YANG-MILLS THEORY AND THE SEGAL-BARGMANN TRANSFORM

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Abstract. Motivated by the results of Landsman and Wren, I consider in this paper the canonical quantization of Yang-Mills theory on a spacetime cylinder, with the goal of obtaining an appropriate Segal-Bargmann transform. Using a Gaussian measure with large variance to approximate the non-existent Lebesgue measure on the space of connections, I consider a variant of the classical Segal-Bargmann transform. A simple but non-rigorous argument shows that this classical transform, restricted to the gauge-invariant subspace, becomes a variant of the generalized Segal-Bargmann transform [H1] for the structure group K. In the infinite-variance limit this yields the invariant form of the Segal-Bargmann transform for K, which may therefore be regarded as the natural Segal-Bargmann transform for the space of connections modulo gauge transformations.

As a consequence of this calculation, I obtain a new version of the generalized Segal-Bargmann transform for Lie groups of compact type, which interpolates between the two previously known versions. Although the infinite-dimensional calculation is non-rigorous, the properties of the predicted finite-dimensional transform may be verified rigorously by finite-dimensional means.

1. Introduction

The Segal-Bargmann transform was developed independently in the early 1960's by Segal [S1, S2, S3] in the infinite-dimensional context of scalar quantum field theories and by Bargmann [B] in the finite-dimensional context of quantum mechanics on $\mathbb{R}^n$. In [H1] I introduced an analog of the Segal-Bargmann transform in the context of quantum mechanics on a compact Lie group. A natural next step is to attempt to combine the compact group with the field theory in order to obtain a transform in the context of quantum gauge theories. One such transform has already been obtained by Ashtekar, *et al* [A], with application to quantum gravity.

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In this paper I consider the canonical quantization of Yang-Mills theory on a spacetime cylinder. The space of connections (on the spatial circle) is an infinite-dimensional linear space, and in this special case the set of connections modulo gauge transformations is a finite-dimensional set, essentially the structure group K. I argue heuristically that (a variant of) the *classical* Segal-Bargmann transform for the space of connections, when restricted to the gauge-invariant subspace, becomes the *generalized* Segal-Bargmann transform for K. Thus the transform for K is not just analogous to the classical transform, it *is* the classical transform on the gauge-invariant subspace. A forthcoming paper [AHS] with Albeverio and Sengupta will consider a Segal-Bargmann transform for the Euclidean approach to the quantization of two-dimensional Yang-Mills theory.

The motivation for this work comes from two sources. The first is the work of Landsman and Wren on the quantization of Yang-Mills on a cylinder, by the method of "Rieffel induction." Their calculations show that the classical coherent states for the space of connections correspond after reduction to the coherent states for the generalized transform on K. My approach to the quantization is quite different from theirs, but yields similar conclusions. The other source of motivation is the work of Gross and Malliavin, who also obtain a version of the generalized Segal-Bargmann transform for K from the infinite-dimensional classical transform. Although the case they consider does not admit an interpretation in terms of Yang-Mills theory, the calculations in this paper are similar to the ones there. See also [HS] for an extension of the Gross-Malliavin method in a different direction.

The infinite-dimensional calculations yield not only a new interpretation of the Segal-Bargmann transform for K (as the transform for the space of connections modulo gauge transformations), but also a new version of that transform. The properties of this new transform that are redicted by the infinite-dimensional argument may then be proved rigorously by finite-dimensional means.

The Segal-Bargmann transform for $\mathbb{R}^n$ may be described as follows. For $t > 0$, consider the density

$$\rho_t(x) = (2\pi t)^{-n/2} e^{-x^2/2t},$$

where $x^2 = x_1^2 + \cdots + x_n^2$. Then define

$$B_t : L^2(\mathbb{R}^n, \rho_t(x)\, dx) \to \mathcal{H}(\mathbb{C}^n)$$

by

$$B_t f(z) = \int_{\mathbb{R}^n} \rho_t(z - x) f(x)\, dx, \quad z \in \mathbb{C}^n.$$

Here $\mathcal{H}(\mathbb{C}^n)$ denotes the space of holomorphic functions on $\mathbb{C}^n$. Note that since ρ_t admits an analytic continuation to $\mathbb{C}^n$, it makes sense to allow $z \in \mathbb{C}^n$. Note also that ρ_t is the fundamental solution at the identity of the heat equation $\partial u/\partial t = 1/2\,\Delta u$. Thus

$$B_t f = \text{ analytic continuation of } e^{t\Delta/2}(f).$$

There is another version C_t of the Segal-Bargmann transform whose domain is the Hilbert space $L^2(\mathbb{R}^n, dx)$, but which is given by the same formula as B_t:

$$C_t f = \text{ analytic continuation of } e^{t\Delta/2}(f).$$

Because we will be concerned with isometry properties of these transforms, it is desirable to give them different names to remind us of the different domain Hilbert spaces.

Both versions of the Segal-Bargmann transform are isometric maps onto certain L^2-spaces of holomorphic functions, as proved in a slightly different form by Bargmann [B]. Segal independently obtained similar results for the infinite-dimensional theory [S1, S2, S3].

Theorem 1 (Bargmann, 1961)**.** *For each $t > 0$, the map B_t is an isometric isomorphism of $L^2(\mathbb{R}^n, \rho_t(x)\ dx)$ onto $\mathcal{H}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, \mu_t(z)\ dz)$, where*

$$\mu_t(z) = (\pi t)^{-n} \exp\left(-|z|^2/t\right),$$

and where dz denotes $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$.

For each $t > 0$, the map C_t is an isometric isomorphism of $L^2(\mathbb{R}^n, dx)$ onto $\mathcal{H}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, \nu_t(z)\ dz)$, where

$$\nu_t(z) = (\pi t)^{-n/2} \exp\left(-(Imz)^2/t\right).$$

The properties of either version of the transform may be obtained from the other by simple changes of variable. Thus in this the $\mathbb{R}^n$ case, the maps B_t and C_t should not be regarded as different transforms, but rather as different normalizations of the same transform. The difference between the two may be regarded as coming from different normalizations of the "coherent states." (See the appendix of [H1].) The B_t normalization allows easy passage to the infinite-dimensional ($n \to \infty$) limit.

Both versions of the Segal-Bargmann transform may be generalized to the setting of compact Lie groups [H1]. (See Section 3 for details.) Let K be a connected compact Lie group. Let $\rho_t(x)$ denote the fundamental solution at the identity of the heat equation $\partial u/\partial t = 1/2\,\Delta_K u$. Let $K_{\mathbb{C}}$ denote the complexification of K, which is a certain complex Lie group containing K as a subgroup. It turns out that ρ_t has a unique analytic continuation from K to $K_{\mathbb{C}}$. Thus we may define, by analogy to the classical case

$$B_t : L^2\left(K, \rho_t\left(x\right)\, dx\right) \to \mathcal{H}\left(K_{\mathbb{C}}\right)$$

by

$$B_t f\left(g\right) = \int_K \rho_t\left(gx^{-1}\right) f\left(x\right)\, dx, \quad g \in K_{\mathbb{C}}.$$

Here dx denotes Haar measure on K. As in $\mathbb{R}^n$, $B_t f$ is the analytic continuation of $e^{t\Delta_K/2}\left(f\right)$. We define C_t by the same formula, but on the Hilbert space $L^2\left(K, dx\right)$.

Theorem 2 (Hall, 1994)**.** *For each $t > 0$, the map B_t is an isometric isomorphism of $L^2\left(K, \rho_t\left(x\right)\, dx\right)$ onto $\mathcal{H}\left(K_{\mathbb{C}}\right) \cap L^2\left(K_{\mathbb{C}}, \mu_t\left(g\right)\, dg\right)$, where μ_t is a suitably defined heat kernel on $K_{\mathbb{C}}$, and where dg denotes Haar measure on $K_{\mathbb{C}}$.*

For each $t > 0$, the map C_t is an isometric isomorphism of $L^2\left(K, dx\right)$ onto $\mathcal{H}\left(K_{\mathbb{C}}\right) \cap L^2\left(K_{\mathbb{C}}, \nu_t\left(g\right)\, dg\right)$, where $\nu_t\left(g\right)$ is given by $\nu_t\left(g\right) = \int_K \mu_t\left(gx\right)\, dx$.

One difference between the situation for $\mathbb{R}^n$ and for K is that on K the two transforms are inequivalent. That is, there is no simple change of variables that will convert one form to the other; if you prefer, they are defined in terms of inequivalent sets of coherent states. (See again the appendix of [H1].) This generalized Segal-Bargmann transform may be regarded as providing a phase space representation for quantum mechanics of a particle with configuration space K, a view developed in [H2, H3]. See [H4] for exposition and a survey of related works.

This paper introduces a new version $B_{s,t}$ of the Segal-Bargmann transform for K, which interpolates continuously between the two previously known versions. In all cases the transform $B_{s,t}$ itself consists of $e^{t\Delta_K/2}$ followed by analytic continuation, but the domain will be an s-dependent Hilbert space, namely, $L^2\left(K, \rho_s\left(x\right)\, dx\right)$. The result is the following.

Theorem 3. *For each $s, t > 0$ satisfying $s > t/2$, the map $B_{s,t}$ given by*

$$B_{s,t} f = \text{ analytic continuation to } K_{\mathbb{C}} \text{ of } e^{t\Delta_K/2}\left(f\right)$$

is an isometric isomorphism of $L^2\left(K, \rho_s\left(x\right)\, dx\right)$ onto $\mathcal{H}\left(K_{\mathbb{C}}\right) \cap L^2\left(K_{\mathbb{C}}, \mu_{s,t}\left(g\right)\, dg\right)$, where $\mu_{s,t}\left(g\right)$ is a suitable heat kernel on $K_{\mathbb{C}}$, and where dg denotes Haar measure on $K_{\mathbb{C}}$.

In Section 2 I consider the quantization of Yang-Mills on a cylinder, and explain how this yields, at least heuristically, the transform $B_{s,t}$ on the gauge-invariant subspace. Then in Section 3 I give a rigorous finite-dimensional proof of Theorem 3. See Theorem 7 for a more precise statement.

In the case $s = t$, the heat kernel $\mu_{s,t} = \mu_{t,t}$ coincides with the heat kernel μ_t of [H1], and so in this case we have simply the transform B_t. In the limit $s \to \infty$, $\rho_s(x)$ converges to the constant function 1, and the heat kernel $\mu_{s,t}$ converges to the "K-averaged" heat kernel ν_t. So we obtain in this limit the transform C_t. I will also consider the opposite extreme, $s \to t/2$. The transform obtained in this case may be regarded as a generalization of the Fourier-Wiener transform. See Section 4 for the limiting cases.

Theorem 3 can (and will) be extended to the case of Lie groups of compact type, a class which includes both compact groups and $\mathbb{R}^n$ (Section 3). This extension allows the classical case and the compact case to be treated in a unified way. For the results of [H1], this extension was given by Driver [D]. In the case $K = \mathbb{R}^n$, the transform $B_{s,t}$ may be converted into the standard Segal-Bargmann transform B_t by simple changes of variable. See Case 2 of the proof of Theorem 7.

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2. Canonical quantization of Yang-Mills on a cylinder

In [LW, W1, W2], N. P. Landsman and K. K. Wren take on the problem of canonically quantizing Yang-Mills theory on a cylinder, using the method of "Rieffel induction." They impose gauge symmetry after quantization, by means of a certain integral over the gauge group with respect to Wiener measure. The unreduced Hilbert space is just an L^2 space over the set of Lie algebra-valued 1-forms with a Gaussian (white noise) measure. Wren's calculations [W1, W2] identify the reduced Hilbert space with $L^2(K, dx)$. (Wren considers at first only "based" gauge transformations; if all gauge transformations are considered, then one would restrict to the space of Ad-K-invariant functions in $L^2(K, dx)$.)

Wren further shows that under the reduction map the usual Gaussian-measure "coherent states" map to those coherent states in $L^2(K, dx)$ that are associated to the transform C_t. Here "coherent state" may be understood as the linear functional that maps an element f of a Hilbert space to the Segal-Bargmann transform of f evaluated at a point. Usually this is a continuous linear functional, which may therefore be identified with an element of the original Hilbert space. In the case of the transform C_t the coherent states are the functions $\psi_g(x) = \overline{\rho_t(gx^{-1})}$, which satisfy $C_t f(g) = \langle \psi_g, f \rangle_{L^2(K,dx)}$. Thus we may say that under Landsman and Wren's reduction map, the classical Segal-Bargmann transform for an infinite-dimensional linear space reduces to the transform C_t for the finite-dimensional non-linear manifold K.

These results raise several interesting questions, among them: 1) Is there a geometric explanation for the occurrence of the heat equation on K, and 2) Can the measure ν_t on $K_{\mathbb{C}}$ be "predicted" by means of infinite-dimensional analysis? Note in regard to 2) that it is not clear what happens to the isometry properties of the Segal-Bargmann transform after reduction. So the isometricity of C_t appears not to be a consequence of the calculations in [W1, W2].

In this section I take a different approach to the quantization problem, but with the goal of obtaining similar results. I use a Gaussian measure with large variance s to approximate the non-existent Lebesgue measure on the space of 1-forms. I then consider an appropriate version of the classical Segal-Bargmann transform, given (of course) by the heat equation for an infinite-dimensional linear space, followed by analytic continuation. I show heuristically that when restricted to the gauge-invariant subspace, this reduces to the heat equation for K, followed by analytic continuation. If the heuristics are correct, then the transform $B_{s,t}$ and its isometry properties follow from properties of the classical transform. Thus in the physically-relevant limit $s \to \infty$, we obtain the isometric transform C_t. This way of thinking about the transform C_t is similar to the work of Gross and Malliavin [GM] and Hall and Sengupta [HS], except that those papers consider only the B_t version of the transform, that is, only the case $s = t$.

Let K be a compact connected Lie group with Lie algebra $\mathfrak{k}$, and fix an Ad-K-invariant inner product on $\mathfrak{k}$. Think of the circle as the interval $[0,1]$ with ends identified. The circle is to be thought of as "space," so that "spacetime" is a cylinder. The unconstrained configuration space is the space $\mathcal{A}$ of $\mathfrak{k}$-valued, square-integrable functions on $[0,1]$. These are to be thought of as $\mathfrak{k}$-valued 1-forms. Let $W(K)$ denote the space of continuous K-valued paths on $[0,1]$, starting at the identity. The Itô map θ maps $\mathcal{A}$ into $W(K)$ by "integrating" a $\mathfrak{k}$-valued 1-form A from 0 to t. More precisely, given $A \in \mathcal{A}$, we define $\theta(A)_t \in W(K)$ to be the solution to the K-valued differential equation

$$\frac{d\theta(A)_t}{dt} = \theta(A)_t \, A_t. \tag{1}$$

$$\theta(A)_0 = e. \tag{2}$$

I am adopting matrix group notation. Geometrically $\theta(A)_t$ is the parallel transport from 0 to t and $\theta(A)_1$ is the holonomy around the circle.

The based gauge group $\mathcal{G}_b$ is the set of absolutely continuous maps $g : [0,1] \to K$ such that $g_0 = g_1 = e$ and such that $\int_0^1 \left|g_t^{-1}\, dg/dt\right|^2 dt < \infty$. A gauge transformation $g \in \mathcal{G}_b$ acts on $\mathcal{A}$ as follows:

$$(g \cdot A)_t = \mathrm{Ad} g_t \left(A_t\right) - \frac{dg}{dt} g_t^{-1}. \tag{3}$$

In terms of the parallel translation θ this action has the simple form

$$\theta(g \cdot A)_t = \theta(A)_t \, g_t^{-1}, \tag{4}$$

as is easily verified.

Formally, we would like to quantize $\mathcal{A}$ by passing to the Hilbert space $L^2(\mathcal{A})$ with respect to Lebesgue measure. Since $\mathcal{A}$ is infinite-dimensional, there is no Lebesgue measure. So instead I consider a Gaussian measure with variance s, and will allow s to tend to infinity. Although I am interested primarily in the Segal-Bargmann transform, I will also discuss the Hamiltonian and the Wilson loop observables.

Now, even for a Gaussian measure we must enlarge $\mathcal{A}$ to a space $\overline{\mathcal{A}}$ of $\mathfrak{k}$-valued distributions. The space $\overline{\mathcal{A}}$ may be taken to be the space of derivatives of continuous $\mathfrak{k}$-valued functions. I then consider the Gaussian measure P_s on $\overline{\mathcal{A}}$ satisfying

$$\int_{\overline{\mathcal{A}}} e^{i(\phi, A)}\, dP_s(A) = e^{-s\|\phi\|^2/2}$$

for all continuous linear functionals ϕ on $\overline{\mathcal{A}}$, where $\|\phi\|$ denotes the norm of ϕ restricted to $\mathcal{A}$. The distribution of A with respect to P_s is a scaled white noise, namely, s times the derivative of standard $\mathfrak{k}$-valued Brownian motion.

The Itô map may be "extended" to an almost-everywhere-defined map of $\overline{\mathcal{A}}$ into $W(K)$, provided that the equation is interpreted as a Stratonovich stochastic differential equation. Alternatively, the Itô map on $\overline{\mathcal{A}}$ may be defined by an Itô s.d.e. with a "correction term" as in [Di, Eq. (36)]. The distribution of $\theta(A)$ is then that of Brownian motion with values in K with the time variable scaled by s. In particular, the distribution of $\theta(A)_1$ is the heat kernel measure ρ_s on K. The action (3) of the based gauge group $\mathcal{G}_b$ on $\mathcal{A}$ extends continuously to an action on $\overline{\mathcal{A}}$ and (4) still holds.

We now consider the Hilbert space $L^2\left(\overline{\mathcal{A}}, P_s\right)$. At a heuristic level, Lebesgue measure on $\overline{\mathcal{A}}$ is an infinite constant times the $s \to \infty$ limit of P_s. We require, then, a version of the Segal-Bargmann transform which will survive the limit $s \to \infty$, at least on the gauge-invariant subspace. So we may not take the standard transform, which involves

$e^{s\Delta/2}$. Instead we take our transform, denoted $\widetilde{B}_{s,t}$, to be $e^{t\Delta/2}$ followed by analytic continuation, where $\Delta = \sum \partial^2/\partial x_k^2$ with the sum ranging over an orthonormal basis for $\mathcal{A}$. Here t is an arbitrary parameter, which may be interpreted as Planck's constant. Let me emphasize that in this the classical setting, the transform $\widetilde{B}_{s,t}$ is not truly new; it is just a different normalization of the transform considered by Segal. See Case 2 of the proof of Theorem 7.

A cylinder function on $\overline{\mathcal{A}}$ is defined to be a function of the form

$$f(A) = g\left(I(A)_{t_1}, \cdots, I(A)_{t_n}\right),$$

where $I(A)_t$ is the $\mathfrak{k}$-valued integral of the white noise A from zero to t. If $s > t/2$, then $e^{t\Delta/2}f$ makes sense for all cylinder functions f. As the target Hilbert space we take $L^2\left(\overline{\mathcal{A}}_{\mathbb{C}}, M_{s,t}\right)$. Here $\overline{\mathcal{A}}_{\mathbb{C}}$ is the space of derivatives of continuous $\mathfrak{k}_{\mathbb{C}}$-valued functions, and $M_{s,t}$ satisfies

$$\int_{\overline{\mathcal{A}}_{\mathbb{C}}} e^{i(\phi,A)}\, dM_{s,t}(A) = e^{(s-t/2)\|\phi_1\|^2/2}\, e^{t\|\phi_2\|^2/4}$$

where ϕ_1 is the restriction of ϕ to $\mathcal{A}$ and ϕ_2 is the restriction of ϕ to $i\mathcal{A}$. Case 2 of the proof of Theorem 7 shows that $M_{s,t}$ is the right Gaussian measure to make $\widetilde{B}_{s,t}$ an isometry. The proof is by reduction to the standard Segal-Bargmann transform. Since cylinder functions are dense, we may extend by continuity to get an isometric map of $L^2\left(\overline{\mathcal{A}}, P_s\right)$ into $L^2\left(\overline{\mathcal{A}}_{\mathbb{C}}, M_{s,t}\right)$. The image of $\widetilde{B}_{s,t}$ is the closure of the space of L^2 holomorphic cylinder functions, which we may optimistically call the "holomorphic" subspace of $L^2\left(\overline{\mathcal{A}}_{\mathbb{C}}, M_{s,t}\right)$.

Gauge symmetry means that we should restrict our attention to gauge-invariant functions, i.e., those such that for all $g \in \mathcal{G}_b$, $f(g \cdot A) = f(A)$ for almost every $A \in \overline{\mathcal{A}}$. Note that I am deliberately not "unitarizing" the action of the gauge group as Wren and Landsman do; the point of letting $s \to \infty$ is to avoid having to do this. So although the map $f(A) \to f(g \cdot A)$ is not unitary for any fixed s, it formally becomes so as s tends to infinity. At a rigorous level, it is easily verified that for functions of the form $f(A) = \psi\left(\theta(A)_{t_1}, \cdots, \theta(A)_{t_n}\right)$ the gauge action becomes unitary in the limit $s \to \infty$.

Now, as a consequence of the ergodicity result of [G, Thm. 2.5] (together with (4)), the functions invariant under based gauge transformations are precisely those of the form $f(A) = \phi\left(\theta(A)_1\right)$, where ϕ is a function on K. That is, f should be a function of the holonomy around the circle. (For invariance under all gauge transformations, not just based gauge transformations, ϕ would be required to be a class function.) We now wish to apply the Segal-Bargmann transform $\widetilde{B}_{s,t}$ to functions of this form, which means roughly that we should compute

$e^{t\Delta/2}f$ and then analytically continue. The key to this computation is the following simple and well-known fact.

Proposition 4. *If A is a white-noise random variable with values in $\overline{\mathcal{A}}$, and B is a fixed element of $\mathcal{A}$, then $\theta(A+B)$ has the same distribution as $\theta(A)\,\theta(B)$.*

Proof. Direct calculation shows that for A and B smooth enough,

$$\theta(A)_t\,\theta(B)_t = \theta\big(\mathrm{Ad}\theta(B)_t^{-1}(A_t) + B_t\big).$$

Standard stochastic techniques show that this remains true almost surely if A is a white noise and B is in $\mathcal{A}$. But the white noise measure is invariant under the pointwise adjoint action, which is just a "rotation" of $\mathcal{A}$. □

Using the proposition, we may formally calculate $e^{t\Delta/2}f$. The measure P_t is the fundamental solution at the identity of the heat equation, which means that $e^{t\Delta/2}$ should be given by convolution with P_t. So if $f(A) = \phi(\theta(A)_1)$ then

$$\begin{aligned} e^{t\Delta/2}f(B) &= \int_{\overline{\mathcal{A}}} \phi\left(\theta(A+B)_1\right)\,dP_t(A) \\ &= \int_{\overline{\mathcal{A}}} \phi\left(\theta(A)_1\,\theta(B)_1\right)\,dP_t(A) \\ &= \int_K \phi\left(x\theta(B)_1\right)\,d\rho_t(x) \\ &= e^{t\Delta_K/2}\phi\left(\theta(B)_1\right). \end{aligned}$$

Here we have used the proposition and the fact that $\theta(A)_1$ is distributed as the heat kernel on K. Analytically continuing, we arrive at the following conjecture.

Conjecture 5. *Suppose $f \in L^2\left(\overline{\mathcal{A}}, p_s\right)$ is of the form $f(A) = \phi(\theta(A)_1)$, where ϕ is a function on K. Then*

$$\widetilde{B}_{s,t}f(A) = \Phi\left(\theta(A)_1\right)$$

where Φ is the analytic continuation to $K_{\mathbb{C}}$ of $e^{t\Delta_K/2}\phi$.

Now, if A is distributed according to the measure P_s, then $\theta(A)_1$ is distributed according to the time s heat kernel measure ρ_s. Thus the space of gauge invariant functions in $L^2\left(\overline{\mathcal{A}}, P_s\right)$ may be identified with the Hilbert space $L^2(K, \rho_s)$. Similarly, the space of functions in $L^2\left(\overline{\mathcal{A}}_{\mathbb{C}}, M_{s,t}\right)$ of the form $F(A) = \Phi(\theta(A)_1)$ with Φ holomorphic may be identified with $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,t})$. If Conjecture 5 is correct, then the transform $\widetilde{B}_{s,t}$ for the infinite-dimensional linear space $\overline{\mathcal{A}}$ reduces, when applied to gauge-invariant functions, to the transform $B_{s,t}$ of Section 3. Thus the isometricity of $B_{s,t}$ is predicted by means of infinite-dimensional analysis.

We may even argue for the surjectivity of $B_{s,t}$ as follows. Consider $F \in L^2\left(\overline{\mathcal{A}}_{\mathbb{C}}, M_{s,t}\right)$ of the form $F(A) = \Phi\left(\theta(A)_1\right)$, with Φ holomorphic. Then F ought to be "holomorphic" as well, and thus in the image of $B_{s,t}$. But F is certainly gauge-invariant, and $B_{s,t}$ commutes with the action of the gauge group, since the gauge action is a combination of rotations and translations, and the Laplacian commutes with such. Thus $B_{s,t}^{-1}F$ should be gauge-invariant as well, hence of the form $f(A) = \phi\left(\theta(A)_1\right)$. But then Conjecture 5 would tell us that Φ is the analytic continuation of $e^{t\Delta_K/2}\phi$, which shows that every Φ comes from a ϕ. This argument is precisely in the spirit of [GM, Cor. 7.9].

Unfortunately, none of these arguments is rigorous. While the conjecture is known to be true [HS, Thm. 17] if $s = t$, even in that case the proof uses some of the results we are trying to derive, namely the "density theorems" of [H1, D]. Similarly, Corollary 7.12 of [GM] is not quite the $s = t$ case of the conjecture, because the authors consider the Itô map only for $\mathcal{A}_{\mathbb{C}}$ and not for $\overline{\mathcal{A}}_{\mathbb{C}}$. (Both [HS] and [GM] express things in terms of the Wiener measure instead of the white noise measure, but it is merely a difference of convention.) What is lacking in either case is a direct proof that for Φ holomorphic on $K_{\mathbb{C}}$, the function $F(A) = \Phi\left(\theta(A)_1\right)$ is in the "holomorphic" subspace of $L^2\left(\overline{\mathcal{A}}_{\mathbb{C}}, M_{t,t}\right)$, that is, the closure of the holomorphic cylinder functions. See the discussion in Section 2.5 of [HS].

In the case $s \neq t$, things are even worse, since then f is in $L^2\left(\overline{\mathcal{A}}, P_s\right)$ but you are trying to integrate $f(A+B)$ with respect to P_t. Since P_s and P_t are mutually singular, this does not make sense for general f. To do things rigorously, one would have to approximate by cylinder functions, and this is non-trivial. Furthermore, when $s \neq t$, it is even less clear how to show that $\Phi\left(\theta(A)_1\right)$ is in the closure of the holomorphic cylinder functions. Despite these difficulties, Conjecture 5 provides motivation for Theorem 7 and also suggests that C_t is the natural Segal-Bargmann transform for K from the point of view of Yang-Mills theory.

Finally, let me consider very briefly the Hamiltonian and the Wilson loop observables. If we could take our Hilbert space to be $L^2(\mathcal{A}, \text{ Lebesgue})$, then the Yang-Mills Hamiltonian should be $-\Delta$ [Di, Sec. 2], the curvature term being zero since the circle is one-dimensional. I would like to take $-\Delta$ as the Hamiltonian even in $L^2\left(\overline{\mathcal{A}}, P_s\right)$. This operator is not self-adjoint, but should become so as $s \to \infty$. We have argued formally that $e^{t\Delta/2}$ becomes $e^{t\Delta_K/2}$ on gauge-invariant functions, and so by differentiating with respect to t, Δ becomes Δ_K on gauge-invariant functions. Thus $-\Delta_K$ should be our Hamiltonian in $L^2(K, \rho_s)$. This operator is self-adjoint in the limiting Hilbert space $L^2(K, dx)$. This conclusion is consistent with [W1, Di]. Similarly, if χ is a character of K, we may consider the Wilson loop $\chi(\theta(A)_1)$ as a multiplication operator on $L^2(\overline{\mathcal{A}}, P_s)$. This corresponds to multiplication by χ on the gauge-invariant subspace, for any s and so also when $s \to \infty$. So as in [Di] we conclude that the Wilson loop should become multiplication by χ on $L^2(K, dx)$. The conclusions of this paragraph are also consistent with those of Rajeev [Ra], who imposes gauge symmetry before quantization. See [H4] for a discussion of how these observables act in the transformed Hilbert space $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$.

3. Rigorous finite-dimensional results

In this section we will prove the results concerning the transform $B_{s,t}$ which were predicted in the previous section. I will consider Lie groups of compact type, a class which contains both compact groups and $\mathbb{R}^n$.

Definition 6. *A Lie group is of* ***compact type*** *if it is locally isomorphic to some compact group.*

Thus $\mathbb{R}^n$ is of compact type, since it is locally isomorphic to an n-torus. Of course, compact groups are also of compact type. It can be shown [D, Cor. 2.2] that every connected Lie group of compact type is the product of a compact group and $\mathbb{R}^n$. See, for example, [He, V] for standard results about Lie groups of compact type.

Let K be a connected Lie group of compact type, and let $\mathfrak{k}$ be the Lie algebra of K. Fix once and for all an inner product on $\mathfrak{k}$ which is invariant under the adjoint action of K. A Lie group admits such an inner product if and only if it is of compact type. Let $X_1, \cdots, X_n$ be an orthonormal basis for $\mathfrak{k}$, and view each X_k as a left-invariant vector field. Let

$$\Delta_K = \sum_{k=1}^{n} X_k^2.$$

This is a bi-invariant differential operator on K and is independent of the choice of orthonormal basis.

Let $\rho_s(x)$ be the solution to the equation

$$\frac{d\rho}{ds} = \frac{1}{2}\Delta_K \rho_s,$$

subject to the initial condition

$$\rho_s \to \delta_e \text{ as } s \to 0.$$

This function is called the heat kernel (at the identity) for K. It is known that ρ_s exists and is unique, and is a strictly positive, C^∞ function on K. With little danger of confusion, we will also let ρ_s denote the associated measure

$$d\rho_s = \rho_s(x)\ dx,$$

where dx denotes Haar measure on K.

Let $K_{\mathbb{C}}$ be the complexification of K, which is a certain connected complex Lie group whose Lie algebra $\mathfrak{k}_{\mathbb{C}}$ is the complexification of $\mathfrak{k}$, and which contains K as a subgroup. (See [H1, Sec. 3] or [Ho] for the definition.) For example, if $K = SU(2)$, then $K_{\mathbb{C}} = SL(2;\mathbb{C})$. The key property of the complexified group $K_{\mathbb{C}}$ is that every finite-dimensional representation of K has a unique analytic continuation to a holomorphic representation of $K_{\mathbb{C}}$. As a consequence of this property, it may be shown [H1, D] that the heat kernel function $\rho_t(x)$ has a unique analytic continuation to $K_{\mathbb{C}}$.

Consider now the Hilbert space $L^2(K,\rho_s)$. For $t < 2s$ define

$$B_{s,t} : L^2(K,\rho_s(x)\ dx) \to \mathcal{H}(K_{\mathbb{C}})$$

by

$$B_{s,t}f(g) = \int_K \rho_t\left(gx^{-1}\right) f(x)\ dx, \quad g \in K_{\mathbb{C}}. \tag{5}$$

Here ρ_t refers to the analytically continued heat kernel and $\mathcal{H}(K_{\mathbb{C}})$ to the space of holomorphic functions on $K_{\mathbb{C}}$. Note that the formula for $B_{s,t}$ involves the time t heat kernel, whereas the domain Hilbert space involves the time s heat kernel. We will prove that the integral is convergent and that $B_{s,t}f$ is in fact holomorphic. Unless K is compact, convergence fails when $t \geq 2s$.

Now consider the operator $A_{s,t}$ on $K_{\mathbb{C}}$ given by

$$A_{s,t} = \left(s - \frac{t}{2}\right)\sum_{k=1}^n X_k^2 + \frac{t}{2}\sum_{k=1}^n JX_k^2. \tag{6}$$

Here J denotes the complex structure on $\mathfrak{k}_{\mathbb{C}}$, and the X_k's and JX_k's are viewed as left-invariant vector fields on $K_{\mathbb{C}}$. For example, if $K = \mathbb{R}^n$, $K_{\mathbb{C}} = \mathbb{C}^n$, then $X_k = \partial/\partial x_k$ and $JX_k = \partial/\partial y_k$. For $t < 2s$ (i.e., for $s > t/2$) $A_{s,t}$ is a second-order, left-invariant, elliptic differential operator on $K_{\mathbb{C}}$. Let $\mu_{s,t}$ be the function on $K_{\mathbb{C}}$ given formally by

$$\mu_{s,t} = e^{A_{s,t}/2}(\delta_e).$$

More precisely, let $\mu_{s,t,r}$ be the solution to the equation

$$\begin{aligned}\frac{d\mu_{s,t,r}}{dr} &= \frac{1}{2}A_{s,t}\mu_{s,t,r} \\ \lim_{r\to 0}\mu_{s,t,r} &= \delta_e.\end{aligned}$$

Then $\mu_{s,t} = \mu_{s,t,1}$. Existence and properties are shown, for example, in [Ro]. In particular $\mu_{s,t}$ is a C^∞, strictly positive function with rapid decay at infinity.

We will let $\mu_{s,t}$ also denote the associated measure, $d\mu_{s,t} = \mu_{s,t}(g)\, dg$, where dg is Haar measure on $K_{\mathbb{C}}$. Finally we will set

$$\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,t}) = \mathcal{H}(K_{\mathbb{C}}) \cap L^2(K_{\mathbb{C}}, \mu_{s,t}).$$

We are now ready to state the main result.

Theorem 7. *Let s and t be positive numbers with $s > t/2$. Then for all $f \in L^2(K, \rho_s)$, the integral*

$$B_{s,t}f(g) = \int_K \rho_t\left(gx^{-1}\right) f(x)\, dx$$

is absolutely convergent for all $g \in K_{\mathbb{C}}$ and depends holomorphically on g. The map $B_{s,t}$ is an isometric isomorphism of $L^2(K, \rho_s)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,t})$.

Proof. We will consider separately the compact and $\mathbb{R}^n$ cases. The general case follows by reduction to those cases.

Case 1. K is compact. Although the method of [H1] can be used with little change, I wish to give a different argument which at least formally does not involve matrix entries. However, the matrix entries still play an important technical role. According to [H1, Prop. 1], the heat kernel ρ_t admits a unique analytic continuation to $K_{\mathbb{C}}$. Although it is possible to use the transform itself to deduce this, as in [D], it is not difficult to obtain the result directly, as in [H1]. In this the compact case there is no trouble with the convergence of the integral that defines $B_{s,t}$ or with the holomorphicity of $B_{s,t}f$. So we need only address isometricity and surjectivity.

Let us first give a heuristic argument for the isometricity of $B_{s,t}$, and then show that this can be made rigorous. Since $\rho_s(x^{-1}) = \rho_s(x)$ [H1], integrating a function ϕ with respect to the measure $\rho_s(x)\, dx$ gives the same result as computing $e^{s\Delta_K/2}(\phi)$ and then evaluating at the identity. Similarly, integrating a function on $K_{\mathbb{C}}$ against $\mu_{s,t}(g)\, dg$ gives the same result as applying $e^{A_{s,t}/2}$ and then evaluating at the identity. Thus the isometricity of $B_{s,t}$ amounts to the statement

$$e^{s\Delta_K/2}\left(\overline{f_1}f_2\right)(e) = e^{A_{s,t}/2}\left(\overline{e^{t\Delta_K/2}f_1}\, e^{t\Delta_K/2}f_2\right)(e). \tag{7}$$

Let us assume that f_1 and f_2 themselves admit an analytic continuation to $K_{\mathbb{C}}$. Now, $\Delta_K = \sum X_k^2$, regarded as a left-invariant differential operator on $K_{\mathbb{C}}$, commutes with complex conjugation and with analytic continuation. Thus

$$\overline{e^{t\Delta_K/2} f_1} = e^{t\Delta_K/2}\overline{f_1}.$$

Note that on the left, we are first applying $e^{t\Delta_K/2}$, then analytically continuing, and then taking the complex conjugate. On the right we are first analytically continuing, then taking the complex conjugate and then applying $e^{t\Delta_K/2}$.

Next consider the operators

$$\begin{aligned} Z_k &= \frac{1}{2}\left(X_k - iJX_k\right) \\ \overline{Z}_k &= \frac{1}{2}\left(X_k + iJX_k\right), \end{aligned}$$

which reduce in the case $K = \mathbb{R}^n$, $K_{\mathbb{C}} = \mathbb{C}^n$ to $\partial/\partial z_k$ and $\partial/\partial \overline{z}_k$. On the holomorphic function f_2 we have $Z_k f_2 = X_k f_2$ and $\overline{Z}_k f_2 = 0$, and on the anti-holomorphic function $\overline{f_1}$, $Z_k \overline{f_1} = 0$ and $\overline{Z}_k f_1 = X_k f_1$. It follows that

$$\overline{e^{t\Delta_K/2} f_1}\, e^{t\Delta_K/2} f_2 = e^{t\sum Z_k^2/2} e^{t\sum \overline{Z}_k^2/2}\left(\overline{f_1} f_2\right).$$

So the desired norm equality becomes

$$e^{s\Delta_K/2}\left(\overline{f_1} f_2\right)(e) = e^{A_{s,t}/2} e^{t\sum Z_k^2/2} e^{t\sum \overline{Z}_k^2/2}\left(\overline{f_1} f_2\right)(e). \tag{8}$$

Now, a holomorphic vector field Z_k automatically commutes with an anti-holomorphic vector field $\overline{Z}_l$. Thus the second and third exponents on the right of (8) may be combined. The exponent that results is

$$\frac{t}{2}\sum_{k=1}^{n}\left(Z_k^2 + \overline{Z}_k^2\right) = \frac{t}{4}\sum_{k=1}^{n}\left(X_k^2 - JX_k^2\right).$$

This is a constant times the Casimir operator for $K_{\mathbb{C}}$, which is bi-invariant and therefore commutes with the left-invariant operator $A_{s,t}$. So in the end all three exponents on the right in (8) may be combined. It thus suffices to have the exponents be equal, and so, multiplying by two and recalling the definition (6) of $A_{s,t}$ it suffices to have

$$s\sum X_k^2 = \left(s - \frac{t}{2}\right)\sum X_k^2 + \frac{t}{2}\sum JX_k^2 + \frac{t}{2}\sum\left(X_k^2 - JX_k^2\right), \tag{9}$$

which is true!

Let us now make this argument rigorous. Let $\mathcal{F}$ denote the space of finite linear combinations of matrix entries for finite-dimensional irreducible representations of K. Since K is compact, $\rho_s(x)$ is bounded and bounded away from zero. Thus $L^2(K,\rho_s)$ is the same space of functions as $L^2(K,dx)$, with a different but equivalent norm. So by the Peter-Weyl theorem, $\mathcal{F}$ is dense in $L^2(K,\rho_s)$. On $\mathcal{F}$ there is little trouble in justifying the above arguments. Indeed, [H1, Lem. 2, Lem. 8] shows that when integrating matrix entries against heat kernels, one may expand everything in power series. Thus $B_{s,t}$ is isometric on $\mathcal{F}$ and so also on $L^2(K,\rho_s)$.

The proof of surjectivity of [H1] applies essentially without change. In particular, the proof of the "averaging lemma" [H1, Lem. 11] applies with μ_t replaced by $\mu_{s,t}$. This lemma is the key to surjectivity.

Case 2: $K=\mathbb{R}^n$. Convergence is an easy calculation, and isometricity can be proved by the method of Case 1. But it is not evident how to prove surjectivity directly. In particular, the arguments of [B] or [D] do not apply if $s\neq t$. Instead, I will reduce everything to properties of the "classical" Segal-Bargmann transform B_t, thus showing that $B_{s,t}$ is, in the $\mathbb{R}^n$ case, just a disguised form of B_t. Because the calculations rely on the explicit Gaussian forms for the heat kernels, no such calculations are possible for general groups.

Let $r=2(s-t/2)$, so that $r>0$. Recalling the definition of $A_{s,t}$ we may compute $\mu_{s,t}$ explicitly:

$$\mu_{s,t}(x+iy)=(\pi r)^{-n/2}(\pi t)^{-n/2}e^{-x^2/r}e^{-y^2/t}.$$

Now consider the map U defined on $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,t})$ given by

$$UF(z)=e^{az^2}F(z).$$

Since $\exp az^2$ is holomorphic and nowhere zero, U is a unitary map of $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,t})$ onto the Hilbert space

$$\mathcal{H}L^2\left(\mathbb{C}^n,\left|e^{az^2}\right|^{-2}\mu_{s,t}\right).$$

Choosing $a=(r-t)/4rt$ gives

$$\left|e^{az^2}\right|^{-2}\mu_{s,t}=c_1e^{-|z|^2/\tau}$$

where $\tau=2rt/(r+t)$ and the value of c_1 is immaterial.

The composite transform $UB_{s,t}$ is

$$UB_{s,t}f(z)=c\int_{\mathbb{R}^n}e^{az^2}e^{-(z-x)^2/2t}f(x)\,dx.$$

By combining the exponents and completing the square we obtain

$$\begin{aligned} UB_{s,t}f(z) &= c\int_{\mathbb{R}^n} \exp\left\{-\frac{1}{2\tau}\left(z-\frac{2r}{r+t}x\right)^2\right\} \\ &\quad \times f(x)\exp\left\{\frac{r-t}{2t(r+t)}x^2\right\} dx. \end{aligned} \tag{10}$$

This is just the time-τ heat operator, combined with a dilation and applied to the function

$$g(x) = f(x)\exp\left\{\frac{r-t}{2t(r+t)}x^2\right\}.$$

Now let $\widetilde{g}(x) = g(2rx/(r+t))$. Making the change-of-variable $y = 2rx/(r+t)$ in (10) gives

$$UB_{s,t}f(z) = c_2\int_{\mathbb{R}^n} \exp\left\{-\frac{1}{2\tau}(z-y)^2\right\}\widetilde{g}(y)\,dy.$$

Furthermore, calculation shows that the norm of f in $L^2(\mathbb{R}^n, \rho_s)$ is the same (up to a constant) as the norm of $\widetilde{g}$ in $L^2(\mathbb{R}^n, \rho_\tau)$. Note that the map $\widetilde{g} \to UB_{s,t}f$ is just the standard Segal-Bargmann transform B_τ, which is isometric from $L^2(\mathbb{R}^n, \rho_\tau)$ onto $\mathcal{H}L^2(\mathbb{C}^n, \mu_\tau)$. Undoing all of these steps shows that $B_{s,t}$ is, up to a constant, an isometric isomorphism of $L^2(\mathbb{R}^n, \rho_s)$ onto $\mathcal{H}L^2(\mathbb{C}^n, \mu_{s,t})$. By considering the case $f = 1$ we see that the constant is one.

Case 3: $K = \mathbb{R}^n \times G$, with G compact and semisimple. Then $K_{\mathbb{C}} = \mathbb{C}^n \times G_{\mathbb{C}}$ and the Lie algebra decomposition $\mathfrak{k} = \mathbb{R}^n \oplus \mathfrak{g}$ is automatically orthogonal with respect to any Ad-invariant inner product. It follows that the heat kernels on both the real side and the complex side will factor. Now (suppressing the measures) we have the standard result

$$L^2(\mathbb{R}^n) \otimes L^2(G) \cong L^2(\mathbb{R}^n \times G),$$

where the isomorphism takes $f_1 \otimes f_2$ to the product function $f_1(x) f_2(y)$. Meanwhile on the complex side, we have

$$\mathcal{H}L^2(\mathbb{C}^n) \otimes \mathcal{H}L^2(G_{\mathbb{C}}) \cong \mathcal{H}L^2(\mathbb{C}^n \times G_{\mathbb{C}}),$$

with $F_1 \otimes F_2$ mapping to $F_1(z) F_2(g)$. This result requires a small proof which is given in the appendix.

Now, since the heat kernel on $\mathbb{R}^n \times G$ factors, the transform applied to a product function will be just the product of the separate transforms. That is, the transform for the product group is (under the above isomorphisms) just the tensor product of the separate transforms. So all the desired properties of the transform for K follow from the corresponding properties of the transforms for $\mathbb{R}^n$ and for G.

Case 4: the general case. Although every connected Lie group of compact type is a product of a compact group with $\mathbb{R}^n$ [D, Cor. 2.2], the Lie algebras of the two factors need not be orthogonal, since the compact factor need not be semisimple. Thus the method of Case 3 is not sufficiently general. But for any K which is connected and of compact type, the universal cover of K is of the form $\widetilde{K} = \mathbb{R}^n \times G$, where G is compact and simply-connected, hence semisimple. (See [He, Chap. II, Cor. 6.5 and Thm. 6.9].) Thus by Case 3, the transform for $\widetilde{K}$ is well-defined, isometric, and surjective.

Now, K itself is $\widetilde{K}/N$, where N is a discrete central subgroup, and

$$\rho_s(x) = \sum_{n \in N} \widetilde{\rho}_s(\widetilde{x}n) \tag{11}$$

where ρ_s is the heat kernel for K, $\widetilde{\rho}_s$ is the heat kernel for $\widetilde{K}$, and $\widetilde{x}$ is any point in $\widetilde{K}$ which projects to x. Note that N is finitely-generated (since it is $\pi_1(K)$) and abelian, hence a product of cyclic groups, and each generator of an infinite cyclic factor must have non-zero $\mathbb{R}^n$-component. So there is no problem with convergence of the sum. It follows that

$$L^2(K, \rho_s) = L^2\left(\widetilde{K}, \widetilde{\rho}_s\right)^N . \tag{12}$$

Here a superscript N indicates the functions which are invariant under the right (or equivalently left) action of N.

The complexification of K will be $K_{\mathbb{C}} = \widetilde{K}_{\mathbb{C}}/N$, where $\widetilde{K}_{\mathbb{C}}$ means the complexification of the universal cover of K, which is the same as the universal cover of the complexification. We have, by analogy to (12)

$$\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,t}) = \mathcal{H}L^2\left(\widetilde{K}_{\mathbb{C}}, \widetilde{\mu}_{s,t}\right)^N . \tag{13}$$

By construction, the transform for $\widetilde{K}$ commutes with the action of N, and so it maps $L^2\left(\widetilde{K}, \widetilde{\rho}_s\right)^N$ into $\mathcal{H}L^2\left(\widetilde{K}_{\mathbb{C}}, \widetilde{\mu}_{s,t}\right)^N$. Furthermore, in light of (11), the transform for $\widetilde{K}$, restricted to the N-invariant subspace, coincides with the transform for K. So convergence and isometricity of the transform for K follow from the corresponding properties for $\widetilde{K}$.

To establish surjectivity for the transform for K, we must prove that the transform for $\widetilde{K}$ maps onto $\mathcal{H}L^2\left(\widetilde{K}_{\mathbb{C}}, \widetilde{\mu}_{s,t}\right)^N$. So suppose $F \in \mathcal{H}L^2\left(\widetilde{K}_{\mathbb{C}}, \widetilde{\mu}_{s,t}\right)^N$. Then by the surjectivity of the transform for $\widetilde{K}$, there exists $f \in L^2\left(\widetilde{K}, \widetilde{\rho}_s\right)$ such that

$$F(g) = \int_{\widetilde{K}} \widetilde{\rho}_t\left(gx^{-1}\right) f(x)\, dx.$$

Since $F(g) = F(gn)$, we have

$$\int_{\widetilde{K}} \widetilde{\rho}_t\left(gx^{-1}\right) f(x)\, dx = \int_{\widetilde{K}} \widetilde{\rho}_t\left(gnx^{-1}\right) f(x)\, dx \tag{14}$$

$$= \int_{\widetilde{K}} \widetilde{\rho}\left(gx^{-1}\right) f(xn)\, dx. \tag{15}$$

Now, if $f(x) \in L^2\left(\widetilde{K}, \widetilde{\rho}_s\right)$, then both $f(x)$ and $f(xn)$ are in $L^2\left(\widetilde{K}, \widetilde{\rho}_r\right)$ for all $r < s$. (There is no trouble on the G factor, and on the $\mathbb{R}^n$ factor this is an easy calculation.) If we take $t/2 < r < s$, then (14) says that $B_{r,t}$ applied to $f(x)$ is the same as $B_{r,t}$ applied to $f(xn)$. But $B_{r,t}$ is isometric and therefore injective, and so we conclude that $f(x) = f(xn)$ for almost every x. Thus f is in the N-invariant subspace, and $B_{s,t}$ is surjective. This completes the proof of Theorem 7. □

4. The limiting cases: $s \to \infty$, and $s \to t/2$

In the case $s = t$, Theorem 7 reduces to the results of [H1, D]. In particular $\mu_{t,t} = \mu_t$, where μ_t is the measure in those papers. This case is special because in this case, and only in this case, the operator $A_{s,t} = A_{t,t}$ annihilates holomorphic functions. This property is central to the development of the "$J^{\perp}$" expansion in [D, DG]. The case $s = t$ is also more amenable to passing to the infinite-dimensional limit, presumably because one has in this case dimension-independent bounds. See [D, Cor. 5.5 and 5.9], [HS, Thm. 4].

In this section we will examine what happens in the limits $s \to t/2$ and $s \to \infty$. Note that in all cases our transform takes f to the analytic continuation of $e^{t\Delta_K/2} f$, but that we are varying the inner product on the domain and range.

Theorem 8. *Let K be compact and normalize Haar measure on K to have mass one. Then $f \in L^2(K, \rho_s)$ if and only if $f \in L^2(K, dx)$, and*

$$\|f\|_{L^2(K,dx)} = \lim_{s\to\infty} \|f\|_{L^2(K,\rho_s)}.$$

The function

$$\nu_{s,t}(g) = \int_K \mu_{s,t}(xg)\, dx, \quad g \in K_{\mathbb{C}}$$

is independent of s and will be denoted $\nu_t(g)$. For all $F \in \mathcal{H}(K_{\mathbb{C}})$, $F \in L^2(K_{\mathbb{C}}, \mu_{s,t})$ if and only if $F \in L^2(K_{\mathbb{C}}, \nu_t)$, and

$$\|F\|_{L^2(K_{\mathbb{C}},\nu_t)} = \lim_{s\to\infty} \|F\|_{L^2(K_{\mathbb{C}},\mu_{s,t})}.$$

Thus the map

$$f \to \text{ analytic continuation } \left(e^{t\Delta_K/2} f\right)$$

is an isometric isomorphism of $L^2(K, dx)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$.

Note that I am using here a different normalization of Haar measure on K, and therefore a different normalization of ν_t, than in [H3]. In that paper I normalize Haar measure on K to coincide with the Riemannian volume measure, so that K need not have volume one.

Proof. Since we are assuming that K is compact, $\rho_s(x)$ will be bounded and bounded away from zero. Thus the L^2 norm with respect to $\rho_s(x)\,dx$ is finite if and only if the L^2 norm with respect to Haar measure is finite. It is an easy and standard result that $\rho_s(x)$ converges uniformly to the constant function 1, which establishes the first limit in the theorem.

Now, let δ_K denote Haar measure on K, viewed as a measure on $K_{\mathbb{C}}$. Then since $A_{s,t}$ is a left-invariant operator, we have formally

$$\nu_{s,t} = e^{A_{s,t}}(\delta_K).$$

But the two terms in the definition of $A_{s,t}$ commute, so

$$\nu_{s,t} = e^{t/2\sum JX_k^2} e^{(s-t/2)\sum X_k^2}(\delta_K).$$

Since δ_K is K-invariant, the exponential involving $\sum X_k^2$ has no effect, and the s-dependence vanishes.

The equivalence of square-integrability with respect to $\mu_{s,t}$ and ν_t is implied by the "averaging lemma" [H1, Lem. 11]. This is stated in [H1] for the case $s = t$, but the same proof applies in general. Using commutativity again we have for $s > t$

$$\mu_{s,t} = e^{(s-t)\sum X_k^2}\, e^{t/2\sum JX_k^2}\, e^{t/2\sum X_k^2}(\delta_e).$$

Thus

$$\mu_{s,t}(g) = \int_K \mu_{t,t}\left(gx^{-1}\right) \rho_{s-t}(x)\, dx,$$

from which it follows that $\lim_{s\to\infty} \mu_{s,t}(g) = \nu_t(g)$ for all g. Furthermore, applying the averaging lemma to $\mu_{t,t}$ we see that for all $s > t$, $\mu_{s,t}(g)$ is dominated by a constant (independent of s) times $\nu_t(g)$. So Dominated Convergence gives the second limit in the theorem. The methods of [H1] are sufficient to make all of this rigorous. ∎

Thus we obtain the unitary transform C_t as the $s \to \infty$ limit of $B_{s,t}$. There is almost certainly a theorem of this sort for general K of compact type, but in the non-compact case one would have to multiply the measures on K and $K_{\mathbb{C}}$ be a suitable function of s in order for the measures to converge, for example, $(2\pi s)^{n/2}$ if $K = \mathbb{R}^n$. Furthermore, in the non-compact case one no longer has the same space of functions for different values of s, so one would have to be careful about how the theorem is formulated.

We now turn to the opposite extreme, the limit $s \to t/2$. In the case $K = \mathbb{R}^n$ the measure $\mu_{s,t}$ collapses as $s \to t/2$ onto the imaginary axis, and our transform becomes a finite-dimensional version of the Fourier-Wiener transform. In the case K compact, we still have when $s = t/2$ an isometric map *into* an L^2-space of holomorphic functions, but at least in the simplest case the map is not onto.

Theorem 9. *Let $K = \mathbb{R}^n$ so that $K_{\mathbb{C}} = \mathbb{C}^n$. As s tends to $t/2$, the measures $\mu_{s,t}$ on $\mathbb{C}^n$ converge in the weak-$*$ topology to the Gaussian measure $(\pi t)^{-n/2} \exp\left(-y^2/t\right) dy = \rho_{t/2}(y)\, dy$ on the "imaginary axis" $i\mathbb{R}^n$. The transform $B_{t/2,t}$ makes sense, say, on polynomials and is given by*

$$\begin{aligned} B_{t/2,t} f(iy) &= \int_{\mathbb{R}^n} \rho_t (iy - x)\, f(x)\, dx \\ &= (2\pi t)^{-n/2} e^{y^2/2t} \int_{\mathbb{R}^n} e^{iyx/t} e^{-x^2/2t} f(x)\, dx. \end{aligned}$$

This transform maps the space of polynomials isometrically into $L^2\left(i\mathbb{R}^n, \rho_{t/2}\right)$. Extending continuously we obtain an isometric isomorphism of $L^2\left(\mathbb{R}^n, \rho_{t/2}\right)$ onto $L^2\left(i\mathbb{R}^n, \rho_{t/2}\right)$.

Note that $B_{t/2,t}$ is essentially the Fourier transform, unitary from $L^2(\mathbb{R}^n, dx)$ to itself, disguised by conversion to Gaussian measure on both sides. Note also that in the limit $s \to t/2$, the analyticity on the range is lost. That is, the range of $B_{t/2,t}$ is not an L^2-space of holomorphic functions, but an ordinary L^2-space, the elements of which need not even be continuous. This loss of analyticity coincides with a loss of convergence in the integral that defines $B_{t/2,t}$.

Proof. Convergence of the measure follows from the explicit formula

$$\mu_{s,t}(x + iy) = (2\pi (s - t/2))^{n/2} e^{-x^2/2(s-t/2)} (\pi t)^{-n/2} e^{-y^2/t}.$$

If f is a polynomial, then $F = e^{t\Delta/2} f$ is also a polynomial. So there is no difficulty in letting $s \to t/2$ to obtain

$$\|f\|_{L^2\left(\mathbb{R}^n, \rho_{t/2}\right)} = \lim_{s \to t/2} \|f\|_{L^2(\mathbb{R}^n, \rho_s)} = \lim_{s \to t/2} \|F\|_{L^2(\mathbb{C}^n, \mu_{s,t})} = \|F\|_{L^2\left(i\mathbb{R}^n, \rho_{t/2}\right)}.$$

So $B_{t/2,t}$ is isometric on polynomials. But $e^{t\Delta/2}$ is invertible on the space of polynomials of degree at most k, and so every polynomial is in the image of $B_{t/2,t}$. Since polynomials are dense in $L^2\left(\mathbb{R}^n, \rho_{t/2}\right)$, $B_{t/2,t}$ is densely defined and has dense image. □

Theorem 10. *Let K be compact. Then as $s \to t/2$ the measures $\mu_{s,t}$ on $K_{\mathbb{C}}$ converge in the weak-$*$ topology to a probability measure, denoted $\mu_{t/2,t}$. If K is semisimple, $\mu_{t/2,t}$ is absolutely continuous with respect to Haar measure on $K_{\mathbb{C}}$. For any compact K, the transform $B_{s,t}$ makes sense with $s = t/2$, and $B_{t/2,t}$ is an isometry of $L^2\left(K, \rho_{t/2}\right)$* into $\mathcal{H}L^2\left(K_{\mathbb{C}}, \mu_{t/2,t}\right)$.

Proof. Every connected compact Lie group is of the form $K = (G \times T)/N$, where G is compact and semisimple, T is a torus, and N is a finite central subgroup. In the semisimple case, the function $\mu_{s,t}$ on $G_{\mathbb{C}}$ exists and is strictly positive [Ro, IV.4] even when $s = t/2$, because in that case the operator $\sum JX_k^2$ is sub-elliptic and therefore hypo-elliptic. Furthermore, as in the proof of Theorem 8, we have for $s > t/2$

$$\mu_{s,t}(g) = \int_G \mu_{t/2,t}\left(gx^{-1}\right) \rho_{s-t/2}(x)\, dx.$$

It follows that the weak-$*$ limit of $\mu_{s,t}$ as s approaches $t/2$ is $\mu_{t/2,t}$. Furthermore,

$$\int_G \mu_{t/2,t}(xg)\, dx = \nu_t(g) = \int_G \mu_{s,t}(xg)\, dx \tag{16}$$

where $\nu_t(g)$ is as in Theorem 8 and decays rapidly at infinity. If f is in the space $\mathcal{F}$ (as in proof of Theorem 7), then the function $F =$ analytic continuation of $e^{t\Delta_K/2}f$ grows only exponentially at infinity. Using (16) we may control the integral of F near infinity uniformly in s, giving

$$\lim_{s\to t/2} \|F\|_{L^2(K_{\mathbb{C}},\mu_{s,t})} = \|F\|_{L^2\left(K_{\mathbb{C}},\mu_{t/2,t}\right)}.$$

Thus, $B_{t/2,t}$ is isometric on $\mathcal{F}$ and so also on $L^2\left(G, \rho_{t/2}\right)$.

In the torus case T, we have $T_{\mathbb{C}} = T \times \mathbb{R}^n$, and the measure $\mu_{s,t}$ is the product of the heat kernel measure $\rho_{(s-t/2)/2}$ on T and the Gaussian measure $\rho_{t/2}$ on $\mathbb{R}^n$. So again there is no trouble in letting s tend to $t/2$. Taking products and periodizing over N pose no problem.

Note, though, that in neither case is there any obvious way to show that square-integrability with respect to $\mu_{t/2,t}$ implies square-integrability with respect to $\mu_{s,t}$. So we cannot obtain the surjectivity of $B_{t/2,t}$ from that of $B_{s,t}$. Compare this to the situation in Theorem 8. □

I will address the surjectivity of $B_{t/2,t}$ only in the simplest case. I conjecture that if K is compact and semisimple, then $B_{t/2,t}$ maps onto $\mathcal{H}L^2\left(K_{\mathbb{C}},\mu_{t/2,t}\right)$.

Theorem 11. *Let $K = S^1 = \mathbb{R}/2\pi\mathbb{Z}$, so that $K_{\mathbb{C}} = \mathbb{C}/2\pi\mathbb{Z}$. Then the image of the map $B_{t/2,t}$ is precisely the space of functions F satisfying:*

1. *F is holomorphic on $\mathbb{C}$ and satisfies $F(z+2\pi) = F(z)$ for all $z \in \mathbb{C}$.*
2. *F is square-integrable with respect to $\mu_{t/2,t}$; that is,*
$$(\pi t)^{-1/2}\int_{\mathbb{R}} |F(iy)|^2 e^{-y^2/t}\,dy < \infty.$$
3. *There exists constants a, b, and c, with $c < 1/2$ such that*
$$|F(x+iy)| \leq a\exp\left(be^{c|y|}\right)$$
for all x, y.

If F is in the image of $B_{t/2,t}$, then for some constant d
$$|F(x+iy)| \leq d\,e^{y^2/2t}$$
for all x, y.

We may not take $c = 1/2$ in Condition 3, as the function
$$F(z) = \exp\left(ie^{iz/2}\right) + \exp\left(-ie^{iz/2}\right)$$
demonstrates. (Observe that F is 2π-periodic, even though neither term is separately.) After all, $|F(iy)| \leq 2$, which means that F is square-integrable, but $F(\pi+iy) = \exp\left(-e^{-y/2}\right) + \exp\left(e^{-y/2}\right)$ which grows too rapidly as $y \to -\infty$ for F to be in the image of $B_{t/2,t}$. Note that in the presence of Conditions 1 and 2 of the theorem, the very weak bounds of Condition 3 imply the stronger bounds given in the last part of the theorem. This is a consequence of the Phragmen-Lindelöf method (e.g., [Ru, Chap. 12]), which allows the values of a holomorphic function in the strip $0 < \mathrm{Re}z < 2\pi$ to be controlled by its values on the boundary, given Condition 3.

Proof. If F is $B_{t/2,t}f$ for some f, then it is clear from the definition of $B_{t/2,t}$ that F satisfies Conditions 1 and 2. Meanwhile,
$$B_{t/2,t}f(x+iy) = \int_0^{2\pi} \rho_t(x+iy-x')\,f(x')\,dx'$$

where

$$\begin{aligned} \rho_t(x+iy-x') &= \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \exp\left\{-(x-x'-2\pi n+iy)^2/2t\right\} \\ &= \frac{1}{\sqrt{2\pi t}} e^{y^2/2t} \sum_{n=-\infty}^{\infty} e^{-i(x-x'-2\pi n)y/t} e^{-(x-x'-2\pi n)^2/2t}. \end{aligned} \tag{17}$$

But the sum in (17) is easily seen to be bounded uniformly in x, x', and y which gives the estimate in the last part of the theorem, and so certainly Condition 3.

Conversely, suppose we have a function F which satisfies Conditions 1, 2, and 3. Then choose α with $c < \alpha < 1/2$, and let

$$h_\varepsilon(z) = \exp\{-\varepsilon\cos\alpha(z-\pi)\}.$$

For $x \in [0, 2\pi]$, $h_\varepsilon(x+iy)$ tends to zero very rapidly as y tends to infinity. Let us apply Cauchy's Formula to the function $F(z)\,e^{z^2/2t}h_\varepsilon(z)$, over the rectangle $0 < \mathrm{Re} z < 2\pi$, $-A < \mathrm{Im} z < A$. The presence of h_ε together with Condition 3 allow us to let $A \to \infty$, giving

$$\begin{aligned} F(z)\,e^{z^2/2t}h_\varepsilon(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(2\pi+iy)\,e^{2\pi^2/t}e^{2\pi iy/t}e^{-y^2/2t}\,h_\varepsilon(2\pi+iy)}{2\pi+iy-z}\,idy \\ &\quad -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(iy)\,e^{-y^2/2t}\,h_\varepsilon(iy)}{iy-z}\,idy. \end{aligned}$$

Now, $F(2\pi+iy) = F(iy)$ is square-integrable with respect to $e^{-y^2/t}dy$, which implies that $F(iy)\,e^{-y^2/2t}$ is square-integrable dy. For $0 < \mathrm{Re} z < 2\pi$, $1/(2\pi+iy-z)$ is also square-integrable dy, which means the product of the two is integrable dy. Moreover, $|h_\varepsilon(iy)| \leq 1$ and $|h_\varepsilon(2\pi+iy)| \leq 1$ for all y, so by dominated convergence we may interchange the limit $\varepsilon \to 0$ with the integrals. So

$$\begin{aligned} F(z)\,e^{z^2/2t} &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(iy)\,e^{2\pi^2/t}e^{2\pi iy/t}e^{-y^2/2t}}{2\pi+iy-z}\,idy \\ &\quad -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(iy)\,e^{-y^2/2t}}{iy-z}\,idy. \end{aligned} \tag{18}$$

Putting the $e^{z^2/2t}$ onto the other side and estimating the integrals using the square-integrability of F gives

$$|F(x+iy)| \leq c\,e^{y^2/2t}\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{2\pi-x}}\right)$$

for $0 < x < 2\pi$. These estimates would apply to any holomorphic function on this strip that satisfied Condition 3 and was square-integrable over the two boundary lines. Given these estimates and the periodicity of F, we may run a similar argument on the strip $0 < \mathrm{Re} z < 4\pi$. The square-root singularity in our estimates for F causes no problem and we obtain a formula similar to (18) but with $4\pi + iy - z$ in the denominator of the first term. This gives good estimates on $F(z)$ for $\mathrm{Re} z$ near 2π, hence also near $\mathrm{Re} z = 0$. So finally

$$|F(x+iy)| \leq c\, e^{y^2/2t}. \tag{19}$$

The estimate (19) implies that F is square-integrable with respect to the measure $\mu_{t/2,r}$ for any $r < t$. Since the transform $B_{t/2,r}$ is surjective for $r > t$, there exists f_r such that F is (the analytic continuation of) $e^{r\Delta/2} f_r$, or $f_r = e^{-r\Delta/2}F$. Now, the L^2 norm on S^1 with respect to Lebesgue measure is bounded by a constant times the L^2 norm with respect to heat kernel measure. Furthermore, a function on the imaginary axis which is square-integrable with respect to $\rho_{t/2}$ will also be square-integrable with respect to $\rho_{r/2}$ for all $r < t$, with bounds on the L^2 norm. So if we apply the transform $B_{r/2,r}$ to f_r we obtain

$$\|f_r\|_{L^2(K,dx)} \leq c_1 \|f_r\|_{L^2\left(K,\rho_{r/2}\right)} = c_1 \|F\|_{\mu_{r/2,r}} \leq c_2 \|F\|_{\mu_{t/2,t}}.$$

Thus the norm of $e^{-r\Delta/2}F$ is not going to infinity as $r \to t$. By considering Fourier series it follows that $f = e^{-t\Delta/2}F$ exists in $L^2(K, dx)$, and hence also in $L^2\left(K, \rho_{t/2}\right)$. So $F = e^{t\Delta/2} f$ and we are done. □

5. Appendix

Let X and Y be two complex manifolds with measures that are given in local coordinates by smooth strictly positive densities with respect to Lebesgue measure. I wish to show that $\mathcal{H}L^2(X) \otimes \mathcal{H}L^2(Y)$ is isomorphic to $\mathcal{H}L^2(X \times Y)$, with the isomorphism taking $F_1 \otimes F_2$ to the product function $F_1(x) F_2(y)$. Now, the conditions on the measures guarantee that in all three cases, the holomorphic subspace is closed in L^2 [D, Lem. 3.2]. Since also $F_1(x) F_2(y)$ is holomorphic if F_1 and F_2 are, it follows that under the usual isomorphism of $L^2(X) \otimes L^2(Y)$ with $L^2(X \times Y)$, $\mathcal{H}L^2(X) \otimes \mathcal{H}L^2(Y)$ maps *into* $\mathcal{H}L^2(X \times Y)$.

We may show that the image is all of $\mathcal{H}L^2(X \times Y)$ as follows. We know [D] that the pointwise evaluation map $F \to F(g)$ is bounded on each of the $\mathcal{H}L^2$-spaces. Thus for each $x \in X$ there exists $k_x \in \mathcal{H}L^2(X)$ such that $\int k_x(x') F(x')\, dx' = F(x)$ for all $F \in \mathcal{H}L^2(X)$, and similarly for Y. So now suppose $F(z, g)$ is holomorphic and square-integrable on $X \times Y$, and that F is orthogonal to the image of $\mathcal{H}L^2(X) \otimes$

$\mathcal{H}L^2(Y)$. Then for all $x \in X$, $y \in Y$

$$\begin{aligned} 0 &= \int_{X\times Y} F(x',y')\, k_x(x')\, k_y(y')\, dx'\, dy' \\ &= \int_Y \left[\int_X F(x',y')\, k_x(x')\ dx'\right] k_y(y')\ dy' \\ &= \int_Y F(x,y')\, k_y(y')\ dy'. \end{aligned}$$

Here we have used that $F(x',y')$ is holomorphic and square-integrable in x' for almost every y'. But now $F(x,y')$ is holomorphic and square-integrable in y' for almost every x, so for almost every x

$$F(x,y) = 0$$

for every y. Since F is continuous and the measure on $X \times Y$ is "nice," this implies that $F \equiv 0$.

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