epsilon^2 = \mathcal{R}_\epsilon$ and $\mathcal{R}_\epsilon \mathcal{Q}_\epsilon = \mathcal{Q}_\epsilon \mathcal{R}_\epsilon = 0$. Since $\mathcal{P}_\epsilon \mathcal{Q}_\epsilon = \mathcal{Q}_\epsilon \mathcal{P}_\epsilon$ we find $\mathcal{P}_\epsilon \mathcal{R}_\epsilon = \mathcal{R}_\epsilon \mathcal{P}_\epsilon$ on locally supported φ. Using a decomposition of the form (2.16), we see that the spectrum of $\mathcal{P}_\epsilon$ is a subset of the spectra of the rank one operator $\mathcal{P}_\epsilon \mathcal{Q}_\epsilon$ (which has 1 as a simple eigenvalue, for the eigenfunction h_ϵ) and the linear operator $\mathcal{P}_\epsilon \mathcal{R}_\epsilon$. Thus, it suffices to check that for any $|z| > \kappa$ the operator $z - \mathcal{P}_\epsilon \mathcal{R}_\epsilon$ has a continuous inverse with domain containing the dense subset of $\mathcal{F}$ formed by locally supported functions.

Let $|z| > \kappa$ and let φ be supported in some finite Λ_φ. For any finite Λ and all finite Λ' containing both Λ and Λ_φ we get from (2.21)

$$\begin{aligned}\|(z - P_{\Lambda'} R_{\Lambda'})^{-1} \varphi_{\Lambda'}\|_\Lambda &\le \frac{1}{|z|} \sum_{n=0}^{\infty} \frac{\|P_{\Lambda'}^n R_{\Lambda'} \varphi_{\Lambda'}\|_\Lambda}{|z|^n} \\ &\le \frac{1}{|z|} e^{C|\Lambda|} \sum_{n=0}^{\infty} \left|\frac{\kappa}{z}\right|^n \|\varphi\|_{\Lambda_\varphi} .\end{aligned} \tag{2.29}$$

Applying now (2.22) for general n (which shows in particular that $\|(P_{\Lambda'}^n R_{\Lambda'} - \mathcal{P}_\epsilon^n \mathcal{R}_\epsilon)\varphi\|_\Lambda \le \kappa^n e^{C|\Lambda|} \|\varphi\|_{\Lambda_\varphi}$ for large enough Λ' and all n), we have that

$$\|(z - \mathcal{P}_\epsilon \mathcal{R}_\epsilon)^{-1} \varphi\|_\Lambda \le \frac{1}{|z|} 2 e^{C|\Lambda|} \sum_{n=0}^{\infty} \left|\frac{\kappa}{z}\right|^n \|\varphi\|_{\Lambda_\varphi} \le C_z(\Lambda) \|\varphi\|_{\Lambda_\varphi} , \tag{2.30}$$

which gives the claim. $\square$

The last result of this section also follows from the key bound in Lemma 2.4 (see [BK1] for similar results):

Corollary 2.8 (Approach to equilibrium and operational correlations). *Fix $\kappa > \kappa_1$ and let $0 \le \epsilon < \epsilon_2$ with $\epsilon_2(\kappa)$ given by Lemma 2.4.*

(1) *For any continuous function $\varphi : \mathcal{X} \to \mathbb{C}$*

$$\lim_{n\to\infty} \int \varphi \circ F_\epsilon^n \, dm^{\mathbb{Z}^d} = \int \varphi \, d\mu_\epsilon \, . \tag{2.31}$$

(2) *There is $C > 0$ such that for any locally supported $\varphi \in \mathcal{F}$ with support Λ_φ, any $\psi \in \mathcal{F}$ for which $\psi = \lim_{\Lambda' \to \mathbb{Z}^d} (\psi_{\Lambda'} | \mathcal{X}_{\Lambda'})$ defines a continuous function on $\mathcal{X}$ and $C_\psi = \sup_{\Lambda'} \|\psi\|_{\Lambda'} < \infty$, and all $n \ge 0$*

$$| \int_{\mathcal{X}} (\varphi \circ F_\epsilon^n) \psi \, dm^{\mathbb{Z}^d} - \int_{\mathcal{X}} \varphi \, d\mu_\epsilon \int_{\mathcal{X}} \psi dm^{\mathbb{Z}^d} | \le e^{C|\Lambda_\varphi|} \kappa^n C_\psi \int_{\mathcal{X}_{\Lambda_\varphi}} |\varphi_{\Lambda_\varphi}| dm^{\Lambda_\varphi} \, . \tag{2.32}$$

Our bounds also give an exponential decay similar to (2.32) for functions $\varphi \in \mathcal{F}$ for which there are $C_\varphi > 0$ and $\upsilon > C$ (with $C > 0$ as in Corollary 2.8 (2)) such that for any finite $\Lambda \subset \Lambda' \subset \mathbb{Z}^d$ we have $\sup_{\mathcal{X}_{\Lambda'}} |\varphi_{\Lambda'} - \varphi_\Lambda| \le C_\varphi e^{-\upsilon|\Lambda|}$.

An alternative to the approach in [BK1] to study the decay of usual correlations

$$| \int (\varphi \circ F_\epsilon^n) \psi \, d\mu_\epsilon - \int \varphi \, d\mu_\epsilon \int \psi d\mu_\epsilon | \tag{2.33}$$

would be to work with a normalised transfer operator $\mathcal{L}_\epsilon$ obtained by taking limits of the normalised operators $L_{\epsilon,\Lambda'}$ where $\mathrm{Det} DF_{\epsilon,\Lambda'}$ in (2.20) is replaced by the Jacobian of $F_{\epsilon,\Lambda'}$ with respect to its SRB measure $\Omega_{\epsilon,\Lambda'} m^{\Lambda'}$. (Then $L_{\epsilon,\Lambda'}$ fixes the constant function 1 on $\mathcal{D}_{\Lambda'}$.) The analogue of Lemma 2.4 for the truncated normalised operators $L_{\epsilon,\Lambda}$ should hold and would yield the same decay for (2.33) as for (2.32) (with φ, ψ as in Corollary 2.8 (2)). Note that our results (for $d = 1$) in Section 3 imply exponential decay of (2.33) with uniform rate κ, close to κ_1 if ϵ is small enough, for continuous $\psi : \mathcal{X} \to \mathbb{C}$ and φ in a Banach space of functions containing locally supported functions.

Proof of Corollary 2.8. We prove (2) (the first claim follows by setting $\psi \equiv 1$ together with a density argument). In the notation of Lemma 2.4, we have (using the local support property of φ to get the third equality and (2.21) in Lemma 2.4 (2) to get the last line):

$$
\begin{aligned}
&| \int_{\mathcal{X}} (\varphi \circ F_\epsilon^n)\psi \, dm^{\mathbb{Z}^d} - \int_{\mathcal{X}} \varphi \, d\mu_\epsilon \int_{\mathcal{X}} \psi dm^{\mathbb{Z}^d} | \\
&\qquad = \lim_{\Lambda' \to \mathbb{Z}^d} | \int_{\mathcal{X}_{\Lambda'}} (\varphi_{\Lambda'} \circ F^n_{\epsilon,\Lambda'})\psi_{\Lambda'} \, dm^{\Lambda'} - \int_{\mathcal{X}_{\Lambda'}} \varphi_{\Lambda'} \, \Omega_{\epsilon,\Lambda'} \, dm^{\Lambda'} \int_{\mathcal{X}_{\Lambda'}} \psi_{\Lambda'} \, dm^{\Lambda'} | \\
&\qquad = \lim_{\Lambda' \to \mathbb{Z}^d} | \int_{\mathcal{X}_{\Lambda'}} \varphi_{\Lambda'} (P^n_{\epsilon,\Lambda'} \psi_{\Lambda'}) \, dm^{\Lambda'} - \int_{\mathcal{X}_{\Lambda'}} \varphi_{\Lambda'} \, \Omega_{\epsilon,\Lambda'} \, dm^{\Lambda'} \int_{\mathcal{X}_{\Lambda'}} \psi_{\Lambda'} \, dm^{\Lambda'} | \\
&\qquad = \lim_{\Lambda' \to \mathbb{Z}^d} | \int_{\mathcal{X}_{\Lambda_\varphi}} \varphi_{\Lambda_\varphi} (P^n_{\epsilon,\Lambda'} R_{\epsilon,\Lambda'} \psi_{\Lambda'})_{\Lambda_\varphi} \, dm^{\Lambda_\varphi} | \\
&\qquad \le e^{C|\Lambda_\varphi|} \kappa^n C_\psi \int_{\mathcal{X}_{\Lambda_\varphi}} |\varphi_{\Lambda_\varphi}| dm^{\Lambda_\varphi} . \quad \square
\end{aligned}
\tag{2.34}
$$

2.B. Proof of the main bound via cluster expansions.

This subsection is devoted to a proof of Lemma 2.4. We first setup the cluster expansion machinery (which will also be used in Section 3).

We recall some definitions from [BK1]. Let $\mathcal{S}$ be the set of finite *spacelike sets* in $\mathbb{Z}^d \times \mathbb{Z}$, that is, $Z \in \mathcal{S}$ is of the form $Z = Y \times \{t\}$ for some finite $Y \subset \mathbb{Z}^d$ and some $t \in \mathbb{Z}$. Let $\mathcal{B}$ be the set of *timelike bonds* in $\mathbb{Z}^d \times \mathbb{Z}$, that is, $b \in \mathcal{B}$ is of the form $b = \{(a,t),(a,t+1)\}$ for some $a \in \mathbb{Z}^d$ and $t \in \mathbb{Z}$. The *support* of a subset of $\mathcal{S} \cup \mathcal{B}$ is the union of its elements (the support is a subset of $\mathbb{Z}^d \times \mathbb{Z}$). A subset of $\mathcal{S} \cup \mathcal{B}$ is *connected* if it can not be written as a union of two non-empty subsets (of $\mathcal{S} \cup \mathcal{B}$) such that the intersection of their supports is empty. A *polymer* γ is a finite connected subset of $\mathcal{S} \cup \mathcal{B}$ such that all the spacelike sets are pairwise disjoint. We write $B(\gamma)$ for the set of timelike bonds in γ and $Z(\gamma)$ for the set of spacelike sets in γ. The support of a polymer γ is denoted by $\overline{\gamma}$ and two polymers are *disjoint,* if their supports do not intersect.

For $Z \subset \mathbb{Z}^d$ and $\xi > 0$ (which will be related below to the constant λ in (1.10)) we shall use the notation

$$
T(Z,\xi) = \sum_{\tau(Z)} e^{-\xi|\tau|} , \tag{2.35}
$$

where the sum is over all trees in Z, i.e., trees in $\mathbb{R}^d$ with set of vertices equal to Z and edges straight line segments between the vertices (an intersection of segments not in Z is not viewed as a vertex), a tree being represented by the set of its edges, and $|\tau| = \sum_{l \in \tau} |l|$ is the *length of the tree* τ, that is, the sum of the lengths of the edges of τ. (All our trees are connected and nonempty.)

The notation $\tilde\epsilon$ is used in Sections 2.B and 3.B to denote a function $\tilde\epsilon(\epsilon) > 0$ which goes to zero as $\epsilon \to 0$, and C denotes a generic positive constant which may vary from place to place, even within the same equation.

Let π_s and π_t be the projections onto the space and time components of $\mathbb{Z}^d \times \mathbb{Z}$, respectively. We start by an ancillary combinatorical lemma (see [BK1, (66-67), p. 393]):

Lemma 2.9. *For each $\epsilon \geq 0$, let $\mathcal{V} = \mathcal{V}_\epsilon$ be a real and positive-valued function, defined on the set of all polymers γ in $\mathbb{Z}^d \times \mathbb{Z}$, such that for some $\xi > 0$ and some $0 < \beta < 1$ there exists $C > 0$ such that*

$$\mathcal{V}_\epsilon(\gamma) \leq C\beta^{|B(\gamma)|} \prod_{Z \in Z(\gamma)} \tilde{\epsilon}^{|Z|} T(Z, \xi), \quad \forall \gamma. \tag{2.36}$$

Then there exists $\tilde{C} > 0$ such that for all small enough ϵ, all finite $\Lambda' \subset \mathbb{Z}^d$, $a \in \Lambda'$, and $t_0 \in \mathbb{Z}$

$$\sum_{\substack{\gamma \\ a \in \pi_s(\overline{\gamma}), \overline{\gamma} \cap \Lambda'_{t_0} \neq \emptyset, Z(\gamma) \neq \emptyset}} \mathcal{V}_\epsilon(\gamma) \leq \tilde{C}\tilde{\epsilon} \quad \textit{and} \quad \sum_{\substack{\gamma \\ a \in \pi_s(\overline{\gamma}), \overline{\gamma} \cap \Lambda'_{t_0} \neq \emptyset}} \mathcal{V}_\epsilon(\gamma) \leq \tilde{C}\,. \tag{2.37}$$

(The constants ξ and β will be related to the coupling decay λ and the single-spin spectral gap κ_1, respectively.)

Proof of Lemma 2.9. We start with some useful estimates. Let $|\cdot|$ be a norm in $\mathbb{Z}^d$. Notice that, if $\xi > 0$, then

$$\sum_{j \in \mathbb{Z}^d} e^{-\xi|j|} < C \tag{2.38}$$

for some constant $C > 0$ (which depends on ξ and the choice of the norm $|\cdot|$), since the corresponding integral converges.

We want to find an upper bound for

$$\sum_{\substack{Z \subset \mathbb{Z}^d \\ a \in Z}} \tilde{\epsilon}^{|Z|} T(Z, \xi)\,, \tag{2.39}$$

where a is an arbitrary point of $\mathbb{Z}^d$ and $\tilde{\epsilon} > 0$ is small. We estimate this sum by first fixing the number $(n+1)$ of points in Z and numbering the points so that a has number 0, then fixing a(n abstract) tree in the set $\{0, \ldots, n\}$ (denoted by $\tau \lhd G_{n+1}$, where G_{n+1} is the full graph of $n+1$ points), summing over all possible n locations of the vertices different from a of a (concrete) tree representing τ in $\mathbb{Z}^d$, and finally summing over all trees and over the n vertices different from a. Note that since we number the points of Z, the same set appears $(|Z|-1)!$ times. Using (2.38) in the second inequality and Cayley's theorem [W, Theorem 10a p. 50] (which says that the number of distinct labelled trees with m vertices is $m^{m-2} < c^m(m-1)!$ for some $c > 0$) in the third inequality, we obtain that

$$\begin{aligned}\sum_{\substack{Z \subset \mathbb{Z}^d \\ a \in Z}} \tilde{\epsilon}^{|Z|} T(Z, \xi) &\leq \sum_{n=0}^{\infty} \frac{\tilde{\epsilon}^{n+1}}{n!} \sum_{\tau \lhd G_{n+1}} \sum_{j_1, \ldots, j_n \in \mathbb{Z}^d} e^{-\xi|\tau(a, j_1, \ldots, j_n)|} \\ &\leq \sum_{n=0}^{\infty} \frac{\tilde{\epsilon}^{n+1}}{n!} \sum_{\tau \lhd G_{n+1}} C^n \leq \sum_{n=0}^{\infty} \frac{(C\tilde{\epsilon})^{n+1}}{n!} c^{n+1} n! \leq C\tilde{\epsilon}\,,\end{aligned} \tag{2.40}$$

where $\tau(a, j_1, \ldots, j_n)$ is a representative of τ such that $a, j_i, \ldots, j_n$ are the locations of its vertices in $\mathbb{Z}^d$. In the same way we see using [S, Lemma V.7.5 p. 456] that for any natural number p

$$\sum_{\substack{Z \subset \mathbb{Z}^d \\ a \in Z}} |Z|^p \tilde{\epsilon}^{|Z|} T(Z, \xi) \leq C \tilde{\epsilon} p! \,. \tag{2.41}$$

Next we estimate the sum

$$\sum_{\substack{\gamma \\ (a,t) \in \overline{\gamma}}} \mathcal{V}(\gamma) \,, \tag{2.42}$$

where (a, t) is an arbitrary point of $\mathbb{Z}^{d+1}$. The graph associated to a polymer γ is the abstract graph with vertices $Z(\gamma)$ and set of edges the set of vertical lines which are chains of time bonds $B(\gamma)$. By definition, a *tree polymer* γ' in $\mathbb{Z}^d \times \mathbb{Z}$ is a polymer whose graph is a (connected) tree. We may associate to each polymer γ such that (a, t) belongs to a spacelike set of γ a (nonuniquely defined) tree polymer γ' whose set of edges is a subset of the vertical lines (i.e., chains of bonds) of γ, and whose set of vertices is the spacelike sets Z of γ. If γ is a polymer with (a, t) belonging to a time-bond of γ but to no spacelike set of γ, we associate to γ a tree polymer whose set of edges is a subset of the vertical lines of γ containing the chain of bonds $l_{(a,t)}$ on which (a, t) lies, and whose set of vertices is $Z(\gamma)$ supplemented with the endpoint(s) of $l_{(a,t)}$. Let $\gamma'(\gamma)$ denote the set of tree polymers associated to γ. We bound the sum over all polymers γ associated with the same tree polymer γ' by noting that from each point of each Z there may be a line going upwards or/and downwards or no line at all. So we replace the complicated sum of all possible sets of bonds by the product of simple sums, taking into account the fact that we can choose as the representative of an edge in γ' a vertical line connecting $(b, t_i) \in Z_i$ to $(b, t_j) \in Z_j$ for any pair of such points. Fixing an arbitrary $\beta < \tilde{\beta} < 1$, we have

$$\begin{aligned}
\sum_{\substack{\gamma \\ (a,t) \in \overline{\gamma}}} \mathcal{V}(\gamma) &\leq C \sum_{\substack{\gamma' \text{ tree polymer} \\ (a,t) \in \overline{\gamma'}}} \sum_{\substack{\gamma \\ \gamma' \in \gamma'(\gamma)}} \beta^{|B(\gamma)|} \prod_{Z \in \gamma} \tilde{\epsilon}^{|Z|} T(Z, \xi) \\
&\leq C \sum_{\substack{\gamma' \text{ tree polymer} \\ (a,t) \in \overline{\gamma'}}} \tilde{\beta}^{|B(\gamma')|} \prod_{Z \in Z(\gamma')} \left(\tilde{\epsilon}\, 2 \sum_{m=0}^{\infty} \left(\frac{\beta}{\tilde{\beta}}\right)^m \right)^{|Z|} T(Z, \xi) \\
&\leq C \sum_{\substack{\gamma' \text{ tree polymer} \\ (a,t) \in \overline{\gamma'}}} \tilde{\beta}^{|B(\gamma')|} \prod_{Z \in Z(\gamma')} (C\tilde{\epsilon})^{|Z|} T(Z, \xi) \,.
\end{aligned} \tag{2.43}$$

Consider first those polymers so that (a, t) belongs to a spacelike set of γ. To obtain an upper bound for their contribution to the sum (2.43), we fix the number of vertices in a(n abstract) labelled tree, then choose one such tree τ, numbering the vertices in such a way that the fixed point (a, t) belongs to a set Z representing vertex number 0, and finally sum over all tree polymers γ' which represent the tree τ (denoted by $\gamma' \sim \tau$), that is,

$$\sum_{\substack{\gamma \\ \exists Z \in Z(\gamma) \\ (a,t) \in Z}} \mathcal{V}(\gamma) \leq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\tau \triangleleft G_{n+1}} \sum_{\substack{\gamma' \sim \tau \\ (a,t) \in \overline{\gamma'}}} \tilde{\beta}^{|B(\gamma')|} \prod_{Z \in Z(\gamma')} (C\tilde{\epsilon})^{|Z|} T(Z, \xi) = \star. \tag{2.44}$$

Next we sum over vertices, that is, over all Z's in γ', and over the lengths of the lines in γ', by stripping the tree τ vertex by vertex. This means that we start from a vertex with degree one, which we may assume to be different from vertex 0 (since there are at least two vertices of degree one in a tree with at least two vertices), and which we call vertex number n by renumbering the vertices. Then the location of the line connecting n to another vertex (say $j = j(n)$) fixes one point (a_n, t_n) in $\mathbb{Z}^{d+1}$ (we may thus use (2.40) to bound the sum over all representatives Z_n of this vertex). Next we sum over the length l_n of this line. We will multiply this sum by 2, since $t_n < t_{j(n)}$ or $t_n > t_{j(n)}$. Thus we take into account all possible values of t_n. Now we have a *stripped tree* which has one vertex less than the tree we started with. If the stripped tree has at least two vertices, we choose again a vertex with degree one distinct from vertex 0, and we repeat the above procedure (including the renumbering). We continue in this way, ending with a tree reduced to the vertex 0 containing (a, t). Note that we will sum over all possible locations of a_n if we sum over the elements $b_{j(n)}$ in the representative $Z_{j(n)}$ of the vertex $j(n)$ to which the vertex n is connected. In general, we denote the point of Z_i corresponding to vertex number i which is determined by the location of the line representing the edge of vertex i connecting Z_i to $Z_j(i)$ by (a_i, t_i), and we denote the corresponding point in $Z_{j(i)}$ by $(b_{j(i),s(i,j(i))}, t_{j(i)})$ (the stripping procedure ensures that $s(i,j)$ ranges between one and $d_j - 1$ where $d_j \geq 1$ is the degree of vertex Z_j for $j \geq 1$ and $1 \leq s(i,0) \leq d_0$); finally, we set $(a_0, t_0) = (a, t)$. (Note that if $j(i_1) = j(i_2) = j$ either $b_{j,s(i_1,j)} \neq b_{j,s(i_2,j)}$ or the vertical heights are different.) We will thus bound the multiple sum with complicated restrictions by an ordered product of simple sums. Since we assumed that there exists at least one Z in each polymer, we obtain, using (2.41) (for $p = d_i - 1 \geq 1$), as well as the version of Cayley's theorem giving the number of trees $\tau(d_0, \ldots, d_n)$ having $n+1$ vertices with given degrees $d_i \geq 1$ (see [S, Lemma V.7.7, p. 457], recall that the sum of the degrees is $2n$ for $n \geq 1$),

$$\star \leq C\tilde{\epsilon} + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{d_0, \ldots, d_n \geq 1 \\ \sum_{i=0}^{n} d_i = 2n}} \sum_{\tau(d_0, \ldots, d_n) \triangleleft G_{n+1}} \sum_{\substack{Z_0 \subset \mathbb{Z}^d \\ a \in Z}} (C\tilde{\epsilon})^{|Z_0|} T(Z_0, \xi) \sum_{\substack{b_{0,s_0} \in Z_0 \\ s_0 = 1, \ldots, d_0}}$$
$$\cdot \prod_{i=1}^{n} 2 \sum_{l_i=1}^{\infty} \tilde{\beta}^{l_i} \sum_{\substack{Z_i \subset \mathbb{Z}^d \\ a_i = b_{j(i), s_{j(i)}} \in Z_i}} (C\tilde{\epsilon})^{|Z_i|} T(Z_i, \xi) \sum_{\substack{b_{i,s_i} \in Z_i \\ s_i = 1, \ldots, d_i - 1}}$$

$$
\begin{aligned}
&\le C\tilde\epsilon + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{d_0,\dots,d_n\ge 1\\ \sum_{i=0}^n d_i=2n}} \sum_{\tau(d_0,\dots,d_n)\lhd G_{n+1}} \sup_{b\in\mathbb{Z}^d} \sum_{\substack{Z_0\subset\mathbb{Z}^d\\ b\in Z_0}} |Z_0|^{d_0}(C\tilde\epsilon)^{|Z_0|}T(Z_0,\xi)\\
&\qquad\cdot\prod_{i=1}^{n}(2\sum_{l_i=1}^{\infty}\tilde\beta^{l_i}) \sup_{b\in\mathbb{Z}^d}\sum_{\substack{Z\subset\mathbb{Z}^d\\ b\in Z}} |Z|^{d_i-1}(C\tilde\epsilon)^{|Z|}T(Z,\xi)\\
&\le C\tilde\epsilon + \sum_{n=1}^{\infty}\frac{(C\tilde\epsilon)^{n+1}c^{n+1}}{n!}\sum_{\substack{d_0,\dots,d_n\ge 1\\ \sum_{i=0}^n d_i=2n}} d_0\big(\prod_{m=0}^{n}(d_m-1)!\big)\frac{(n-1)!}{\prod_{i=0}^n(d_i-1)!}\\
&\le C\tilde\epsilon + \sum_{n=1}^{\infty}\frac{(C\tilde\epsilon)^{n+1}c^{n+1}}{n}2^{2n-1}\cdot n \le C\tilde\epsilon\,, \qquad (2.45)
\end{aligned}
$$

where the product in the first two lines is an ordered product (we use here that $j(i) < i$ by the stripping construction) and we used [S, Proposition V.7A.1, p. 464] to get the first inequality in the last line of (2.45).

The case where (a,t) belongs to a line of γ' is similar. Thus, we find

$$
\sum_{\substack{\gamma\\ (a,t)\in\overline{\gamma}\\ Z(\gamma)\ne\emptyset}} \mathcal{V}(\gamma) \le C\tilde\epsilon\,. \qquad (2.46)
$$

If we also include polymers γ with $Z(\gamma)=\emptyset$ in the sum (i.e., polymers consisting of a single vertical line), we find $\sum_{(a,t)\in\overline{\gamma}}\mathcal{V}(\gamma)\le C$, without the $\tilde\epsilon$ factor.

In the same way as we proved (2.46) we may see that

$$
\sum_{\substack{\gamma\\ a\in\pi_s(\overline{\gamma}),\overline{\gamma}\cap\Lambda'_{t_0}\ne\emptyset\\ Z(\gamma)\ne\emptyset}} \mathcal{V}(\gamma) \le \tilde C\tilde\epsilon\,. \qquad (2.47)
$$

Indeed, if the fixed point belongs to a set at level t, we can extract a factor $\hat\beta^{|t-t_0|}$, and the sum over t changes only the constant C. $\square$

Before we prove Lemma 2.4, we need more notation. For $\Lambda\subset\mathbb{Z}^d$ we define $\Lambda_{j,k} = \Lambda\times\{j,\dots,k\}$ for all $j<k\in\mathbb{Z}$ and $\Lambda_n=\Lambda\times\{n\}$ for all $n\in\mathbb{Z}$ and for all $\Lambda\subset\mathbb{Z}^d$. A polymer γ is said to be in $\Lambda_{j,k}$ if $\overline{\gamma}\subset\Lambda_{j,k}$ and no spacelike set of γ intersects Λ_j; a polymer γ is called a *vacuum polymer in* $\Lambda_{j,k}$ if $\overline{\gamma}\subset\Lambda_{j,k}$ and $\overline{\gamma}\cap\Lambda_j=\overline{\gamma}\cap\Lambda_k=\emptyset$. For a polymer γ, let $\partial\gamma$ be the set of $(a,t)\in\overline{\gamma}$ such that (a,t) belongs to exactly one bond of γ and to no spacelike set of γ.

Note that for any polymer γ the set $\pi_t(\overline{\gamma})=[t_-,t_+]$ is a connected interval. Let $\gamma_\pm=\pi_s(\overline{\gamma}\cap\pi_t^{-1}(t_\pm))$. A *weight* $V(\gamma)$ of a polymer γ with $B(\gamma)\ne\emptyset$ is a bounded linear operator, with norm denoted by $\|V(\gamma)\|$, from $\mathcal{H}(\mathcal{D}(\alpha')_{\gamma_-})$ to $\mathcal{H}(\mathcal{D}(\alpha)_{\gamma_+})$ for

$\alpha' < \alpha$ as in Proposition A. If γ is just a single spacelike set, a *weight* $V(\gamma)$ is a bounded operator from $\mathcal{H}(\mathcal{D}(\alpha'')_{\gamma_-})$ to $\mathcal{H}(\mathcal{D}(\alpha)_{\gamma_+})$ for some $\alpha < \alpha''$ (i.e., the domain of analyticity shrinks slightly due to the coupling, see [BK1, Proposition 2, p. 383]). We write $V(\gamma)(h) = V(\gamma)(h_{\gamma_-})$ and $lV(\gamma) = l_{\gamma_+}V(\gamma)$.

We shall use the following important result from [BK1] (which itself is based on Proposition A above, note that we will use Proposition A more explicitly in Section 3):

Proposition B (Bricmont–Kupiainen [BK1, Proposition 3, Proposition 4]). *There is a family of polymer weights $W(\gamma)$, $V(\gamma)$, and $V^b(\gamma)$, so that:*
Combinatorics: *We have the two expansions for $P^n_{\Lambda'} = P^n_{\epsilon,\Lambda'}$:*

$$P^n_{\Lambda'} = \sum_{\Gamma} \prod_{\gamma\in\Gamma_v} lW(\gamma)(h) \bigotimes_{\gamma\in\Gamma_{0,n}} W(\gamma) \otimes \big[(h_{\Lambda'_+} \bigotimes_{\gamma\in\Gamma_n} W(\gamma)(h)) \cdot (l_{\Lambda'_-} \bigotimes_{\gamma\in\Gamma_0} lW(\gamma))\big] \quad (2.48)$$

and

$$P^n_{\Lambda'} = \sum_{\substack{\Gamma \\ \Gamma_v=\emptyset}} \bigotimes_{\gamma\in\Gamma_{0,n}} V^b(\gamma) \otimes \big[(h_{\Lambda'_+} \bigotimes_{\gamma\in\Gamma_n} V^b(\gamma)(h)) \cdot (l_{\Lambda'_-} \bigotimes_{\gamma\in\Gamma_0} lV(\gamma))\big]\,, \quad (2.49)$$

where in (2.49) *the sum is over sets Γ (possibly empty) of disjoint polymers γ in $\Lambda'_{0,n}$ with $\overline{\gamma}\cap(\Lambda'_0\cup\Lambda'_n)\neq\emptyset$ and $\partial\gamma\subset\Lambda'_0\cup\Lambda'_n$, the set $\Gamma_{0,n}$ is the set of polymers γ in Γ such that their supports intersect both Λ'_0 and Λ'_n and $\partial\gamma\cap\Lambda'_0=\overline{\gamma}\cap\Lambda'_0\neq\emptyset$, the set Γ_n is the set of polymers in Γ such that their supports intersect only Λ'_n, and Γ_0 is the set of polymers in Γ such that their supports intersect only Λ'_0 and $\partial\gamma\cap\Lambda'_0=\overline{\gamma}\cap\Lambda'_0\neq\emptyset$; also, $\Lambda'_+=\Lambda'\setminus\cup_{\gamma\in\Gamma_{0,n}\cup\Gamma_n}\gamma_+$ and $\Lambda'_-=\Lambda'\setminus\cup_{\gamma\in\Gamma_{0,n}\cup\Gamma_0}\gamma_-$. In* (2.48) *the sum ranges over the sets Γ of (possibly empty) sets of disjoint polymers γ in $\Lambda'_{0,n}$ with $\partial\gamma\subset\Lambda'_0\cup\Lambda'_n$, and for each Γ we write $\Gamma_v=\Gamma_v(0,n)\subset\Gamma$ for the subset of vacuum polymers in $\Lambda'_{0,n}$ (i.e., those for which $\overline{\gamma}\cap(\Lambda'_0\cup\Lambda'_n)=\emptyset$).*
Bounds: *For any $\kappa_1<\kappa<1$ with κ_1 defined in* (1.5), *there is $C(\kappa)$, independent of Λ' and n, such that the norms $\mathcal{V}(\gamma)=\|V(\gamma)\|$ and $\mathcal{V}^b(\gamma)=\|V^b(\gamma)\|$ of the weights in* (2.49) *satisfy* (2.36) *for $\xi=\lambda/4$ and $\beta=\kappa$ and the norms $\|W(\gamma)\|$ of the weights in* (2.48) *satisfy* (2.36) *for $\xi=\lambda/2$ and $\beta=\kappa$.*

The decomposition (2.49) is obtained from (2.48) in [BK1] by dividing away the contribution of vacuum polymers: we shall use a similar technique in the proof of Lemma 2.10 below. Obtaining the bound (2.36) for the norms of the weights $V(\gamma)$ in (2.49) is maybe the most important step in the proof of Proposition B. (In [BK1], the analogous bounds are stated for operator weights between $\mathcal{H}(\mathcal{D}(\alpha)_{\gamma_-})$ to $\mathcal{H}(\mathcal{D}(\alpha)_{\gamma_+})$ but since all the relevant ingredients are analyticity improving we get the above-claimed bounds from (1.18) up to slightly tuning the constants $\tilde{\epsilon}$ in (2.36) and $\alpha'<\alpha$ from Proposition A.) Note that the definitions [BK1, (13),(14),(19), and (22), pp. 383-384] imply that the weights $W(\gamma)$, and thus $V(\gamma)$, do not depend on Λ'. (Of course, if a polymer γ is not in Λ'_{0n} it will just not appear in the sums (2.48-2.49).)

We next prove a lemma which is needed in the proof of Lemma 2.4 and in Section 3.B. To state it, we introduce two additional notations. First, for fixed finite $\Lambda'\subset\mathbb{Z}^d$ and

$n \geq 0$, consider the weights W from (2.48) and define $U(v) \in \mathbb{C}$ for any vacuum polymer v in $\Lambda'_{0,n+1}$ by

$$U(v) = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\substack{\gamma_1,\dots,\gamma_m \\ \cup_{i=1}^m \gamma_i \sim v}} \prod_{i=1}^{m} lW(\gamma_i)(h) \sum_{G \lhd G_m} \prod_{(i,j)\in G} \chi_{ij}\,, \tag{2.50}$$

where the second sum is over vacuum polymers $\gamma_1, \dots, \gamma_m$ in $\Lambda'_{0,n+1}$ (not necessarily disjoint or even distinct) satisfying the condition $\cup\gamma_i \sim v$ to be explained below, the third sum is over all full connected graphs on the set $\{1, \dots, m\}$, and $\chi_{ij} = -1$ if $\overline{\gamma}_i \cap \overline{\gamma}_j \neq \emptyset$ and 0 otherwise. The notation $\cup\gamma_i \sim v$ means the following. We assume that $\gamma_1, \dots, \gamma_m$ are such that the union of the supports of the γ_i is connected and thus defines a polymer v as follows: the union of the sets of bonds in the γ_i's is the set of bonds of v and if $Z_1, \dots, Z_k$ is a maximal set of spatial sets from the γ_i's such that $Z = \cup_{\ell=1}^{k} Z_\ell$ is connected, then Z is a spacelike set of v. Using the bounds on W from Proposition B, one can show that the $|U(\gamma)|$ satisfy the bound (2.36) for $\xi = \lambda/3$ and $\beta = \kappa$ ([BK1, (65), p. 393]).

Second, using the weights $V^b(\gamma)$ from (2.49), define for any finite Λ' an element $\tilde{\Omega}_{\Lambda'}$ of $\mathcal{H}^{\Lambda'}(D(\alpha))$ by

$$\tilde{\Omega}_{\Lambda'} = \sum_{\Gamma} h_{\Lambda'_+} \underset{\gamma\in\Gamma}{\otimes} V^b(\gamma)(h)\,, \tag{2.51}$$

where Γ ranges over all finite sets (possibly empty) of disjoint polymers in $\Lambda'_{-\infty,n}$ with $\overline{\gamma} \cap \Lambda'_n \neq \emptyset$ and $\partial\gamma \subset \Lambda'_n$ (the infinite sum (2.51) converges by virtue of the bounds from Lemma 2.9, and it clearly does not depend on n; note that the weights $V(\gamma)$ for single space-like sets γ occuring in (2.51) are evaluated at h which is analytic in an annulus $D(\alpha'')$ for $\alpha'' > \alpha$ so that we may afford to lose analyticity).

Lemma 2.10. *Let $\Lambda \subset \Lambda' \subset \mathbb{Z}^d$ be finite. Then for any $n \in \mathbb{Z}^+$, there are polymer weights $\widehat{V}^b(\gamma)$ so that using the notations from Proposition B and (2.50), we have*

$$\begin{aligned} l_{\Lambda'\setminus\Lambda} P^n_{\Lambda'} = \sum_{\substack{\Gamma \\ \Gamma'_v = \emptyset}} \exp\Big(- \sum_{\substack{v\in\Upsilon_v^{0,n+1} \\ \overline{v}\cap\Lambda_n\neq\emptyset \\ \overline{v}\cap\overline{\Gamma}=\emptyset}} U(v)\Big) \underset{\gamma\in\Gamma_{0,\Lambda_n}}{\otimes} l_{\Lambda'\setminus\Lambda} \widehat{V}^b(\gamma) \\ \otimes\big[(h_{\Lambda_+} \underset{\gamma\in\Gamma_{\Lambda_n}}{\otimes} l_{\Lambda'\setminus\Lambda}\widehat{V}^b(\gamma)(h)) \cdot (l_{\Lambda'_-} \underset{\gamma\in\Gamma'_0}{\otimes} lV(\gamma))\big]\,, \end{aligned} \tag{2.52}$$

where the sum is over sets Γ (possibly empty) of disjoint polymers γ in $\Lambda'_{0,n}$ with $\overline{\gamma} \cap (\Lambda'_0 \cup \Lambda_n) \neq \emptyset$ and $\partial\gamma \subset \Lambda'_0 \cup \Lambda_n$, the set Γ'_0 are the set of polymers in Γ whose supports intersect only Λ'_0, the sets Γ_{0,Λ_n} and Γ_{Λ_n} are like Γ_n and $\Gamma_{0,n}$ in Proposition B with Λ'_n replaced by Λ_n, the set $\Upsilon_v^{0,n+1}$ is the set of vacuum polymers in $\Lambda'_{0,n+1}$, the set $\Lambda_+ = \Lambda \setminus \cup_{\gamma\in\Gamma_{0,\Lambda_n}\cup\Gamma_{\Lambda_n}} \gamma_+$, and $\overline{\Gamma} = \cup_{\gamma\in\Gamma}\overline{\gamma}$. The norms of the weights $\widehat{V}^b(\gamma)$ satisfy (2.36) for $\xi = \lambda/4$ and $\beta = \kappa$.

Using the notation (2.50-2.51)*, we have for all* $n \geq 0$

$$l_{\Lambda'\setminus\Lambda}(\tilde{\Omega}_{\Lambda'} \cdot l_{\Lambda'}) = \sum_{\substack{\Gamma \\ \Gamma'_{v,-\infty}=\emptyset}} \exp\Big(- \sum_{\substack{v\in\Upsilon_v^{-\infty,n+1} \\ \overline{v}\cap\Lambda_n\neq\emptyset \\ \overline{v}\cap\overline{\Gamma}=\emptyset}} U(v)\Big)$$
$$\sum_{\Gamma_0^n}\Big[(h_{\Lambda_+} \bigotimes_{\gamma\in\Gamma_{\Lambda_n}} l_{\Lambda'\setminus\Lambda}\widehat{V}^b(\gamma)(h)) \cdot (l_{\Lambda'_-} \bigotimes_{\zeta\in\Gamma_0^n} lV(\zeta))\Big]\,, \tag{2.53}$$

where Γ *ranges over finite sets of disjoint polymers in* $\Lambda'_{-\infty,n}$ *with* $\overline{\gamma}\cap\Lambda_n\neq\emptyset$ *and* $\partial\gamma\subset\Lambda_n$*,* Γ_0^n *ranges over finite sets of disjoint polymers* ζ *in* $\Lambda'_{0,n}$ *with* $\emptyset\neq\partial\zeta\cap\Lambda'_0=\overline{\zeta}\cap\Lambda'_0$*,* $\Upsilon_v^{-\infty,n+1}$ *is the set of finite vacuum polymers* v *in* $\Lambda'_{-\infty,n+1}$*, finally* $\Lambda_+=\Lambda\setminus\cup_{\gamma\in\Gamma}\gamma_+$*, and* $\Lambda'_-=\Lambda'\setminus\cup_{\zeta\in\Gamma_0^n}\zeta_-$*.*

Remarks 2.11. a) At the end of the proof of Lemma 2.4 we shall prove that $\tilde{\Omega}_{\Lambda'}=\Omega_{\Lambda'}$.

b) Note in (2.52) that at level n the polymers intersect Λ, not just Λ', except if $\gamma\in\Gamma'_0$ in which case $\overline{\gamma}$ may intersect $(\Lambda'\setminus\Lambda)_n$ but not Λ_n. In (2.53), $\Gamma_{\Lambda_n}=\Gamma$.

c) One could prove a result analogous to Lemma 2.10 based on (2.49) instead of (2.48), but one would obtain weights $\widehat{W}^b$ with norms decaying more slowly (with $\xi=\lambda/5$ instead of $\xi=\lambda/4$ in (2.36)).

Proof of Lemma 2.10. Our method is the same as used in the proof of [BK1, Proposition 4], but for the sake of easier reading we give the main steps here. The main idea, which is also used to obtain (2.49) from (2.48), is to divide away the contribution of vacuum polymers. Roughly speaking, since we integrate only in $\Lambda'\setminus\Lambda$, we are left with the contribution of vacuum polymers which intersect Λ_n (cf. the exponential factor in (2.52) and (2.53)).

Our starting point is the identity

$$1 = l_{\Lambda'}(P^n_{\Lambda'}h_{\Lambda'}) = \sum_{\Gamma_v}\prod_{\gamma\in\Gamma_v} lW(\gamma)(h)\,, \tag{2.54}$$

where Γ_v ranges over sets of disjoint vacuum polymers in $\Lambda'_{0,n+1}$, which is obtained by replacing P^{n-1} by P^n in [BK1, (34) p. 387]. The basic result of the **polymer expansion formalism** from statistical mechanics says that estimates (2.36) for the weight-norms from Proposition B together with the subsequent bounds (2.37) from Lemma 2.9 guarantee that for small enough $\tilde{\epsilon}$ we can "compute the logarithm of the sum" in the right-hand-side of (2.54). More precisely, this means that we can write

$$\sum_{\Gamma_v}\prod_{\gamma\in\Gamma_v} lW(\gamma)(h) = \exp\Big(\sum_{v\in\Upsilon_v^{0,n+1}} U(v)\Big) \tag{2.55}$$

with $U(v)$ as in (2.50) (see [BK1, p. 393]). (In principle, the sum over v in (2.55) involves also disconnected $v\subset\mathbb{Z}^d$. However, because of the signs χ_{ij}, the expression

(2.50) defining $U(v)$ vanishes unless v is connected, that is, v is a polymer.) A detailed proof of (2.55) can be found for example in [J, Lemma 4.5], where a situation slightly different from ours is considered (on the polymer expansion method see also [S, Chapter V.7, pp. 447–470]).

We now calculate the integral $l_{\Lambda'\setminus\Lambda}$ of $P^n_{\Lambda'}$ using the formula (2.48). Since $l(\mathrm{Id}-Q) = (\mathrm{Id}-Q)(h) = 0$, where Q is the spectral projection of P corresponding the eigenvalue 1 (see (1.4) and (1.5); see also [BK1, p. 387], noting that R there is just our $P-Q$) polymers in (2.48) with $\partial\gamma \cap (\Lambda'_n \setminus \Lambda_n) \neq \emptyset$ have a vanishing contribution to the integral. The sum in (2.48) thus becomes a sum $\sum^*$ over (possibly empty) sets Γ of disjoint polymers, which are either in $\Lambda'_{0,n}$, or are vacuum polymers in $\Lambda'_{0,n+1}$ whose support does not intersect Λ_n (but could meet the set $\Lambda'\setminus\Lambda$ at level n):

$$l_{\Lambda'\setminus\Lambda} P^n_{\Lambda'} = \sum_{\Gamma}^{*} \prod_{\gamma\in\Gamma_v(\Gamma)} lW(\gamma)(h) \bigotimes_{\gamma\in\Gamma_{0,\Lambda_n}} l_{\Lambda'\setminus\Lambda} W(\gamma) \\ \otimes\big[(h_{\Lambda_+} \bigotimes_{\gamma\in\Gamma_{\Lambda_n}} l_{\Lambda'\setminus\Lambda} W(\gamma)(h)) \cdot (l_{\Lambda'_-} \bigotimes_{\gamma\in\Gamma'_0} lW(\gamma))\big]\,, \tag{2.56}$$

where we still have denoted again $\Gamma_v = \Gamma_v(\Gamma)$ the union of the "old" and "new" vacuum polymers in a given set Γ. (Note that the polymers in Γ_{Λ_n} satisfy $\partial\gamma \subset \Lambda_n$ but $\overline{\gamma}$ can intersect $\Lambda'_n \setminus \Lambda_n$.) Next, letting Γ range over all sets of disjoint polymers in $\Lambda'_{0,n}$ with $\partial\gamma \subset \Lambda'_0 \cup \Lambda_n$, and for any given Γ letting $\Gamma^*_v = \Gamma^*_v(\Gamma)$ range over sets of disjoint vacuum polymers in $\Lambda'_{0,n+1}$ intersecting neither $\overline{\Gamma}$ nor Λ_n, we find:

$$\begin{aligned} l_{\Lambda'\setminus\Lambda} P^n_{\Lambda'} &= \sum_{\substack{\Gamma\\ \Gamma'_v=\emptyset}} \bigotimes_{\gamma\in\Gamma_{0,\Lambda_n}} l_{\Lambda'\setminus\Lambda} W(\gamma) \otimes\big[(h_{\Lambda_+} \bigotimes_{\gamma\in\Gamma_{\Lambda_n}} l_{\Lambda'\setminus\Lambda} W(\gamma)(h)) \cdot (l_{\Lambda'_-} \bigotimes_{\gamma\in\Gamma'_0} lW(\gamma))\big] \\ &\qquad \cdot \sum_{\substack{\Gamma^*_v \\ \overline{\zeta}\cap\overline{\Gamma}=\emptyset,\, \overline{\zeta}\cap\Lambda_n=\emptyset \\ \forall\zeta\in\Gamma^*_v}} \prod_{\zeta\in\Gamma_v} lW(\zeta)(h) \\ &= \sum_{\substack{\Gamma\\ \Gamma'_v=\emptyset}} \bigotimes_{\gamma\in\Gamma_{0,\Lambda_n}} l_{\Lambda'\setminus\Lambda} W(\gamma) \otimes\big[(h_{\Lambda_+} \bigotimes_{\gamma\in\Gamma_{\Lambda_n}} l_{\Lambda'\setminus\Lambda} W(\gamma)(h)) \cdot (l_{\Lambda'_-} \bigotimes_{\gamma\in\Gamma'_0} lW(\gamma))\big] \\ &\qquad \cdot \exp\Big(\sum_{\substack{v\in\Upsilon^{0,n+1}_v \\ \overline{v}\cap\overline{\Gamma}=\emptyset,\, \overline{v}\cap\Lambda_n=\emptyset}} U(v)\Big)\,, \end{aligned} \tag{2.57}$$

where we have applied the polymer formalism (2.55), as we may, to the sum over Γ^*_v.

Now we divide the right-hand-side of (2.57) by the expression for 1 given by (2.54)–(2.55), and we obtain (2.52) by following the proof of [BK1, Proposition 4, pp. 393–394].

In order to prove (2.53), we note that since $P_{\Lambda'}$ preserves the Lebesgue integral of a function on $\mathcal{X}_{\Lambda'}$ (by definition (2.20)), equation (2.52) for $\Lambda = \emptyset$ implies that for all n

$$l_{\Lambda'} = l_{\Lambda'} P^n_{\Lambda'} = \sum_{\Gamma^n_0} l_{\Lambda'_-} \bigotimes_{\zeta\in\Gamma^n_0} lV(\zeta)\,. \tag{2.58}$$

By definition (2.51) of $\tilde\Omega_{\Lambda'}$, the calculation leading to (2.52) sketched above then also proves (2.53). $\square$

Proof of Lemma 2.4. The proof combines ideas from the proofs of [BK1, Proposition 5 (43), and Proposition 6].

For a fixed finite $\Lambda \subset \mathbb{Z}^d$ and a given $\varphi \in \mathcal{F}$, we are going to estimate the $\|\cdot\|_\Lambda$ norm of the difference

$$
\begin{aligned}
&l_{\Lambda'\setminus\Lambda} P^n_{\Lambda'} - l_{\Lambda'\setminus\Lambda}(\tilde\Omega_{\Lambda'} \cdot l_{\Lambda'}) \\
&\quad = \sum^{*}_{\substack{\Gamma \\ \Gamma'_v=\emptyset}} \exp\Big(-\sum_{\substack{v\in\Upsilon^{0,n+1}_v \\ \overline{v}\cap\Lambda_n\neq\emptyset \\ \overline{v}\cap\overline{\Gamma}=\emptyset}} U(v)\Big) \bigotimes_{\gamma\in\Gamma_{0,\Lambda_n}} l_{\Lambda'\setminus\Lambda}\widehat{V}^b(\gamma) \\
&\qquad\qquad\qquad \otimes\Big(h_{\Lambda_+} \bigotimes_{\gamma\in\Gamma_{\Lambda_n}} l_{\Lambda'\setminus\Lambda}\widehat{V}^b(\gamma)(h)\Big)\cdot\Big(l_{\Lambda'_-}\bigotimes_{\gamma\in\Gamma'_0} lV(\gamma)\Big) \\
&\quad - \sum^{*}_{\substack{\Gamma \\ \Gamma'_{v,-\infty}=\emptyset}} \exp\Big(-\sum_{\substack{v\in\Upsilon^{-\infty,n+1}_v \\ \overline{v}\cap\Lambda_n\neq\emptyset \\ \overline{v}\cap\overline{\Gamma}=\emptyset}} U(v)\Big) \sum_{\Gamma^n_0}\Big(h_{\Lambda_+} \bigotimes_{\gamma\in\Gamma_{\Lambda_n}} l_{\Lambda'\setminus\Lambda}\widehat{V}^b(\gamma)(h)\Big)\cdot\Big(l_{\Lambda'_-}\bigotimes_{\zeta\in\Gamma^n_0} lV(\zeta)\Big)
\end{aligned}
\tag{2.59}
$$

(where we have used the expansions from Lemma 2.10), evaluated at $\varphi_{\Lambda'}$. We will then show that $\tilde\Omega_{\Lambda'} = \Omega_{\Lambda'}$.

We make first a preliminary observation. According to (1.19), if Γ is a finite set of polymers γ, containing each at least one time bond,

$$
\begin{aligned}
\|(\bigotimes_{\gamma\in\Gamma} V(\gamma)\otimes l_{\Lambda'_-})(\varphi_{\Lambda'})\|_\Lambda &\le \prod_{\gamma\in\Gamma}\|V(\gamma)\|\cdot\|l_{\Lambda'_-}(\varphi_{\Lambda'})\|_{\otimes_{i\in\Gamma_-}\mathcal{H}(D(\alpha'))} \\
&\le C^{|\Gamma_-|}\Big(\prod_{\gamma\in\Gamma}\|V(\gamma)\|\Big)\cdot\|l_{\Lambda'_-}(\varphi_{\Lambda'})\|_{\Gamma_-} \\
&\le C^{|\Gamma_-|}\Big(\prod_{\gamma\in\Gamma}\|V(\gamma)\|\Big)\cdot\|\varphi\|_{\Lambda'}\,,
\end{aligned}
\tag{2.60}
$$

where $\Gamma_- = \cup_{\gamma\in\Gamma}\gamma_-$. The bounds (2.36) guarantee that every $\|V(\gamma)\|$ contributes to a factor at most $\tilde\epsilon^{|\gamma_-|}$ except if γ contains only bonds connecting Λ_0 to Λ_n. Thus, by changing $\tilde\epsilon$ we can embed the factor $C^{|\Gamma_-|}$ into the bounds of $\|V(\gamma)\|$'s, except possibly for a factor $C^{|\Lambda|}$.

In a nutshell, Lemma 2.4 comes from the fact that the difference (2.59) can be written as a sum such that in every term there is a polymer or a pair of polymers whose support intersects Λ'_0, Λ_n, and all time levels between 0 and n. Namely, in the first summand there are those terms with $\Gamma_{0,\Lambda_n} \neq \emptyset$ which have a polymer whose support intersects

Λ_0', Λ_n. Those terms in the first summand with $\Gamma_{0,\Lambda_n} = \emptyset$ have a counterpart in the second summand which differs only in the sum over υ's. So we can take this first term as a common factor and are left with a factor

$$|1 - \exp(-\sum_{\substack{\upsilon \in \Upsilon_\upsilon^{-\infty,n+1} \\ \overline{\upsilon}\cap\Lambda_n\neq\emptyset \\ \overline{\upsilon}\cap\overline{\Gamma}=\emptyset, \overline{\upsilon}\cap\Lambda_0'\neq\emptyset}} U(\upsilon))| \leq \sum_{\substack{\upsilon \in \Upsilon_\upsilon^{-\infty,n+1} \\ \overline{\upsilon}\cap\Lambda_n\neq\emptyset \\ \overline{\upsilon}\cap\overline{\Gamma}=\emptyset, \overline{\upsilon}\cap\Lambda_0'\neq\emptyset}} |U(\upsilon)| \exp(\sum_{\substack{\upsilon' \in \Upsilon_\upsilon^{-\infty,n+1} \\ \overline{\upsilon}'\cap\Lambda_n\neq\emptyset \\ \overline{\upsilon}'\cap\overline{\Gamma}=\emptyset, \overline{\upsilon}'\cap\Lambda_0'\neq\emptyset}} |U(\upsilon')|). \tag{2.61}$$

Note that all polymers υ, υ' appearing in (2.61) intersect both Λ_0' and Λ_n. We are left with the terms in the second summand which have not appeared as "counterparts" of terms in the first summand. They either contain a polymer γ in $\Lambda'_{-\infty,n}$ with $\overline{\gamma}\cap\Lambda_n \neq \emptyset$ but $\gamma \notin \Lambda'_{0,n}$ (and thus $\overline{\gamma}\cap\Lambda_0' \neq \emptyset$), or a polymer $\zeta \in \Gamma_{0,n}$ intersecting a polymer $\gamma \in \Gamma$, with $\overline{\zeta}\cap\Lambda_0' \neq \emptyset$ and $\overline{\gamma}\cap\Lambda_n \neq \emptyset$, i.e., $\overline{\gamma}\cup\overline{\zeta}$ intersects both Λ_0' and Λ_n.

We will estimate the sum in (2.52) over terms with $\Gamma_{0,\Lambda_n} \neq \emptyset$. The other terms in (2.59) (including those which produce (2.61)) can be handled similarly and are left to the reader.

First we want to expand the exponential in (2.52) into a sum over polymers that will be merged with the sum over Γ. More precisely, for fixed Γ we write the exponential as a power series and collect intersecting υ's to form families of vacuum polymers in $\Lambda'_{0,n+1}$ such that each polymer intersects Λ_n (see [BK1 (72)–(74) p. 394]). Unlike in [BK1], all the polymers υ are disjoint from Γ, so we can add the resulting families of vacuum polymers to Γ, obtaining a new set $\Gamma_U(\Gamma)$ of disjoint polymers containing now a subset Γ_{v,Λ_n} of vacuum polymers. So we want to estimate

$$\sum_{\Gamma_U} \prod_{\gamma\in\Gamma_{v,\Lambda_n}} U^v(\gamma) \bigotimes_{\gamma\in\Gamma_{0,\Lambda_n}} l_{\Lambda'\setminus\Lambda}\widehat{V}^b(\gamma)$$
$$\otimes(h_{\Lambda_+} \bigotimes_{\gamma\in\Gamma_{\Lambda_n}} l_{\Lambda'\setminus\Lambda}\widehat{V}^b(\gamma)(h)) \cdot (l_{\Lambda'_-} \bigotimes_{\gamma\in\Gamma_0'} lV(\gamma)), \tag{2.62}$$

where the moduli of the complex weights $U^v(\gamma)$ satisfy the bound (2.36) for $\xi = \lambda/4$ and $\beta = \kappa$.

We say that two polymers γ and γ' are *strongly space-connected* if $\pi_s(\overline{\gamma})\cap\pi_s(\overline{\gamma}') \neq \emptyset$, and we define the *space connectedness* equivalence relation on finite sets of polymers by completing the strong space-connectedness relation (which is not transitive). Let $\Gamma_Y = \Gamma_Y(\Gamma_U)$ be the set of polymers in (2.62) which belongs to an equivalence class of Γ_U which is space-connected to Λ_0 (or, equivalently, Λ_n), and call the union of their space projections $Y = Y(\Gamma_Y)$. (Note that all polymers in Γ_{v,Λ_n} belong to Γ_Y.) With this notation, we may rewrite the sum (2.62) restricted to those Γ_U with $\Gamma_{0,\Lambda_n} \neq \emptyset$ as follows:

$$\sum_{Y\subset\Lambda'} \sum_{\substack{\Gamma_Y:\pi_s(\overline{\Gamma_Y})=Y\\ \Gamma_{0,\Lambda_n,Y}\neq\emptyset}} \prod_{\gamma\in\Gamma_{v,\Lambda_n,Y}} U^v(\gamma)$$

$$\bigotimes_{\gamma\in\Gamma_{0,\Lambda_n,Y}} l_{\Lambda'\setminus\Lambda}\widehat{V}^b(\gamma)\otimes\Big[(h_{\Lambda_+}\bigotimes_{\gamma\in\Gamma_{\Lambda_n,Y}} l_{\Lambda'\setminus\Lambda}\widehat{V}^b(\gamma)(h))\cdot\big[l_{(\Lambda\cup Y)_-}\bigotimes_{\gamma\in\Gamma_{0,Y}} lV(\gamma)$$

$$\otimes \sum_{\substack{\Gamma_0\\ \pi_s(\overline{\Gamma}_0)\subset\Lambda'\setminus(\Lambda\cup Y)}} l_{(\Lambda'\setminus(\Lambda\cup Y))_-}\bigotimes_{\gamma'\in\Gamma_0} lV(\gamma')\big]\Big], \tag{2.63}$$

where the notations $\Gamma_{v,\Lambda_n,Y}$, $\Gamma_{0,\Lambda_n,Y}$, $\Gamma_{\Lambda_n,Y}$, $\Gamma_{0,Y}$, $(\Lambda\cup Y)_-$, and $(\Lambda'\setminus(\Lambda\cup Y))_-$ are analogous to the notations in Proposition B, and the rightmost sum is over all sets Γ_0 of polymers in $(\Lambda'\setminus(\Lambda\cup Y))_{0,n}$ with nonempty boundary at level 0. (The sum over Γ_0 depends on Y but not on Γ_Y.)

We first bound the norm of the rightmost sum in (2.63), for fixed Y. Since $l_{\tilde\Lambda}P_{\tilde\Lambda}=l_{\tilde\Lambda}$ for each finite $\tilde\Lambda$, we get from (2.52) in Lemma 2.10 with $\Lambda=\emptyset$ (i.e., the formula $l_{\tilde\Lambda}P^n_{\tilde\Lambda}=\sum_{\Gamma:\Gamma=\Gamma_0'(\Gamma)} l_{\tilde\Lambda_-}\otimes_{\gamma\in\Gamma_0'} lV(\gamma)$, which we apply to $\tilde\Lambda=\Lambda'\setminus(\Lambda\cup Y)$) that

$$\sum_{\substack{\Gamma_0\\ \pi_s(\overline{\Gamma}_0)\subset\Lambda'\setminus(\Lambda\cup Y)}} l_{(\Lambda'\setminus(\Lambda\cup Y))_-}\bigotimes_{\gamma'\in\Gamma_0} lV(\gamma')=l_{\Lambda'\setminus(\Lambda\cup Y)}\,. \tag{2.64}$$

Thus the norm of this sum is just 1 for any Y.

We are left with the sum over Y and Γ_Y. The contribution of the norm of each factor h_{Λ_+} is bounded above by $e^{C|\Lambda|}$. Also, the norm of $\otimes_{\gamma\in\Gamma_{\Lambda_n,Y}} l_{\Lambda'\setminus\Lambda}\widehat{V}^b(\gamma)(h))$ can be uniformly estimated using (2.36), since $\sup_{D(\alpha)}|h|\leq C$ produces a bound $C\tilde\epsilon$. Thus, using (2.60), it suffices to bound the sum, over Y and Γ_Y, of the product of the norms of polymer weights $\widetilde{V}(\gamma)$ satisfying (2.36) for $\xi=\lambda/4$ and $\beta=\kappa$.

We define a *long polymer* to be a polymer in Γ_{0,Λ_n} (by (2.36) from Lemma 2.10, any long polymer produces a factor $\tilde\kappa^n$, where $\kappa<\tilde\kappa<1$). There is at least one long polymer in each term of the double sum, and we now reorganise this sum as follows. Fix one point a in Λ_n, and consider those terms in the sum such that a belongs to a long polymer. For such a (Y,Γ_Y) term, let $\Gamma_L=\Gamma_L(a,\Gamma_Y)$ be the space-connectedness equivalence class in Γ_Y including this long polymer. We sum next over the $\Gamma_L(a)$ for fixed a, forgetting for a moment the other equivalence classes of Γ_Y.

We associate a tree to Γ_L with vertices the polymers in Γ_L, and edges between two vertices only if the corresponding polymers are strongly space-connected. As in the proof of Lemma 2.9 (note however, that vertices now represent polymers), we estimate the sum over Γ_L by first fixing the number (k) of vertices and numbering them, then fixing a tree with given degrees of vertices $(d_1,\ldots,d_k)$, and summing over vertices. If there is an edge joining vertex i to vertex j, then there are $|\pi_s(\overline{\gamma}_j)|$ possibilities for the spatial position of the fixed point in the sum over vertex i. Unlike in the proof of Lemma 2.9 we do not have vertical lines of time-bonds between two space-connected polymers; however we may use the fact that the polymers which appear satisfy $\overline{\gamma} \cap ((\Lambda'_0 \cup \Lambda_n) \neq \emptyset$ and apply (2.37) from Lemma 2.9 for $t_0 = 0$ and $t_0 = n$. Note that if $Z(\gamma) \neq \emptyset$ then

$$|\pi_s(\overline{\gamma})| \leq \sum_{Z\in\gamma} |Z| \,. \tag{2.65}$$

If there are no spacelike sets in γ, then $|\pi_s(\overline{\gamma})| = 1$, but then there is only one γ in Γ_L. Thus, we obtain, using Lemma 2.9 and (2.41) in the first inequality, and then [S, Lemma V.7.7 and Proposition V.7A.1] as for the bound (2.45), that

$$\begin{aligned}
&\sum_{\substack{\Gamma_L \\ (a,n)\in\overline{\Gamma}_L}} \prod_{\gamma\in\Gamma_L} \|\widetilde{V}(\gamma)\| \\
&\qquad \leq C\kappa^n + \tilde{\kappa}^n \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\substack{d_1,\ldots,d_k\geq 1 \\ \sum_{i=1}^k d_i = 2(k-1)}} \sum_{\tau(d_1,\ldots,d_k)} (\tilde{C}\tilde{\epsilon})^k d_1 \prod_{i=1}^{k} (d_i-1)! \\
&\qquad \leq C\tilde{\kappa}^n \,,
\end{aligned} \tag{2.66}$$

where the term $C\kappa^n$ comes from the case where there is only a single line in Γ_L. Since there are $|\Lambda|$ ways to choose a, and since $C|\Lambda| \leq \exp(C|\Lambda|)$, we find

$$\sum_{a\in\Lambda} \sum_{\substack{\Gamma_L \\ (a,n)\in\overline{\Gamma}_L}} \prod_{\gamma\in\Gamma_L} \|\widetilde{V}(\gamma)\| \leq \tilde{\kappa}^n e^{C|\Lambda|} \,. \tag{2.67}$$

The sum over the other equivalence classes in Γ_Y is estimated as follows. By definition of Γ_Y, there is at least one polymer γ in each equivalence class such that $\pi_s(\overline{\gamma}) \cap \Lambda \neq \emptyset$ and $\overline{\gamma} \cap \Lambda'_t \neq \emptyset$, where t is either 0 or n. Then, using a computation analogous to (2.66), the sum over all classes different from Γ_L can be bounded above by

$$\left(\prod_{b\in\Lambda} (1 + \sum_{\substack{\Gamma_S \\ b\in\pi_s(\overline{\Gamma}_S)}} \prod_{\gamma\in\Gamma_S} \|\widetilde{V}(\gamma)\|) \right)^2 \leq e^{C|\Lambda|} \,. \tag{2.68}$$

Combining all the estimates, we find a bound for the norm of (2.59) acting on $\varphi_{\Lambda'}$:

$$\| \left[l_{\Lambda'\setminus\Lambda} P^n_{\Lambda'} - l_{\Lambda'\setminus\Lambda}(\tilde{\Omega}_{\Lambda'} \cdot l_{\Lambda'}) \right] \varphi_{\Lambda'} \|_{\Lambda} \leq \tilde{\kappa}^n e^{C|\Lambda|} \|\varphi\|_{\Lambda'} \,. \tag{2.69}$$

It follows from the exponentially decaying bound (2.69) in the case $\Lambda = \Lambda'$ that $P_{\Lambda'}(\tilde{\Omega}_{\Lambda'}) = \tilde{\Omega}_{\Lambda'}$, and that 1 is a simple eigenvalue of $P_{\Lambda'}$ and the only eigenvalue on the unit circle. Therefore $\Omega_{\Lambda'} = \tilde{\Omega}_{\Lambda'}/(l_{\Lambda'}\tilde{\Omega}_{\Lambda'})$ is a normalised eigenvector of $P_{\Lambda'}$ for the eigenvalue 1. In fact, $l_{\Lambda'}\tilde{\Omega}_{\Lambda'} = 1$, i.e., $\tilde{\Omega}_{\Lambda'} = \Omega_{\Lambda'}$. (To see this, use the fact that

$P_{\Lambda'}$ does not change the Lebesgue integral of a function, and observe that by formula (2.49) and definition (2.51) we have $\tilde{\Omega}_{\Lambda'} = \lim_{n\to\infty} P^n_{\Lambda'}(1)$.) Therefore, the difference (2.59) coincides with $l_{\Lambda'\setminus\Lambda} P^n_{\Lambda'} R_{\Lambda'}$, so that we have both proved the inequality (2.21), and showed statement (1) of Lemma 2.4. (The restriction of $\Omega_{\Lambda'}$ to $\mathcal{X}_{\Lambda'}$ is positive because it is the density of the absolutely continuous invariant measure of an analytic expanding torus map $F_{\Lambda'}$.)

The proof of (2.22) can be done using the same ideas and is left to the reader. (The bounds (2.36) in Proposition B and Lemma 2.10 yield a factor decaying exponentially with the distance between Λ and $\mathbb{Z}^d \setminus \Lambda'$, since only those families of polymers survive which connect Λ to $\mathbb{Z}^d \setminus \Lambda'$.) $\square$

3. Spectrum on a Banach space of observables

3.A Preliminaries and statements of results.

We use here the notation introduced in Sections 1 and 2. We shall first introduce a Banach space of complex Borel measures on $\mathcal{X}$ which are absolutely continuous with respect to the SRB measure μ_ϵ for F_ϵ from the previous section. We will then see, assuming $d = 1$, that perturbation theory can be applied to describe the spectrum of a (normalised) coupled transfer operator $\mathcal{L}_\epsilon$ acting on the densities ϕ of these measures following (2.7), i.e., $(\mathcal{L}_\epsilon\phi)\mu_\epsilon = (F_\epsilon)_*(\phi\mu_\epsilon)$. Both coupled and uncoupled operators act on the same Banach space of densities although the corresponding spaces of measures are different (since the reference measures μ_ϵ and μ_0 are singular with respect to each other).

For k a *multi-index* on $\mathbb{Z}^d$, that is,

$$k \in \mathbb{Z}^{\mathbb{Z}^d}\,, \quad \text{i.e.,} \qquad k = (k_i)_{i\in\mathbb{Z}^d}\,, \qquad k_i \in \mathbb{Z} \quad \forall i\,, \tag{3.1}$$

the *support* of k, denoted by $\mathrm{supp}\,(k)$, is by definition the set of $i \in \mathbb{Z}^d$ such that $k_i \neq 0$. For a multi-index k with finite support, we write

$$|k| = \sum_{i\in\mathrm{supp}\,(k)} |k_i| \quad \text{and } z^k = \prod_{i\in\mathrm{supp}\,(k)} z_i^{k_i}\,, \tag{3.2}$$

where $z_i \in \mathbb{C}$.

If τ is a tree in $\mathbb{Z}^d$, that is, the set of vertices of τ is a subset of $\mathbb{Z}^d$, we denote the set of edges of τ by $E(\tau)$, and for $e \in E(\tau)$ we write $|e|$ for the euclidean length of e viewed as a subset of $\mathbb{Z}^d$. For an arbitrary finite $X \subset \mathbb{Z}^d$, we define $L_{\surd}(X)$, the *square-root length of the shortest tree* on X, to be $L_{\surd} = l(\tau)$, where τ is a tree which minimises $l(\tau) = \sum_{e\in E(\tau)} \sqrt{|e|}$ among the set of trees in $\mathbb{R}^d$ with straight edges whose set of vertices is X.

Definition 3.1 (Banach space of densities). For fixed $0 < \alpha < 1$ as above and fixed $\delta, \eta > 0$, let $\mathcal{V} = \mathcal{V}(\delta, \eta)$ be the complex Banach space of formal power series with variables indexed by $\mathbb{Z}^d$:

$$\phi = \sum_{k\in\mathcal{I}} a_k z^k\,, \tag{3.3}$$

where $a_k \in \mathbb{C}$ and $\mathcal{I} = \mathbb{Z}^{\mathbb{Z}^d}$ is the set of multi-indices, for which the following norm is finite

$$\|\phi\|_{\delta,\eta} = \sup_{k\in\mathcal{I}} e^{\delta L_{\surd}(\mathrm{supp}\,(k)\cup\{0\})}(\tfrac{1}{\eta})^{|\mathrm{supp}\,(k)|}(\tfrac{1}{1-\alpha})^{|k|}|a_k|\,. \tag{3.4}$$

(Note that (3.4) implies that $a_k \neq 0$ only if k has a finite support.) The support of $\phi \in \mathcal{V}$ is by definition $\cup_{k\in\mathcal{I},a_k\neq 0}\,\mathrm{supp}\,(k)$. For each finite Λ we define Banach spaces $\mathcal{B}_\Lambda = \mathcal{B}_\Lambda(\delta,\eta)$ by considering those elements of $\mathcal{V}$ which are supported in Λ, with the $\|\cdot\|_{\delta,\eta}$ norm. Finally, we set $\mathcal{B} = \mathcal{B}(\delta,\eta) \subset \mathcal{V}$ to be the $\|\cdot\|_{\delta,\eta}$-closure of locally supported elements of $\mathcal{V}$ (i.e., those ϕ which have finite support) with the $\|\cdot\|_{\delta,\eta}$ norm.

It is easy to see that $(\mathcal{V},\|\cdot\|_{\delta,\eta})$ is a Banach space, which in turn implies that $\mathcal{B}$ is a Banach space. To see that $\mathcal{B} \neq \mathcal{V}$ observe that setting $a_k = e^{-\delta L_{\surd}(\mathrm{supp}\,(k)\cup\{0\})}\eta^{|\mathrm{supp}\,(k)|}(1-\alpha)^{|k|}$ for each k with finite support in (3.3) defines an element of $\mathcal{V}$ which is not the limit of a sequence in $\mathcal{B}$ for the $\|\cdot\|_{\eta,\delta}$ norm.

A vector $\phi \in \mathcal{B}_\Lambda(\delta,\eta)$ defines an analytic function on $\mathcal{D}_\Lambda$ (also denoted by ϕ) and thus a bounded analytic function on $\mathcal{D}(\tilde\alpha)_\Lambda$ for any $0 < \tilde\alpha < \alpha$. The coordinates $\{a_k\,, k \in \mathcal{I}\}$ of ϕ are just the Laurent series coefficients of ϕ and thus uniquely determined by the restriction of ϕ to $\mathcal{X}_\Lambda$ (by uniqueness of analytic extensions).

In the case $d = 1$, the following lemma will allow us to view elements $\phi \in \mathcal{B}$ as bounded functions on $\mathcal{D}_{\tilde\alpha} := \mathcal{D}(\tilde\alpha)_{\mathbb{Z}}$ for any $0 < \tilde\alpha < \alpha$:

Lemma 3.2. *Assume $d = 1$. Fix $\delta > 0$ and $0 < \tilde\alpha < \alpha$. There exist $\eta_0(\tilde\alpha,\delta) > 0$ and $C = C(\tilde\alpha,\delta) > 0$ such that for all $\eta \le \eta_0$, each $\phi \in \mathcal{B}(\delta,\eta)$ and all finite $X \subset \mathbb{Z}$*

$$\sum_{\substack{k\in\mathcal{I}\\ X\subset\mathrm{supp}\,(k)}} \|a_k z^k\|_{\mathcal{H}(\mathcal{D}(\tilde\alpha)_{\mathrm{supp}\,(k)})} \le e^{C\eta(|X|+1)}\, e^{-\delta L_{\surd}(X\cup\{0\})}\, \eta^{|X|}\, \|\phi\|_{\delta,\eta}\,. \tag{3.5}$$

Lemma 3.2 applied to the case $X = \emptyset$ shows that an element ϕ of $\mathcal{B}$ is a bounded function on $\mathcal{D}_{\tilde\alpha}$ (also denoted by ϕ) for any $0 < \tilde\alpha < \alpha$, with $\sup_{\overline{\mathcal{D}}_{\tilde\alpha}} |\phi| \le C(\tilde\alpha,\delta,\eta)\,\|\phi\|_{\delta,\eta}$. The function associated to $\phi \in \mathcal{B}$ vanishes on $\mathcal{X}$ if and only if all the a_k, $k \in \mathcal{I}$, are zero. The supremum norm on $\mathcal{D}(\tilde\alpha)_\Lambda$ and the $\|\cdot\|_{\delta,\eta}$ norm are equivalent on any finite Λ, but on the whole space the $\|\cdot\|_{\delta,\eta}$ norm cannot be bounded by a multiple of the supremum norm.

Proof of Lemma 3.2. Let $X = \{x\} \subset \mathbb{Z}$ be such that $x = l > 0$. Then using the substitution $t \to s^2$, the fact that $\sqrt{l-s^2} \ge \sqrt{l} - s/2$ for $s \le 4\sqrt{l}/5$, and integration by parts, we have that

$$\sum_{\substack{w\in\mathbb{Z}\\ 0<w<x}} e^{-\delta L_{\surd}(\{0,w,x\})} = \sum_{m=1}^{l-1} e^{-\delta(\sqrt{m}+\sqrt{l-m})} \le 2\int_0^{\frac{l}{2}} e^{-\delta(\sqrt{t}+\sqrt{l-t})}dt \le (\tfrac{4}{\delta})^2 e^{-\delta\sqrt{l}}\,. \tag{3.6}$$

Also, if $x = \ell \ge 0$,

$$\sum_{\substack{w\in\mathbb{Z}\\ x<w}} e^{-\delta L_{\surd}(\{0,w,x\})} \le e^{-\delta\sqrt{l}} \sum_{m=1}^{\infty} e^{-\delta\sqrt{m}} \le \tfrac{2}{\delta^2} e^{-\delta\sqrt{l}}\,. \tag{3.7}$$

Using (3.6), we obtain that

$$\sum_{j=1}^{l-1} \sum_{\substack{w_1,\dots,w_j\in\mathbb{Z}\\ 0<w_1<\dots<w_j<x}} e^{-\delta L_{\surd}(\{0,w_1,\dots,w_j,x\})}\eta^{1+j} \le e^{-\delta L_{\surd}(\{0,x\})}\eta \sum_{j=1}^{\infty}((\tfrac{4}{\delta})^2\eta)^j$$

$$\le c\eta e^{-\delta L_{\surd}(\{0,x\})}\eta\,. \tag{3.8}$$

In the same way, using (3.7), we see that for $x \ge 0$

$$\sum_{j=1}^{\infty} \sum_{\substack{w_1,\dots,w_j\in\mathbb{Z}\\ x<w_1<\dots<w_j}} e^{-\delta L_{\surd}(\{0,x,w_1,\dots,w_j\})}\eta^{1+j} \le e^{-\delta L_{\surd}(\{0,x\})}\eta \sum_{j=1}^{\infty}(\tfrac{2}{\delta^2}\eta)^j$$

$$\le c\eta e^{-\delta L_{\surd}(\{0,x\})}\eta\,. \tag{3.9}$$

The statement of the lemma for $|X| \le 1$ follows from by (3.9).

Let X be a finite set containing at least two elements. Since between each successive points of X (separated by a distance l) one may put j extra points for some $0 \le j < l$, and an arbitrary number of points both on the left and right of X, we get from (3.8)–(3.9) (using $1 + C\eta < e^{C\eta}$) that

$$\sum_{\substack{Y\subset\mathbb{Z}\\ X\subset Y,Y \text{ finite}}} e^{-\delta L_{\surd}(Y\cup\{0\})}\eta^{|Y|} \le e^{C\eta(|X|+1)}e^{-\delta L_{\surd}(X\cup\{0\})}\eta^{|X|}\,. \tag{3.10}$$

Thus for $|X| \ge 2$ we find

$$\sum_{\substack{k\in\mathcal{I}\\ X\subset \operatorname{supp}(k)}} \|a_k z^k\|_{\mathcal{H}(\mathcal{D}(\tilde\alpha)_{\operatorname{supp}(k)})} \le \sum_{\substack{Y\subset\mathbb{Z}\\ X\subset Y,Y \text{ finite}}} \sum_{\substack{k\in\mathcal{I}\\ \operatorname{supp}(k)=Y}} \|a_k z^k\|_{\mathcal{H}(\mathcal{D}(\tilde\alpha)_{\operatorname{supp}(k)})}$$

$$\le \|\phi\|_{\delta,\eta} \sum_{Y\supset X} e^{-\delta L_{\surd}(Y\cup\{0\})}\eta^{|Y|} \sum_{\substack{k\in\mathcal{I}\\ \operatorname{supp}(k)=Y}} (1-\alpha)^{|k|}\left|\frac{1}{1-\tilde\alpha}\right|^{|k|}$$

$$\le \|\phi\|_{\delta,\eta} e^{-\delta L_{\surd}(X\cup\{0\})}\eta^{|X|}e^{C\eta(|X|+1)}\,. \quad \square \tag{3.11}$$

We view elements of $\mathcal{B}$ as "analytic" extensions of densities of complex measures on $\mathcal{X}$ which are absolutely continuous with respect to the coupled SRB measure μ_ϵ from Theorem 2.6 for $0 \le \epsilon < \epsilon_2$. In other words, we shall work with a normalised transfer operator, which is just the coupled lattice version of the single-site normalised operator $L : \mathcal{H}(D) \to \mathcal{H}(D)$ defined by $(L\varphi)(x) = \frac{1}{h(x)}\sum_{y\in D, f(y)=x}\frac{\varphi(y)h(y)}{Df(y)}$ (the constant function 1 on D is fixed by L, and the dual of L fixes $\mu_0 = h\,m$).

We need some preparation. Let $\Lambda \subset \mathbb{Z}$ be a finite interval. We denote as before the Perron-Frobenius operator (2.20) associated to our cutoff map $F_{\epsilon,\Lambda}$ on $\mathcal{D}_\Lambda$ by $P_{\epsilon,\Lambda}$.

Let $L_\Lambda = L_{\epsilon,\Lambda}$ be the (normalised) transfer operator with respect to the SRB-measure $\Omega_{\epsilon,\Lambda}\, m^\Lambda$ of the cutoff map $F_{\epsilon,\Lambda}$, i.e., for any bounded $\phi : \overline{\mathcal{D}}_\Lambda \to \mathbb{C}$:

$$L_{\epsilon,\Lambda}(\phi) = \frac{1}{\Omega_{\epsilon,\Lambda}} P_{\epsilon,\Lambda}(\phi\, \Omega_{\epsilon,\Lambda})\,. \tag{3.12}$$

Since the constant function $\equiv 1$ on $\overline{\mathcal{D}}_\Lambda$ is fixed by $L_{\epsilon,\Lambda}$ for any ϵ and since $L_{\epsilon,\Lambda}$ acting on functions on $\mathcal{X}_\Lambda$ is positive, we find

$$\sup_{\mathcal{X}_\Lambda} |L_{\epsilon,\Lambda}\phi| \le \sup_{\mathcal{X}_\Lambda} |\phi|\,. \tag{3.13}$$

For any finite Λ and each $0 < \tilde\alpha < \alpha$, we have $\mathcal{H}(\mathcal{D}(\alpha)_\Lambda) \subset \mathcal{B}_\Lambda(\delta,\eta) \subset \mathcal{H}(\mathcal{D}(\tilde\alpha)_\Lambda)$, where $\|\varphi\|_{\mathcal{B}_\Lambda} \le C(\Lambda)\|\varphi\|_{\mathcal{H}(\mathcal{D}(\alpha)_\Lambda)}$ and $\|\varphi\|_{\mathcal{H}(\mathcal{D}(\tilde\alpha)_\Lambda)} \le C(\tilde\alpha)\|\varphi\|_{\mathcal{B}_\Lambda}$, whenever the right hand side of the inequality considered is well defined. From now on, we assume that $\alpha' < \alpha$ are as in Proposition A from Section 2, and δ and $\eta(\delta,\tilde\alpha)$ have been chosen as in Lemma 3.2 for $\tilde\alpha = \alpha'$. (The result stated in Proposition A for $P_{\epsilon,\Lambda}$ holds also for $L_{\epsilon,\Lambda}$ by (3.12) since $|\Omega_{\epsilon,\Lambda}|$ is bounded away from zero.) Since $L_{\epsilon,\Lambda} : \mathcal{H}(\mathcal{D}(\alpha')_\Lambda) \to \mathcal{H}(\mathcal{D}(\alpha)_\Lambda)$ is bounded and thus compact (by Montel's theorem), the operator $L_{\epsilon,\Lambda}$ is compact both on $\mathcal{H}(\mathcal{D}(\tilde\alpha)_\Lambda)$ for $\alpha' \le \tilde\alpha < \alpha$ (with spectrum given by the zeroes of a Fredholm determinant, see [Ru1, Ru2]) and thus independent of $\tilde\alpha$) and on $\mathcal{B}_\Lambda$. Clearly, the spectra of the compact operators $L_{\epsilon,\Lambda}$ and $P_{\epsilon,\Lambda}$ on any $\mathcal{H}(\mathcal{D}(\tilde\alpha)_\Lambda)$ with $\alpha' \le \tilde\alpha < \alpha$ coincide. The compactness property together with the inclusion of Banach spaces stated above imply that the spectra of $L_{\epsilon,\Lambda}$ acting on $\mathcal{B}_\Lambda$ and $\mathcal{H}^\Lambda$ coincide for $0 \le \epsilon < \epsilon_2$. In particular, the spectrum of $L_{0,\Lambda}$ on $\mathcal{B}_\Lambda$ consists of the set Σ_Λ introduced above (2.12).

In this section, we shall denote by $R_{\epsilon,\Lambda}$ the bounded projection on $\mathcal{B}_\Lambda$ defined by

$$R_{\epsilon,\Lambda}(\phi) = \phi - l_\Lambda(\phi\Omega_{\epsilon,\Lambda}) \tag{3.14}$$

(there is no risk of confusion with the unnormalised projection operator noted R_Λ in Section 2). For a fixed $0 < \kappa < 1$ such that no element of Σ_∞ (defined by (2.10)) has modulus κ, we denote by $R_{0,\Lambda,<\kappa} : \mathcal{B}_\Lambda \to \mathcal{B}_\Lambda$ the spectral projection corresponding to the part of the spectrum of $L_{0,\Lambda}$ inside the disc of radius κ. Note that $\mathrm{Id} - R_{0,\Lambda,<\kappa}$ is a bounded (finite rank) operator on $\mathcal{B}_\Lambda$, and since each generalised eigenfunctional of $L^*_{0,\Lambda}$ associated to an eigenvalue ρ belongs in fact to the dual of $\mathcal{H}(\mathcal{D}(\tilde\alpha)_\Lambda)$ for any $0 < \tilde\alpha < \alpha$, and thus to the dual of $\mathcal{B}_\Lambda$, the operator $\mathrm{Id} - R_{0,\Lambda,<\kappa}$ is in fact a finite rank operator from $\mathcal{H}(\mathcal{D}(\alpha')_\Lambda)$ to $\mathcal{H}(\mathcal{D}(\alpha)_\Lambda)$.

The inclusion $\mathcal{H}(D(\tilde\alpha)_\Lambda) \subset \mathcal{H}(\mathcal{D}(\tilde\alpha)_{\Lambda'})$ will often be implicit. Finally, we shall consider the Banach spaces $\mathcal{C} = \mathcal{C}(\mathcal{X})$ and $\mathcal{C}(\mathcal{X}_\Lambda)$ of continuous functions $\phi : \mathcal{X} \to \mathbb{C}$ and $\phi : \mathcal{X}_\Lambda \to \mathbb{C}$, respectively, endowed with the supremum norm.

Lemma 3.3 (Main Banach bounds). *Assume $d = 1$. Let ϵ_2 be as in Lemma 2.4 and $\delta > 0$. Then there exist positive η_0, λ_0, and for any $0 < \kappa_1 < \kappa$ with κ_1 defined* *in* (1.5)*, a positive constant* ϵ_3 *such that for all* $\lambda \geq \lambda_0$ *(in the coupling decay* (1.10)*),* $0 \leq \epsilon \leq \epsilon_3$*, and* $0 < \eta \leq \eta_0$*:*

(1) *There exist a function* $\tilde{\epsilon}(\epsilon) \geq 0$*, which tends to zero as* ϵ *tends to zero, and* $n_0(\kappa) \in \mathbb{Z}^+$ *such that for all* $n \geq n_0$ *and all finite* $\Lambda \subset \mathbb{Z}$

$$\|L^n_{\epsilon,\Lambda} R_{\epsilon,\Lambda} - L^n_{0,\Lambda} R_{0,\Lambda}\|_{\delta,\eta} \leq \tilde{\epsilon}\kappa^n \tag{3.15}$$

and

$$\|L^n_{\epsilon,\Lambda} R_{\epsilon,\Lambda}\|_{\delta,\eta} \leq \kappa^n \,. \tag{3.16}$$

(2) *There are* $C > 0$ *and* $\tilde{\lambda} > 0$ *so that for any* $n \geq n_0$*, any finite* $\Lambda \subset \Lambda' \subset \Lambda'' \subset \mathbb{Z}$*, and any* $\phi \in \mathcal{B}$ *supported in* Λ*,*

$$\|L^n_{\epsilon,\Lambda'} R_{\epsilon,\Lambda'}(\phi) - L^n_{\epsilon,\Lambda''} R_{\epsilon,\Lambda''}(\phi)\|_{\delta,\eta} \leq Ce^{-\tilde{\lambda} d(\Lambda,\mathbb{Z}\setminus\Lambda')}\|\phi\|_{\delta,\eta} \,. \tag{3.17}$$

For $n < n_0$ *the inequality* (3.17) *holds, but replacing the factor* $Ce^{-\tilde{\lambda} d(\Lambda,\mathbb{Z}\setminus\Lambda')}$ *in the right-hand side by* $e^{C\eta|\Lambda|}e^{-\tilde{\lambda} d(\Lambda,\mathbb{Z}\setminus\Lambda')}$*. Also for all* $\alpha' < \alpha$ *there is* $C(\Lambda, \alpha') > 0$ *such that for any* $\phi \in \mathcal{H}(\mathcal{D}(\alpha')_\Lambda)$

$$\sup_{\mathcal{X}_{\Lambda''}} |L_{\epsilon,\Lambda'}(\phi) - L_{\epsilon,\Lambda''}(\phi)| \leq C(\Lambda, \alpha')e^{-\tilde{\lambda} d(\Lambda,\mathbb{Z}\setminus\Lambda')}\|\phi\|_{\mathcal{H}(\mathcal{D}(\alpha')_\Lambda)} \,. \tag{3.18}$$

(3) *For any* $\rho \in \Sigma_\infty$*, there is* $C = C(\rho) > 0$ *such that for each finite* Λ *with* $\rho \in \Sigma_\Lambda$ *the spectral projection* $\Pi_{\Lambda,\rho}$ *associated to the eigenvalue of finite multiplicity* ρ *of* $L_{0,\Lambda}$ *satisfies*

$$\|\Pi_{\Lambda,\rho}\|_{\delta,\eta} \leq C \,. \tag{3.19}$$

(4) *Fix* $0 < \kappa < 1$ *such that no element of* Σ_∞ *has modulus* κ*. Then for* $\tilde{\kappa} > \kappa$ *there is* $n_1(\tilde{\kappa}) \geq 1$ *so that for any* $n \geq n_1$ *and any finite* Λ *in* $\mathbb{Z}$

$$\|L^n_{0,\Lambda} R_{0,\Lambda,<\kappa}\|_{\delta,\eta} \leq \tilde{\kappa}^n \,. \tag{3.20}$$

Recall that the algebraic (and thus geometric) multiplicity of any $1 \neq \rho \in \Sigma_\infty$ as an eigenvalue of $L_{0,\Lambda}$ tends to ∞ as $\Lambda \to \mathbb{Z}$.

The proof of the key Lemma 3.3 will be given in Section 3.B. We now state and prove the main result of this section.

Theorem 3.4 (Coupled and uncoupled operator spectra on Banach space). *Assume* $d = 1$*. For any* $\delta > 0$ *and* $\kappa > \kappa_1$*, if* η*,* ϵ_3*,* λ*, and* $n_0(\kappa)$ *are as given by Lemma 3.2 and Lemma 3.3, we have*

(1) 1 *is a simple eigenvalue of* $L_{\epsilon,\Lambda}$ *for each finite* Λ *and all* $0 \leq \epsilon < \epsilon_0$*. There is a nonnegative function* $\tilde{\epsilon}(\epsilon)$ *with* $\tilde{\epsilon} \to 0$ *as* $\epsilon \to 0$ *such that for any finite* Λ *the nonzero spectrum of the normalised coupled operator* $L_{\epsilon,\Lambda}$ *(acting on either of the Banach spaces* $\mathcal{B}_\Lambda$ *or* $\mathcal{H}^\Lambda$*) is contained in a* $\tilde{\epsilon}$ *neighbourhood of the nonzero*

spectrum of the uncoupled normalised operator $L_{0,\Lambda}$. More precisely, for $\rho \in \Sigma_\infty$ and any disc Δ, centered at ρ and which does not intersect $\{0\} \cup (\Sigma_\infty \setminus \rho)$, there is $\epsilon(\Delta)$ such that for all $0 \le \epsilon < \epsilon(\Delta)$ and all finite Λ, the eigenvalues in Δ of the compact operator $L_{\epsilon,\Lambda}$ have algebraic multiplicities summing to the algebraic multiplicity of ρ for $L_{0,\Lambda}$.

(2) *For all $0 \le \epsilon < \epsilon_3$, the condition $(\mathcal{L}_\epsilon \phi)\mu_\epsilon = (F_\epsilon)_*(\phi\mu_\epsilon)$ defines a bounded operator $\mathcal{L}_\epsilon : \mathcal{C} \to \mathcal{C}$, and a closable operator $\mathcal{L}_\epsilon$ on $Dom\,\mathcal{L}_\epsilon \subset \mathcal{B}$, the vector space of locally supported functions in $\mathcal{B}$. Furthermore, $\mathcal{L}_\epsilon^n$ extends to a bounded operator on $\mathcal{B}$ for all $n \ge n_0$.*

(3) *For all $n \ge n_0$ the nonzero spectrum of $\mathcal{L}_0^n$ on $\mathcal{B}$ consists of a simple eigenvalue at 1, together with isolated eigenvalues of infinite multiplicity at each ρ^n for $1 \ne \rho \in \Sigma_\infty$. In particular, the eigenvalues of the closable operator $\mathcal{L}_0$ on $Dom\,\mathcal{L}_0 \subset \mathcal{B}$ consist of the simple eigenvalue 1 and eigenvalues of infinite multiplicity at each $1 \ne \rho \in \Sigma_\infty$.*

(4) *For all $n \ge n_0$ there exists a function $\tilde\epsilon_n(\epsilon) \ge 0$, which tends to zero as ϵ tends to zero, such that the spectrum of the bounded operator $\mathcal{L}_\epsilon^n$ on $\mathcal{B}$ contains the simple eigenvalue 1 and the rest of the nonzero spectrum is inside a $\tilde\epsilon_n$-neighbourhood of $\mathrm{sp}(\mathcal{L}_0^n) \setminus \{1\}$. There is $\tilde\epsilon_1(\epsilon)$ tending to zero when $\epsilon \to 0$ such that the nonzero spectrum of the closable operator $\mathcal{L}_\epsilon$ on $Dom\,\mathcal{L}_\epsilon \subset \mathcal{B}$, except for a simple eigenvalue at 1, and maybe for continuous spectrum, is contained in a $\tilde\epsilon_1(\epsilon)$ neighbourhood of $\mathrm{sp}(\mathcal{L}_0) \setminus \{1\}$.*

For $A \subset \mathbb{C}$ and $n \ge 1$ we write A^n for the set $\{z^n \mid z \in A\}$.

Proof of Theorem 3.4. (1) We start with a preliminary bound on the resolvent of $L_{0,\Lambda}^n R_{0,\Lambda}$. We then use it to apply standard perturbation theory in the parameter ϵ to the compact operators $L_{\epsilon,\Lambda}^n R_{\epsilon,\Lambda}$ and $L_{0,\Lambda}^n R_{0,\Lambda}$ *uniformly in* Λ for each fixed $n \ge n_0$.

Let $N(\kappa, \Lambda)$ be the cardinality of the set $\Sigma_{\Lambda,>\kappa}$ defined in (2.13), and for $\rho \in \Sigma_{\Lambda,>\kappa}$ let $R_{0,\Lambda,<\kappa}$ and $\Pi_{\Lambda,\rho}$ be as in Lemma 3.3. (Recall that $N(\kappa) = \sup_\Lambda N(\Lambda,\kappa)$ is finite.) Let also $D_{\Lambda,\rho}$ be the nilpotent operator $D_{\Lambda,\rho} = (L_{0,\Lambda} - \rho)\,\Pi_{\Lambda,\rho}$ associated to $\rho \in \Sigma_{\Lambda,>\kappa} \setminus \{1\}$ (the nilpotent operator of $\rho = 1$ is zero). Writing now $D_{\Lambda,\rho,(n)}$ for the nilpotent operator $(L_{0,\Lambda}^n - \rho^n)\,\Pi_{\Lambda,\rho}$ associated to ρ^n and $L_{0,\Lambda}^n$ (or equivalently, ρ^n and $L_{0,\Lambda}^n R_{0,\Lambda}$, since $\rho \in \Sigma_\infty \setminus \{1\}$), we observe that (3.19) together with (3.16) imply that there is a constant $C > 0$ so that for all $n \ge n_0$ the operator norm $\|D_{\Lambda,\rho,(n)}\|_{\delta,\eta} \le C$, uniformly in Λ. Finally, for any $\rho \in \Sigma_\infty$ and $n \ge 1$ it is not difficult to verify that the order $q(\Lambda,\rho,n)$ of $D_{\Lambda,\rho,(n)}$ (i.e., the smallest nonnegative integer q such that $D_{\Lambda,\rho,(n)}^q \equiv 0$) for the eigenvalue ρ^n of $L_{0,\Lambda}^n$ does not depend on n (we write $q(\Lambda,\rho)$) and is uniformly bounded in Λ.

Let $z \in \mathbb{C} \setminus \{0\}$ be such that $\zeta_n = d(z^n, \Sigma_\infty^n)$ is nonzero for $n = n_0$. We claim that there are $C(|z|) > 0$ and $Q(|z|) \ge 1$ such that for all finite $\Lambda \subset \mathbb{Z}$ and all $n \ge n_0$ with $\zeta_n > 0$

$$\|(z^n - L_{0,\Lambda}^n R_{0,\Lambda})^{-1}\|_{\delta,\eta} \le C^n/\zeta_n^Q\,. \tag{3.21}$$

Let κ be such that $|z| > 2\kappa$ and that no element of Σ_∞ has modulus κ. The spectral decomposition (see e.g. [Ka, III.6.5, (6.35)]) associated to the finite system of eigenvalues

$\Sigma_{\Lambda,>\kappa}$ gives

$$(z^n - L_{0,\Lambda}^n R_{0,\Lambda})^{-1} = (z^n - L_{0,\Lambda}^n R_{0,\Lambda,<\kappa})^{-1} R_{0,\Lambda,<\kappa} + \sum_{\rho\in\Sigma_{\Lambda,>\kappa}} \left[\frac{\Pi_{\Lambda,\rho}}{z^n-\rho^n} + \sum_{j=1}^{q(\Lambda,\rho)} \frac{D^j_{\Lambda,\rho,(n)}}{(z^n-\rho^n)^{j+1}} \right], \tag{3.22}$$

so that Lemma 3.3 (3–4) yield (3.21) for $n \ge n_1(|z|)$. From

$$(z^{n_0} - L_{0,\Lambda}^{n_0} R_{0,\Lambda})^{-1} = \big[\sum_{j=0}^{n_2-1} z^{(n_2-1-j)n_0} (L_{0,\Lambda} R_{0,\Lambda})^{n_0 j}\big] (z^{n_2 n_0} - L_{0,\Lambda}^{n_2 n_0} R_{0,\Lambda})^{-1} \tag{3.23}$$

for $n_2 = [n_1/n_0] + 1$, and since $\|(L_{0,\Lambda} R_{0,\Lambda})^{n_0 j}\|_{\delta,\eta} \le 1$ for $j \ge 0$ (by Lemma 3.3 (1)), we obtain the claimed bound (3.21) for any $n \ge n_0$.

Now, using the formal equality $(z-(M+N))^{-1} = (z-M)^{-1}(1-(z-M)^{-1}N)^{-1}$ and (3.15) from Lemma 3.3, we find for any $n \ge n_0$

$$\begin{aligned} \|(z^n - L_{\epsilon,\Lambda}^n R_{\epsilon,\Lambda})^{-1}\|_{\delta,\eta} &\le \|(z^n - L_{0,\Lambda}^n R_{0,\Lambda})^{-1}\|_{\delta,\eta} \\ &\qquad \times \sum_{j=0}^{\infty} \|(z^n - L_{0,\Lambda}^n R_{0,\Lambda})^{-1}\|^j_{\delta,\eta} \, \|L_{0,\Lambda}^n R_{0,\Lambda} - L_{\epsilon,\Lambda}^n R_{\epsilon,\Lambda}\|^j_{\delta,\eta} \\ &\le \frac{C^n}{\zeta_n^Q} \sum_{j=0}^{\infty} (\tilde\epsilon \kappa^n)^j \left(\frac{C^n}{\zeta_n^Q}\right)^j , \end{aligned} \tag{3.24}$$

where $\tilde\epsilon \to 0$ as $\epsilon \to 0$. Clearly, the sum (3.24) converges if $\tilde\epsilon < \zeta_n^Q/(C^n\kappa^n)$ which can be guaranteed by taking ϵ small enough (depending only on $|z|$, ζ_n, and n). Essentially the same computation shows that for any $n \ge n_0$ and all finite Λ

$$\|(z^n - L_{\epsilon,\Lambda}^n R_{\epsilon,\Lambda})^{-1} - (z^n - L_{0,\Lambda}^n R_{0,\Lambda})^{-1}\|_{\delta,\eta} \le C(\zeta_n, |z|^n)\tilde\epsilon , \tag{3.25}$$

so that (using the Cauchy representation for spectral projections, see e.g. [Ka, III.6.4]) the difference between the considered spectral projections for $L_{\epsilon,\Lambda}^n R_{\epsilon,\Lambda}$ and $L_{0,\Lambda}^n R_{0,\Lambda}$ goes to zero with ϵ for any fixed $n \ge n_0$, uniformly in Λ, in particular the spectral dimensions coincide for small ϵ.

If subsets A and A_ϵ of the complex plane have the property that there is $n \ge 1$ such that A_ϵ^n is in a $\hat\epsilon(\epsilon)$ neighbourhood of A^n, and A_ϵ^{n+1} is in a $\hat\epsilon(\epsilon)$ neighbourhood of A^{n+1}, with $\hat\epsilon \to 0$ as $\epsilon \to 0$, then A_ϵ lies in a $\bar\epsilon(\epsilon)$ neighbourhood of A with $\bar\epsilon(\epsilon) \to 0$ as $\epsilon \to 0$. Thus, applying the above bounds to $n = n_0$ and $n_0 + 1$, and using the fact that the operators are compact, we obtain claim (1) of the theorem. (Note that there is no claim regarding the geometric multiplicities.)

(2) For any finite $\Lambda \subset \mathbb{Z}$ and for any $\phi \in \mathcal{H}(\mathcal{D}(\alpha')_\Lambda)$, using (3.18) in Lemma 3.3, we may define a continuous function $\mathcal{L}_\epsilon(\phi \circ \pi_\Lambda) : \mathcal{X} \to \mathbb{C}$ by the limit $\mathcal{L}_\epsilon(\phi \circ \pi_\Lambda) = \lim_{\Lambda' \to \mathbb{Z}} L_{\epsilon,\Lambda'}(\phi \circ \pi_\Lambda^{\Lambda'})|_{\mathcal{X}_{\Lambda'}}$. Therefore, since locally supported continuous function are dense in $\mathcal{C}$ and real analytic functions are dense in $C_\mathbb{R}(\mathcal{X}_\Lambda)$ for the supremum norm, the inequality (3.13) implies that we may define a bounded operator $\mathcal{L}_\epsilon$ on $\mathcal{C}$ by setting (here $\phi = \lim_{\Lambda \to \mathbb{Z}} \phi_\Lambda$)

$$\mathcal{L}_\epsilon(\phi) = \lim_{\Lambda \to \mathbb{Z}} \mathcal{L}_\epsilon(\phi_\Lambda) = \lim_{\Lambda \to \mathbb{Z}} \big(\lim_{\Lambda' \to \mathbb{Z}} L_{\epsilon,\Lambda'}(\phi_\Lambda) \big) . \tag{3.26}$$

We now show that the transfer operator $\mathcal{L}_\epsilon$ defined on $\mathcal{C}$ by (3.26) satisfies (2.7). Fixing $\phi \in \mathcal{C}$ supported in some finite Y (general $\phi \in \mathcal{C}$ are handled by a continuity argument) and using the fact that $\psi \circ F_{\epsilon,\Lambda}$ converges to $\psi \circ F_\epsilon$ as $\Lambda \to \mathbb{Z}$ in the sup-norm for all finite $Y' \subset \mathbb{Z}$ and all $\psi \in C(\mathcal{X}_{Y'})$, we obtain just like in (2.26) that

$$\int_\mathcal{X} (\psi \circ F_\epsilon)\, \phi \, d\mu_\epsilon = \int_\mathcal{X} \psi \, (\mathcal{L}_\epsilon \phi) \, d\mu_\epsilon . \tag{3.27}$$

Clearly, for any $\phi \in \mathcal{C}$ we have $\mathcal{L}_\epsilon \phi = \int \phi \, d\mu_\epsilon + \mathcal{N}_\epsilon \phi$ with

$$\mathcal{N}_\epsilon \phi = \lim_{\Lambda \to \mathbb{Z}} \lim_{\Lambda' \to \mathbb{Z}} L_{\epsilon,\Lambda'} R_{\epsilon,\Lambda'}(\phi_\Lambda) , \tag{3.28}$$

where ϕ_Λ tends to ϕ in the supremum norm. Also, $\mathcal{N}_\epsilon = \mathcal{L}_\epsilon \mathcal{R}_\epsilon$ where we write $\mathcal{R}_\epsilon \phi = \phi - \int_\mathcal{X} \phi \, d\mu_\epsilon$. ($\mathcal{R}_\epsilon$ is a bounded operator on both Banach spaces $\mathcal{B}$ and $\mathcal{C}$.)

We now define $\mathcal{L}_\epsilon$ on locally supported elements ϕ of $\mathcal{B}$. Assume ϕ is supported in a finite Y. For any finite $\Lambda \supset Y$, the function $L_{\epsilon,\Lambda} R_{\epsilon,\Lambda}(\phi)$ is a locally supported element of $\mathcal{B}$. From (3.17) in Lemma 3.3, we see that the limit $\mathcal{N}_\epsilon(\phi) = \lim_{\Lambda \to \mathbb{Z}} L_{\epsilon,\Lambda} R_{\epsilon,\Lambda}(\phi)$ exists in $\mathcal{B}$. We may thus define $\mathcal{L}_\epsilon \phi \in \mathcal{B}$ by

$$\mathcal{L}_\epsilon(\phi) = \int \phi \, d\mu_\epsilon + \mathcal{N}_\epsilon(\phi) = \int \phi \, d\mu_\epsilon + \lim_{\Lambda \to \mathbb{Z}} L_{\epsilon,\Lambda} R_{\epsilon,\Lambda}(\phi) . \tag{3.29}$$

The restriction of $\mathcal{L}_\epsilon(\phi)$ to $\mathcal{X}$ coincides with the image of $\phi|_\mathcal{X} \in \mathcal{C}$ by the transfer operator defined by (3.26) so that it is legitimate to use the same notation.

The above considerations on the action of the normalised transfer operator on continuous functions show that $\mathcal{L}_\epsilon \mathcal{R}_\epsilon = \mathcal{N}_\epsilon$ is a closable operator. Indeed, we need to check that whenever ϕ_m is a sequence of locally supported elements of $\mathcal{B}$, tending to zero for the $\|\cdot\|_{\delta,\eta}$ norm, and such that $\psi = \lim_{m \to \infty} \mathcal{N}_\epsilon \phi_m$ exists in $\mathcal{B}$, then $\psi = 0$. This is true because the ϕ_m restricted to $\mathcal{X}$ will tend to zero in the supremum norm by Lemma 3.2, so that $\lim_{m \to \infty} \mathcal{N}_\epsilon(\phi_m|_\mathcal{X}) = \psi|_\mathcal{X}$ is identically zero as an element of $\mathcal{C}$, so that ψ itself vanishes by analyticity. It is not clear that the first projection of the closure of the graph of $\mathcal{N}_\epsilon$ contains $\mathcal{B}$, so that we do not know whether $\mathcal{L}_\epsilon$ is closed (and thus bounded) on the whole Banach space $\mathcal{B}$.

The claims on boundedness for high iterates follow from Lemma 3.3. More precisely: we extend the definition of $\mathcal{L}_\epsilon^n \mathcal{R}_\epsilon$ to the Banach space $\mathcal{B}$, for any $n \geq n_0$, and show that it is a bounded operator on $\mathcal{B}$. (We may then set $\mathcal{L}_\epsilon^n \phi = \mathcal{L}_\epsilon^n \mathcal{R}_\epsilon \phi + \int \phi \, d\mu_\epsilon$.) To define $\mathcal{L}_\epsilon^n \mathcal{R}_\epsilon \phi$ for $\phi \in \mathcal{B}$, we consider a $\| \cdot \|_{\delta,\eta}$ Cauchy sequence ϕ_m of locally supported elements in $\mathcal{B}$ converging to $\phi \in \mathcal{B}$, and we want to check that for any fixed $n \geq n_0$ the sequence $\mathcal{L}_\epsilon^n \mathcal{R}_\epsilon \phi_m$ is Cauchy (this will allow us to set $\mathcal{L}_\epsilon^n \mathcal{R}_\epsilon \phi = \lim_{m\to\infty} \mathcal{L}_\epsilon^n \mathcal{R}_\epsilon \phi_m$). Fix $\zeta > 0$. Then there is m_0 so that for any $p \geq m \geq m_0$ we have $\|\phi_p - \phi_m\|_{\delta,\eta} < \zeta$. By (3.17) in Lemma 3.3 we can define $\mathcal{L}_\epsilon^n \mathcal{R}_\epsilon$ for locally supported elements just like $\mathcal{L}_\epsilon \mathcal{R}_\epsilon$ above:

$$\mathcal{L}_\epsilon^n \mathcal{R}_\epsilon(\phi_p) = \lim_{\Lambda\to\infty} L_{\epsilon,\Lambda}^n R_{\epsilon,\Lambda}(\phi_p) \, . \tag{3.30}$$

Therefore, using (3.16) from Lemma 3.3,

$$\begin{aligned} \|\mathcal{L}_\epsilon^n \mathcal{R}_\epsilon(\phi_p - \phi_m)\|_{\delta,\eta} &\leq \lim_{\Lambda\to\infty} \|L_{\epsilon,\Lambda}^n R_{\epsilon,\Lambda}(\phi_p - \phi_m)\|_{\delta,\eta} \\ &\leq \kappa^n \|\phi_p - \phi_m\|_{\delta,\eta} \\ &\leq \kappa^n \zeta \, , \end{aligned} \tag{3.31}$$

proving the desired Cauchy property. Clearly, if ϕ is such that $\mathcal{L}_\epsilon \psi$ is defined for $\psi = \mathcal{L}_\epsilon^k \mathcal{R}_\epsilon \phi$, for all $k = 1, \ldots, n-1$, then this definition coincides with $\mathcal{L}_\epsilon^n \mathcal{R}_\epsilon \phi$.

To show that $\mathcal{L}_\epsilon^n \mathcal{R}_\epsilon$ is bounded on $\mathcal{B}$ for $n \geq n_0$, we note that for any $\phi \in \mathcal{B}$ and any $\zeta > 0$ there is a locally supported ϕ_Λ with $\|\phi_\Lambda\|_{\delta,\eta} \leq \|\phi\|_{\delta,\eta} + \zeta$ and $\|\mathcal{L}_\epsilon^n \mathcal{R}_\epsilon \phi\|_{\delta,\eta} \leq \|\mathcal{L}_\epsilon^n \mathcal{R}_\epsilon \phi_\Lambda\|_{\delta,\eta} + \zeta$. It then suffices to use the uniform bound (3.16) once more.

(3) Clearly, 1 is an eigenvalue of $\mathcal{L}_0$ and each other point in Σ_∞ is an eigenvalue of infinite multiplicity (see the proof of Proposition 2.3). Let $z \in \mathbb{C} \setminus \{0\}$ be such that $d(z^n, \Sigma_\infty^n) > 0$. Using the bounds in Lemma 3.3 (2–4), and proceeding as in part (2) of the present proof, we see that the spectral decomposition (3.22) for the resolvent of $(z^n - L_{0,\Lambda}^n R_{0,\Lambda})^{-1}$ passes to the limit $\Lambda \to \mathbb{Z}$ for $n \geq n_0$, showing in particular that $\|(z^n - \mathcal{L}_0^n \mathcal{R}_0)^{-1}\|_{\delta,\eta}$ is finite. This shows that the spectrum of $\mathcal{L}_0$ on $\mathcal{B}$ is Σ_∞. The multiplicity of 1 is simple, because the limit of (3.22) gives a spectral decomposition for $\mathcal{L}_0^n \mathcal{R}_0$.

(4) Applying again the method from part (2) of the current proof (using (3.15)) to see that $L_{0,\Lambda}^n R_{0,\Lambda} - L_{\epsilon,\Lambda}^n R_{\epsilon,\Lambda}$ converges to a bounded operator $\mathcal{L}_0^n \mathcal{R}_0 - \mathcal{L}_\epsilon^n \mathcal{R}_\epsilon$ on $\mathcal{B}$ for $n \geq n_0$, one proves the first claim by letting $\Lambda \to \infty$ in (3.24–3.25) to show that perturbation theory applies in $\mathcal{B}$. To get the second claim, it is enough to show that $(z - \mathcal{L}_\epsilon \mathcal{R}_\epsilon)^{-1} \phi$ is well defined for locally supported ϕ and suitable z. Since $\mathcal{L}_\epsilon^j \mathcal{R}_\epsilon \phi$ is a well defined element of $\mathcal{B}$ for each $j \geq 0$, this can be obtained from the discussion yielding the first claim, together with a formula similar to (3.23). □

We formulate a consequence of Theorem 3.4 on decay of correlations, which may be obtained from standard arguments (see also the proof of Corollary 2.8):

Corollary 3.5 (Decay of correlations). *Assume $d = 1$ and let μ_ϵ be the SRB measure of F_ϵ constructed in Section 2 for small enough ϵ. For any $\kappa > \kappa_1$ there are $\epsilon_3 > 0$, $\delta > 0$, $\eta > 0$, $\lambda > 0$, and $n_0 \geq 1$ so that for any $0 \leq \epsilon < \epsilon_3$, any bounded $\psi : \mathcal{X} \to \mathbb{C}$ and any $\phi \in \mathcal{B}$ we have for all $n \geq n_0$:*

$$\left| \int_{\mathcal{X}} \psi \circ F_\epsilon^n \cdot \phi \, d\mu_\epsilon - \int_{\mathcal{X}} \psi \, d\mu_\epsilon \int_{\mathcal{X}} \phi \, d\mu_\epsilon \right| \leq \sup_{\mathcal{X}} |\psi| \cdot \|\phi\|_{\delta,\eta} \kappa^n \, . \tag{3.32}$$

3.B Proof of the main bounds via cluster expansions.

Proof of Lemma 3.3. For the sake of brevity, we omit the finite set $\Lambda \subset \mathbb{Z}$ from the notation in the present proof. We will also use the notation from Section 2.B.

(1) For $\phi \in \mathcal{B}_\Lambda$ (or $\phi \in \mathcal{B}$) defined by (3.3) and $X \subset \mathbb{Z}$ we shall use the notation

$$\phi_X(z) = \sum_{\substack{k\in\mathcal{I} \\ \mathrm{supp}\,(k)=X}} a_k z^k \,. \tag{3.33}$$

We divide the proof into two parts. First (in part 1.A) we derive an expression for $(L_\varepsilon^n R_\varepsilon - L_0^n R_0)(\phi)_Y$ for all $Y \subset \Lambda$ using polymers, and then (in part 1.B) we obtain bounds for this function and its Laurent series coefficients in order to control the $\|\cdot\|_{\eta,\delta}$ norm.

1.A) Fixing ϕ and X, we first derive an expression for $P_\epsilon^n(\phi_X\,\Omega_\epsilon) - \Omega_\epsilon \cdot l_\Lambda(\phi_X\,\Omega_\epsilon)$. For P_ϵ^n we use (2.49):

$$P_\epsilon^n = \sum_{\substack{\Gamma \\ \Gamma_v=\emptyset}} \underset{\gamma\in\Gamma_{0,n}}{\otimes} V^b(\gamma) \otimes \left[(h_{\Lambda_+(\Gamma)} \underset{\gamma\in\Gamma_n}{\otimes} V^b(\gamma)(h)) \cdot (l_{\Lambda_-} \underset{\gamma\in\Gamma_0}{\otimes} lV(\gamma)) \right], \tag{3.34}$$

and for Ω_ϵ we use (2.51) with $n = 0$ (recall that $\tilde{\Omega}_\epsilon = \Omega_\epsilon$):

$$\Omega_\epsilon = \sum_{\Gamma'} h_{\Lambda_+(\Gamma')} \underset{\gamma'\in\Gamma'}{\otimes} V^b(\gamma')(h)\,. \tag{3.35}$$

We drop from now on the superscript b in order to simplify notation, since there is no danger of confusion.

In the expansion for $P_\epsilon^n(\phi_X\Omega_\epsilon)$, we will "glue" together sets of polymers γ_i and γ'_j which intersect at level 0 to form polymers $\gamma = (\cup_i\gamma_i)\bigcup(\cup_j\gamma'_j)$ in $\Lambda_{-\infty,n}$. In particular, vacuum polymers in $\Lambda_{-\infty,n}$ will appear when terms $V(\gamma')(h)$ from the expansion (3.35) for Ω_ϵ are glued only with terms $lV(\gamma)$ with $\gamma \in \Gamma_0$ from the expansion (3.34) for P_ϵ^n. More precisely, we write

$$P_\epsilon^n(\phi_X\Omega_\epsilon) = \sum_{\Gamma} \prod_{\gamma\in\Gamma_{v,0}} l\tilde{V}(\gamma)(h) \underset{\gamma\in\Gamma_X}{\otimes} *\tilde{V}_X(\gamma)(*\phi) \otimes (h_{\Lambda_+} \underset{\gamma\in\Gamma'_n}{\otimes} \tilde{V}(\gamma)(h))\,, \tag{3.36}$$

where the sum is over sets Γ (possibly empty) of disjoint polymers in $\Lambda_{-\infty,n}$ with $\partial\gamma \subset \Lambda_0 \cup \Lambda'_n$ and the other notation is as follows. We decompose each set Γ as $\Gamma = \Gamma_{v,0} \cup \Gamma_X \cup \Gamma'_n$, where $\Gamma_{v,0}$ is the set of vacuum polymers (in $\Lambda_{-\infty,n}$) of Γ whose supports intersect level 0 but not X_0, the set Γ_X contains all polymers in Γ whose supports intersect X at level 0, and Γ'_n is those polymers in Γ whose support intersect level n but do not intersect X at level 0. The suprema of the functions $\tilde{V}(\gamma)(h)$ and $l\tilde{V}(\gamma)(h)$ in the relevant polyannuli satisfy the bounds (2.36) for $\xi = \lambda/4$ and some $\kappa_1 < \beta = \hat{\kappa} < 1$. The $*$-notation for polymers in Γ_X is defined as follows: The arguments of polymers γ in Γ_X are $\phi_{X\cap\overline{\gamma}_0} h_{\overline{\gamma}_0}$ at level 0 and h elsewhere (note that $\phi_{X\cap\overline{\gamma}_0}$ is just the sum (3.33) restricted to the corresponding multi-indices, in particular Lemma 1.1 does not intervene here). At those points of X at level 0 which do not intersect $\overline{\Gamma}$ we have $l(\phi_{X\setminus\overline{\gamma}_0} h)$. If the support of γ does not intersect level n, the result is integrated with respect to l_{γ_+} if $t_+ \neq 0$, and with respect to $l_{\gamma_+}(\phi_X \cdot)$ at level 0 if $t_+ = 0$. The function obtained finally has supremum in a suitable polyannulus satisfying bounds following from putting together (2.36) for $\xi = \lambda/4$ with the bounds from (3.4) on the $|a_k|$, for the relevant multi-indices k.

Note that the functions $\tilde{V}(\gamma)(h)$ (respectively $l\tilde{V}(\gamma)(h)$) associated to glued polymers γ generally do not coincide with $V(\gamma)(h)$ (respectively $lV(\gamma)(h)$) since we have not summed over those vacuum polymers inside γ whose support intersect level 0 (see [BK1, proof of Proposition 4, pp. 393–394] for the definition of $V(\gamma)$).

As an example, let $\Gamma = \{\gamma' \cup \gamma\}$, where $\gamma'_+ = Y$ and $\gamma_- = Z$ such that $Y \cap Z \cap X \neq \emptyset$. Then $V(\gamma')(h)$ is a function at level 0 depending on coordinates in Y. We multiply this function by $\phi_X h_{\Lambda\setminus Y}$ and obtain a function on Λ_0. Then we act on this function by $V(\gamma) \otimes l_{\Lambda\setminus Z}$ and obtain an analytic function Ψ on $D(\alpha)_{\gamma_+}$ at level t_+, which is the level where γ ends. If $t_+ = n$, then $*\tilde{V}_X(\gamma' \cup \gamma)(*\phi)$ is just this function Ψ; if $t_+ < n$, then $*\tilde{V}_X(\gamma' \cup \gamma)(*\phi)$ is the constant function $l_{\gamma_+}\Psi$.

Similarly, using (2.58) (with $n-1$) for l_Λ and (2.51) (with n and 0) for Ω_ϵ, we get (for the same $l\tilde{V}(\gamma)$ and similarly defined $l\tilde{V}_X(\gamma)$ as in (3.36))

$$\Omega_\epsilon \cdot l_\Lambda(\phi_X \Omega_\epsilon) = \sum_{\Gamma, \Gamma_n^u} \prod_{\gamma\in\Gamma_{v,0}} l\tilde{V}(\gamma)(h) \bigotimes_{\gamma\in\Gamma_X} l\tilde{V}_X(\gamma)(*\phi) \otimes (h_{\Lambda_+^u} \bigotimes_{\gamma^u\in\Gamma_n^u} V(\gamma^u)(h)), \quad (3.37)$$

where $\Gamma = \Gamma_{v,0} \cup \Gamma_X$ runs over sets (possibly empty) of disjoint polymers in $\Lambda_{-\infty,n-1}$, and Γ_n^u runs over sets of disjoint polymers in $\Lambda_{-\infty,n}$ with $\overline{\gamma}^u \cap \Lambda_n \neq \emptyset$ and $\partial\gamma^u \subset \Lambda_n$; finally $\Lambda_+^u = \Lambda \setminus \cup_{\gamma^u\in\Gamma_n^u} \gamma_+^u$. (There is no "tilde" in $V(\gamma^u)$ since γ^u is never obtained by gluing polymers coming below level 0.)

Using (2.58) for $l_\Lambda P_\epsilon^{n-1}$, we see that

$$1 = l_\Lambda \Omega_\epsilon = l_\Lambda P_\epsilon^{n-1}(\Omega_\epsilon) = \sum_{\Gamma_{v,0}} \prod_{\gamma\in\Gamma_{v,0}} l\tilde{V}(\gamma)(h), \quad (3.38)$$

where $\Gamma_{v,0}$ runs over sets of disjoint vacuum polymers in $\Lambda_{-\infty,n}$ such that the support of each of them intersects level 0. Using (3.38), we can divide the vacuum polymers intersecting level 0 away both in (3.36) and (3.37) (just like in the proof of [BK1, Proposition 4, pp. 393–394] or in the proof of Lemma 2.10 from Section 2). The weights resulting from this operation (which compensates the absence of sums over vacuum polymers in the construction of the $\tilde{V}$, writing P_ϵ^n as $P_\epsilon^k P_\epsilon^\ell$ for $k+\ell=n$) are again the $V(\gamma)$ and $V^b(\gamma)$ from Proposition B.

Observe that if in (3.36) there is no $\gamma \in \Gamma_X$ such that its support intersects level n (we write, $\Gamma_{X,n} = \emptyset$, defining implicitly $\Gamma_{X,n}$) then the same term is also in (3.37). So when we subtract (3.37) from (3.36), only those terms from (3.36) survive for which

$\Gamma = \Gamma_X \cup \Gamma'_n$ and such that there exists $\gamma \in \Gamma_{X,n}$, and from (3.37) only those terms with $\Gamma = \Gamma_X$ such that there exists $\gamma^u \in \Gamma^u_n$ with $\overline{\gamma}^u \cap (\overline{\Gamma}_X \cup (X,0)) \neq \emptyset$. To simplify notation we denote from now on $\overline{\Gamma}_X \cup (X,0)$ simply by $\overline{\Gamma}_X$.

Defining $\tilde{\psi}_Y = \Omega_\epsilon(L_\epsilon R_\epsilon \phi)_Y$, for all nonempty $Y \subset \Lambda$, the above remarks yield

$$\tilde{\psi}_Y = \sum_{\substack{X\subset\Lambda \\ X\neq\emptyset}} \Bigg(\sum_{\substack{\Gamma \\ \Gamma^+_X = Y}} \underset{\gamma\in\Gamma_X}{\otimes} *V_X(\gamma)(*\phi) \otimes (h_{\Lambda_+} \underset{\gamma\in\Gamma'_n}{\otimes} V(\gamma)(h))$$

$$- \sum_{\substack{\Gamma_X,\Gamma^u_n \\ \overline{\Gamma}_X\cap\overline{\Gamma}^u_n\neq\emptyset, \Gamma^{u+}_X = Y}} \underset{\gamma\in\Gamma_X}{\otimes} lV_X(\gamma)(*\phi) \otimes (h_{\Lambda^u_+} \underset{\gamma^u\in\Gamma^u_n}{\otimes} V(\gamma^u)(h)) \Bigg), \tag{3.39}$$

where $\Gamma^+_X = \overline{\Gamma}_X \cap \Lambda_n$ and $\Gamma^{u+}_X = \cup_{\gamma^u\in\Gamma^u_n, \overline{\gamma}^u\cap\overline{\Gamma}_X\neq\emptyset} \overline{\gamma}^u \cap \Lambda_n$. It follows from Proposition A in Section 2, the decay properties of the Laurent coefficients of $\phi \in \mathcal{B}$, and the domains and ranges of the relevant operator weights, that the function $\tilde{\psi}_Y$ is a bounded analytic function on $\mathcal{D}(\alpha)_\Lambda$ (for small enough ϵ). Note that since all the terms with $\Gamma^+ = \emptyset$ are cancelled, we have $\tilde{\psi}_\emptyset = 0$.

Now we have to divide $\tilde{\psi}_Y$ by Ω_ϵ to obtain an expansion for $\psi_Y = (L^n_\varepsilon R_\varepsilon \phi)_Y$. In order to keep track of the powers of h appearing in the denominator, it is convenient to introduce new weights

$$V^h(\gamma) = \frac{1}{h_{\gamma_+}} V(\gamma)(h_{\gamma_-}). \tag{3.40}$$

Clearly,

$$h_{\Lambda_+} \underset{\gamma\in\Gamma_n}{\otimes} V(\gamma)(h) = h_\Lambda \cdot (\underset{\gamma\in\Gamma_n}{\otimes} V^h(\gamma)) \tag{3.41}$$

for any family Γ_n of polymers (with Λ_+ and the other notations are as in Proposition B). Since there exists $C > 0$ such that $|h| > C$ on $D(\alpha)$, the weight-norms of the $V^h(\gamma)$ satisfy (2.36) for $\xi = \lambda/4$ but with $\tilde{\epsilon}$ replaced by $C^{-1}\tilde{\epsilon}$ (which we still denote by $\tilde{\epsilon}$). Using the V^h, we may write

$$\Omega_\epsilon = h_\Lambda \cdot (\sum_{\Gamma_n} \underset{\gamma\in\Gamma_n}{\otimes} V^h(\gamma)). \tag{3.42}$$

Using the polymer expansion formalism, we express (3.42) as an exponential (compare (2.55) and (2.50)), that is, we write

$$\Omega_\epsilon = h_\Lambda \exp\big(\sum_{\upsilon\in\Upsilon^{-\infty,n}} \widehat{U}(\upsilon)\big), \tag{3.43}$$

where $\Upsilon^{-\infty,n}$ is the set of polymers in $\Lambda_{-\infty,n}$ whose supports intersect level n, and the supremum norms of the functions $\widehat{U}(\upsilon)$ in the relevant polyannuli satisfy (2.36) with $\xi = \lambda/5$.

The sum over Γ_n' in (3.39) almost gives Ω_ϵ: we are only missing those γ's which intersect $\overline{\Gamma}_X$. (We use here ideas similar to those applied in the proof of Lemma 2.4, see in particular (2.63), namely we apply the polymer formalism to take the logarithm of the sum over all $\Gamma_n'(\Gamma_X)$ for fixed Γ_X.) Similarly, in the sum over Γ_n^u, we may take those $\gamma^u \in \Gamma_n^u$ such that $\overline{\gamma}^u \cap \overline{\Gamma}_X \neq \emptyset$ as a common factor (call from now on the set of these γ^u's Γ_X^u), and we have to compensate those polymers whose supports intersect the supports of these γ^u's. Finally, using the notation V^h from (3.41) we find the desired expression for $\psi_Y = (L_\epsilon^n R_\epsilon \phi)_Y$:

$$\begin{aligned}\psi_Y = \sum_{\substack{X\subset\Lambda\\ X\neq\emptyset}} \Bigg(& \sum_{\substack{\Gamma_X\\ \Gamma_X^+=Y}} \bigotimes_{\gamma\in\Gamma_X} *V_X^h(\gamma)(*\phi) \exp\Big(-\sum_{\substack{\upsilon\in\Upsilon^{-\infty,n}\\ \overline{\upsilon}\cap\overline{\Gamma}_X\neq\emptyset}} \widehat{U}(\upsilon)\Big) \\ & - \sum_{\substack{\Gamma_X,\Gamma_X^u\\ \overline{\Gamma}_X\cap\overline{\gamma}^u\neq\emptyset,\Gamma_X^{u+}=Y}} \bigotimes_{\gamma\in\Gamma_X} lV_X(\gamma)(*\phi) \bigotimes_{\gamma^u\in\Gamma_X^u} V^h(\gamma^u)(h) \exp\Big(-\sum_{\substack{\upsilon\in\Upsilon^{-\infty,n}\\ \overline{\upsilon}\cap(\overline{\Gamma}_X\cup\overline{\Gamma}_X^u)\neq\emptyset}} \widehat{U}(\upsilon)\Big)\Bigg).\end{aligned} \tag{3.44}$$

We still have $\psi_\emptyset = 0$, which was to be expected since we are restricting L_ϵ to the subspace complementary to the one-dimensional subspace spanned by the constant function.

1.B) First note that $(L_0^n R_0 \phi)_Y$ contains only polymers which are vertical lines from X to Y, where $X \supset Y$. Moreover $\widehat{U}(\upsilon) = 0$ for all υ, and $V(\gamma)(h) = 0$ for all γ in the expansion for $(L_0^n R_0\phi)_Y$. Thus the difference $(L_\varepsilon^n R_\epsilon\phi)_Y - (L_0^n R_0\phi)_Y$ is like $\psi_Y = (L_\epsilon^n R_\epsilon\phi)_Y$, except that there is always at least one polymer which carries at least one $\tilde{\epsilon}$, or we have a factor $\exp(-\sum \widehat{U}(\upsilon)) - 1$, which can be bounded above by $\sum |\widehat{U}(\upsilon)| \exp(\sum |\widehat{U}(\upsilon)|)$ (see (2.61), and note, in view of the remark after (2.46), that there is always at least one $\tilde{\epsilon}$ in each relevant υ, so we do get a factor $C\tilde{\epsilon}$). Therefore we can extract a factor $\sqrt{\tilde{\epsilon}}$ (which we just call $\tilde{\epsilon}$) and estimate the rest by bounding ψ_Y.

We will estimate only the first summand in (3.44). The second summand and the $\sum |\widehat{U}(\upsilon)|$-terms can be bounded similarly. We now briefly sketch the main step, which is to find the factors $\eta^{|Y|}$ and $e^{-\delta L_\surd(Y\cup\{0\})}$ in order to control the $\|\cdot\|_{\delta,\eta}$-norm: A factor $\eta^{|X|}e^{-\delta L_\surd(X\cup\{0\})}$ is obtained from ϕ_X. If $|Y| > |X|$ or $L_\surd(Y\cup\{0\}) > L_\surd(X\cup\{0\})$ we will be able to extract "missing small" factors from the coupling, since we can take $\tilde{\epsilon}$ smaller than η and λ larger than δ. Then we will use the results from Section 2.B which guarantee the convergence of the sums over polymers. All these sums will give us some extra factors, which we can kill by taking a large enough iterate n_0, using the bound (2.36).

We now go into more details. Since the sum over all polymers containing a given point is smaller than $C\tilde{\epsilon}$ (see (2.46)), the sum over $\widehat{U}$'s in the first summand of (3.44) changes only slightly $\tilde{\epsilon}$ and β in the bounds (2.36). More precisely, $\tilde{\epsilon}$ is replaced by $e^{C\tilde{\epsilon}}\tilde{\epsilon}$ and β by $e^{C\tilde{\epsilon}}\beta$, and also the constant C in the correction term in Lemma 3.2 changes slightly. Similarly the sum over those $\gamma \in \Gamma_X$ such that $\overline{\gamma} \cap \Lambda_n = \emptyset$ (i.e., the "short" elements in Γ_X), just replaces η by $e^{C\tilde{\epsilon}}\eta$ in the weight of ϕ_X and again changes only the constant C in Lemma 3.2.

We are thus left with those (long) polymers γ in Γ_X with $\gamma_- \subset X$ (at level 0) and $\gamma_+ \subset Y$ (at level n) and such that $\cup_{\gamma\in\Gamma_X}\gamma_+ = Y$ (recall that Y is fixed and we sum over X). We will perform the sum over these long polymers similarly as in the proof of Lemma 2.9, as we explain now. Before this, we observe that we can extract a factor κ^n from each term of the first summand of (3.44), since there is always at least one long polymer, and because β in (2.36) can be taken as close to κ_1 as wanted if ϵ is small enough.

We first setup the combinatorical picture. Fix X and a set Γ_X. Let $\widehat{X} \subset X \cap Y$ be the set of points of X connected by polymers $\gamma \in \Gamma_X$ which are straight vertical lines to the corresponding points of Y. Let $\gamma \in \Gamma_X$ be such that it is not a vertical line. We associate a set of tree polymers γ' to our polymer $\gamma \in \Gamma_X$ just like in the paragraph before (2.43), but imposing the requirement that the support of each such tree intersects X_0 and $\overline{\gamma}' \cap Y_n = \overline{\gamma} \cap Y_n$. To insure this property when there are no spacelike sets of γ at level 0 or n, i.e., when only the boundary of γ intersects level 0 or n, we sometimes need to include one extra line downwards and several extra lines upwards. (When summing over the stripped polymer trees, we will also have to sum over all extra vertical lines.)

For each polymer tree γ' we choose one point from $\overline{\gamma}' \cap X_0$ and one point from $\overline{\gamma}' \cap Y_n$. These points together with the straight vertical lines determine sets $Y' \subset Y$ and $X' \subset X$ with $|X'| = |Y'|$. (The point selected in Y_n will play the part of the "fixed point" used in the stripping procedure from Section 2; the point in $\widehat{X}$, or more precisely the set X', will be used to control the decay coming from the bounds on the Laurent coefficients of ϕ_X.)

Now that the combinatorical tools are defined, the second point is to observe that, since $d = 1$, for any finite sets $W \subset Z \subset \mathbb{Z}$, we have $|\tau(Z)| \geq |\tau_{\min}(W)|$ for each tree $\tau(Z)$ over Z (see (2.35) for definitions). Therefore, when $W \subset Z$,

$$T(Z, \tfrac{\lambda}{4}) = \sum_{\tau(Z)} e^{-\frac{\lambda}{4}|\tau|} \leq e^{-\frac{\lambda}{8}|\tau_{\min}(W)|} T(Z, \tfrac{\lambda}{8}) \leq e^{-\delta L_{\surd}(W)} T(Z, \tfrac{\lambda}{8}), \tag{3.45}$$

if $8\delta \leq \lambda$. (In the corresponding bounds for $\widehat{U}(v)$'s, which we mentioned in the first paragraph of part 1.B, we would need $10\delta < \lambda$.)

The third step decomposes into two substeps which mirror the fact that the bounds on the supremum norms (in the "large" $(1-\alpha)^{\pm 1}$-polyannuli) of the summed terms consist of two factors. The first factor comes from the bound on the supremum norm of ϕ (whose topological tensor product decomposition is furnished by its Laurent series), restricted to "small" polyannuli of polyradius $(1-\alpha')^{\pm 1}$ (this bounds crucially relies on Lemma 3.2). The second factor originates from the bounds (2.36) on the relevant (analyticity improving) operators, from $\mathcal{H}(\mathcal{D}(\alpha')_{\gamma_-})$ to $\mathcal{H}(\mathcal{D}(\alpha)_{\gamma_+})$.

We perform now the first substep, taking into account the contribution from $\|\phi\|_{\delta,\eta} < \infty$: By Lemma 3.2 the sum of $\|\phi_X\|_{\mathcal{H}(D(\alpha'))_X}$ over all $X \supset X'$, for a fixed X', is bounded above by

$$e^{C\eta(|X'|+1)} e^{-\delta L_{\surd}(X' \cup \{0\})} \eta^{|X'|} \|\phi\|_{\delta,\eta} \,. \tag{3.46}$$

Up to multiplying $\tilde{\epsilon}$ by a constant in (2.36), we may extract a factor $\eta^{|\gamma'_+|}$ from the bound (2.36) satisfied by the function associated to each tree polymer γ'. Since, by construction $Y = \widehat{X} \cup (\cup_{\gamma'} \gamma'_+)$, and since the space projection of each polymer tree γ' connects all the points of γ'_+ to each other and to the set X', we get from the inequality $\sum_{i=1}^{k} L_{\surd}(W_1) \geq L_{\surd}(W)$ (which holds for any $W \subset \cup_{i=1}^{k} W_i$ such that $W_1, \ldots, W_k \subset \mathbb{Z}$ form a connected set with respect to intersection), combined with (3.45), that we can extract a factor

$$e^{-\delta L_{\surd}(Y \cup \{0\})} \eta^{|Y|} \tag{3.47}$$

from the bound for each term of the sum over long polymers in Γ_X. (We took $W = Y \cup \{0\}$, the integer $k-1$ to be the number of different spacelike sets Z in all the γ' appearing for a given X', setting W_i, for $i = 1, \ldots k-1$, to be these sets Z, and finally $W_k = X' \cup \{0\}$.) Using a fixed X' has given us the connectedness we needed above, and by (3.46) the sum over all X containing X' produced by this procedure costs us a factor (recall that $|X'| = |Y'| \leq |Y|$)

$$e^{C\eta(|X'|+1)} \leq e^{c\eta|Y|} . \tag{3.48}$$

where $c > C$ (we use here that $|Y| \neq 0$).

Moving now to the second substep, we consider the sum over all possible Y' for our fixed set Y. From the lines connecting X' to Y' we can extract a factor $(C\tilde{\kappa}^{n_0})^{|Y'|}$ for some $\tilde{\kappa} < 1$ and $n_0 \leq n$, and then there are at least $|Y| - |Y'|$ $\tilde{\epsilon}$'s, which give $(\tilde{\epsilon} C \tilde{\kappa}^{n_0})^{|Y|-|Y'|}$. Thus we find for the contribution of Y' to the sum over X a factor

$$(C\tilde{\kappa}^{n_0})^{|Y|} \tilde{\epsilon}^{|Y|-j} , \tag{3.49}$$

where $j = |Y'|$. Therefore the sum over all choices of $Y' = Y'(Y)$ of the bound (3.49) is not larger than

$$(C\tilde{\kappa}^{n_0})^{|Y|} \sum_{j=0}^{|Y|} \binom{|Y|}{j} \tilde{\epsilon}^{|Y|-j} = (C\tilde{\kappa}^{n_0})^{|Y|} (1+\tilde{\epsilon})^{|Y|} , \tag{3.50}$$

which, when combined with the cost factor (3.48), produces a bound $(C\tilde{\kappa}^{n_0})^{|Y|} e^{C\eta|Y|}$. Taking a large enough iterate n_0, we may kill the $e^{C\eta|Y|}$, and even extract a factor $(\hat{\kappa}^{n_0})^{|Y|}$ for some $\kappa_1 < \hat{\kappa} < 1$.

The fourth and final step consists in summing over trees, just like in the proof of (2.46), where each tree has a fixed point which belongs to Y, and we sum over all choices of X' (the sum over X for a given X' has already been taken into account).

Collecting the above estimates, we obtain

$$\begin{aligned} \|(L_\epsilon^n R_\epsilon \phi)_Y - (L_0^n R_0 \phi)_Y\|_{\mathcal{H}(\mathcal{D}(\alpha)_\Lambda)} &\leq \|(L_\epsilon^n R_\epsilon \phi)_Y - (L_0^n R_0 \phi)_Y\|_{\hat{\otimes}_{i\in\Lambda} \mathcal{H}(D(\alpha))} \\ &\leq \tilde{\epsilon} \kappa^n e^{-\delta L_{\surd}(Y \cup \{0\})} \eta^{|Y|} (\hat{\kappa}^{n_0})^{|Y|} \|\phi\|_{\delta,\eta} . \end{aligned} \tag{3.51}$$

To translate this bound for the supremum of functions $L_\epsilon^n R_\epsilon \phi$ on polyannuli into a bound for their Laurent coefficients, we use the Cauchy formula, obtaining for any multi-index k with $\mathrm{supp}\,(k) \subset \Lambda$,

$$
\begin{aligned}
|a'_k(\epsilon)| &= \Big| \sum_{Z \supset \mathrm{supp}\,(k)} (\frac{1}{2\pi i})^{|Z|} \oint_{|z|=r} \frac{\psi_Z}{\prod_{i\in Z} z_i^{k_i+1}} \prod_{i\in Z} dz_i \Big| \\
&\le \sum_{Z \supset \mathrm{supp}\,(k)} \|\psi_Z\|_{\mathcal{H}(\mathcal{D}(\alpha)_\Lambda)} (1-\alpha)^{|k|} ,
\end{aligned} \tag{3.52}
$$

where $|z| = r$ means $r_i = 1/(1-\alpha)$ if $k_i > 0$, and $r_i = 1-\alpha$ if $k_i < 0$, for $k_i = 0$ we set $r_i = 1$. Using (3.10) in the second inequality, and taking n_0 large enough and replacing $C\tilde{\epsilon}$ by $\tilde{\epsilon}$ in the third inequality, we find

$$
\begin{aligned}
|a'_k(\epsilon) - a'_k(0)| &\le \tilde{\epsilon}\kappa^n \|\phi\|_{\delta,\eta} (1-\alpha)^{|k|} \sum_{Z \supset \mathrm{supp}\,(k)} (\hat{\kappa}^{n_0})^{|Z|} e^{-\delta L_{\vee}(Z\cup\{0\})} \eta^{|Z|} \\
&\le \tilde{\epsilon}\kappa^n \|\phi\|_{\delta,\eta} (1-\alpha)^{|k|} (\hat{\kappa}^{n_0})^{|\mathrm{supp}\,(k)|} e^{-\delta L_{\vee}(\mathrm{supp}\,(k)\cup\{0\})} \eta^{|\mathrm{supp}\,(k)|} e^{C\eta(|\mathrm{supp}\,(k)|+1)} \\
&\le \tilde{\epsilon}\kappa^n e^{-\delta L_{\vee}(\mathrm{supp}\,(k)\cup\{0\})} \eta^{|\mathrm{supp}\,(k)|} (1-\alpha)^{|k|} \|\phi\|_{\delta,\eta} .
\end{aligned} \tag{3.53}
$$

This proves (3.15) and also (3.16) since, as we indicated at the beginning of part 1.B of this proof, the inequality (3.51) without the $\tilde{\epsilon}$ factor in the right-hand-side applies to $\|(L_\epsilon^n R_\epsilon \phi)_Y\|_{\delta,\eta}$.

(2) The proof of (3.17) is similar to the proof of (3.15). Expanding the exponential factors in the the expression for $(L_\epsilon^n R_\epsilon \phi)_Y$ in (3.44) (see [BK1, (72) and (73), p. 394] or (2.62) in the proof of Lemma 2.4), we see that only those families of polymers survive which connect Λ to $\mathbb{Z} \setminus \Lambda'$, and we can extract a factor $\tilde{\lambda}$ from the tree decay λ of the coupling. (We can actually obtain a κ^n decay in the right-hand-side of (3.17), but we will not need this.)

In the case $n < n_0$ we need to use $\tilde{\epsilon}$'s to kill part of the cost factor $e^{C\eta(|\mathrm{supp}\,(k)|+1)}$ in (3.53). As a result, we are left with a large factor $e^{C\eta|\Lambda|}$.

For (3.18) we write ϕ as an element of $\otimes_{i\in\Lambda} \mathcal{H}(D(\tilde{\alpha}))$ for $\tilde{\alpha} < \alpha'$ in order to use an expansion similar to (3.44). By Lemma 1.1, this gives a factor $e^{C|\Lambda|}$. Then we just sum over all families of polymers, noting that Λ is always connected to $\mathbb{Z} \setminus \Lambda'$, and all the polymers are connected to Λ.

(3) We start with the simplest case where ρ has a unique representation as a product of eigenvalues $\rho = \prod_{i=1}^{\ell} \rho_i$, with $\rho_i \neq \rho_j$ for all $i \neq j$ and $\rho_i \neq 1$ for all $i = 1, \dots, \ell$. According to the discussion after (3.14) each $\Pi_{\{a\},\rho_i} : \mathcal{H}(D(\alpha')_{\{a\}}) \to \mathcal{H}(D(\alpha)_{\{a\}})$, for $i = 1, \dots, \ell$ and $a \in \Lambda$, is a bounded operator (with $\alpha' < \alpha$). Recall also that the projections $\Pi_{\{a\},1}$ corresponding the eigenvalue 1 are just integration on the circle at $\{a\}$ with respect to the single-site SRB-measure, that is,

$$
\Pi_{\{a\},1}(\varphi) = l_{\{a\}}(h_{\{a\}}\varphi) . \tag{3.54}
$$

For $Z \subset \Lambda$ we use the notation $Q_Z(\varphi) = l_Z(h_Z\, \varphi)$.

The following decomposition formula for the spectral projection is easily obtained:

$$\Pi_{\Lambda,\rho} = \sum_{\substack{Y\subset\Lambda \\ |Y|=\ell}} Q_{\Lambda\setminus Y} \otimes A_{Y,\rho}\,, \tag{3.55}$$

where $A_{Y,\rho}$ is defined by writing $Y = \{y_1, \dots, y_\ell\}$ and setting

$$A_{Y,\rho} = \sum_{\sigma} \Pi_{\{y_1\},\rho_{\sigma(1)}} \otimes \cdots \otimes \Pi_{\{y_\ell\},\rho_{\sigma(\ell)}}\,, \tag{3.56}$$

where the sum is over all permutations σ of the set $\{1, \dots, \ell\}$.

Let $\phi \in \mathcal{B}_\Lambda(\delta, \eta)$. Since $\Pi_{\{a\},\rho_i}(1) = 0$ for all $i = 1, \dots, \ell$, we have (using the notation ϕ_X from (3.33))

$$\Pi_{\Lambda,\rho}\phi = \sum_{X\subset\Lambda} \Pi_{\Lambda,\rho}\phi_X = \sum_{\substack{Y\subset\Lambda \\ |Y|=\ell}} \psi_Y\,, \tag{3.57}$$

where

$$\psi_Y = \sum_{Z\supset Y} Q_{\Lambda\setminus Y} \otimes A_{Y,\rho}\phi_Z\,. \tag{3.58}$$

By Lemma 3.2, and using the fact that all the single-site projections $\Pi_{\{a\},\rho_i}$ involved are analyticity improving (cf. (3.51) for a similar situation), we have (the finite-site h's in Q_Z change only the constant C in (3.5) since they are bounded)

$$\|\psi_Y\|_{\mathcal{H}(\mathcal{D}(\alpha)_\Lambda)} \le C(\rho) e^{-\delta L_{\surd}(Y\cup\{0\})} \eta^{|Y|} \|\phi\|_{\delta,\eta}\,, \tag{3.59}$$

where we have embedded $e^{C\eta(\ell+1)}$ (recall $|Y| = \ell$) from Lemma 3.2 into $C(\rho)$ (the value of ℓ here depends only on ρ). Then proceding like in (3.52)–(3.53), we find

$$\|\Pi_{\Lambda,\rho}\|_{\delta,\eta} \le C(\rho)\,. \tag{3.60}$$

Although the estimate (3.60) increases going inward in the spectrum, i.e., when the number of factors ρ_j giving ρ increases, it is nevertheless uniform in Λ.

To complete the proof of the statement on the norms of the $\Pi_{\Lambda,\rho}$ we make two remarks. First, the case in which the eigenvalue ρ is given by a product of not necessarily distinct single-site eigenvalues, say $\rho = \prod_{i=1}^{k} \rho_i^{m_i}$, with $m_i \ge 1$ and $\sum_i m_i = \ell$, can be easily treated along the lines indicated above: The only significant difference is that we do not sum over all permutations σ in (3.56), but only over those which modify the product of the ρ_i. Second, one has to take into account the possibility that a given ρ can be written in more than one way as a product of single-site eigenvalues ρ_j. But then $\Pi_{\Lambda,\rho}$ is just the sum over the corresponding terms, and since the number of different possibilities is bounded uniformly in Λ the bound (3.19) is established.

(4) Let $0 < \kappa < 1$ be such that κ is not the modulus of any element of Σ_∞. We write $\{1 = \rho_0, \rho_1, \ldots, \rho_p\} = \Sigma \cap \{|z| > \kappa\}$ (recall that Σ is just the spectrum of L, see (1.4) and note that $p = p(\kappa)$ is the cardinality of $\Sigma_{\{a\},>\kappa}$ defined in (2.13), for any $a \in \mathbb{Z}$). Writing $N(\kappa)$ for the cardinality of the sets $\Sigma_{\Lambda,\kappa}$ when $|\Lambda| \geq \ell(\kappa)$ (see above (2.13)), we denote

$$\Sigma_{\infty,>\kappa} = \{\xi_1, \ldots, \xi_{N(\kappa)}\} = \{\xi = \prod_{j=1}^{m_j} \rho_j^{\alpha_j} \mid \quad |\xi| > \kappa\}. \tag{3.61}$$

We use the following decomposition for the single-site transfer operator L

$$L^n = L^n R_{<\kappa} + Q + \sum_{j=1}^{p} L^n \Pi_{\rho_j}, \tag{3.62}$$

where $R_{<\kappa}$ is the spectral projection inside the disk of radius κ. For each $1 \leq j \leq p$ and any $\tilde\rho_j > |\rho_j|$ there is a constant $C = C(j, \tilde\rho_j)$ such that for each $n \geq 1$ the norm of the operator $L^n \Pi_{\rho_j} : \mathcal{H}(D(\alpha')) \to \mathcal{H}(D(\alpha))$ satisfies

$$\|L^n \Pi_{\rho_j}\| \leq C \tilde\rho_j^n. \tag{3.63}$$

The following decomposition is now straightforward

$$L^n_{0,\Lambda} = \sum_{\substack{Y_0,\ldots,Y_p \\ Y_i \subset \Lambda,\, Y_i \cap Y_j = \emptyset}} Q_{\Lambda \setminus (\cup_{j=0}^p Y_j)} \otimes T^n_{Y_0} \otimes S^n_{Y_1} \otimes \cdots \otimes S^n_{Y_p}, \tag{3.64}$$

where $T_{Y_0} = \otimes_{y \in Y_0} L R_{<\kappa}$ and $S_{Y_j} = \otimes_{y \in Y_j} L \Pi_{\rho_j}$. Thus, we find

$$L^n_{0,\Lambda} R_{0,\Lambda,<\kappa} = L^n_{0,\Lambda}\left(\mathrm{Id} - \Pi_{\Lambda,1} - \sum_{j=1}^{N(\kappa)} \Pi_{\Lambda,\xi_j}\right) = \sum_{Y \subset \Lambda} Q_{\Lambda \setminus Y} \otimes B_Y(\kappa, n), \tag{3.65}$$

with $B_Y(\kappa, n) : \mathcal{H}(D(\alpha')_Y) \to \mathcal{H}(D(\alpha)_Y)$ defined for $Y \subset \Lambda$ and $n \geq 1$ by

$$B_Y(\kappa, n) = \sum_{\substack{Y_0, Y_1, \ldots, Y_p \\ Y = \cup_{j=0}^p Y_j, Y_i \cap Y_j = \emptyset \\ Y_0 \neq \emptyset}} T^n_{Y_0} \otimes S^n_{Y_1} \otimes \cdots \otimes S^n_{Y_p} + \sum_{\substack{Y_1, \ldots, Y_p \\ Y = \cup_{j=1}^p Y_j, Y_i \cap Y_j = \emptyset \\ \prod_{j=1}^p \rho_j^{|Y_j|} < \kappa}} S^n_{Y_1} \otimes \cdots \otimes S^n_{Y_p}. \tag{3.66}$$

Therefore, fixing $\tilde\kappa > \bar\kappa > \kappa$, there is $C > 0$ such that for all $n \geq 1$

$$\|B_Y(\kappa, n)\| \leq \sum_{\substack{Y_0, \ldots, Y_p \\ Y = \cup_{j=0}^p Y_j}} C^{|Y|} \bar\kappa^{n|Y_0|} \tilde\rho_1^{n|Y_1|} \cdots \tilde\rho_p^{n|Y_p|} = C^{|Y|}\left(\bar\kappa^n + \sum_{j=1}^{p} \tilde\rho_j^n\right)^{|Y|} \tag{3.67}$$

where $C = C(\bar{\kappa}, \tilde{\rho}_1, \dots, \tilde{\rho}_p)$.

If we set $M = M(\kappa) \geq 1$ to be the smallest integer such that $\bar{\kappa}^M < \kappa$ and $\tilde{\rho}_j^M < \kappa$ for all $j = 1, \dots, p$, then for any $n \geq 1$

$$\left(\bar{\kappa}^n + \sum_{j=1}^{p} \tilde{\rho}_j^n \right)^{|Y|} \leq \left((p+1)\, \kappa^{(n/M)} \right)^{|Y|}, \tag{3.68}$$

so that

$$\|B_Y(\kappa, n)\| \leq C^{|Y|}(p+1)^{|Y|} \left(\kappa^{(n/M)} \right)^{|Y|} \leq C^{|Y|} \left(\kappa^{(n/M)} \right)^{|Y|}. \tag{3.69}$$

From (3.69), we see that if $|Y| > M$ there is $n_2(\tilde{\kappa}) \geq 1$ so that for any $n \geq n_2$

$$\|B_Y(\kappa, n)\| \leq C^{|Y|} \left(\frac{\kappa^{n/M}}{\tilde{\kappa}^{n/|Y|}} \right)^{|Y|} \tilde{\kappa}^n \leq \hat{\kappa}^{n_2}\, \tilde{\kappa}^n, \tag{3.70}$$

for some $\kappa_1 < \hat{\kappa} < 1$.

By (3.66), there is $n_1(\tilde{\kappa}) \geq n_2$ and a constant $C = C(M)$, such that for each $|Y| \leq M$ and all $n \geq n_1$, the norm of $B_Y(\kappa, n)$ is bounded by $C\kappa^n \leq \tilde{\kappa}^n$.

The estimate $\|L^n_{0,\Lambda} R_{0,\Lambda,<\kappa}\|_{\delta,\eta} \leq \tilde{\kappa}^n$ in Lemma 3.3 (4) now follows from the above bounds on the norm of the $B_Y(\kappa, n)$, as in the proof of statement (3) already given. (The factor $\hat{\kappa}^{n_2}$ in (3.70) is helpful to get rid of the factor $\exp(|\mathrm{supp}\,(k)| + 1)$ implicit in (3.59) when using (3.53) for $|Y| > M$.) $\square$

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