LECTURES ON GLAUBER DYNAMICS FOR DISCRETE SPIN MODELS

Fabio Martinelli

Dipartimento di Energetica, Università dell'Aquila, Italy
e-mail: martin@mat.uniroma3.it

Abstract.
These notes have been the subject of a course I gave in the summer 1997 for the school in probability theory in Saint Flour. I review in a self–contained way the state of the art, sometimes providing new and simpler proofs of the most relevant results, of the theory of Glauber dynamics for classical lattice spin models of statistical mechanics. The material covers the dynamics in the one phase region, in the presence of boundary phase transitions, in the phase coexistence region for the two dimensional Ising model and in the so–called Griffiths phase for random systems.

Contents

1. Introduction

The aim of these notes is to present in a unified way progresses made in the last years in the theory of a special class of symmetric Markovian evolutions for lattice Gibbsian random fields, better known as Glauber dynamics or, in case of ± 1 variables, stochastic Ising models. Such evolutions, besides being interesting as models of non–equilibrium statistical mechanics, are important in many other different areas like images reconstruction, Monte Carlo Markov Chains methods and stochastic optimization.

The fact that in the thermodynamic limit the number of degrees of freedom diverges, poses new and challenging mathematical problems when one studies e.g. the ergodic properties or the large deviations in presence of a phase transition, and it requires the development of a new techniques like, for example, infinite dimensional logarithmic Sobolev inequalities for Gibbsian random fields. Moreover a detailed analysis of the dynamics poses new interesting questions in the theory of Gibbsian random fields itself.

This interplay between dynamical and equilibrum problems turned out to be quite fruitful in both directions and, in my opinion, it represents one of the most interesting aspect of the subject.

Although the theory of Gibbsian random fields, *i.e.* equilibrium statistical mechanics, is clearly very important for our purposes, by no means these notes are intended as an introduction to this difficult subject. Rather, I have tried to present (but in general not to derive with the notable exception of a selfcontained and elementary discussion of finite size mixing conditions) as clearly as possible all the results concerning Gibbs measure of lattice spin models needed to study our Markov processes. Then, based on these results, I develop in a selfcontained way the analysis of the ergodic properties of the Glauber dynamics, its behaviour in the presence of a phase transition or when the interactions in the original model are random. Most of the results presented here are not new but some of them are quite recent and many of their proofs have been revisited and, hopefully, considerably simplified.

Unfortunately, due to space–time problems, I have not been able to cover all the interesting aspects of Markovian dynamics in models of classical statistical mechanics. I have in fact left out important and very lively topics like the microscopic description of metastability and the difficult subject of conservative Kawasaki–type dynamics. The choice of the topics presented here is largely based on my own research in the last few years.

I would like to finish this brief introduction with a short overview of the notes.

In section 2 I recall some basic facts of lattice Gibbsian random fields and I introduce two important notion of mixing for them. In particular I provide a new, elementary proof that exponential decay of covariences in a finite large enough region implies the same property on any regular (e.g. cubic) region of the lattice.

In section 3 I introduce the Glauber dynamics, its graphical representation and two relevant analytic quantities for the study of its ergodic behaviour, namely the spectral gap and the logarithmic Sobolev constant. For the necessary background in the theory of finite Markov chains, I definitely suggest the excellent notes by Saloff–Coste [SC].

In section 4 I discuss and provide a new proof for the main results concerning the exponential ergodicity in the absence of a phase transition.

In section 5 I discuss without proofs the question of a boundary phase transition for the solid–on–solid approximation of the three dimensional Ising model and its consequences on the rate of exponential convergence to equilibrium for the associated Glauber dynamics.
In section 6 I discuss the behaviour of Glauber dynamics for the two dimensional Ising model in the presence of a phase transition. In particular I provide a new and simpler proof of a sharp lower bound on the spectral gap with plus of free boundary conditions.
In section 7 I apply all the results of section 4 and 6 to analyze the behaviour of the dynamics for a particular random ferromagnet, namely the dilute Ising model.
In section 8 I conclude the notes with a list of open problems.

Acknowledgments

I wish to warmly thank my hosts, the summer school of probability theory and Prof. Bernard in particular, for their very kind hospitality in Saint Flour and the whole staff of the school for providing an excellent environment for the meeting. During the preparation of these notes and also during their presentation in Saint Flour I benefit from the comments and criticism of several colleagues. To all of them go my hearthly thanks.

2. Gibbs Measures of Lattice Spin Models

2.1 Notation

The lattice. We consider the d dimensional lattice $\mathbb{Z}^d$ with *sites* $x = \{x_1, \dots, x_d\}$ and norm

$$|x| = \max_{i \in \{1,\dots,d\}} |x_i|$$

The associated distance function is denoted by $d(\cdot,\cdot)$. Sometimes we will need another distance, in the sequel denoted by $d_2(\cdot,\cdot)$, defined by

$$d_2(x,y) = \Big(\sum_{i=1}^{d} |x_i - y_i|^2\Big)^{1/2}$$

By Q_L we denote the cube of all $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ such that $x_i \in \{0, \dots, L-1\}$. If $x \in \mathbb{Z}^d$, $Q_L(x)$ stands for $Q_L + x$. We also let B_L be the ball of radius L centered at the origin, *i.e.* $B_L = Q_{2L+1}((-L, \dots, -L))$.

A finite subset Λ of $\mathbb{Z}^d$ is said to be a *multiple* of Q_L if there exists $y \nu Q_l$ such that Λ is the union of a finite number of cubes $Q_L(x_i + y)$ where $x_i \in L\mathbb{Z}^d$.

If Λ is a finite subset of $\mathbb{Z}^d$ we write $\Lambda \subset\subset \mathbb{Z}^d$. The cardinality of Λ is denoted by $|\Lambda|$. $\mathbb{F}$ is the set of all nonempty finite subsets of $\mathbb{Z}^d$. We define the exterior *n–boundary* as $\partial_n^+ \Lambda = \{x \in \Lambda^c : d(x,\Lambda) \le n\}$. Given $r \in \mathbb{Z}_+$ we say that a subset V of $\mathbb{Z}^d$ is *r-connected* if, for all $y, z \in V$ there exist $\{x_1, \dots, x_n\} \subset V$ such that $x_1 = y$, $x_n = z$ and $|x_{i+1} - x_i| \le r$ for $i = 2, \dots, n$.

The configuration space. Our *configuration space* is $\Omega = S^{\mathbb{Z}^d}$, where $S = \{-1, 1\}$, or $\Omega_V = S^V$ for some $V \subset \mathbb{Z}^d$. The single spin space S is endowed with the discrete topology and Ω with the corresponding product topology. Given $\sigma \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ we denote by σ_Λ the natural projection over Ω_Λ. If U, V are disjoint, $\sigma_U \eta_V$ is the configuration on $U \cup V$ which is equal to σ on U and η on V.

If f is a function on Ω, Λ_f denotes the smallest subset of $\mathbb{Z}^d$ such that $f(\sigma)$ depends only on σ_{Λ_f}. f is called *local* if Λ_f is finite. $\mathcal{F}_\Lambda$ stands for the $\sigma-$algebra generated by the set of projections $\{\pi_x\}$, $x \in \Lambda$, from Ω to $\{-1, 1\}$, where $\pi_x : \sigma \mapsto \sigma(x)$. When $\Lambda = \mathbb{Z}^d$ we set $\mathcal{F} \equiv \mathcal{F}_{\mathbb{Z}^d}$ and $\mathcal{F}$ coincides with the Borel $\sigma-$algebra on Ω with respect to the topology introduced above. By $\|f\|_\infty$ we mean the supremum norm of f. The *gradient* of a function f is defined as

$$(\nabla_x f)(\sigma) = f(\sigma^x) - f(\sigma)$$

where $\sigma^x \in \Omega$ is the configuration obtained from σ, by flipping the spin at the site x. If $\Lambda \in \mathbb{F}$ we let

$$|\nabla_\Lambda f|^2 = \sum_{x \in \Lambda} (\nabla_x f)^2$$

We also define

$$|||f||| = \sum_{x \in \mathbb{Z}^d} \|\nabla_x f\|_\infty$$

2.2 Gibbs Measures

Definition 2.1. *A finite range, translation–invariant potential* $\{J_\Lambda\}_{\Lambda\in\mathbb{F}}$ *is a real function on the set of all non empty finite subsets of* $\mathbb{Z}^d$ *with the following properties*

(1) $J_A = J_{A+x}$ *for all* $A \in \mathbb{F}$ *and all* $x \in \mathbb{Z}^d$

(2) *There exists* $r > 0$ *such that* $J_A = 0$ *if* $\operatorname{diam} A > r$. r *is called the range of the interaction.*

(3) $\|J\| \equiv \sum_{A\ni 0} |J_A| < \infty$

Given a *potential* or *interaction* J with the above two properties and $V \in \mathbb{F}$, we define the Hamiltonian $H_V^J : \Omega \mapsto \mathbb{R}$ by

$$H_V^J(\sigma) = -\sum_{A:\, A\cap V\neq\emptyset} J_A \prod_{x\in A} \sigma(x)$$

For $\sigma, \tau \in \Omega$ we also let $H_V^{J,\tau}(\sigma) = H_V^J(\sigma_V \tau_{V^c})$ and τ is called the *boundary condition.* For each $V \in \mathbb{F}$, $\tau \in \Omega$ the (finite volume) conditional Gibbs measure on $(\Omega, \mathcal{F})$, are given by

$$\mu_V^{J,\tau}(\sigma) = \begin{cases} \left(Z_V^{J,\tau}\right)^{-1} \exp[-H_V^{J,\tau}(\sigma)] & \text{if } \sigma(x) = \tau(x) \text{ for all } x \in V^c \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

where $Z_V^{J,\tau}$ is the proper normalization factor called partition function. Notice that in (2.1) we have adsorbed in the interaction J the usual inverse temperature factor β in front of the Hamiltonian. In most notation we will drop the superscript J if that does not generate confusion.

Given a measurable bounded function f on Ω, $\mu_V(f)$ denotes the function $\sigma \mapsto \mu_V^\sigma(f)$ where $\mu_V^\sigma(f)$ is just the average of f w.r.t. μ_V^σ. Analogously, if $X \in \mathcal{F}$, $\mu_V^\tau(X) \equiv \mu_V^\tau(\mathbb{1}_X)$, where $\mathbb{1}_X$ is the characteristic function on X. $\mu_V^\tau(f,g)$ stands for the covariance or *truncated correlation* (with respect to μ_V^τ) of f and g. The set of measures (2.1) satisfies the DLR compatibility conditions

$$\mu_\Lambda^\tau(\mu_V(X)) = \mu_\Lambda^\tau(X) \qquad \forall\, X \in \mathcal{F} \qquad \forall\, V \subset \Lambda \subset\subset \mathbb{Z}^d \tag{2.2}$$

Definition 2.2. *A probability measure* μ *on* $(\Omega, \mathcal{F})$ *is called a Gibbs measure for* J *if*

$$\mu(\mu_V(X)) = \mu(X) \qquad \forall\, X \in \mathcal{F} \qquad \forall\, V \in \mathbb{F} \tag{2.3}$$

Remark. In the above definition we could have replaced the $\sigma-$algebra $\mathcal{F}$ with $\mathcal{F}_V$ (see section 2.3.2 in [EFS]).

The set of all Gibbs measures relative to a fixed given potential J will be denoted by $\mathcal{G}$. It can be proved that $\mathcal{G}$ is a nonempty, convex compact set. We will say that the discrete spin system described by the potential J exhibits a phase transition if $\mathcal{G}$ contains more than one element. The reader is refered to [Geo] and [EFS] for a much more advanced discussion of Gibbs measures.

The most famous example of lattice spin system is certainly the Ising model. For this model the potential J takes the following values

$$J_A = \beta \begin{cases} 1 & \text{if } A = \{x,y\} \text{ with } d_2(x,y) \le 1 \\ h & \text{if } A = \{x\} \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

where β represents the inverse temperature and h the external magnetic field.

For the Ising model in dimension greater than one it is well known (see e.g [Pf]) that there exists a finite value β_c, called the critical point, such that there is a unique Gibbs measure for any $h \neq 0$ or any $\beta < \beta_c$. If instead $h = 0$ and $\beta > \beta_c$ there is a phase transition. In particular the so called spontaneous magnetization is non zero. Namely there exist two Gibbs measures denoted by $\mu^{\beta}_{\pm}$, that can be obtained as thermodynamic limit as $L \to \infty$ of the finite volume Gibbs measures $\mu^{\beta,\pm}_{B_L}$ respectively and such that $m^*(\beta) \equiv \mu^{\beta}_{+}\big(\sigma(0)\big) = -\mu^{\beta}_{-}\big(\sigma(0)\big) > 0$. Here the supercript $\pm$ in $\mu^{\beta,\pm}_{B_L}$ denotes the special boundary conditions for which all the boundary spins are all equal to either plus or minus one. In this notes we will refer to $\mu^{\beta}_{\pm}$ as the *plus and minus phase* respectively.

2.3 Weak and Strong Mixing Conditions

As a next step we define two similar, but at the same time deeply different, notion of *weak dependence of the boundary conditions* for finite volume Gibbs measures (see [MO]). These notion will be denoted in the sequel *weak* and *strong mixing* (not to be confused with the classical notion of strong-mixing for random fields) respectively. They both imply that there exists a unique infinite volume Gibbs state with exponentially decaying truncated correlations functions. Actually the validity of our strong mixing condition on e.g. all squares implies much more, namely analyticity properties of the Gibbs measure, the existence of a convergent cluster expansion (see [O] and [OP]) and good behaviour under the renormalization–group transformation known as the "decimation transformation" (see [MO3] and [MO4]). Moreover, and this is our main motivation, both notion play a key role in the discussion of the exponential ergodicity of a Glauber dynamics for discrete lattice spin systems.

Roughly speaking, the *weak mixing* condition implies that if in a finite volume V we consider the Gibbs state with boundary condition τ, then a local (e.g. in a single site $y \in V^c$) modification of the boundary condition τ has an influence on the corresponding Gibbs measure which decays exponentially fast inside V with the *distance from the boundary* $\partial^+ V$.

Strong mixing condition, instead, implies, in the same setting as above, that the influence of the perturbation decays in V exponentially fast with the distance from the *support of the perturbation* (e.g. the site y).

This distinction is very important since, even if we are in the one phase region with a unique infinite volume Gibbs state with exponentially decaying truncated correlation functions, it may happen that, if we consider the same Gibbs state in a finite volume V, a local perturbation of the boundary condition radically modifies the Gibbs measure close to the boundary while leaving it essentially unchanged in the bulk and this "long range order effect" at the boundary persists even when V becomes arbitrarily large. We will refer to this phenomenon as a "boundary phase transition". It is clear that if a "boundary phase transition" takes place, then our Gibbs measure may satisfy a weak mixing condition but not a strong one.

A "boundary phase transition" is apparently not such an exotic phenomenon since, besides being proved for the so called Czech models [Sh] (in dimension 3 and higher), it is also expected (see section 5) to take place for the three dimensional ferromagnetic Ising model at low temperatures and small enough magnetic field (depending on the temperature). On the contrary, for finite range two dimensional systems and for regular volumes (e.g squares) we do not expect any "boundary phase transition" since the boundary is one–dimensional and, unless the interaction is long range, no phase transition occurs. Thus in two dimensions weak mixing should be equivalent to strong mixing. That is precisely the content of theorem 2.5 below.

We conclude this short introduction with a warning. It may happen, also for very natural model like the Ising model at low temperature and positive external field, that strong mixing holds for "regular" volumes, like all multiples of a given large enough cube, but fails for other sets (see [MO1]). This fact led to a revision of the theory of "completely analytical Gibbsian random fields" (see [DS]) and it plays an important role in the recently much debated issue of pathologies of renormalization group transformations in statistical mechanics (see [EFS]).

Let us now define our two conditions. We first recall that the *variation distance* between two probability measures μ_1, μ_2 on a finite set Y is defined as :

$$\begin{aligned} \|\mu_1 - \mu_2\| &= \frac{1}{2}\sum_{y\in Y} |\mu_1(y) - \mu_2(y)| \\ &= \sup_{X\subset Y} |\mu_1(X) - \mu_2(X)| \end{aligned} \tag{2.5}$$

Given $\Delta \subset V \subset\subset \mathbb{Z}^d$ and a Gibbs measure μ_V^τ on Ω_V, we denote by $\mu_{V,\Delta}^\tau$ the projection of the measure μ_V^τ on Ω_Δ, *i.e.*

$$\mu_{V,\Delta}^\tau(\sigma) = \sum_{\eta;\, \eta_\Delta = \sigma_\Delta} \mu_V^\tau(\eta)$$

We are now in a position to define strong mixing and weak mixing.

Definition 2.3. *We say that the Gibbs measures μ_V satisfy the weak mixing condition in V with constants C and m if for every subset $\Delta \subset V$*

$$\sup_{\tau,\tau'} \|\mu_{V,\Delta}^\tau - \mu_{V,\Delta}^{\tau'}\| \le C \sum_{x\in\Delta,\, y\in\partial_r^+ V} e^{-md(x,y)} \tag{2.6}$$

We denote this condition by $WM(V, C, m)$.

Definition 2.4. *We say that the Gibbs measures μ_V satisfy the strong mixing condition in V with constants C and m if for every subset $\Delta \subset V$ and every site $y \in V^c$*

$$\sup_{\tau} \|\mu_{V,\Delta}^\tau - \mu_{V,\Delta}^{\tau^y}\| \le C e^{-md(\Delta,y)} \tag{2.7}$$

We denote this condition by $SM(V, C, m)$.

Remark. It is clear that either one of the above properties becomes interesting when it holds with the *same* constants C and m for an infinite class of finite subsets of $\mathbb{Z}^d$ e.g. all cubes.

It is a relatively easy task to show that strong mixing is more stringent than weak mixing in the sense that, for example, strong mixing for all cubes implies weak mixing for all cubes.
Let us in fact fix a cube Q_L, a subset $\Delta \subset Q_L$ and a pair of boundary conditions τ and τ'. Let also $\{\tau_i\}_{i=1}^n$ be a sequence of interpolating configurations between τ and τ' such that $n \le \partial_r^+ Q_L$, τ_{i+1} differs from τ_i at exactly one site $y_i \in \partial_r^+ Q_L$, $\tau_1 = \tau$ and τ_n agrees with τ' on $\partial_r^+ Q_L$. Then, using $SM(Q_L, C, m)$, we write

$$\begin{aligned}
\|\mu_{Q_L,\Delta}^{\tau} - \mu_{Q_L,\Delta}^{\tau'}\| &\le \sum_i \|\mu_{Q_L,\Delta}^{\tau_{i+1}} - \mu_{Q_L,\Delta}^{\tau_i}\| \\
&\le \sum_i Ce^{-md(\Delta, y_i)} \\
&\le \sum_{x\in\Delta} \sum_{y\in\partial_r^+ Q_L} Ce^{-md(x,y)}
\end{aligned}$$

i.e. $WM(Q_L, C, m)$.
The converse of the above result, namely weak mixing implies strong mixing, is in general expected to be false in dimension greater than two. In two dimensions we have instead the following (see [MOS])

Theorem 2.5. *In two dimensions, $WM(V, C, m)$ for every $V \subset\subset \mathbb{Z}^d$ implies $SM(Q_L, C', m')$ for every square Q_L, for suitable constants C' and m'.*

Remark. It is very important to notice that it is known, by means of explicit examples, that the above result becomes false if we try to replace in the above theorem *for all squares* with *for all finite subsets of* $\mathbb{Z}^2$ (see [MO1]).

We conclude this paragraph by discussing the validity of the above conditions for the Ising model (2.4). In two dimensions it has been proved (see [Hi], [MO1] and particularily [ShSch]) that condition $WM(V, C, m)$ holds true for any set $V \subset\subset \mathbb{Z}^2$ everywhere in the one–phase region *i.e.* whenever the external magnetic field h is different from zero or the inverse temperature β is smaller than the critical value β_c, with constants C and m depending on β and h. Thanks to theorem 2.5 the same holds for $SM(Q_L, C', m')$ for all integers L.

In higher dimensions weak mixing for every finite set V is known to holds for any $\beta < \beta_c$ (see [Hi]) or for large enough β and arbitrary non–zero external field h (see [MO1]). Strong mixing for all cubes has been proved instead for small enough β or large enough βh (see [MO1]). Moreover, as we have already anticipated, there is strong evidence if not a proof that strong mixing for all cubes is false if β is large and h is small enough depending on β (see section 5).

2.4 Mixing properties and bounds on relative densities

In this paragraph we first show that strong mixing is equivalent to the exponential decay of finite–volume covariances, uniformly in the boundary conditions (see condition $SMT(V, C, m)$ below). Then we prove two useful equilibrium results on finite volume Gibbs measures. The first one is

what was called in [MO1] "effectiveness" of the strong mixing condition and it implies that in order to verify strong mixing on, e.g. all cubes, it is sufficient to verify it for a finite class of subsets. The second result says that a power–law decay of covariances with a large enough power depending on the dimension implies exponential decay (absence of intermediate rates of decay of covariances). We finally conclude with a simple bound on the relative density between the projection over certain sets of two different Gibbs measures, once one assumes exponential decay of correlations.
Before going on we need the following definition:

Definition 2.6. *Given $V \subset\subset \mathbb{Z}^d$, $n, \alpha > 0$, we say that the condition $SMT(V, n, \alpha)$ holds if for all local functions f and g on Ω such that $d(\Lambda_f, \Lambda_g) \geq n$ we have*

$$\sup_{\tau\in\Omega} |\mu_V^\tau(f,g)| \leq |\Lambda_f||\Lambda_g| \, \|f\|_\infty \|g\|_\infty \exp[-\alpha \, d(\Lambda_f, \Lambda_g)]$$

With the above notation we have

Theorem 2.7. *The following are equivalent*

i) There exist C, m and L_0 such that $SM(\Lambda, C, m)$ holds for all Λ multiples of Q_{L_0}

ii) There exist l, m and L_0 such that $SMT(\Lambda, l, m,)$ holds for all Λ multiples of Q_{L_0}

Proof. Let us first show that $i) \to ii)$. For this purpose we choose $L \in \mathbb{Z}_+$ in such a way that Q_L is a multiple of Q_{L_0}. Thanks to proposition 2.9 below it is enough to prove condition $SMT(V, L/2, m')$ for some $m' > 0$ independent of L and all sets $\Lambda \subset B_{2L}$ which are multiple of Q_L, provided that L was chosen large enough.

We choose a multiple l_1 of L_0 in such a way that $L^{2/3} \leq l_1 \leq L/8$, we fix a set $\Lambda \subset B_{2L}$ multiple of Q_L and we partition it into disjoint cubes as follows

$$\Lambda = \cup_{i=1}^n B_i \,, \qquad B_i = Q_{l_1}(x_i) \cap \Lambda; \qquad x_i \in l_1\mathbb{Z}^d + \bar{x}$$

for a suitable $\bar{x}$.
We also set $\Lambda_I = \cup_{i\in I} B_i \quad \forall I \subset \{1 \dots n\}$. Given now two arbitrary functions f and g, with $d(\Lambda_f, \Lambda_g) \geq L/2$, let

$$I_f = \{i \in \{1 \dots n\} : \; B_i \cap \Lambda_f \neq \emptyset\} \qquad I_f^c = \{1 \dots n\} \setminus I_f$$

and similarly for g. Notice that, because of our choice of l_1, we have

$$d(\Lambda_{I_f}, \Lambda_{I_g}) \geq L/2 - 2l_1 \geq L/4 \tag{2.8}$$

Now we write, using the DLR equations,

$$\begin{aligned} \mu_\Lambda^\tau(f,g) &= \sum_{(\sigma,\sigma')} \mu_\Lambda^\tau(\sigma)\mu_\Lambda^\tau(\sigma') f(\sigma)[\mu^\sigma_{\Lambda_{I_f^c}}(g) - \mu^{\sigma'}_{\Lambda_{I_f^c}}(g)] \\ &\leq \|f\|_\infty \|g\|_\infty \sup_{\tau,\tau'} \|\mu^\tau_{\Lambda_{I_f^c}, \Lambda_{I_g}} - \mu^{\tau'}_{\Lambda_{I_f^c}, \Lambda_{I_g}}\| \end{aligned} \tag{2.9}$$

so that, using a simple telescopic argument

$$\sup_{\tau} |\mu^{\tau}_{\Lambda}(f,g)| \le \|f\|_\infty \|g\|_\infty \sum_{y\in\Lambda_{I_f}} \sup_{\tau} \|\mu^{\tau}_{\Lambda_{I_f^c},\Lambda_{I_g}} - \mu^{\tau^y}_{\Lambda_{I_f^c},\Lambda_{I_g}}\| \tag{2.10}$$

We can now apply property $SM(C,m,\Lambda_{I_f^c})$ to estimate each term in the sum appearing in the r.h.s of (2.10). We get

$$\text{RHS of (2.10)} \ \le \|f\|_\infty \|g\|_\infty C L^d e^{-mL/4} \le \|f\|_\infty \|g\|_\infty |\Lambda_f||\Lambda_g| e^{-md(\Lambda_f,\Lambda_g)/8} \tag{2.11}$$

namely $SMT(\Lambda, L/2, m/8)$, provided that L is large enough depending on m.

In order to prove the converse, namely $ii) \to i)$, we first need some additional notation.

Given any two measures μ and ν on $(\Omega,\mathcal{F})$, and given $V\in\mathbb{F}$ such that for, each $X\in\mathcal{F}_V$, $\nu(X)=0$ implies $\mu(X)=0$, we define the $\mathcal{F}_V-$measurable function

$$\frac{d\mu}{d\nu}\Big|_V : \eta \mapsto \frac{\mu\{\sigma\in\Omega:\, \sigma_V=\eta_V\}}{\nu\{\sigma\in\Omega:\, \sigma_V=\eta_V\}} \qquad \eta\in\Omega \tag{2.12}$$

where 0/0 means 0. We have, of course,

$$\mu(f) = \nu(f\,\frac{d\mu}{d\nu}\Big|_V) \qquad \forall f\in\mathcal{F}_V \tag{2.13}$$

With this notation we have the following result

Lemma 2.8. *Let $\Delta\subset V\subset\subset\mathbb{Z}^d$, and let $x\in V^c$ such that $d(x,\Delta)>r$. If $U\equiv V\setminus\Delta$, we have*

$$\sup_{\tau\in\Omega} \Big\|1-\frac{d\mu_V^{\tau^x}}{d\mu_V^{\tau}}\Big|_\Delta\Big\|_\infty \le e^{16\|J\|} \sum_{y\in\Delta} \sup_{\tau\in\Omega} \big|\,\mu_U^{\tau}\big(\,e^{-\nabla_x H_U}\,,\,e^{-\nabla_y H_U}\,\big)\big| \tag{2.14}$$

Assuming the lemma let us complete the proof of the theorem.
Let V be a multiple of Q_{L_0}, let $\Delta\subset V$ and let $\hat\Delta$ be the smallest subset of V containing Δ which is also a multiple of L_0. Thanks to (2.14), property $SMT(V,l,m)$ and the simple bound

$$\big|\,\mu_U^{\tau}\big(\,e^{-\nabla_x H_U}\,,\,e^{-\nabla_y H_U}\,\big)\big| \le (2r+1)^{2d} e^{4\|J\|} e^{m(l-d(x,y))}$$

we get

$$\begin{aligned} \|\mu^{\tau}_{V,\Delta} - \mu^{\tau^y}_{V,\Delta}\| &\le \|\mu^{\tau}_{V,\hat\Delta} - \mu^{\tau^y}_{V,\hat\Delta}\| \\ &\le \Big\|1-\frac{d\mu_V^{\tau^y}}{d\mu_V^{\tau}}\Big|_{\hat\Delta}\Big\|_\infty \\ &\le (2r+1)^{2d} e^{20\|J\|} \sum_{x\in\hat\Delta} e^{ml-md(x,y)} \\ &\le A(2r+1)^{2d} e^{20\|J\|} \sum_{x\in\Delta} e^{ml-md(x,y)} \end{aligned} \tag{2.15}$$

for a suitable constant A depending on L_0 and m. Clearly (2.15) proves $SM(V,C,m)$ with $C = A(2r+1)^{2d}e^{20\|J\|}\,e^{ml}$. $\square$

Proof of lemma 2.8 Let $\underline{1} \in \Omega$ be the configuration with all spins equal to $+1$, and let

$$W_{V,\Delta}(\sigma) = \log Z_U^{\sigma} - \log Z_U^{\sigma_{\Delta^c}\underline{1}_\Delta}$$

It is easy to show that

$$\Big\| 1 - \frac{d\mu_V^{\tau^x}}{d\mu_V^{\tau}}\Big|_\Delta \Big\|_\infty \le e^{2\|\nabla_x W_{V,\Delta}\|_\infty}\|\nabla_x W_{V,\Delta}\|_\infty \qquad \text{for all } \tau\in\Omega$$

which, using the trivial bound $\|\nabla_x W_{V,\Delta}\|_\infty \le 4\|J\|$, gives

$$\Big\| 1 - \frac{d\mu_V^{\tau^x}}{d\mu_V^{\tau}}\Big|_\Delta \Big\|_\infty \le e^{8\|J\|}\|\nabla_x W_{V,\Delta}\|_\infty \qquad \text{for all } \tau\in\Omega$$

By proceeding as in Lemma 3.1 of [MO2] one can show that

$$\|\nabla_x W_{V,\Delta}\|_\infty \le e^{6\|J\|}\sum_{y\in\Delta} e^{2\|J\|}\sup_{\tau\in\Omega}\big|\,\mu_U^{\tau}\big(\,e^{-\nabla_x H_U}\,,\,e^{-\nabla_y H_U}\,\big)\big|$$

which completes the proof of the lemma. $\square$

The next key result (see also [DS], [O], [MO1] and [Yo1]) says that in order to prove exponential decay of covariances for all sets V multiple of a given cube Q_l it is sufficient to verify such property only for a finite class of sets.

Let $\mathcal{F}_l$ be the class of all subsets of $\mathbb{Z}^d$ which are multiple of Q_l and denote by $\mathcal{F}_l^0$ the class of all sets $V\in\mathcal{F}$ such that $V\subset B_{2l}$. Then we have

Proposition 2.9. *Given $\alpha>0$ there exist positive integers $\bar{l}(d,r,\alpha,\|J\|)$, $\gamma_1(d,r,\|J\|)$, $\widehat{m}(d,r,\alpha)$ such that if $l>\bar{l}$ and $SMT(V,l/2,\alpha)$ holds for all $V\in\mathcal{F}_l^0$, then $SMT(V,\gamma_1 l,\widehat{m}(d,r,\alpha))$ holds for all $V\in\mathcal{F}_l$.*

Proof. Given a positive integer l define the non–negative, bounded, translation invariant function $f_l(x-y)$ on $\mathbb{Z}^d$ by

$$f_l(x-y) \equiv \sup_{V\in\mathcal{F}_l}\sup_{\tau\in\Omega}|\mu_V^{\tau}(h_x,h_y)| \tag{2.16}$$

The proof is based on the following key lemma

Lemma 2.10. *Given $\alpha>0$ there exist $\bar{l}(d,r,\alpha)$ such that, if $l>\bar{l}$ and $SMT(V,l/2,\alpha)$ holds for all $V\in\mathcal{F}_l^0$, then there exists $C(\alpha,d,r,\|J\|)$ such that*

$$f_l(x-y)\le Ce^{-\frac{\alpha}{10}d(x,y)} \qquad \forall\ d(x,y)\ge 4l+r$$

Let us assume for a moment the lemma and let us complete the proof of the proposition.
We fix an integer l, a set $V \in \mathcal{F}_l$, a boundary condition τ and two functions f and g with zero $\mu_V^\tau(\cdot)$ mean and $d(\Lambda_f, \Lambda_g) \geq 6l + 2r$. Without loss of generality we assume $V = \cup_{i=1}^n Q_l(x_i)$ with $x_i \in l\mathbb{Z}^d$. Let also V_1 (V_2) be the union of those cubes $\{Q_l(x_i)\}_{i=1}^n$ that intersect the support of f (g). Clearly $d(V_1, V_2) \geq 4l + r$. Then we write

$$
\begin{aligned}
|\mu_V^\tau(fg)| &= |\mu_V^\tau\big(f\mu_{V\setminus V_1}(g)\big)| \\
&\leq \|f\|_\infty \|\mu_{V\setminus V_1}(g)\|_\infty^\tau
\end{aligned}
\tag{2.17}
$$

where in $\|\mu_{V\setminus V_1}(g)\|_\infty^\tau$ the supremum over the boundary condition in $\partial_r^+(V \setminus V_1)$ has to be taken only over configurations that agree with τ in $\partial_r^+ V$. In order to bound $\|\mu_{V\setminus V_1}(g)\|_\infty^\tau$ we use the following simple formula valid for any pairs of sets $A \subset B$ and any function f that depends only on the spins in A

$$
\begin{aligned}
\sup_\tau \sup_{\sigma;\, \sigma_{B^c} = \tau_{B^c}} |\mu_A^\sigma(f) - \mu_B^\tau(f)| &= \sup_{\sigma;\, \sigma_{B^c} = \tau_{B^c}} |\sum_\eta \mu_B^\tau(\eta)[\mu_A^\sigma(f) - \mu_A^\eta(f)| \\
&\leq \sup_{\substack{\sigma,\eta; \\ \sigma_{B^c} = \eta_{B^c}}} |\mu_A^\sigma(f) - \mu_A^\eta(f)| \\
&\leq \sup_\tau \sum_{x \in \partial_r^+ A \cap B} |\mu_A^{\tau^x}(f) - \mu_A^\tau(f)| \\
&= \sup_\tau \sum_{x \in \partial_r^+ A \cap B} \frac{|\mu_A^\tau(h_x, f)|}{\mu_A^\tau(h_x)} \\
&\leq e^{8\|J\|} \sup_\tau \sum_{x \in \partial_r^+ A \cap B} |\mu_A^\tau(h_x, f)|
\end{aligned}
\tag{2.18}
$$

where $h_x = e^{-\nabla_x H}$.
If we apply (2.18) to (2.17) we get that the r.h.s. of (2.17) is bounded from above by

$$
|\mu_V^\tau(fg)| \leq \|f\|_\infty e^{8\|J\|} \sup_\tau \sum_{x \in V_1} |\mu_{V\setminus V_1}^\tau(\bar{h}_x g)|
\tag{2.19}
$$

where $\bar{h}_x \equiv h_x - \mu_{V\setminus V_1}^\tau(h_x)$. We can at this point repeat the previous steps for each term $\mu_{V\setminus V_1}^\tau(\bar{h}_x g)|$ and write

$$
\begin{aligned}
\mu_{V\setminus V_1}^\tau(\bar{h}_x g)| &= \mu_{V\setminus V_1}^\tau(g\mu_{V\setminus V_0}(\bar{h}_x))| \\
&\leq \|g\|_\infty \sup_\tau \sum_{y \in V_2} |\mu_{V\setminus V_0}^\tau(h_x, h_y)|
\end{aligned}
\tag{2.20}
$$

where $V_0 = V_1 \cup V_2$.
Notice that, by construction, $V \setminus V_0$ is still a multiple of Q_l. Therefore we can use lemma 2.10 in

order to estimate $|\mu^\tau_{V\setminus V_0}(h_x,h_y)|$. We get

$$|\mu^\tau_V(f,g)| \le \|f\|_\infty\,\|g\|_\infty e^{16\|J\|}\sum_{\substack{x\in V_1\\ y\in V_2}} f_l(x-y) \tag{2.21}$$

and property $SMT(V,5l,\alpha/11)$ follows from lemma 2.10 provided that l is taken large enough. $\square$

Proof of Lemma 2.10. Let

$$F_l(n)\equiv \sup_{x;\,|x|\ge n} f_l(x)$$

Then the proof follows from the following recursive formula valid for $l>2r$, any integer $n\ge 4l+r$ and a suitable constant $k(d,r)$:

$$F_l(n)\le e^{16\|J\|}k(d,r)A(l)l^{d-1}F_l(n-4l-r) \tag{2.22}$$

where

$$A(l)=\sup_{V\in\mathcal{F}^0_l}\ \sup_{x\in Q_l\cap\partial^+_r Q_l}\ \sup_{\tau\in\Omega}\ \sum_{z\in\partial^+_r B_{2l}}|\mu^\tau_V(h_x,h_z)| \tag{2.23}$$

Let us in fact assume (2.22). Then it follows immediately that

$$\begin{aligned}F_l(n)&\le\left[k(d,r)e^{16\|J\|}A(l)l^{d-1}\right]^{\lfloor\frac{n}{4l+r}\rfloor}\|F_l\|_\infty\\ &\le\left[k(d,r)e^{16\|J\|}A(l)l^{d-1}\right]^{\lfloor\frac{n}{4l+r}\rfloor}\|h_0\|^2_\infty\end{aligned} \tag{2.24}$$

where $\lfloor\cdot\rfloor$ denotes the integer part.
Let us now assume $SMT(V,l/2,\alpha)$ for all $V\in\mathcal{F}^0_l$. Then $A(l)\le e^{-\frac{\alpha}{2}l}$ for all l large enough and therefore it is possible to find $\bar l(d,r,\alpha)$ and $C(\alpha,d,r,\|J\|)$ such that for all $l\ge\bar l$

$$F_l(n)\le Ce^{-\frac{\alpha}{10}n}$$

Thus the sought exponential decay of $f_l(x)$ follows.

Let us prove (2.22). We fix $n\ge 4l+r$, two sites x,y with $d(x,y)\ge n$, a set $V\in\mathcal{F}_l$ and a boundary configuration τ. Without loss of generality we assume that $V=\cup_{i=1}^n Q_l(x_i)$, $x_i\in l\mathbb{Z}^d$, $x_1=0$ and that $x\in Q_l(x_1)\cup\partial^+_r Q_l(x_1)$. Next we set

$$\begin{aligned}A&=V\cap B_{2l}\\ B&=B_{3l}\setminus B_{2l}\\ C&=V\cap B^c_{3l}\end{aligned}$$

and we observe that by construction $x\in A$, $y\in C$ and that neither h_x nor h_y depend on the spins in B. Therefore, using the DLR equations together with the Markov property of our Gibbsian field and by denoting with $\bar h_x$ the function $h_x-\mu^\tau_V(h_x)$ and similarly for $\bar h_y$, we can write

$$\begin{aligned} |\mu_V^\tau(h_x, h_y)| &= |\mu_V^\tau(\bar h_x \bar h_y)| \\ &= |\mu_V^\tau\big(\mu_C(\bar h_y)\mu_A(\bar h_x)\big)| \\ &\le \|\mu_A(\bar h_x)\|_\infty^\tau \|\mu_C(\bar h_y)\|_\infty^\tau \end{aligned} \tag{2.25}$$

where $\|\cdot\|_\infty^\tau$ means that we take the supremum over all configuration σ that agree with τ outside V. We claim at this point that

$$\begin{aligned} \|\mu_A(\bar h_x)\|_\infty^\tau &\le e^{8\|J\|} A(l) \\ \|\mu_C(\bar h_y)\|_\infty^\tau &\le e^{8\|J\|} \sum_{z\in\partial_r^+ C\cap B} f_l(z-y) \\ &\le e^{8\|J\|} k(d,r) l^{d-1} F_l(n-4l-r) \end{aligned} \tag{2.26}$$

for a suitable constant $k(d,r)$, so that (2.22) follows from (2.25) and the arbitrariness of x, y, V and τ.
In turn (2.26) follows at once from formula (2.18) and the obvious bound $|B_{3l} \setminus B_{2l}| \le k(d,r)l^{d-1}$. $\square$

Remark. In the sequel for any given $\alpha > 0$ we will denote by $\widehat{m}(\alpha)$ the constant $\widehat{m}(d,r,\alpha)$ given in the above proposition.

Remark. With some extra work and using once more the Markov property, one can actually replace the basic assumption "$SMT(V, l/2, \alpha)$ holds for all $V \in \mathcal{F}_l^0$" of proposition 2.9 with "$SMT(Q_l, l/4, \alpha)$ holds".

Remark. It is clear from the proof of lemma 2.10 that if we would have replaced in the hypothesis of the lemma "for all $V \in \mathcal{F}_l^0$" with "for all $V \subset B_{2l}$" we would have derived the exponential decay of the covariances $\mu_V^\tau(h_x, h_y)$, uniformly for all V and τ and not just for all $V \in \mathcal{F}_l$.

It clearly follows from (2.23), (2.24) that in order to get the exponential decay of the function $f_l(x)$ it is enough to assume that $k(d,r)A(l)l^{d-1} < 1$. In particular this is the case if $\sum_{x\in\mathbb{Z}^d} |x|^{(d-1)} f_l(x) < \infty$ since, by the definition of $f_l(x-y)$,

$$A(l) \le \sup_{x\in Q_l} \sum_{z\in\partial_r^+ B_{2l}} f_l(x-z)$$

Let us formalize all that in the next proposition

Proposition 2.11. *Fix $L_0 < \infty$ and assume that for all sets V multiple of Q_{L_0} and all functions f and g on Ω such that $d(\Lambda_f, \Lambda_g) \ge n$ we have*

$$\sup_{\tau\in\Omega} |\mu_V^\tau(f,g)| \le |\Lambda_f||\Lambda_g| \, \|f\|_\infty \|g\|_\infty \phi(n)$$

with $\lim_{n\to\infty} n^{2(d-1)}\phi(n) = 0$. *Then there exists* l *and* m *such that* $SMT(V, l, m)$ *holds for all* V *multiple of* Q_{L_0}.

We conclude this part concerning Gibbsian fields with a last useful result

Proposition 2.12. *For each* $m > 0$ *there exists* $C(d, r, m)$ *such that the following holds. Let* $A \subset\subset \mathbb{Z}^d$, $A_0 \subset A$ *and* $B_0 \subset \partial_r^+ A$. *Let* $\bar{A} = A \cup \partial_r^+ A$ *and assume that*

(i) $md_0 \equiv md(A_0, B_0) \geq \max\{\, C\,,\, 100\|J\|\,,\, 10\,(\log|B_0| + 1)\,\}$

(ii) $SMT(A\backslash A_0,\, d_0 - 2r,\, m)$ *holds*

then for each pair of configurations $\sigma, \tau \in \Omega$ *which agree on* $\partial_r^+ A\backslash B_0$, *we have*

$$\left\| 1 - \frac{d\mu_A^\tau}{d\mu_A^\sigma}\Big|_{A_0} \right\|_\infty \leq e^{-(m/4)d_0} \tag{2.27}$$

Proof. For each $\eta \in \Omega_{A_0}$, consider the event $F_\eta = \{\sigma \in \Omega : \sigma_{A_0} = \eta\}$. Choose a pair of configurations σ, τ which agree on $\partial_r^+ A\backslash B_0$. Then there exists a sequence of interpolating configurations $\gamma_i \in \Omega$ for $i = 1, \ldots, n$ such that $n \leq |B_0|$, γ_{i+1} differs from γ_i at exactly one site, $\gamma_1 = \sigma$ and γ_n agrees with τ on $\partial_r^+ A$. Thus, for each $\eta \in \Omega_{A_0}$, we can write

$$\left| 1 - \frac{\mu_A^\tau(F_\eta)}{\mu_A^\sigma(F_\eta)} \right| = \left| 1 - \prod_{i=2}^n \frac{\mu_A^{\gamma_i}(F_\eta)}{\mu_A^{\gamma_{i-1}}(F_\eta)} \right| \tag{2.28}$$

If we define

$$a = \sup_{\zeta\in\Omega,\, x\in B_0,\, \eta\in\Omega_{A_0}} \left| 1 - \frac{\mu_A^\zeta(F_\eta)}{\mu_A^{\zeta^x}(F_\eta)} \right|$$

then it is easy to check that, if $a \leq \frac{1}{10}$ and $a|B_0| \leq 1$ then the RHS of (2.28) cannot exceed $ea|B_0|$, so if we show that, for instance, $a \leq e^{-(m/2)d_0}$, the proposition follows.

Let then, for $z \in \mathbb{Z}^d$, $g_z = \exp(-\nabla_z H_{A\backslash A_0})$. By lemma 2.8, and the SMT property given in the hypotheses, we find

$$\begin{aligned} a &\leq \sup_{x\in B_0} e^{14\|J\|} \sum_{y\in A_0} e^{2\|J\|} |\Lambda_{g_x}||\Lambda_{g_y}|\, \|g_x\|_\infty \|g_y\|_\infty\, e^{-md(\Lambda_{g_x}, \Lambda_{g_y})} \leq \\ &\leq\ (2r+1)^{2d}\, e^{20\|J\|} \sup_{x\in B_0} \sum_{y\in A_0} e^{-m\,(|x-y|-2r)} \end{aligned}$$

In the second inequality we have used the fact that Λ_{g_x} is contained in a ball of center x and radius r, and the fact that $\|g_x\|_\infty \leq \exp(2\|J\|)$. Finally, using the hypothesis on d_0, we easily get

$$a \leq e^{-(m/2)d_0}$$

□

3. The Glauber Dynamics

In this section we define the Markov process that will be the main object of our investigation. In what follows the interaction J is fixed and, whenever confusion does not arise, it will not appear in our notation.

3.1 The Dynamics in Finite Volume.

The stochastic dynamics we want to study is determined by the Markov generators L_V, $V \subset\subset \mathbb{Z}^d$, defined by

$$(L_V f)(\sigma) = \sum_{x\in V} c_J(x,\sigma)(\nabla_x f)(\sigma) \qquad \sigma \in \Omega \tag{3.1}$$

The nonnegative real quantities $c_J(x,\sigma)$ are the *transition rates* for the process.

The general assumptions on the transition rates are

(H1) *Finite range interactions.* If $\sigma(y) = \sigma'(y)$ for all y such that $d(x,y) \le r$, then $c_J(x,\sigma) = c_J(x,\sigma')$

(H2) *Detailed balance.* For all $\sigma \in \Omega$ and $x \in \mathbb{Z}^d$,

$$\exp\left[-H_{\{x\}}(\sigma)\right] c_J(x,\sigma) = \exp\left[-H_{\{x\}}(\sigma^x)\right] c_J(x,\sigma^x) \tag{3.2}$$

(H3) *Positivity and boundedness.* There exist positive real numbers c_m and c_M such that

$$0 < c_m \le \inf_{x,\sigma} c_J(x,\sigma) \quad \text{and} \quad \sup_{x,\sigma} c_J(x,\sigma) \le c_M \tag{3.3}$$

(H4) *Translation invariance.* If, for some $k \in \mathbb{Z}^d$, $\sigma'(y) = \sigma(y+k)$ for all $y \in \mathbb{Z}^d$ then $c_J(x,\sigma') = c_J(x+k,\sigma)$ for all $x \in \mathbb{Z}^d$

Three cases one may want to keep in mind are

$$c_J(x,\sigma) = \min\left\{e^{-(\nabla_x H_{\{x\}})(\sigma)}\,,\,1\right\} \tag{3.4}$$

$$c_J(x,\sigma) = \mu^\sigma_{\{x\}}(\sigma^x) = \left[1 + e^{(\nabla_x H_{\{x\}})(\sigma)}\right]^{-1} \tag{3.5}$$

$$c_J(x,\sigma) = \frac{1}{2}\left[1 + e^{-(\nabla_x H_{\{x\}})(\sigma)}\right] \tag{3.6}$$

The first two examples correspond to the so called Metropolis and heat–bath dynamics respectively.

We denote by L_V^τ the operator L_V acting on $L^2(\Omega, d\mu_V^\tau)$ (this amounts to choose τ as the boundary condition). Assumptions (1), (2) and (3) guarantee that there exists a unique Markov process whose generator is L_V^τ, and whose semigroup we denote by $\{T_V^\tau(t)\}_{t\ge 0}$. L_V^τ is a bounded selfadjoint operator on $L^2(\Omega, d\mu_V^\tau)$. The process has a unique invariant measure given by μ_V^τ which is moreover *reversible* for the process.

3.2. Infinite Volume Dynamics.

Let μ be a Gibbs measure for the interaction J. Since the transition rates are bounded and of finite range, the infinite volume generator L obtained by choosing $V = \mathbb{Z}^d$ in (3.1) is well defined on the set of functions f such that $|||f|||$ is finite. We can then take the closure of L in $C(\Omega)$, the metric space of all continuous functions on Ω with the sup–distance, and get a Markov generator (see, for instance Theorem 3.9 in Chapter I in [L]) or take the closure in $L^2(\Omega, d\mu)$ and get a selfadjoint Markov generator in $L^2(\Omega, \mu)$ (see Theorem 4.1 in Chapter IV of [L]) that will be denoted by L. In the latter case, since the generator is self–adjoint on $L^2(\Omega, d\mu)$ the associated Markov process is reversible w.r.t. the Gibbs measure μ. We conclude with a general result relating the set of invariant measures of the infinite volume Glauber dynamics with the set of Gibbs meaures for the given interaction J (see Theorems 2.14, 2.15, 2.16, 5.12 and 5.14 in [L])

Theorem 3.1. *Assume (H1)...(H4). Then*

a) *If $d = 1$ or $d = 2$ the set of invariant measures for the above Markov process coincides with the set of Gibbs measures $\mathcal{G}$*

b) *If $d \geq 3$ then*

   i) *any invariant measure which is also translation invariant is a Gibbs measure*

   ii) *the set of Gibbs measures coincides with the set of reversible invariant measures*

   iii) *If the process is attractive (see paragraph 3.4 below) then the process is ergodic if and only if there is no phase transition*

3.3. Graphical Construction.

We describe here a very convenient way, introduced in [Sch], to realize simultaneously on the same probability space all Markov processes whose generator is L_V^τ, as the initial and the boundary conditions vary in Ω. As a byproduct of the construction we will get, in a rather simple way, a key result which shows that "information" propagates through the system at finite speed.

We associate to each site $x \in \mathbb{Z}^d$ two independent Poisson processes, each one with rate c_M, and we assume independence as x varies in $\mathbb{Z}^d$. We denote the successive arrival after time $t = 0$ of the two processes by $\{t^+_{x,n}\}_{n=1,2\ldots}$ and $\{t^-_{x,n}\}_{n=1,2\ldots}$. We say that at time t there has been an *upmark* at x if $t^+_{x,n} = t$ for some n and similarly for a *downmark*. Notice that with probability one all the arrival times are different. Next we associate to each arrival time $t^*_{x,n}$, where $*$ stands for either $+$ or $-$, a random variable $U^*_{x,n}$ uniformly distributed in the interval $[0, 1]$. We assume that these random variables are mutually independent and independent from the Poisson processes. This completes the construction of the probability space. The corresponding probability measure and expectation are denoted by $\mathbb{P}$ and $\mathbb{E}$ respectively.

Given now $V \subset\subset \mathbb{Z}^d$, a boundary condition $\tau \in \Omega$ and an initial condition $\eta \in \Omega_V$, we construct a Markov process $\{\sigma_t^{V,J,\tau,\eta}\}_{t\geq 0}$ on the above probability space according to the following updating rules. Let us suppose that $t = t^*_{x,n}$ for some $x \in V$ and $n \in \mathbb{Z}_+$ and that the configuration immmediately before t was σ. Then:

i) The spin $\sigma(y)$ with $y \neq x$ does not change

ii) If $\sigma(x) = -1$ and the mark was a downmark then $\sigma(x)$ does not change. Similarly if $\sigma(x) = +1$ and the mark was an upmark

iii) If $\sigma(x) = -1$ and the mark was an upmark then we flip $\sigma(x)$ if and only if $c_J^\tau(x, \sigma) \geq U^+_{x,n} c_M$. Similarly if $\sigma(x) = +1$ and the mark was a downmark.

One can easily check that the above continuous Markov chain on Ω_V has indeed the correct flip rates $c_J^\tau(x,\sigma)$ so that the above construction represents a global coupling among all processes generated by L_V^τ as the boundary condition τ and the initial condition η vary. In order to investigate how the process $\sigma_t^{V,J,\tau,\eta}(x)$ at site x is affected by a faraway change either in the boundary condition τ or in the initial configuration η, let us introduce, for any given integer l, the event $E(x,t,l)$ that there exists a collection of sites $\{x_0,\dots x_n\}$ and times $\{t_0,\dots t_n\}$ such that

i) $0 < t_0 < \dots < t_n \le t$ and at each time t_i there is a mark at site x_i

ii) $d(x_0,x) \ge l$, $d(x_i,x_{i+1}) \le r$ and $x_n = x$, where r denotes the range of the local flip rates.

By construction, if the event $E(x,t,l)^c$ occurred, then the spin $\sigma_t^{V,J,\tau,\eta}(x)$ does not depend on the spin $\eta(y)$ if $d(x,y) > l$ and similarly for $\tau(y)$. The probability of the event $E(x,t,l)$ can be estimated from above by

$$\mathbb{P}(E(x,t,l)) \le \sum_{n\ge[l/r]} (2r+1)^{dn}\, \mathbb{P}(Z\ge n) \tag{3.7}$$

where Z is a Poisson random variable with mean $2tc_M$. An elementary calculation shows that

$$\mathbb{P}(Z\ge n) \le e^{-n(\log(n/2c_M t)-1)} \tag{3.8}$$

As a consequence we get that the r.h.s of (3.7), as a function of l, for any given fixed time t tends to zero, as $l\to\infty$, faster than an exponential. Moreover there exists a constant k_0, depending only on the dimension d, the range r and c_M, such that the r.h.s of (3.7) is smaller than e^{-t} for all $l \ge k_0 t$.

Let us formalize all what we said in a lemma.

Lemma 3.2. *There exists a constant k_0 depending on d, r and c_M, such that for all local function f and all $t\ge 0$*

i) for all pairs $V_1 \subset\subset \mathbb{Z}^d$ and $V_2 \subset\subset \mathbb{Z}^d$, with $d(V_i^c,\Lambda_f) \ge k_0 t$, $i=1,2$,

$$\sup_{\tau_1,\tau_2\in\Omega} \|T_{V_1}^{\tau_1}(t)f - T_{V_2}^{\tau_2}(t)f\|_\infty \le |||f|||\, e^{-t}$$

ii) for all $V \subset\subset \mathbb{Z}^d$ with $d(V^c,\Lambda_f) \ge k_0 t$ and all $\sigma_1,\sigma_2 \in \Omega_V$, with $\sigma_1(x)=\sigma_2(x)$ for all x such that $d(x,\Lambda_f)\le k_0 t$,

$$\sup_{\tau\in\Omega} |T_V^\tau(t)f(\sigma_1) - T_V^\tau(t)f(\sigma_2)| \le |||f|||\, e^{-t}$$

Proof. We use the global coupling defined above. More precisely we write

$$T_{V_1}^{\tau_1}(t)f(\sigma_1) - T_{V_2}^{\tau_2}(t)f(\sigma_2) = \mathbb{E}\big(f(\sigma_t^{V_1,J,\tau_1,\sigma_1}) - f(\sigma_t^{V_2,J,\tau_2,\sigma_2})\big) \tag{3.9}$$

and notice that, by a simple telescoping argument,

$$\sup_{\sigma,\eta} |f(\sigma)-f(\eta)| \le \sum_x \|\nabla_x f\|_\infty \chi\big(\sigma(x)\neq\eta(x)\big)$$

Therefore the absolute value of the r.h.s of (3.9) can be bounded from above by

$$|||f||| \sup_{x\in\Lambda_f} \mathbb{P}\big(\sigma_t^{V_1,J,\tau_1,\sigma_1}(x) \neq \sigma_t^{V_2,J,\tau_2,\sigma_2}(x)\big) \tag{3.10}$$

Using the hypotheses of the lemma we get that each term in the sum appearing in the r.h.s of (3.10) can be bounded from above by

$$\mathbb{P}\big(E(x,t,l)\big)$$

with $l \geq k_0 t$. Thus the lemma follows from (3.7) and (3.8). $\square$

We conclude by observing that, in our particular framework, one could construct directly the infinite volume dynamics using the global coupling constructed above together with lemma 3.2, without appealing to the much more general results of [L].

3.4 Attractive Dynamics for Ferromagnetic Interactions.

Let us introduce a partial order on the configuration space Ω by saying that $\sigma \leq \eta$ iff $\sigma(x) \leq \eta(x)\ \forall x \in \mathbb{Z}^d$. A function $f : \Omega_V \mapsto \mathbb{R}$ is called *monotone increasing (decreasing)* if $\sigma \leq \sigma'$ implies $f(\sigma) \leq f(\sigma')$ $(f(\sigma) \geq f(\sigma'))$. An event is called *positive* (*negative*) if its characteristic function is increasing (decreasing). Given two probability measures μ, μ' on Ω_V we write $\mu \leq \mu'$ if $\mu(f) \leq \mu'(f)$ for all increasing functions f (with $\mu(f)$ we denote the expectation with respect μ).

Then the dynamics defined by the transition rates $c_J(x,\sigma)$ is said to be *attractive* if $\sigma \leq \eta$ and $\sigma(x) = \eta(x)$ imply that

$$\eta(x)c_J(x,\eta) \leq \sigma(x)c_J(x,\sigma)$$

An example of attractive dynamics are the Metropolis and Heat Bath dynamics for a (generalized) Ising model, in which the interaction $J = \{J_A\}_{A\in F}$ is a one and two-body ferromagnetic interaction of the form

$$J_A = \begin{cases} J \geq 0 & \text{if } A = \{x,y\} \text{ with } d_2(x,y) \leq r \\ h & \text{if } A = \{x\} \\ 0 & \text{otherwise} \end{cases}$$

An very useful consequence of attractivity is the following *monotonicity* property in the boundary condition and in the initial configuration of the Markov process $\sigma_t^{V,J,\tau,\eta}$ given by the graphical construction

$$\mathbb{P}\big(\sigma_t^{V,J,\tau_1,\eta_1} \leq \sigma_t^{V,J,\tau_2,\eta_2}\big) = 1 \quad \forall\, \tau_1 \leq \tau_2 \text{ and } \forall\, \eta_1 \leq \eta_2 \tag{3.11}$$

Such property follows immediately from the updating rules which govern the evolution of the Markov process $\sigma_t^{V,J,\tau,\eta}$. In turn (3.11) implies

(1) If f is an increasing function on Ω_V then $T_V^\tau(t)f$ is also increasing for all $t \geq 0$
(2) If ρ_1, ρ_2 are two probability measures on Ω_V such that $\rho_1 \leq \rho_2$ then
$\rho_1 T_V^\tau(t) \leq \rho_2 T_V^\tau(t)$ for all $t \geq 0$

Moreover (3.11) allows us to define a *standard* coupling between two finite volume Gibbs measures which preserve the order of the b.c. Take in fact two boundary conditions $\tau_1 \leq \tau_2$ and let $\nu_V^{\tau_1,\tau_2}$ be the unique invariant measure of the coupled process $(\sigma_t^{V,J,\tau_1}, \sigma_t^{V,J,\tau_2})$ on the set $\{ (\sigma,\eta) :\ \sigma \le \eta \}$. Then we have

(1) $\nu_V^{\tau_1,\tau_2}\{(\sigma_1,\sigma_2) : \sigma_1 = \sigma\} = \mu_V^{\tau_1}(\sigma)$ for all $\sigma \in \Omega_V$
(2) $\nu_V^{\tau_1,\tau_2}\{(\sigma_1,\sigma_2) : \sigma_2 = \sigma\} = \mu_V^{\tau_2}(\sigma)$ for all $\sigma \in \Omega_V$
(3) $\nu_V^{\tau_1,\tau_2}\{(\sigma_1,\sigma_2) : \sigma_1 \le \sigma_2\} = 1$ so that $\mu_V^{\tau_1} \le \mu_V^{\tau_2}$.

In particular, from (3) it follows that any finite volume Gibbs measure μ_V^τ is an FKG–measure i.e. satisfies (see chapter II.2 of [L])

$$\mu_V^\tau(fg) \ge \mu_V^\tau(f)\mu_V^\tau(g)$$

for any increasing functions f and g.

3.5 Spectral Gap and Logarithmic Sobolev Constant

A fundamental quantity associated with the dynamics of a reversible system is the gap of the generator, *i.e.*

$$\mathrm{gap}(L_V^\tau) = \inf \mathrm{spec}\,(-L_V^\tau \restriction \mathbb{1}^\perp)$$

where $\mathbb{1}^\perp$ is the subspace of $L^2(\Omega, d\mu_V^\tau)$ orthogonal to the constant functions. The gap can also be characterized as

$$\mathrm{gap}(L_V^\tau) = \inf_{f\in L^2(\Omega,d\mu_V^\tau)} \frac{\mathcal{E}_V^\tau(f,f)}{\mathrm{Var}_V^\tau(f)} \tag{3.12}$$

where $\mathcal{E}$ is the Dirichlet form associated with the generator L,

$$\mathcal{E}_V^\tau(f,f) = \frac{1}{2}\sum_{\sigma\in\Omega_V}\sum_{x\in V}\mu_V^\tau(\sigma)\,c(x,\sigma)\,[(\nabla_x f)(\sigma)]^2 \tag{3.13}$$

and Var_V^τ is the variance relative to the probability measure μ_V^τ. When the transition rates are chosen as in (3.6), it is easy to verify that the Dirichlet form takes a particularly simple form

$$\mathcal{E}_V^\tau(f,f) = \frac{1}{2}\,\mu_V^\tau(|\nabla_V f|^2) \tag{3.14}$$

By simple spectral theory, for any $f \in L^2(\Omega, d\mu_V^\tau)$, one gets the following bound on the Markov semigroup $T_V^\tau(t)$ in terms of the spectral gap

$$\|T_V^\tau(t)f - \mu_V^\tau(f)\|^2_{2,\mu_V^\tau} \ \le\ e^{-2\,\mathrm{gap}(L_V^\tau)t}\,\|f\|^2_{2,\mu_V^\tau}$$

where $\|f\|_{2,\mu_V^\tau}$ denotes the $L^2(\Omega, d\mu_V^\tau)$ norm.

Quite bad is the analogous bound in the $\|\cdot\|_\infty$ norm. In finite volume and for general f, the best one can get using only the spectral gap is

$$\begin{aligned}\|T_V^\tau(t)f - \mu_V^\tau(f)\|_\infty &\le \big[\inf_\sigma \mu_V^\tau(\sigma)\big]^{-1/2}\|T_V^\tau(t)f - \mu_V^\tau(f)\|_{2,\mu_V^\tau}\\ &\le \big[\inf_\sigma \mu_V^\tau(\sigma)\big]^{-1/2}e^{-\,\mathrm{gap}(L_V^\tau)t}\,\|f\|_{2,\mu_V^\tau}\end{aligned} \tag{3.15}$$

Notice that the factor $\left[\inf_\sigma \mu_V^\tau(\sigma)\right]^{-1/2}$ is usually exponentially large in the size of V and therefore the above bound becomes meaningful only for extremely large times. For local functions the above estimate can be slightly improved. Let in fact f be a local function. Then, using lemma 3.2, one can safely replace the original set V by a "ball" of radius $k_0 t$ around the support of f, Λ_f (see (3.20)...(3.21) below for details). However the prefactor multiplying the negative exponential is still very large, of the order of $\exp(Ct^d)$ so that the resulting bound is practically useless.

Much more powerful are the estimates obtained through "hypercontractive bounds" on the Markov semigroup. For this purpose, we define the *logarithmic Sobolev constant* $c_s(L_V^\tau)$ associated with the generator L_V^τ as the infimum over all c such that, for all positive functions f

$$\mu_V^\tau(f^2 \log f) \leq c\,\mathcal{E}_V^\tau(f,f) + \mu_V^\tau(f^2)\log\sqrt{\mu_V^\tau(f^2)} \tag{3.16}$$

More generally, given a probability measure ν on Ω, we denote by $c_s(\nu)$ the smallest constant c such that, for all positive local functions f

$$\nu(f^2 \log f) \leq \frac{c}{2}\,\nu(|\nabla_V f|^2) + \nu(f^2)\log\sqrt{\nu(f^2)} \tag{3.17}$$

Notice that if we take $\nu = \mu_V^\tau$ then $c_s(\mu_V^\tau)$ coincides with the logarithmic Sobolev constant associated with the generator L_V^τ corresponding to the rates (3.6). Even if the transition rates are different from those given in (3.6) we can estimate $c_s(L_V^\tau)$ in terms of $c_s(\mu_V^\tau)$ as follows:

$$\frac{c_m}{c_M}\,c_s(L_V^\tau) \leq c_s(\mu_V^\tau) \leq \frac{c_M}{c_m}\,c_s(L_V^\tau) \tag{3.18}$$

Finally we set

$$\begin{aligned} c_s(L_V) &= \sup_\tau c_s(L_V^\tau) \\ c_s(\mu_V) &= \sup_\tau c_s(\mu_V^\tau) \end{aligned} \tag{3.19}$$

The role of the logarithmic Sobolev constant in the analysis of the exponential ergodicity of the dynamics is clarified by the following theorem

Theorem 3.3. *Assume*

$$\sup_L \sup_\tau c_s(L_{B_L}^\tau) < c_s^\infty < +\infty$$

Then:

a) *there exists a positive constant m and for any local function f there exists a constant C_f such that*

$$\sup_L \sup_\tau \|T_{B_L}^\tau(t)f - \mu_{B_L}^\tau(f)\|_\infty \leq C_f e^{-mt}$$

b) *There exist positive α and l_0 such that $SMT(B_L, l_0, \alpha)$ holds for all L*

c) The infinite volume dynamics is exponentially ergodic in the $\|\cdot\|_\infty$–norm and there exists a unique Gibbs measure μ with covariances decaying exponentially fast.

Proof.
a) Let us fix a cube B_L, a boundary condition $\tau \in \Omega$ and a local function f such that $\mu^\tau_{B_L}(f) = 0$. Without loss of generality we assume that $0 \in \Lambda_f$. For any $t \geq 0$ let also $l_t = \min\{l : d(\Lambda_f, B_l^c) \geq k_0 t\}$ where k_0 is the constant appearing in lemma 3.2 and let $\Lambda_t \equiv B_{l_t \wedge L}$. Then we write

$$\|T^\tau_{B_L}(t)f\|_\infty \leq \|T^\tau_{B_L}(t)f - T^\tau_{\Lambda_t}(f)\|_\infty + \|T^\tau_{\Lambda_t}(t)f\|_\infty \tag{3.20}$$

The first term in the r.h.s of (3.20) is smaller than $|||f|||\, e^{-t}$ because of lemma 3.2. In order to estimate from above the second term we use the following trivial bound valid for any $q \geq 1$, any $V \subset\subset \mathbb{Z}^d$, any positive probability measure ν on Ω_V and any function f

$$\|f\|_\infty \leq \left[\inf_\sigma \nu(\sigma)\right]^{-\frac{1}{q}} \|f\|_{q,\nu} \tag{3.21}$$

where $\|f\|_{q,\nu}$ denotes the $L^q(\Omega_V, d\nu)$ norm.

Given $\varepsilon \in (0,1)$, we take ν as the marginal on Ω_{Λ_t} of the Gibbs measure $\mu^\tau_{B_L}$ and apply (3.21) to $T^\tau_{\Lambda_t}(t)f$. Using once more lemma 3.2 and the trivial bound $\inf_\sigma \mu^\tau_V(\sigma) \geq e^{-2\|J\|_\infty |V|}$, we get

$$\begin{aligned}
\|T^\tau_{\Lambda_t}(t)f\|_\infty &\leq A(t,q)\|T^\tau_{\Lambda_t}(t)f\|_{q,\mu^\tau_{B_L}} \\
&\leq A(t,q)\|T^\tau_{B_L}(t)f\|_{q,\mu^\tau_{B_L}} + A(t,q)|||f|||\, e^{-t} \\
&= A(t,q)\|T^\tau_{B_L}((1-\varepsilon)t)f_t\|_{q,\mu^\tau_{B_L}} + A(t,q)|||f|||\, e^{-t}
\end{aligned} \tag{3.22}$$

where

$$A(t,q) = e^{2\|J\|_\infty |\Lambda_t|/q} \qquad \text{and} \qquad f_t = T^\tau_{B_L}(\varepsilon t)f$$

Notice that so far we never used the key assumption of the theorem, namely boundedness of the logarithmic Sobolev constant. It is at this stage that such hypothesis becomes crucial. Thanks in fact to Gross's integration lemma (see e.g [DeSt]) we have that the Markov semigroup $T^\tau_V(t)$ maps $L^2(\Omega, d\mu^\tau_V)$ contractively into $L^q(\Omega, d\mu^\tau_V)$ for any $2 \leq q \leq 1 + e^{t/c_s(L^\tau_V)}$. Thus

$$\begin{aligned}
A(t,q)\|T^\tau_{B_L}((1-\varepsilon)t)f_t\|_{q,\mu^\tau_{B_L}} &\leq A(t,q)\|f_t\|_{2,\mu^\tau_{B_L}} \\
&\leq A(t,q)\|f\|_{2,\mu^\tau_{B_L}}\, e^{-\operatorname{gap}(L^\tau_{B_L})\varepsilon t}
\end{aligned} \tag{3.23}$$

for all $2 \leq q \leq 1 + e^{(1-\varepsilon)t/c_s(L^\tau_{B_L})}$. Since $\sup_\tau \sup_L c_s(L^\tau_{B_L}) < c_s^\infty < +\infty$ we can apply the above inequality with $q = q(t) = 1 + e^{(1-\varepsilon)t/c_s^\infty}$ and get that $\lim_{t\to\infty} A(t, q(t)) = 1$. In order to conclude the proof of the first part of the theorem it is enough to recall a well known result (see e.g [SC]) that says that

$$\operatorname{gap}(L^\tau_V) \geq c_s(L^\tau_V)^{-1}$$

b) To prove exponential decay of covariances we proceed as in Lemma 3.1 of [SZ]. Let fix a box B_L and a boundary condition τ and let f and g be two local functions with zero $\mu^\tau_{B_L}-$mean and supports Λ_f, Λ_g contained in B_L. Let $l = \lfloor d(\Lambda_f, \Lambda_g)/2 \rfloor$ and let $t = \frac{l}{k_0}$, k_0 being the constant appearing in lemma 3.2. Let also $\Lambda_1 = \{x \in B_L : d(x, \Lambda_f) \le l\}$ and take $\Lambda_2 = B_L \setminus \Lambda_1$. Then, using reversibility, we write

$$\begin{aligned} |\mu^\tau_{B_L}(fg)| &= |\mu^\tau_{B_L}\big(T^\tau_{B_L}(t)(fg)\big)| \\ &\le |\mu^\tau_{B_L}\big(g\, T^\tau_{B_L}(2t)f\big)| + \|T^\tau_{B_L}(t)(fg) - T^\tau_{B_L}(t)f\, T^\tau_{B_L}(t)g\|_\infty \end{aligned} \tag{3.24}$$

The first term in the r.h.s of (3.24) is bounded from above by

$$\|g\|_{2,\mu^\tau_{B_L}}\ \|f\|_{2,\mu^\tau_{B_L}}\, e^{-t\,\mathrm{gap}(L^\tau_{B_L})} \tag{3.25}$$

Since $\mathrm{gap}(L^\tau_{B_L}) \ge 1/c_s^\infty$ and $\|g\|_{2,\mu^\tau_{B_L}} \le \|g\|_\infty$ we get, remembering the definition of t, that (3.25) is bounded from above by

$$\|g\|_\infty\ \|f\|_\infty e^{-\frac{l}{k_0 c_s^\infty}} \tag{3.26}$$

The second term in the r.h.s of (3.24), thanks to lemma approx and our choice of time t is bounded from above by

$$\begin{aligned} \|T^\tau_{B_L}(t)fg - T^\tau_{\Lambda_1}(t)f\, T^\tau_{\Lambda_2}(t)g\|_\infty &+ \|T^\tau_{B_L}(t)f - T^\tau_{\Lambda_1}(t)f\|_\infty \|g\|_\infty \\ &+ \|T^\tau_{B_L}(t)g - T^\tau_{\Lambda_2}(t)g\|_\infty \|f\|_\infty \\ &\le \big(|||fg||| + \|g\|_\infty |||f||| + \|f\|_\infty |||g|||\big)e^{-t} \\ &\le 4\|g\|_\infty\ \|f\|_\infty |\Lambda_f||\Lambda_g| e^{-\frac{l}{k_0}} \end{aligned} \tag{3.27}$$

Take now l_0 such that

$$e^{-\frac{l_0}{2k_0 c_s^\infty}} + 4e^{-\frac{l_0}{2k_0}} \le 1$$

and set $\alpha = \min[(2k_0 c_s^\infty)^{-1}, (2k_0)^{-1}]$. With this choice and using (3.26), (3.27) the r.h.s of (3.24) is bounded from above by

$$\|g\|_\infty\ \|f\|_\infty |\Lambda_f||\Lambda_g| e^{-\alpha l}$$

if $l \ge l_0$. Thus $SMT(B_L, l_0, \alpha)$ follows.

c) To prove exponential ergodicity of the infinite volume dynamics we fix a local function f with $0 \in \Lambda_f$ and set, for any time t, $\Lambda_t = B_{l_t}$ where l_t was defined at the beginning of the proof. Then, thanks once more to lemma 3.2, for any $\tau \in \Omega$ we have

$$\begin{aligned} \sup_{\sigma,\eta} |T(t)f(\sigma) - T(t)f(\eta)| &\le \sup_{\sigma,\eta} |T^\tau_{\Lambda_t}(t)f(\sigma) - T^\tau_{\Lambda_t}(t)f(\eta)| + 2|||f|||e^{-t} \\ &\le 2C_f e^{-mt} + 2|||f|||e^{-t} \end{aligned}$$

where in the last step we used the bound (3.23). Thus the infinite volume dynamics is exponentially ergodic and therefore (see theorem 3.1) there in no phase transition. □

Remark. Notice that, in the proof of the exponential decay of covariances, we could have assumed the (apparently) weaker condition

$$\inf_L \inf_\tau \operatorname{gap}(L^\tau_{B_L}) > 0$$

It is remarkable that such a weaker hypotheses implies the stronger one used in the theorem above (see [MO1], [MO2] and [SZ]). This issue will be illustrated in the next section (see theorem 4.6).

Remark. It is worthwhile to observe that exponential ergodicity in the $\|\cdot\|_\infty$−norm for all local functions f, *i.e.*

$$\|T(t)f - \mu(f)\|_\infty \leq C_f e^{-mt}$$

for some finite constant C_f and some $m > 0$ independent of f, implies that the spectral gap of the infinite volume generator L on $L^2(\Omega, d\mu)$ is larger or equal to m. Here μ denotes the unique invariant measure of the Glauber dynamics and therefore the unique Gibbs state for the interaction J.

Let us in fact assume exponential ergodicity in the $\|\cdot\|_\infty$−norm with rate $m > 0$ for all local functions and let $\delta \in (0,1)$. Let also $P_{[0,\delta m]}$ be the spectral projection of the interval $[0, \delta m]$ associated to the selfadjoint generator L. Then necessarily $P_{[0,\delta m]} f = 0$ for all local functions f with zero μ−mean. If not we would get the contradiction

$$\|P_{[0,\delta m]} f\|_{2,\mu} e^{-\delta m t} \leq \|T(t)f\|_{2,\mu} \leq C_f e^{-mt} \qquad \forall\, t \geq 0$$

for any local function f with zero μ−mean. To finish the argument it is sufficient to observe that, since the continuous functions are dense in $L^2(\Omega, d\mu)$, for any $f \in L^2(\Omega, d\mu)$ there exists a sequence $f^{(n)}$ of local functions such that $\limsup_{n\to\infty} \|f - f^{(n)}\|_{2,\mu} = 0$.

3.6 From Single Spin Dynamics to Block Dynamics.

We will also consider a more general version of the finite volume dynamics discussed so far in which more than one spin can flip at once. Let $\mathcal{D} = \{V_1, \ldots, V_n\}$ be an arbitrary collection of finite sets $V_i \in \mathbb{F}$ and let $V = \cup_i V_i$. Then we will denote by *block dynamics* with blocks $\{V_1, \ldots, V_n\}$ the continuos time Markov chain in which each block waits an exponential time of mean one and then the configuration inside the block is replaced by a new configuration distributed according to the Gibbs measure of the block given the previous configuration outside the block. More precisely, the generator of the Markov process corresponding to $\mathcal{D}$ is defined as

$$L^\tau_{\mathcal{D}} f = \sum_{i=1}^n (\mu^\tau_{V_i}(f) - f) \tag{3.28}$$

From the DLR condition (2.2) it follows that $L^\tau_{\mathcal{D}}$ is self–adjoint on $L^2(\Omega, d\mu^\tau_V)$, *i.e.* the block dynamics is reversible w.r.t. the Gibbs measure μ^τ_V, so that μ^τ_V represents its unique (in finite volume) invariant measure.

3.7 General Results on the Spectral Gap

In this section we collect some technical results to be used in the next sections. Although most of the results presented here are rather simple and some of them can actually be found in the literature, we thought however useful, also for future purposes, to put them together in a sort of primitive tool–box for the subject.

We begin by proving a simple, but important result relating the spectral gap for the single–site Glauber dynamics to the spectral gap of a block–dynamics defined on the same set V. This simple, but in our approach very important result can be roughly understood as follows. The relaxation time ($\equiv$ inverse of the spectral gap) of a single–site Glauber dynamics for a given finite volume Gibbs measure is not larger than the largest among the relaxation times of the same dynamics restricted to each of the blocks of some block–dynamics for the same Gibbs measure times the relaxation time of the block–dynamics itself. More precisely

Proposition 3.4. *Let $\mathcal{D} = \{V_1, \ldots, V_n\}$ be an arbitrary collection of finite sets $V_i \in \mathbb{F}$ and let $V = \cup_i V_i$. For any given boundary condition $\tau \in \Omega$, let L^τ_V be given by (3.1) with transition rates satisfying $(H1), (H2), (H3)$ and let $L^\tau_{\mathcal{D}}$ be given by (3.28). Then for any $f \in L^2(\Omega, d\mu^\tau_V)$ we have*

$$\begin{aligned} &\mathrm{Var}^\tau_V(f) \le \\ &\left(\mathrm{gap}(L^\tau_{\mathcal{D}})^{-1} \inf_i \inf_{\varphi\in\Omega} \mathrm{gap}(L^\varphi_{V_i})\right)^{-1} \frac{1}{2} \sum_{\sigma\in\Omega} \mu^\tau_V(\sigma) \sum_{x\in V} N_x c_J(x,\sigma)\,[f(\sigma^x) - f(\sigma)]^2 \end{aligned} \tag{3.29}$$

where $N_x = \#\{i : V_i \ni x\}$. In particular

$$\mathrm{gap}(L^\tau_V) \ge \mathrm{gap}(L^\tau_{\mathcal{D}}) \inf_i \inf_{\varphi\in\Omega} \mathrm{gap}(L^\varphi_{V_i}) \left(\sup_{x\in V} \#\{i : V_i \ni x\}\right)^{-1} \tag{3.30}$$

Proof. Let

$$g = \inf_i \inf_{\varphi\in\Omega} \mathrm{gap}(L^\varphi_{V_i})$$

Thanks to (3.12), (3.30) follows immediately from (3.29). In turn (3.29) is proven if we can show that

$$\mathcal{E}^\tau_{\mathcal{D}}(f,f) \le g^{-1} \frac{1}{2} \sum_{\sigma\in\Omega} \mu^\tau_V(\sigma) \sum_{x\in V} \#\{i : V_i \ni x\} c_J(x,\sigma)\,[f(\sigma^x) - f(\sigma)]^2 \tag{3.31}$$

for all $f \in L^2(\Omega, d\mu^\tau_V)$. But, using (3.28), we find

$$\begin{aligned} \mathcal{E}^\tau_{\mathcal{D}}(f,f) &= \sum_{\sigma'\in\Omega} \mu^\tau_V(\sigma') \sum_i \mathrm{Var}^{\sigma'}_{V_i}(f) \\ &\le g^{-1} \sum_{\sigma'\in\Omega} \mu^\tau_V(\sigma') \sum_i \mathcal{E}^{\sigma'}_{V_i}(f,f) \end{aligned} \tag{3.32}$$

On the other side, again because of (2.2),

$$\sum_{\sigma'\in\Omega}\mu_V^\tau(\sigma')\sum_i \mathcal{E}_{V_i}^{\sigma'}(f,f) =$$

$$= \frac{1}{2}\sum_{\sigma'\in\Omega}\mu_V^\tau(\sigma')\sum_i\sum_{\sigma\in\Omega}\mu_{V_i}^{\sigma'}(\sigma)\sum_{x\in V_i}c_J(x,\sigma)\,[f(\sigma^x)-f(\sigma)]^2$$

$$\leq \frac{1}{2}\sum_{\sigma\in\Omega}\mu_V^\tau(\sigma)\sum_{x\in V}\#\{i : V_i \ni x\}c_J(x,\sigma)\,[f(\sigma^x)-f(\sigma)]^2$$

(3.31) and the proposition follows. $\square$

Next we provide three lower bounds on the spectral gap of the block dynamics with just two blocks.

Proposition 3.5. *Let $V \subset\subset \mathbb{Z}^d$, and let A, B be two (possibly intersecting) subsets of V such that $V = A\cup B$. Let $\mathcal{D} = \{A, B\}$. Assume that*

$$\sup_{\tau\in\Omega}\Big\|1-\frac{d\mu_A^\tau}{d\mu_V^\tau}\Big|_{\partial_r^+ B}\Big\|_\infty \leq \varepsilon < 1 \tag{3.33}$$

Then the gap for the block dynamics on $\mathcal{D}$ satisfies

$$\inf_{\tau\in\Omega}\operatorname{gap}(L_{\mathcal{D}}^\tau) \geq 1-\sqrt{\varepsilon}$$

Proof. The action of the semigroup $T_{\mathcal{D}}(t)$ associated to the block dynamics is given by

$$T_{\mathcal{D}}(t)f = \sum_{n=0}^\infty \frac{t^n}{n!}(L_{\mathcal{D}})^n f$$

Using the explicit expression for $L_{\mathcal{D}}$ and some elementary combinatorics, it is not difficult to show that

$$T_{\mathcal{D}}(t)f = \sum_{n=0}^\infty \frac{(2t)^n}{n!}e^{-2t}\frac{1}{2^n}\sum_{X\in\{A,B\}^n}\mu_{X_1}\cdots\mu_{X_n}(f) \tag{3.34}$$

Since $(\mu_A)^2 = \mu_A$ (and similarly for B) the last summation (over X) in (3.34) can be written as

$$\sum_{k=0}^{n-1}\binom{n-1}{k}(\widehat{A}_{k+1}+\widehat{B}_{k+1})f \tag{3.35}$$

where

$$\widehat{A}_k = (\mu_A\circ\mu_B)^{\lfloor k/2\rfloor}\circ\mu_A^{k-2\lfloor k/2\rfloor} \qquad \widehat{B}_k = (\mu_B\circ\mu_A)^{\lfloor k/2\rfloor}\circ\mu_B^{k-2\lfloor k/2\rfloor}$$

If now g is an arbitrary bounded measurable function on Ω, such that $\mu_V(g) = 0$, we get

$$\|\mu_A\mu_B\mu_A g\|_\infty \leq \|\mu_V\mu_B\mu_A g\|_\infty + \|\mu_V\mu_B\mu_A g - \mu_A\mu_B\mu_A g\|_\infty \tag{3.36}$$

By the DLR property (2.2) the first term on the RHS of (3.36) is equal to $\mu_V(g) = 0$. Furthermore, since the interaction has range r, the function $h \equiv \mu_B\mu_A g$ is $\mathcal{F}_{V^c \cup \partial_r^+ B}$ measurable. This fact together with hypothesis (3.33) and the trivial observation that μ_A and μ_V agree on $\mathcal{F}_{V^c}$ implies

$$\|\mu_V\mu_B\mu_A g - \mu_A\mu_B\mu_A g\|_\infty \leq \varepsilon\|\mu_B\mu_A g\|_\infty \leq \varepsilon\|\mu_A g\|_\infty \tag{3.37}$$

Iterating this inequality we get, for each bounded measurable f with $\mu_V(f) = 0$,

$$\|\widehat{A}_k f\|_\infty \leq (\sqrt{\varepsilon})^{k-3}\|f\|_\infty \qquad \|\widehat{B}_k f\|_\infty \leq (\sqrt{\varepsilon})^{k-3}\|f\|_\infty \tag{3.38}$$

Thus, we get that the sup norm of (3.35) is not greater than

$$\|f\|_\infty \frac{2}{\varepsilon^{3/2}}(1+\sqrt{\varepsilon})^{n-1}$$

which, inserted back into (3.34) yields

$$\|T_{\mathcal{D}}(t)f\|_\infty \leq \|f\|_\infty 4\varepsilon^{-3/2}e^{-2t}\sum_{n=0}^\infty \frac{t^n}{n!}(1+\sqrt{\varepsilon})^n = \|f\|_\infty 4\varepsilon^{-3/2}e^{-(1-\sqrt{\varepsilon})t} \quad \square$$

Corollary 3.6. *Let V, A and B be as in Proposition 3.5. Let also $A_0 = A \cap \partial_s^+ B$, with $s \geq r$, $B_0 = B \cap \partial_r^+ A$ and $\bar{A} = A \cup \partial_r^+ A$. For each $m > 0$ there exists $C(d, r, m)$ such that if*

(i) $md_0 \equiv md(A_0, B_0) \geq \max\{\, C\,,\, 100\|J\|\,,\, 10\,(\log|B_0| + 1)\,\}$
(ii) $SMT(A\backslash A_0,\, d_0 - 2r,\, m)$ *holds*

then

$$\inf_{\tau\in\Omega} \operatorname{gap}(L^\tau_{\{A,B\}}) \geq \frac{1}{2}$$

Proof. Thanks to Proposition 3.5 it is sufficient to show that

$$\sup_{\tau\in\Omega}\Big\|1 - \frac{d\mu^\tau_A}{d\mu^\tau_V}\Big|_{A_0}\Big\|_\infty \leq \frac{1}{4} \tag{3.39}$$

By the DLR property (2.2) we have

$$\text{LHS of (3.39)} \;\leq \sup_{\tau,\sigma\in\Omega\,:\,\tau_{V^c}=\sigma_{V^c}}\Big\|1 - \frac{d\mu^\tau_A}{d\mu^\sigma_A}\Big|_{A_0}\Big\|_\infty \tag{3.40}$$

At this point we can use Proposition 2.12 and obtain the result. $\square$

Remark. It is important to observe that, while it is very likely that a block dynamics with few large blocks will have a spectral gap larger than the spectral gap of a single spin flip dynamics, precisely the opposite may happen for the logarithmic Sobolev constant $c_s(L^\tau_{\mathcal{D}})$. To see this fix a finite set V and consider the covering $\mathcal{D}$ with just one block eqaul to V itself. Then clearly $\mathrm{gap}(L^\tau_{\mathcal{D}}) = 1$ since $\mathcal{E}^\tau_V(f,f) = \mathrm{Var}^\tau_V(f)$ but $c_s(L^\tau_{\mathcal{D}}) \ge \sup_\eta -\frac{1}{2}\log\bigl(\mu^\tau_V(\eta_V)\bigr)\bigl(1-\mu^\tau_V(\eta_V)\bigr)$. To prove it let us consider, for any $\eta \in \Omega$, the test function

$$f_\varepsilon(\sigma) = \begin{cases} \bigl(\mu^\tau_V(\eta_V)\bigr)^{-1/2} & \text{if } \sigma_V = \eta_V \\ \varepsilon & \text{otherwise} \end{cases}$$

where $\varepsilon > 0$ is a positive number that eventually will be sent to zero. Then an explicit computation shows that

$$\lim_{\varepsilon\to 0} \mathrm{Var}^\tau_V(f_\varepsilon) = 1 - \mu^\tau_V(\eta_V)$$
$$\lim_{\varepsilon\to 0} \mu^\tau_V\bigl(f^2_\varepsilon \log(f_e)\bigr) = -\frac{1}{2}\log\bigl(\mu^\tau_V(\eta_V)\bigr)$$
$$\lim_{\varepsilon\to 0} \mu^\tau_V(f^2_\varepsilon) = 1$$

so that, necessarily, $c_s(L^\tau_{\mathcal{D}}) \ge -\frac{1}{2}\log\bigl(\mu^\tau_V(\eta_V)\bigr)\bigl(1-\mu^\tau_V(\eta_V)\bigr)$ for all $\eta \in \Omega$. We conclude by observing that, for non pathological models, the above lower bound is of the order of $\log(|V|)$, in agreement with the general uppr bound proved in proposition 3.9.

Proposition 3.7. *Let V, A, B be as in Proposition 3.5. Let*

$$N = |\partial^+_r A \cap B| \wedge |\partial^+_r B \cap A|$$

Then there exists $k = k(d,r)$ such that

$$\inf_{\tau\in\Omega} \mathrm{gap}(L^\tau_{\{A,B\}}) \ge \exp[\,-k\|J\|_V N\,]$$

Proof. We can assume $N = |\partial^+_r A \cap B|$. Consider a new interaction J^0 such that A and $V\backslash A$ are decoupled, *i.e.*

$$J^0_X = \begin{cases} J_X & \text{if } X \cap \partial^+_r A \cap B = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

We have clearly $\sum_{X\ni x} \|J_X - J^0_X\| = 0$ unless x is in a neighborhood of radius r of $\partial^+_r A \cap B$, hence

$$\sum_{x\in V}\sum_{X\ni x} \|J_X - J^0_X\| \le k_1 N \|J\|$$

for some k_1 which depends on d and r. This implies that for all functions f on Ω,

$$\mathcal{E}^{J,\tau}_{\{A,B\}}(f,f) \ge \exp(-4k_1\|J\|N)\,\mathcal{E}^{J^0,\tau}_{\{A,B\}}(f,f) \tag{3.41}$$

$$\mathrm{Var}^{J,\tau}_V(f) \le \exp(4k_1\|J\|N)\ \mathrm{Var}^{J^0,\tau}_V(f) \tag{3.42}$$

From (3.41), (3.42) and the variational characterization of the gap (3.12), it follows that

$$\operatorname{gap}(L^{J,\tau}_{\{A,B\}}) \geq \exp[-8k_1\|J\|N] \operatorname{gap}(L^{J^0,\tau}_{\{A,B\}}) \tag{3.43}$$

In order to estimate the gap for the block–dynamics with couplings J^0, we just notice that the hypotheses of Proposition 3.5 are satisfied with $\varepsilon = 0$, and thus $\operatorname{gap}(L^{J^0,\tau}_{\{A,B\}}) \geq 1$. □

The second important result is a very general lower bound on the spectral gap of Glauber dynamics in an *arbitrary* set $V \subset\subset \mathbb{Z}^d$. It says that the spectral gap is always larger than a negative exponential of $|V|^{\frac{d-1}{d}}$. Notice that if V is cube then $|V|^{\frac{d-1}{d}}$ is simply its surface. In this case the bound is certainly optimal, at least in our general setting, since it is known that for several models of lattice *discrete spins* in the phase coexistence region, the activation energy between different stable phases is proportional to the *surface* of the region in consideration (see [M] and [CGMS] for more precise statements for the Ising model). Apparently the situation for *continuous* spins systems can be very different. For Heisenberg models, in fact, it is believed on the basis of spin–wave theory (see [B1], [B2]) that, at least for cubic regions, the spectral gap does not go to zero faster than the inverse of the volume. It is a challenging problem to actually prove it!

Theorem 3.8. *Let $d \geq 2$. There exist $k(d, r, \kappa_1)$, such that, for each $\Lambda \subset\subset \mathbb{Z}^d$ and for each $\tau \in \Omega$, we have (c_m was defined in (3.3))*

$$\operatorname{gap}(L^\tau_\Lambda) \geq c_m \exp\left[\, -k\,\|J\|\,|\Lambda|^{\frac{d-1}{d}}\,\right] \tag{3.44}$$

Proof. For each non–negative integer n, let

$$(K_n) = \text{ the inequality (3.44) holds for all } \Lambda \in \mathbb{F} \text{ such that } |\Lambda| \leq (3/2)^n$$

We want to show, that (K_n) holds for all $n \in \mathbb{Z}_+$, by proving that there exists $n_0(d,r) \in \mathbb{Z}_+$ such that (K_{n_0}) holds, and such that, for all $n \geq n_0$, (K_n) implies (K_{n+1}).

Assume then that K_{n-1} holds, and take any Λ such that $(3/2)^{n-1} < |\Lambda| \leq (3/2)^n$. Let $v = |\Lambda|$. Using a geometric lemma proved in [CMM1] (see Proposition A1.1 there), it is possible to write Λ as the disjoint union of two subsets X and Y, such that

(a) $v/2 - k_1 v^{\frac{d-1}{d}} \leq |X| \leq v/2$
(b) $|\delta_r(X,Y)| \leq k_1 v^{\frac{d-1}{d}}$

where

$$\delta_r(A,B) = (\partial_r^+ A \cap B) \cup (\partial_r^+ B \cap A)$$

and k_1 depends only on d and r. There exists then $n_0(d,r)$ such that if $n > n_0$ (and thus $v > (3/2)^{n_0-1}$), then $|Y| \leq (2/3)|\Lambda|$. So we can apply the inductive hypothesis to both X and Y. Moreover proposition 3.4 shows that

$$\inf_{\tau\in\Omega} \operatorname{gap}(L^\tau_\Lambda) \geq \inf_{\tau\in\Omega} \inf_{W\in\{X,Y\}} \operatorname{gap}(L^\tau_W) \inf_{\tau\in\Omega} \operatorname{gap}(L^\tau_{\{X,Y\}}) \tag{3.45}$$

where, as usual, the last term refers to the block dynamics. By Proposition 3.7 we know that

$$\inf_{\tau\in\Omega} \operatorname{gap}(L^{\tau}_{\{X,Y\}}) \geq e^{-k_2\|J\| v^{\frac{d-1}{d}}}$$

for some $k_2 = k_2(d,r)$. Together with the inductive hypothesis on X and Y, this gives

$$\inf_{\tau\in\Omega} \operatorname{gap}(L^{\tau}_{\Lambda}) \geq c_m \exp\big[-k\|J\||Y|^{\frac{d-1}{d}} - k_2\|J\| v^{\frac{d-1}{d}} \big] \tag{3.46}$$

Since $|Y| \leq (2/3)v$, we have

$$\inf_{\tau\in\Omega} \operatorname{gap}(L^{\tau}_{\Lambda}) \geq c_m \exp\big[-k\|J\| v^{\frac{d-1}{d}} \big] \qquad \text{if } k \geq \frac{k_2}{1-(2/3)^{\frac{d-1}{d}}}$$

In this way we have shown that (K_n) implies (K_{n+1}) for all $n \geq n_0(d,r)$.

All is left is to prove (K_{n_0}). For this purpose we observe that

$$e^{-2\|J-J'\|_{\Lambda}|\Lambda|} \leq \frac{\mu^{J,\tau}_{\Lambda}(\sigma)}{\mu^{J',\tau}_{\Lambda}(\sigma)} \leq e^{2\|J-J'\|_{\Lambda}|\Lambda|} \qquad \text{for all } \tau, \sigma, \Lambda, J, J' \tag{3.47}$$

Choose now any Λ with volume not exceeding $(3/2)^{n_0}$ and let $\widetilde{L}_{\Lambda}$ be the generator of the heat–bath dynamics with $J = 0$, *i.e.*

$$\widetilde{L}_{\Lambda} = \sum_{x\in\Lambda} \widetilde{L}_{\{x\}} = \sum_{x\in\Lambda} (\mu^{J=0}_{\{x\}} - \mathbb{1})$$

Since all $\widetilde{L}_{\{x\}}$ commute, it follows that $\operatorname{gap}(\widetilde{L}_{\Lambda}) = \operatorname{gap}(\widetilde{L}_{\{x\}}) = 1$ (the last equality can be checked via an explicit calculation). From (3.3), (3.12) and (3.47), it now follows that

$$\operatorname{gap}(L^{J,\tau}_{\Lambda}) \geq \operatorname{gap}(\widetilde{L}_{\Lambda})\, e^{-6\|J\||\Lambda|}\, c_m e^{-\kappa_1\|J\|} \geq c_m \exp\big[-(6+\kappa_1)\|J\|(3/2)^{n_0} \big]$$

which implies (K_{n_0}) (and then (3.44)), if we take $k \geq (6+\kappa_1)(3/2)^{n_0}$. □

3.8 General Results on the Logarithmic Sobolev Constant

Here we recall two well known results concerning the logarithmic Sobolev constant.

Proposition 3.9. *For each $\Lambda \subset\subset \mathbb{Z}^d$ we have*

$$c_s(L^{\tau}_{\Lambda}) \leq \Big[4 + (4\|J\| + 2\log 2)|\Lambda| \Big] (\operatorname{gap}(L^{\tau}_{\Lambda}))^{-1}$$

Proof. The proposition follows from (3.12), Proposition 3.10 below, and from a trivial estimate on $\inf_{\sigma} \mu^{\tau}_{\Lambda}(\sigma)$.

□

Proposition 3.10. *Let Ω be a finite set, let μ be a probability measure on $(\Omega, 2^\Omega)$ and assume*

$$\mu_0 \equiv \inf_{x\in\Omega} \mu(x) > 0$$

Then, for each positive function f on Ω, we have

$$\mu(f^2 \log f) \le (4 + 2\log \mu_0^{-1}) \operatorname{Var}(f) + \mu(f^2) \log \sqrt{\mu(f^2)}$$

Proof. We can assume $\mu(f^2) = 1$. If we let $f = \mu(f)(1+g)$, we find $\mu(g) = 0$ and $\mu(g^2) = \operatorname{Var}(f)/\mu(f)^2$. Let A be the set of all $x \in \Omega$ such that $|g(x)| < 1$. We can then write

$$\mu(f^2 \log f) = \mu(f^2 \log f\, \mathbb{I}_A) + \mu(f^2 \log f\, \mathbb{I}_{A^c}) \tag{3.48}$$

Let's denote by X_1 respectively X_2 the first and the second term in the RHS of (3.48). Using the inequalities $\log(1+g) \le g$ and $\log \mu(f) \le \log \mu(f^2) \le 0$, we get

$$\begin{aligned} X_1 &\le \mu(f)^2 \mu[(g + 2g^2 + g^3)\mathbb{I}_A] \\ &\le \mu(f)^2 \big[\, 3\mu(g^2) + \mu(g\mathbb{I}_A) \,\big] = 3\operatorname{Var}(f) + \mu(f)^2 \mu(g\mathbb{I}_A) \end{aligned} \tag{3.49}$$

To take care of the last term we remember that $\mu(g) = 0$, so $\mu(g\mathbb{I}_A) = -\mu(g\mathbb{I}_{A^c})$ which implies

$$|\mu(g\mathbb{I}_A)| \le \mu(|g|\mathbb{I}_{A^c}) \le (\mu(g^2)\mu(\mathbb{I}_{A^c}))^{1/2} \le \mu(g^2)$$

Thus we get $X_1 \le 4 \operatorname{Var} f$. As for X_2, we write

$$X_2 \le \big(\sup_{x\in\Omega} \log f(x)\big)\, \mu(f^2 \mathbb{I}_{A^c}) \le \log \|f\|_\infty\, \mu(f^2\mathbb{I}_{A^c}) \tag{3.50}$$

Finally we observe that $\|f\|_\infty$ is bounded by $(\mu(f^2)/\mu_0)^{1/2} = \mu_0^{-(1/2)}$ while

$$\mu(f^2\mathbb{I}_{A^c}) = \mu(f)^2 \mu((1 + 2g + g^2)\mathbb{I}_{A^c}) \le 4\mu(f)^2\mu(g^2) = 4\operatorname{Var}(f)$$

This concludes the proof. □

Proposition 3.11. *Given a finite set $V \subset\subset \mathbb{Z}^d$ and two probability measures on Ω_V, ν_1 and ν_2, assume $\|\frac{d\nu_1}{d\nu_2}\|_\infty \vee \|\frac{d\nu_2}{d\nu_1}\|_\infty \le \varepsilon$. Then*

$$c_s(\nu_1) \le \varepsilon^2 c_s(\nu_2) \tag{3.51}$$

Proof. Inequality (3.51) follows immediately from the bound

$$\nu_2(|\nabla_V f|^2) \le \varepsilon \nu_1(|\nabla_V f|^2)$$

and the identity

$$\nu_1(f^2 \log f^2) - \nu_1(f^2)\log(\nu_1(f^2)) = \min_{a>0} \nu_1\big(f^2 \log f^2 - \log(a) f^2 - f^2 + a\big)$$

together with the observation that the expression $f^2 \log f^2 - \log(a) f^2 - f^2 + a$ is non–negative for all positive f and a (see [HS]). $\square$

3.9 Possible Rates of Convergence to Equilibrium

We conclude this general section on the dynamics with a discussion of the possible rates of convergence to equilibrium when the infinite volume process is ergodic. The flavour of the results discussed below is that, for local functions f, either the convergence of $T(t)f$ to $\mu(f)$ is very slow (e.g a small inverse power of the time) or it is exponentially fast.

Let us first consider the attractive case. Define

$$\rho(t) = \mathbb{P}\big(\sigma_t^+(0) \neq \sigma_t^-(0)\big) \tag{3.52}$$

where $\mathbb{P}$ is the global coupling constructed in section 3.3, σ_t^+ denotes the dynamics in infinite volume starting from the configuration identically equal to $+1$ and similarly for σ_t^-. It is easy to see that if $\rho(t)$ tends to zero as $t \to \infty$ then the infinite volume dynamics is ergodic. In fact, for any local function f, we can write

$$\begin{aligned}
\sup_{\sigma,\eta} |T_t(f)(\sigma) \;-\; T_t(f)(\eta)| &= \sup_{\sigma,\eta} |\,\mathbb{E}\big(f(\sigma_t^\sigma) - f(\sigma_t^\eta)\big)| \\
&\leq |||f||| \sup_{x \in \Lambda_f} \mathbb{P}\big(\sigma_t^+(x) \neq \sigma_t^-(x)\big) \\
&= |||f|||\, \rho(t)
\end{aligned}$$

where we used (3.10) and (3.11) together with translation invariance.

The main result on the absence of intermediate speed of relaxation to equilibrium in the attractive case can then be formulated as follows (see [Ho1])

Theorem 3.12. *There exists two constants ε and t_0 depending only on the flip rates and on the range of the interaction r such that, $t^d \rho(t) \leq \varepsilon$ for some $t \geq t_0$ implies that $\rho(t)$ decays to zero exponentially fast.*

Proof. The proof is based on the following key inequality valid for all $t \geq 0$:

$$\rho(2t) \;\leq\; Ct^d \rho(t)^2 + e^{-t} \tag{3.53}$$

for a suitable constant C.

Let us in fact assume (3.53) and define

$$u(t) \equiv 2^d (Ct^d + 1)\big(\rho(t) \vee e^{-t/2}\big)$$

Then, by explicit calculation, $u(2t) \leq u(t)^2$ and thus, if there exists a time t_0 such that $u(t_0) < 1$, then $u(t)$ tends to zero exponentially fast. The existence of a time t_0 with the above property is precisely the assumption appearing in the theorem.

To prove (3.53) let $\nu_t^{+,-}$ be the joint distribution of σ_t^+ and σ_t^- given by the graphical construction. Because of (3.11) $\nu_t^{+,-}$ is above the diagonal i.e. $\nu_t^{+,-}((\xi,\eta);\ \xi \geq \eta) = 1$. Let now χ_t be the characteristic function of the event that $\sigma_t^+(x) = \sigma_t^-(x)$ for all $|x| \leq k_0 t$, where k_0 appears in lemma 3.2. Then, using the Markov property, we can write $\rho(2t)$ as:

$$\begin{aligned}\rho(2t) &= \mathbb{P}\big(\sigma_{2t}^+(0) \neq \sigma_{2t}^-(0)\big)\\ &= \int d\nu_t^{+,-}(\xi,\xi')\,\chi_t\ \mathbb{P}\big(\sigma_t^{\xi}(0) \neq \sigma_t^{\xi'}(0)\big)\\ &+ \int d\nu_t^{+,-}(\xi,\xi')(1-\chi_t)\,\mathbb{P}\big(\sigma_t^{\xi}(0) \neq \sigma_t^{\xi'}(0)\big)\end{aligned} \tag{3.54}$$

The first term in the r.h.s of (3.54) is bounded from above by e^{-t} because of lemma 3.2. To bound the second term we observe that, because of monotonicity in the starting configuration,

$$\mathbb{P}\big(\sigma_t^{\xi}(0) \neq \sigma_t^{\xi'}(0)\big) \leq \rho(t)$$

for any $\xi \geq \xi'$. Moreover

$$\int d\nu_t^{+,-}(\xi,\xi')(1-\chi_t) \leq \frac{1}{2}(2k_0t+1)^d\rho(t)$$

because of translation invariance and the definition of $\rho(t)$. Thus (3.53) follows and the theorem is proved. $\square$

Let us now examine the more difficult, non-attractive case. The analogous of theorem 3.12 reads as follows (see [Yo1])

Theorem 3.13. *Assume that for some L_0, some positive non–increasing function $\phi(t)$ satisfying $\sum_n n^{2(d-1)}\phi(n) < \infty$, any set V multiple of Q_{L_0} and any local function f*

$$\sup_\tau \mu_V^\tau\big(|T_V^\tau(t)f - \mu_V^\tau(f)|\big) \leq \|f\|_\infty\,|\Lambda_f|\,\phi(t) \tag{3.55}$$

Then there exists L_0', a positive constant m and for any local function f there exists a constant $C'(f) = C'(|\Lambda_f|, \|f\|_\infty)$ such that for any set V multiple of $Q_{L_0'}$

$$\sup_\tau \|T_V^\tau(t)f - \mu_V^\tau(f)\|_\infty \leq C'(f)\,e^{-mt}$$

Proof. The idea of the proof is to show that the assumption of the theorem is sufficient to guarantee that there exists L_0' such that property $SMT(V,l,m)$ for some l and m and all sets V which are multiple of $Q_{L_0'}$ holds. Once the exponential decay of covariances is available then we get, thanks to theorem 4.6 below, that there exits a finite constant c_s^∞ such that

$$\sup_\tau c_s(L_V^\tau) < \infty$$

for all sets V multiple of $Q_{L'_0}$. Thus the thesis follows from theorem 3.3.

In order to prove strong mixing we first establish some form of decay for covariances of the Gibbs measure μ_V^τ. For this purpose let f and g be two local functions with supports contained in the set V multiple of Q_{L_0} and let $l = d(\Lambda_f, \Lambda_g)$. Formula (3.24) written for μ_V^τ, together with (3.55), gives the following bound

$$|\mu_V^\tau(f;g)| \le \|g\|_\infty \, \|f\|_\infty \, |\Lambda_f| \phi(2k_0^{-1}l) + |||f||| \, |||g||| \, |\Lambda_f| \, |\Lambda_g| e^{-k_0^{-1}l}$$

i.e. the decay of covariances for large values of l is governed by the function $\phi(2l) \wedge e^{-l}$. To conclude the proof we appeal to proposition 2.11. □

The above result indicates that a power law decay to equilibrium in the L^1 sense is enough to get exponential ergodicity. One possibility to establish such a weaker decay is through generalized Nash inequalities (see e.g [SC] and [BZ]).

Let $c_N(L_0, \alpha)$, $\alpha > 0$ and $L_0 < \infty$, be the smallest constant c such that the following inequality holds for all sets V multiple of Q_{L_0} and all local function f:

$$\mathrm{Var}_V^\tau(f)^{1+\alpha} \le c\, \mathcal{E}_V^\tau(f,f)\, |||f|||^{2\alpha} \qquad \forall\, \tau \in \Omega \tag{3.56}$$

With the above notation we have the following result

Theorem 3.14. *Assume that $c_N(L_0, \alpha) < \infty$ for some finite L_0 and some $\alpha < (6d-2)^{-1}$. Then the conclusions of theorem 3.13 hold.*

Proof. Let us fix a set V multiple of Q_{L_0}, a boundary condition $\tau \in \Omega$ and a local function f with zero mean w.r.t. the Gibbs measure μ_V^τ.

Thanks to theorem 3.13 it is sufficient to bound the decay rate to equilibrium of $T_V^\tau(t)f$ in the $L^1(d\mu_V^\tau)$–sense, uniformly in V. For this purpose define

$$f_t = T_V^\tau(t) f$$
$$u_t = \mathrm{Var}_V^\tau(f_t)$$

An explicit computation gives

$$\dot u_t = -2\mathcal{E}_V^\tau(f_t, f_t) \le -\frac{2}{c_N(\alpha) |||f_t|||^{2\alpha}} u_t^{1+\alpha} \tag{3.57}$$

for some $\alpha < (6d-2)^{-1}$, because of (3.56) together with the assumption of the theorem.

Unfortunately, contrary to the L^p–norms, the seminorm $|||f_t|||$ is not obviously bounded from above by $|||f|||$. However, thanks to lemma 3.2, it is quite simple to check that

$$|||f_t||| \le A t^d |||f|||$$

for a suitable constant A. We have in fact that

$$
\begin{aligned}
|||f_t||| &= \sum_y \sup_\eta |\,\mathbb{E}\big(f(\sigma_t^{V,J,\tau,\eta^y}) - f(\sigma_t^{V,J,\tau,\eta})\big)| \\
&\le \sum_y \sum_x \|\nabla_x f\|_\infty \sup_\eta \mathbb{P}\big(\sigma_t^{V,J,\tau,\eta^y}(x) \neq \sigma_t^{V,J,\tau,\eta}(x)\big) \\
&\le (kt)^d |||f||| + \sum_x \|\nabla_x f\|_\infty \sum_{y:\, d(x,y)\ge kt} \sup_\eta \mathbb{P}\big(\sigma_t^{V,J,\tau,\eta^y}(x) \neq \sigma_t^{V,J,\tau,\eta}(x)\big) \\
&\le 2(kt)^d |||f|||
\end{aligned}
$$

if the constant k is chosen large enough. In the last step we used the bound (3.8) in order to estimate $\mathbb{P}\big(\sigma_t^{V,J,\tau,\eta^y}(x) \neq \sigma_t^{V,J,\tau,\eta}(x)\big)$ when $d(x,y) \ge kt$.

Thus (3.57) reads

$$\dot{u}_t \;\le\; -2\big(c_N(\alpha) A |||f|||^{2\alpha}\big)^{-1} t^{-2d\alpha} u_t^{1+\alpha}$$

Since $2d\alpha < 1$, this differential inequality implies

$$u_t \le C_f t^{-(1-2d\alpha)/\alpha} \tag{3.58}$$

for a suitable constant C_f depending only on $\|f\|_\infty$ and $|\Lambda_f|$.

To conclude the proof it is enough to write

$$\mu_V^\tau\big(|T_V^\tau(t) f)|\big) \;\le\; u_t^{1/2} \;\le\; \big[C_f t^{-(1-2d\alpha)/\alpha}\big]^{1/2} \tag{3.59}$$

and notice that the r.h.s of (3.59) satisfies the hypothesis of theorem 3.13. □

We conclude with an example of a spin system whose infinite volume dynamics is ergodic but with an algebraic decay to equilibrium (see [Ho2]).

Let us consider the two dimensional Ising model (see (2.4)) at the critical temperature *i.e.* we take $d = 2$ and a two-body ferromagnetic interaction of the form

$$J_A = \begin{cases} \beta_c & \text{if } A = \{x,y\} \text{ with } d_2(x,y) \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$.

It is well known (see e.g [MCoy]) that the above system has a unique infinite volume Gibbs measure μ with polynomially decaying correlations functions. More precisely

$$\lim_{x_1\to\infty} \frac{\log\big[\mu\big(\sigma((0,0))\sigma((x_1,0))\big)\big]}{\log(x_1)} = -\frac{1}{4}$$

where $\sigma((x_1,x_2))$ denotes the spin at the site $x = (x_1,x_2)$. Given now a time t, let $x = (\lfloor k_0 t\rfloor, 0)$. Then, using reversibility together with lemma 3.2, we can write

$$
\begin{aligned}
\mu\big(\sigma(0)\sigma(x)\big) &= \mu\big(T(t/2)\sigma(0)\sigma(x)\big) \\
&\le \mu\big((T(t/2)\sigma(0))\,(T(t/2)\sigma(x))\big) + 2e^{-t} \\
&\le \|\,T(t/2)\sigma(0)\|_{2,\mu}\,\|\,T(t/2)\sigma(x)\|_{2,\mu} + 2e^{-t} \\
&= \|\,T(t/2)\sigma(0)\|_{2,\mu}^2 + 2e^{-t}
\end{aligned}
$$

i.e.

$$\lim_{t\to\infty} \frac{\log(\| T(t/2)\sigma(0)\|_{2,\mu})}{\log(t)} \geq -\frac{1}{4}$$

Thus the relaxation to equilibrium is not exponential.

4. One Phase Region

In this section we discuss the main results on the exponential ergodicity in the absence of a phase transition. In this case equilibrium is reached by an homogeneous process: far apart regions equilibrate in finite time without exchanging almost any information, very much like an infinite collection of non–interacting continuous time ergodic Markov chains.

We first discuss the attractive case and we prove exponential ergodicity of the infinite volume dynamics under the weak mixing condition (see definition 2.3). Then we turn to the general case and prove exponential ergodicity uniformly in all cubes under the strong mixing condition (see definition 2.4). We refer the reader to [MO1], [SZ], [LY], [La], [MO2], [Yo1] and references therein to have a complete overview of related results under similar assumptions.

4.1 The Attractive Case

In this first paragraph we assume that the dynamics is attractive. In this case it is not difficult to check that condition $WM(V, C, m_0)$ for all $V \subset\subset \mathbb{Z}^d$ is equivalent to the following

There exist positive constants C and m such that, for any integer L

$$\mu^+_{B_L}(\sigma(0)) - \mu^-_{B_L}(\sigma(0)) \;\le\; Ce^{-mL} \tag{4.1}$$

Indeed, by the very definition of variation distance, $WM(V, C, m)$ for all $V \subset\subset \mathbb{Z}^d$ trivially implies (4.1). To see the converse, let $\nu^{+,-}_V$ be the joint representation above the diagonal of the two Gibbs measures μ^+_V and μ^-_V given by attractivity (see section 3.4) and set, for any $x \in V$, $l_x = d(x, \partial^+_r V)$. Then, thanks to attractivity and finite volume ergodicity, we can write

$$\begin{aligned}
\sup_{\tau_1,\tau_2} \|\mu^{\tau_1}_{V,\Delta} - \mu^{\tau_2}_{V,\Delta}\| &= \sup_{\tau_1,\tau_2}\sup_{X\in\mathcal{F}_\Delta} |\mu^{\tau_1}_{V,\Delta}(X) - \mu^{\tau_2}_{V,\Delta}(X)| \\
&\le \sup_{\tau_1,\tau_2}\lim_{t\to\infty} \mathbb{P}\big(\sigma^{V,\tau_1,+}_t(x) \neq \sigma^{V,\tau_2,+}_t(x) \;\text{ for some } x\in\Delta\big) \\
&\le \lim_{t\to\infty}\sum_{x\in\Delta} \mathbb{P}\big(\sigma^{V,+,+}_t(x) \neq \sigma^{V,-,+}_t(x)\big) \\
&= \sum_{x\in\Delta} \nu^{+,-}_V(\sigma(x)\neq\eta(x)) \\
&\le \sum_{x\in\Delta} \nu^{+,-}_{B_{l_x}+x}(\sigma(x)\neq\eta(x)) \\
&\le \sum_{x\in\Delta} Ce^{-md(x,\partial^+_r V)} \\
&\le \sum_{x\in\Delta,\, y\in\partial^+_r V} Ce^{-md(x,y)}
\end{aligned}$$

Thus $WM(V, C, m)$ holds.

Theorem 4.1. *In the attractive case the following are equivalent:*

i) $WM(V,C,m)$ *for all* $V \subset\subset \mathbb{Z}^d$

ii) There exists a positive constant m *and for any local function* f *there exists a constant* C_f *such that:*

$$\|T_t(f) - \mu(f)\|_\infty \le C_f\, e^{-mt}$$

Proof. Let us first show that **i)** $\to$ **ii)**. For this purpose and as in section 3.9 we define

$$\rho(t) = \mathbb{P}\big(\sigma_t^+(0) \neq \sigma_t^-(0)\big)$$

As in theorem 3.12, if $\rho(t)$ decays exponentially fast to zero then the theorem follows. Moreover, thanks to theorem 3.12, the exponential decay of $\rho(t)$ follows once one is able to show that $\rho(t)$ goes to zero faster than t^{-d}. In order to prove such a weaker decay of $\rho(t)$ let us first show, as a preliminary step, that under our assumption $\rho(t) \to 0$ as $t \to \infty$.

Lemma 4.2. *Under the same assumption of theorem 4.1* $\rho(t) \to 0$ *as* $t \to \infty$.

Proof. Using the attractivity of the dynamics, for any $V \subset\subset \mathbb{Z}^d$ such that $0 \in V$ and any $t \ge 0$

$$\rho(t) \;\le\; \mathbb{P}\big(\sigma_t^{V,+,+}(0) \neq \sigma_t^{V,-,-}(0)\big) \tag{4.2}$$

where $\sigma_t^{V,+,+}$ denotes the dynamics in V with plus boundary condition and initial condition identically equal to $+1$ and similarily for $\sigma_t^{V,-,-}$. If we pass to the limit $t \to \infty$ in (4.2) and use finite volume ergodicity, we get

$$\limsup_{t\to\infty} \rho(t) \;\le\; 2\big[\mu_V^+(\sigma(0)) - \mu_V^-(\sigma(0))\big] \tag{4.3}$$

Finally, by taking V as a large cube centered at the origin and using weak mixing, we get that the r.h.s of (4.3) can be made arbitrarily small. $\square$

As a second step, using weak mixing, we will establish a powerful recursive inequality satisfied by $\rho(t)$ which, combined with lemma 4.2, will show that $t^d\rho(t) \to 0$ as $t \to \infty$.

Lemma 4.3. *Under the same assumption of theorem 4.1, for all* $t \ge 0$ *and all* $L \in \mathbb{Z}_+$

$$\rho(2t) \;\le\; 2(2L+1)^d \rho(t)^2 + 2Ce^{-mL}$$

Remark. Although our inequality is apparently very similar to the one appearing in the proof of theorem 3.12 on the absence of intermediate speed of convergence to equilibrium, the differences between the two are actually quite deep and mainly consists in the freedom of the parameter L that in the proof of theorem 3.12 was fixed and of the order of the time t.

Proof. We write $\rho(2t)$ as:

$$\rho(2t) \;=\; \int d\mu(\eta)\left[\mathbb{P}\left(\sigma_{2t}^{+}(0) - \sigma_{2t}^{\eta}(0)\right)\right] \;+\; \int d\mu(\eta)\left[\mathbb{P}\left(\sigma_{2t}^{\eta}(0)\right) - \sigma_{2t}^{-}(0)\right)\right] \tag{4.4}$$

and we show that each one of the two integrals is bounded by a half of the r.h.s. of the recursive inequality.

Let $\nu_t^{+,\eta}$ be the joint distribution of σ_t^+ and σ_t^η given by the graphical construction. Because of (3.11) $\nu_t^{+,\eta}$ is above the diagonal i.e. $\nu_t^{+,\eta}((\xi,\eta);\; \xi \geq \eta) \;=\; 1$. Then, using the Markov property, we can write:

$$\begin{aligned}\int d\mu(\eta)\left[\mathbb{P}\left(\sigma_{2t}^{+}(0) \neq \sigma_{2t}^{\eta}(0)\right)\right] = &\int d\mu(\eta)\int d\nu_t^{+,\eta}(\xi,\xi')\,\chi_L\;\mathbb{P}\left(\sigma_t^{\xi}(0) \neq \sigma_t^{\xi'}(0)\right)\\ &+\int d\mu(\eta)\int d\nu_t^{+,\eta}(\xi,\xi')(1\,-\,\chi_L)\,\mathbb{P}\left(\sigma_t^{\xi}(0) \neq \sigma_t^{\xi'}(0)\right)\end{aligned} \tag{4.5}$$

where χ_L is the characteristic function of the event that $\xi(x) = \xi'(x)$ for all $|x| \,\leq\, L$. By attractivity and traslation invariance, the second term in the r.h.s. of (4.5) can be bounded from above by:

$$(2L+1)^d\rho(t)\int d\mu(\eta)\nu_t^{+,\eta}(\xi(0)\;\neq\;\xi'(0))\;\;\leq\;\;(2L+1)^d\rho(t)^2 \tag{4.6}$$

Let now, for any given $\tau \in \Omega_{B_L}$, $\hat{\chi}_{L,\tau}$ be the characteristic function of the event :

$$\xi(x) \;=\; \xi'(x) \;=\; \tau(x) \qquad \forall\, x \in Q_L$$

Thus

$$\chi_L \;=\; \sum_{\tau\in\Omega_{B_L}} \hat{\chi}_{L,\tau}$$

and therefore the first term in the r.h.s. of (4.6) can be written as:

$$\int d\mu(\eta) \sum_{\tau\in\Omega_{B_L}} \int d\nu_t^{+,\eta}(\xi,\xi')\;\hat{\chi}_{L,\tau}\;\;\mathbb{P}\left(\sigma_t^{\xi}(0) \neq \sigma_t^{\xi'}(0)\right) \tag{4.7}$$

Attractivity allows us to bound the quantity $\mathbb{E}\left(\sigma_t^{\xi}(0) \neq \sigma_t^{\xi'}(0)\right)$ by imposing extra plus and minus boundary conditions outside the cube B_L. More precisely :

$$\mathbb{P}\left(\sigma_t^{\xi}(0) \neq \sigma_t^{\xi'}(0)\right) \leq \mathbb{P}\left(\sigma_t^{B_L,+,\xi}(0) \neq \sigma_t^{B_L,-,\xi'}(0)\right) \tag{4.8}$$

Thus (4.7) is bounded from above by :

$$\int d\mu(\eta) \sum_{\tau\in\Omega_{B_L}} \int d\nu_t^{+,\eta}(\xi,\xi')\hat{\chi}_{L,\tau}\,\mathbb{P}\left(\sigma_t^{B_L,+,\tau}(0) \neq \sigma_t^{B_L,-,\tau}(0)\right)\;\leq$$

$$\int d\mu(\eta) \sum_{\tau\in\Omega_{B_L}} \nu_t^{+,\eta}(\xi'(j)\;=\;\tau(j)\;\forall\, j \,\in\, Q_L)\left[\mathbb{P}\left(\sigma_t^{B_L,+,\tau}(0) \neq \sigma_t^{B_L,-,\tau}(0)\right)\right] \tag{4.9}$$

We use at this point attractivity to write

$$\mathbb{P}\big(\sigma_t^{B_L,+,\tau}(0) \neq \sigma_t^{B_L,-,\tau}(0)\big) = 2\,\mathbb{E}\big(\sigma_t^{B_L,+,\tau}(0) - \sigma_t^{B_L,-,\tau}(0)\big)$$

and we notice that, since $f(\tau) \equiv \mathbb{E}\big(\sigma_t^{B_L,+,\tau}(0)\big)$ is an increasing function of τ and $\mu \le \mu^+_{B_L}$, then

$$\begin{aligned}\int d\mu(\eta) \int d\nu_t^{+,\eta}(\xi,\xi') f(\xi') &= \int d\mu(\eta) f(\eta) \\ &\le \int d\mu^+_{B_L}(\eta)\,\mathbb{E}\big(f(\sigma_t^{B_L,+,\eta})\big) \\ &= \int d\mu^+_{B_L}(\eta)\,\mathbb{E}\big(\sigma_{2t}^{B_L,+,\eta}(0)\big) \\ &= \mu^+_{B_L}(\sigma(0))\end{aligned}$$

where we used the Markov property and the reversibility of $\sigma_t^{B_L,+,\eta}$ w.r.t $\mu^+_{B_L}$. A similar reasoning shows that

$$\int d\mu(\eta) \int d\nu_t^{+,\eta}(\xi,\xi') f(\xi') \ge \mu^-_{B_L}(\sigma(0))$$

Thus the r.h.s of (4.9) is bounded from above by

$$\mu^+_{B_L}(\sigma(0)) \;-\; \mu^-_{B_L}(\sigma(0)) \;\le\; Ce^{-mL} \tag{4.10}$$

because of the weak mixing assumption. □

As a final step we combine together lemma 4.2 and lemma 4.3.

Lemma 4.4. *If, for some positive* m *and* C, $\rho(2t) \;\le\; 2(2L+1)^d\rho(t)^2 + 2Ce^{-mL}$ *for all* $t \ge 0$ *and all* $L \in \mathbb{Z}_+$, *and if* $\lim_{t\to\infty}\rho(t) = 0$, *then*

$$\lim_{t\to\infty} t^d \rho(t) = 0$$

Proof. Assume that t is so large that

$$2(-\frac{4}{m}\log(\rho(t)) + 1)^d + Ce^m)\sqrt{\rho(t)} \;\le\; 1 \tag{4.11}$$

and set $L(t) = \big[-\frac{2}{m}\log(\rho(t))\big]$. Then

$$\rho(2t) \;\le\; 2\big(-\frac{4}{m}\log(\rho(t)) + 1\big)^d \rho(t)^2 + 2\,Ce^{2\log(\rho(t))+m} \;\le\; \rho(t)^{\frac{3}{2}}$$

Let now t_0 be so large that $\rho(t_0) < 1$ and (4.11) holds for all $t \ge t_0$. One gets at once that for all $n \in \mathbb{Z}_+$

$$\rho(2^n t_0) \;\le\; \rho(t_0)^{(\frac{3}{2})^n}$$

from which the lemma follows. □

Once lemma 4.4 is available then we can apply theorem 3.12 and conclude that $\rho(t)$ must decay exponentially fast.

ii) $\rightarrow$ **i)**.

Clearly *ii)* together with attractivity imply that the infinite volume dynamics is ergodic with a unique invariant measure μ (see Corollary 2.4 in chapter III of [L]). Thus we write:

$$\mu^+_{B_L}(\sigma(0)) \; - \; \mu^-_{B_L}(\sigma(0)) \; = \; \left[\mu^+_{B_L}(\sigma(0)) \; - \; \mu(\sigma(0))\right] \; + \; \left[\mu(\sigma(0)) \; - \; \mu^-_{B_L}(\sigma(0))\right] \tag{4.12}$$

Let us estimate $\mu^+_{B_L}(\sigma(0)) \; - \; \mu(\sigma(0))$. By adding and subtracting the term $\mathbb{E}\big(\sigma^+_t(0)\big)$ and using the exponential convergence to equilibrium together with attractivity, we get

$$0 \; \leq \; \mu^+_{B_L}(\sigma(0)) \; - \; \mu(\sigma(0)) \; \leq \; Ce^{-mt} \; + \; \mu^+_{B_L}(\sigma(0)) - \mathbb{E}\big(\sigma^+_t(0)\big) \tag{4.13}$$

We now choose the time t as t $= k_0^{-1}L$ where the constant k_0 appears in lemma 3.2 and add and subtract the term $\mathbb{E}\big(\sigma_t^{B_L,+,+}(0)\big)$. Using lemma 3.2 the r.h.s. of (4.13) can be bounded from above by :

$$Ce^{-k_0^{-1}mL} + 2e^{-k_0^{-1}L} + \mu^+_{B_L}(\sigma(0)) \; - \; \mathbb{E}\big(\sigma_t^{B_L,+,+}(0)\big) \; \leq \; Ce^{-k_0^{-1}mL} + 2e^{-k_0^{-1}L}$$

since, by attractivity, $\mu^+_{B_L}(\sigma(0)) \; \leq \; \mathbb{E}\big(\sigma_t^{B_L,+,+}(0)\big)$.

In conclusion we have shown that $\mu^+_{B_L}(\sigma(0)) \; - \; \mu(\sigma(0))$ is smaller than $Ce^{-k_0^{-1}mL} + 2e^{-k_0^{-1}L}$. The same argument applies also to the other term in the r.h.s. of (4.12) $\mu(\sigma(0)) \; - \; \mu^-_{B_L}(\sigma(0))$. □

4.2 The General Case: Recursive Analysis.

Here we prove exponential ergodicity under a strong mixing assumption on the family of all parallelepipeds with ratio between the smallest and the largest side greater than a given fixed constant. For simplicity we carry out our analysis in two dimensions but the extension to higher dimension is straightforward.

Let $R(l_1, l_2)$ denote the rectangle

$$R(l_1, l_2) = \big\{(x_1, x_2) \in \mathbb{Z}^2; \; 0 \leq x_1 \leq l_1 - 1, \; 0 \leq x_2 \leq l_2 - 1 \; \big\} \; ; \qquad l_1, l_2 \in \mathbb{Z}_+$$

and let $\mathcal{R}_L$ be the family of "fat" rectangles with "size" smaller than L, namely those rectangles $R(l_1, l_2, x) \equiv R(l_1, l_2) + x$, $x \in \mathbb{Z}^2$, with $l_1 \wedge l_2 \geq 0.1(l_1 \vee l_2)$ and $l_1 \vee l_2 \leq L$.

Let also

$$g(L) = \min_{R \in \mathcal{R}_L} \; \min_{\tau} \operatorname{gap}(L_R^\tau)$$

$$c_s(L) = \sup_{R \in \mathcal{R}_L} \; c_s(\mu_R^J)$$

where $\operatorname{gap}(L_R^\tau)$ and $c_s(\mu_R^J)$ have been defined in (3.12) and (3.19) respectively.

With the above notation we will prove the following key recursive bounds.

Theorem 4.5. *Assume $SMT(R, l, m)$ for all $R \in \bigcup_{L \geq l} \mathcal{R}_L$. Then there exists a positive constant $k = k(d, r, \|J\|)$ such that*

$$g(2L) \geq (1 - \frac{k}{\sqrt{L}})g(L)$$

for all L large enough. In particular $\inf_L g(L) > 0$.

Theorem 4.6. *Assume $SMT(R, l, m)$ for all $R \in \bigcup_{L \geq l} \mathcal{R}_L$. Then there exists a positive constant $k = k(d, r, \|J\|)$ such that*

$$c_s(2L) \leq (1 + \frac{k}{\sqrt{L}})c_s(L)$$

for all L large enough. In particular $\sup_L c_s(L) < \infty$.

Proof of theorem 4.5. The second part of the theorem is a straightforward consequence of the recursive bound. Thus we concentrate on what happens to the gap when one doubles the length scale. In order to have a clear understanding of what will follow let us consider first the trivial case of zero interaction, $J_A = 0 \,\forall\, A \subset\subset \mathbb{Z}^d$, in e.g two dimensions. Let us take a cube Q_{2L} and divide it into two equal vertical non–overlapping rectangles $R_i(L, 2L)$. Then formula (3.30) naturally allows us to compute the ratio between the spectral gap in Q_{2L} and the spectral gap in $R_i(L, 2L)$, $i = 1, 2$, in terms of the spectral gap of the block dynamics in Q_{2L} with blocks $R_i(L, 2L)$, $i = 1, 2$. Since the interaction is zero this last quantity is exactly equal to one either by exact computation or because of proposition 3.5, and we get that spectral gap in Q_{2L} is equal to that in either one of the two rectangles. We can at this point repeat the whole procedure for each one of the two rectangles and prove finally that the gap does not change when we double the scale. When the interaction is present and strong mixing holds one could be tempted to proceed in exactly the same way. However in the interacting case a lower bound on the spectral gap of the block dynamics with just two blocks is no longer that simple unless the two blocks overlap a little bit and one can appeal to proposition 3.5. On the other hand if the two rectangles overlap then an error term due precisely to the overlap appears in the formula comparing the single site dynamics with the block dynamics (see (3.29) and (3.30)). In particular, without any particular effort, one would get immediately a bound like

$$g(2L) \geq \frac{1}{16}g(L)$$

which already implies that $g(L) \geq L^{-\alpha}$ for a suitable $\alpha > 0$ and any L large enough. The way out to show that the factor $\frac{1}{16}$ can be actually replace by $1 - \frac{k}{\sqrt{L}}$ is to "spread out" over a non-zero fraction of the starting volume Q_{2L} the error term in (3.29). This appears to be a new idea and in our opinion leads to a rather simple proof of both theorems 4.5, 4.6. Let us now explain the details.

If $g(L) = g(2L)$ there is nothing to prove so we suppose that $g(2L) < g(L)$. Let us consider a rectangle $R \equiv R(l_1, l_2) \in \mathcal{R}_{2L}$ and assume, without loss of generality, that the longest side is l_2. Since we are assuming that $g(2L) < g(L)$ necessarily $L < l_2 \leq 2L$.

We set $d=[\sqrt{L}]$ and, for any integer $1 \le n \le \frac{L}{10d}$, we cover R with the following two rectangles

$$\begin{aligned} R_n^{top} &= \{x \in R;\ l_2/2+(n-1)d < x_2 \le l_2-1\} \\ R_n^{bot} &= \{x \in R;\ 0 \le x_2 \le l_2/2+nd\} \end{aligned}$$

Notice that, since $l_1 \ge 0.1 l_2$ and $l_2 > L$, we can use our assumption and apply proposition 2.12 to get that

$$\begin{aligned} \sup_{\tau\in\Omega} \Big\| 1-\frac{d\mu^\tau_{R_n^{top}}}{d\mu^\tau_R}\Big|_{\partial_r^+ R_n^{bot}\cap R_n^{top}}\Big\|_\infty &\le \sup_{\tau,\eta\in\Omega:\,\tau_{R^c}=\eta_{R^c}} \Big\| 1-\frac{d\mu^\tau_{R_n^{top}}}{d\mu^\eta_{R_n^{top}}}\Big|_{\partial_r^+ R_n^{bot}\cap R_n^{top}}\Big\|_\infty \\ &\le e^{-2C\sqrt{L}} \end{aligned}$$

for any L large enough and a suitable positive constant C independent of L.
Thus, thanks to (3.12), (3.29), (3.33) and proposition 3.5, for any $f \in L^2(\Omega, d\mu^\tau_V)$ we get

$$\mathrm{Var}^\tau_R(f) \le \lambda(\tau,L,R_n^{top},R_n^{bot})\big\{\mathcal{E}^\tau_R(f,f)+\frac{1}{2}\mu^\tau_R\big(\sum_{x\in R_n^{top}\cap R_n^{bot}} c_J(x,\sigma)|\nabla_x f|^2\big)\big\} \tag{4.14}$$

where

$$\lambda(\tau,L,R_n^{top},R_n^{bot}) = (1-e^{-C\sqrt{L}})^{-1}\big(\mathrm{gap}(L^\tau_{R_n^{top}})\wedge \mathrm{gap}(L^\tau_{R_n^{bot}})\big)^{-1}$$

Notice that, as n varies, the strips $R_n^{top}\cap R_n^{bot}$ are disjoint. Therefore, if we average (4.14) over $n \in \big[1, \frac{L}{10d}\big]$ and use the trivial bound

$$\sum_{n=1}^{[L/10d]} \frac{1}{2}\mu^\tau_R\big(\sum_{x\in R_n^{top}\cap R_n^{bot}} c_J(x,\sigma)|\nabla_x f|^2\big) \le \mathcal{E}^\tau_R(f,f)$$

we get

$$\mathrm{Var}^\tau_R(f) \le \big(1+[\frac{L}{10d}]^{-1}\big)\sup_{1\le n\le \frac{L}{10d}} \lambda(\tau,L,R_n^{top},R_n^{bot})\mathcal{E}^\tau_R(f,f) \tag{4.15}$$

i.e

$$\mathrm{gap}(L^\tau_R) \ge (1-\frac{C_1}{\sqrt{L}}) \min_{1\le n\le [L/10d]}\big(\mathrm{gap}(L^\tau_{R_n^{top}})\wedge \mathrm{gap}(L^\tau_{R_n^{bot}})\big) \tag{4.16}$$

for a suitable positive constant C_1.

Let us now fix $n \in \big[1, \frac{L}{10d}\big]$ and examine the spectral gap of the bottom rectangle R_n^{bot}, the reasoning being similar for the top one.
There are two cases to analyze:

a) $l_1 \leq \frac{3}{2}L$. In this case one easily verifies that $R_n^{bot} \in \mathcal{R}_{\frac{3}{2}L}$ and thus, by definition,
$\mathrm{gap}(L^\tau_{R_n^{bot}}) \geq g(\frac{3}{2}L)$

b) $l_1 > \frac{3}{2}L$. In this case $R_n^{bot} \in \mathcal{R}_{2L}$ but now the *longest* side is l_1 and the *shortest* one is smaller than $\frac{l_2}{2} + nd + 1$ which in turn is smaller than $1.2L$ since $l_2 \leq 2L$ and
$nd \leq \frac{L}{10}$

In conclusion we obtain that the r.h.s of (4.16) is larger than

$$[1 - \frac{C_1}{\sqrt{L}}]\Big(g(\frac{3}{2}L) \wedge \min_{\substack{R(l_1,l_2)\,\in\,\mathcal{R}_{2L} \\ l_1\,\leq\,1.2L,\, l_2 \geq 3/2L}} \mathrm{gap}(L^\tau_{R(l_1,l_2)})\Big) \tag{4.17}$$

In (4.17), without loss of generality, we kept the longest side always along the second coordinate direction.
Let us finally apply the lower bound (4.16) to an arbitrary element $R(l_1,l_2) \in \mathcal{R}_{2L}$ such that $l_1 \leq 1.2L$ and $l_2 \geq 3/2L$ and let us repeat the previous two–cases analysis. Since $l_1 \leq 1.2L$, case *b*) above is no longer possible and thus we get immediately from (4.16) that

$$\min_{\substack{R(l_1,l_2)\,\in\,\mathcal{R}_{2L} \\ l_1\,\leq\,1.2L,\, l_2 \geq 3/2L}} \mathrm{gap}(L^\tau_{R(l_1,l_2)})) \geq (1 - \frac{C_1}{\sqrt{L}})g(\frac{3}{2}L) \tag{4.18}$$

By combining (4.17) with (4.18) we finally get

$$g(2L) = \min_{R\in\mathcal{R}_{2L}} \mathrm{gap}(L^\tau_R) \geq \big(1 - \frac{C_1}{\sqrt{L}}\big)^2 g\big(\frac{3}{2}L\big) \tag{4.19}$$

In order to conclude the proof of the theorem it is enough to iterate two more times (4.19). □

Proof of theorem 4.6. We proceed exactly as in the proof of theorem 4.5 but instead of using block–dynamics with two blocks in order to relate the logarithmic Sobolev constant on scale $2L$ to that on scale L we use conditional expectation. The reason for this is that, contrary to what happens for the spectral gap, the Dirichlet form of a block dynamics with few large blocks may have a logarithmic Sobolev constant much larger than the logarithmic Sobolev constant of a corresponding single spin flip dynamics (see the remark at the end of the proof of corollary 3.6).

If $c_s(2L) = c_s(L)$ there is nothing to prove so we suppose that $c_s(2L) > c_s(L)$. Let us consider a rectangle $R \equiv R(l_1,l_2) \in \mathcal{R}_{2L}$ and assume, without loss of generality, that the longest side is l_2. Since we are assuming that $c_s(2L) > c_s(L)$ necessarily $L < l_2 \leq 2L$. Let also d, R_n^{top}, R_n^{bot} be as in the proof of proposition 4.5. Then, for any non–negative $f \in L^2(\Omega, \mu^\tau_R)$ such that $\mu^\tau_R(f^2) = 1$, we write

$$\begin{aligned} \mu^\tau_R(f^2\log f) &= \mu^\tau_R\big(\mu^\tau_{R_n^{bot}}(f^2\log f)\big) \\ &\leq \big[\sup_{n\in[1,L/10d]} c_s(\mu^\tau_{R_n^{bot}})\big]\frac{1}{2}\mu^\tau_R\big(|\nabla_{R_n^{bot}} f|^2\big) + \mu^\tau_R(g_n^2\log g_n) \end{aligned} \tag{4.20}$$

where $g_n \equiv \big(\mu^\tau_{R_n^{bot}}(f^2)\big)^{\frac{1}{2}}$.

Notice that $\Lambda_{g_n} \subset R_n^{top} \setminus R_n^{bot}$ and that, as in the proof of theorem 4.5, our assumption implies

$$\sup_{\tau\in\Omega} \Big\| 1 - \frac{d\mu^\tau_{R_n^{top}}}{d\mu^\tau_R}\Big|_{R_n^{top}\setminus R_n^{bot}} \Big\|_\infty \le e^{-2C\sqrt{L}}$$

for any L large enough and a suitable positive constant C independent of L.

Thus we can use proposition 3.11 to estimate from above the second term in the r.h.s of (4.20) by

$$\mu^\tau_R(g_n^2 \log g_n) \le (1 + e^{-C\sqrt{L}}) c_s(\mu^\tau_{R_n^{top}}) \frac{1}{2} \mu^\tau_R\big(|\nabla_{R_n^{top}} g_n|^2\big) + \mu^\tau_R(g_n^2) \log\big(\sqrt{\mu^\tau_R(g_n^2)}\,\big) \tag{4.21}$$

Notice that the last term in the r.h.s of (4.21) is zero because of the identity $\mu^\tau_R(g_n^2) = \mu^\tau_R(f^2) = 1$. In conclusion the r.h.s. of (4.20) is bounded from above by

$$\begin{aligned} &\big[\sup_{n\in[1,L/10d]} c_s(\mu^\tau_{R_n^{bot}})\big] \frac{1}{2}\mu^\tau_R\big(|\nabla_{R_n^{bot}} f|^2\big) + \\ &(1+e^{-C\sqrt{L}})\big[\sup_{n\in[1,L/10d]} c_s(\mu^\tau_{R_n^{top}})\big] \frac{1}{2}\mu^\tau_R\big(|\nabla_{R_n^{top}} g_n|^2\big) \end{aligned} \tag{4.22}$$

In order to conclude the proof we need an estimate of the term $|\nabla_{R_n^{top}} g_n|^2$ in term of quantities like $|\nabla_{R_n^{top}} f|^2$. This estimate is provided by the following key lemma.

Lemma 4.7. *In the same hypotheses of theorem 4.6 there exist two positive constants k_1 and k_2 such that*

$$|\nabla_{R_n^{top}} g_n|^2 \le \mu^\tau_{R_n^{bot}}(|\nabla_{R_n^{top}} f|^2) + k_1 \mu^\tau_{R_n^{bot}}(|\nabla_{R_n^{bot}\cap R_n^{top}} f|^2) + e^{-k_2\sqrt{L}} \mu^\tau_{R_n^{bot}}(|\nabla_{R_n^{bot}} f|^2)$$

for all L large enough.

Let us postpone the proof of lemma 4.7 and let us conclude the proof of proposition 4.6. If we insert the bound of lemma 4.7 into (4.22) we get our final estimate

$$\begin{aligned} \mu^\tau_R(f^2 \log f) \le& \\ &(1+e^{-k_3\sqrt{L}}) \sup_{n\in[1,L/10d]} \big[c_s(\mu^\tau_{R_n^{bot}}) \vee c_s(\mu^\tau_{R_n^{top}})\big] \frac{1}{2}\mu^\tau_R(|\nabla_R f|^2) + \\ &k_4\big[\sup_{n\in[1,L/10d]} c_s(\mu^\tau_{R_n^{top}})\big] \frac{1}{2}\mu^\tau_R\big(|\nabla_{R_n^{bot}\cap R_n^{top}} f|^2\big) \end{aligned} \tag{4.23}$$

for suitable constants k_3, k_4 and all L large enough independent of f.
By averaging inequality (4.23) over $n \in [1, L/10d]$ as in the proof of theorem 4.5, we get our final bound

$$\mu^\tau_R(f^2 \log f) \le (1 + \frac{k_5}{\sqrt{L}}) \sup_{n\in[1,L/10d]} \big[c_s(\mu^\tau_{R_n^{bot}}) \vee c_s(\mu^\tau_{R_n^{top}})\big] \frac{1}{2}\mu^\tau_R(|\nabla_R f|^2) \tag{4.24}$$

for a suitable constant k_5, any L large enough and any non–negative $f \in L^2(\Omega, \mu^\tau_R)$ with $\mu^\tau_R(f^2) = 1$. By definition, such inequality implies that

$$c_s(2L) \le (1 + \frac{k_5}{\sqrt{L}}) \sup_{n \in [1, L/10d]} \left[c_s(\mu^\tau_{R^{bot}_n}) \vee c_s(\mu^\tau_{R^{top}_n}) \right] \tag{4.25}$$

for any L large enough.

At this point we continue exactly as in the proof of theorem 4.5 after (4.16) and conclude that

$$c_s(2L) \le (1 + \frac{k_7}{\sqrt{L}}) c_s(\frac{3}{2}L) \tag{4.26}$$

for any L large enough and a suitable constant k_7. Two more iterations of (4.26) suffice to conclude the proof of the theorem. □

In order to conclude our analysis of the logarithmic Sobolev constant we have to prove lemma 4.7. This seemingly technical point is however a key part of the whole approach and its proof requires a new recursive argument that we present in the following lemma.

Lemma 4.8. *Given $R \in \mathcal{R}_L$ and $m > 0$, define $k_m(R)$ as the smallest constant k such that*

$$|\nabla_x \sqrt{\mu^\tau_R(f^2)}|^2 \le k\big(\mu^\tau_R(|\nabla_x f|^2) + \sum_{y \in R} e^{-m|x-y|} \mu^\tau_R(|\nabla_y f|^2)\big) \tag{4.27}$$

for all $\tau \in \Omega$, $x \in \partial^+_r R$ and $f \in L^2(\Omega, d\mu^\tau_R)$, r being the range of the interaction. Let also $k_m(L) \equiv \sup_{R \in \mathcal{R}_L} k_m(R)$.
Then, in the same hypothesis of theorem 4.6 there exist two positive constants m_0 and a such that

$$k_m(2L) \le (1 + e^{-aL}) k_m(L)$$

for any $m \le m_0$ and any L large enough.

Corollary 4.9. *In the same hypothesis of theorem 4.6 there exist two positive constants k and m such that, for any rectangle $R(l_1, l_2)$ with $l_1 \wedge l_2 \ge 0.1\, l_1 \vee l_2$, for all $\tau \in \Omega$, $x \in \partial^+_r R$ and $f \in L^2(\Omega, d\mu^\tau_R)$*

$$|\nabla_x \sqrt{\mu^\tau_R(f^2)}|^2 \le k\big(\mu^\tau_R(|\nabla_x f|^2) + \sum_{y \in R} e^{-m|x-y|} \mu^\tau_R(|\nabla_y f|^2)\big) \tag{4.28}$$

Proof of the Corollary. It follows immediately from lemma 4.8 that the $\sup_L k_m(L) < \infty$ if m is small enough. Thus the statement for "fat" rectangles with large enough size follows from lemma 4.8. The statement for "fat" rectangles with small size is obviously true if we take the constant k large enough. □

Proof of Lemma 4.7. Take $x \in R_n^{top}$. Then, if $x \notin \partial_r^+ R_n^{bot}$, we compute directly $|\nabla_x g_n|^2$ to found that

$$|\nabla_x g_n|^2 = \mu^\tau_{R_n^{bot}}(|\nabla_x f|^2)$$

If instead $x \in \partial_r^+ R_n^{bot}$ we apply corollary 4.9 and get

$$|\nabla_x g_n|^2 \le k\big(\mu^\tau_{R_n^{bot}}(|\nabla_x f|^2) + \sum_{y \in R_n^{bot}} e^{-m|x-y|}\mu^\tau_{R_n^{bot}}(|\nabla_y f|^2)\big)$$

for a suitable constant k and m. By summing over $x \in R_n^{top}$ we get the sought bound. □

Proof of Lemma 4.8. Given $m > 0$ and L large enough, we suppose that $k_m(2L) > k_m(L)$. Let us consider a rectangle $R \equiv R(l_1, l_2) \in \mathcal{R}_{2L}$ and assume, without loss of generality, that the longest side is l_2. Since we are assuming that $k_m(2L) > k_m(L)$ necessarily $L < l_2 \le L$. Let us also fix $x \in \partial_r^+ R$ and $f \in L^2(\Omega, d\mu^\tau_R)$. A simple computation (see e.g [Z] or [MO2]) shows that

$$|\nabla_x f^2| = 2|\langle\, f\,\rangle_x|\,|\nabla_x f| \tag{4.29}$$

where $\langle g\rangle_x = \frac{1}{2}\big(g(\sigma^x) + g(\sigma)\big)$.

Thus, if

$$|\nabla_x \mu^\tau_R(f^2)| \le 2|\langle \sqrt{\mu^\tau_R(f^2)}\,\rangle_x|\,A$$

then

$$|\nabla_x \sqrt{\mu^\tau_R(f^2)}\,| \le A \tag{4.30}$$

Let us compute $|\nabla_x \mu^\tau_R(f^2)|$. For this purpose let R_i, $i = 1,2,3$ be the rectangles in R defined by

$$R_1 = \big\{(x_1,x_2) \in \mathbb{Z}^2;\ 0 \le x_1 \le l_1 - 1,\ 0 \le x_2 \le \frac{l_2}{2}\ \big\}$$
$$R_2 = \big\{(x_1,x_2) \in \mathbb{Z}^2;\ 0 \le x_1 \le l_1 - 1,\ \frac{l_2}{4} \le x_2 \le \frac{3l_2}{4}\ \big\}$$
$$R_3 = \big\{(x_1,x_2) \in \mathbb{Z}^2;\ 0 \le x_1 \le l_1 - 1,\ \frac{l_2}{2} \le x_2 \le l_2\ \big\}$$

Then , depending whether the second coordinate x_2 of the site x lies in the interval $I_1 = (-\infty, \frac{3}{8}l_2]$, $I_2 = (\frac{3}{8}l_2, \frac{5}{8}l_2]$ or $I_3 = (\frac{5}{8}l_2, \infty)$, we write, using (2.2),

$$|\nabla_x \mu^\tau_R(f^2)| = |\nabla_x \mu^\tau_R\big(\mu^\tau_{R_i}(f^2)\big)| \qquad \text{if} \qquad x_2 \in I_i \qquad i = 1,2,3 \tag{4.31}$$

Let us consider for shortness only the first case, $x_2 \in I_1$, the other ones being similar. A simple computation shows that

$$|\nabla_x \mu^\tau_R\big(\mu^\tau_{R_1}(f^2)\big)| \le |\,\mu^\tau_R\big((h_x - 1)\mu^{\tau^x}_{R_1}(f^2)\big)| + |\,\mu^\tau_R\big(\nabla_x \mu^\tau_{R_1}(f^2)\big)| \tag{4.32}$$

where $h_x \equiv \frac{d\mu_R^{\tau^x}}{d\mu_R^\tau}\Big|_{R\setminus R_1}$. Notice that, since $d(x, R\setminus R_1) \geq 0.1 l_2 \geq 0.1L$, for any L large enough our assumption together with proposition 2.12 implies

$$\|h_x - 1\|_\infty \;\leq\; e^{-CL}$$

for a suitable constant C independent of τ, L and x. Thus the first term in the r.h.s of (4.32) can be bounded from above by

$$\begin{aligned}
&\sqrt{2}e^{-CL}\sqrt{\mu_R^\tau(\mu_{R_1}^{\tau^x}(f^2))}\sqrt{\mathrm{Var}_R^\tau\big(\sqrt{\mu_{R_1}^{\tau^x}(f^2)}\big)} \\
&\qquad \leq\; ke^{-CL}\sqrt{\mu_R^\tau(f^2)}\sqrt{\mathrm{Var}_R^{\tau^x}\big(\sqrt{\mu_{R_1}^{\tau^x}(f^2)}\big)} \\
&\qquad \leq\; ke^{-CL}\langle\, \sqrt{\mu_R^\tau(f^2)}\,\rangle_x\{\mathrm{Var}_R^\tau(f)\}^{\frac{1}{2}} \\
&\qquad \leq\; \bar{k}e^{-CL}\langle\, \sqrt{\mu_R^\tau(f^2)}\,\rangle_x\{\mu_R^\tau(|\nabla_R f|^2)\}^{\frac{1}{2}}
\end{aligned} \tag{4.33}$$

for a suitable constant k. In the above steps we used the simple bounds valid for any non–negative g

$$\begin{aligned}
|\,\mu_R^\tau(h;g)\,| &\leq \|h\|_\infty\sqrt{2\mu_R^\tau(g)}\sqrt{\mathrm{Var}_R^\tau(\sqrt{g})} \\
\|\frac{d\mu_R^{\tau^x}}{d\mu_R^\tau}\|_\infty \;&\leq\; e^{2\|J\|_\infty} \\
\mathrm{Var}_R^{\tau^x}\big(\sqrt{\mu_{R_1}^{\tau^x}(f^2)}\big) \;&\leq\; \mathrm{Var}_R^{\tau^x}(f) \\
\mathrm{Var}_R^{\tau^x}(g) \;&\leq\; \|\frac{d\mu_R^{\tau^x}}{d\mu_R^\tau}\|_\infty^2\,\mathrm{Var}_R^\tau(g)
\end{aligned}$$

(3.12) and the fact that $\sup_L \sup_{R\in\mathcal{R}_L} \mathrm{gap}(L_R^\tau)^{-1} < \infty$.

Let us bound from above the second term in the r.h.s of (4.32). We observe that each R_i, $i = 1, 2, 3$, either belongs to $\mathcal{R}_{\frac{3}{2}L}$ or it belongs to $\mathcal{R}_{2L}$ but with the shortest side smaller than L, depending whether l_1 is smaller or larger than $\frac{3}{2}L$. Thus we can use the definition of $k_m(L)$, $k_m(R)$ together with (4.29) to write

$$\begin{aligned}
&|\nabla_x\mu_{R_1}^\tau(f^2)| \\
&\leq\; 2\langle\, \sqrt{\mu_{R_1}^\tau(f^2)}\,\rangle_x\,\sqrt{k'_m(L)}\{\,\mu_{R_1}^\tau(|\nabla_x f|^2) + \sum_{y\in R_1} e^{-m|x-y|}\mu_{R_1}^\tau(|\nabla_y f|^2)\,\}^{\frac{1}{2}}
\end{aligned} \tag{4.34}$$

with

$$k'_m(L) = \begin{cases} \sup_{\substack{R(l_1,l_2)\in\mathcal{R}_{2L} \\ l_1\leq L}} k_m(R) & \text{if } l_1 > \frac{3}{2}L \\ k_m(\frac{3}{2}L) & \text{otherwise} \end{cases}$$

Above, as in the proof of theorem 4.5, without loss of generality we kept the longest side always in the second direction. A final Schwartz inequality together with (2.2) gives

$$\begin{aligned}&\mu_R^\tau\big(|\nabla_x \mu_{R_1}^\tau(f^2)|\big)\\ &\le 2\langle \sqrt{\mu_R^\tau(f^2)}\,\rangle_x \sqrt{k'_m(L)}\big\{\mu_R^\tau(|\nabla_x f|^2)+\sum_{y\in R_1} e^{-m|x-y|}\mu_R^\tau(|\nabla_y f|^2)\big\}^{\frac12}\end{aligned} \tag{4.35}$$

In conclusion, if we combine together (4.30), (4.33) and (4.34) we get

$$\begin{aligned}&|\nabla_x \sqrt{\mu_R^\tau(f^2)}\,|^2 \le\\ &\Big\{\bar k e^{-CL}\{\mu_R^\tau(|\nabla_R f|^2)\}^{\frac12}+\sqrt{k'_m(L)}\big\{\mu_R^\tau(|\nabla_x f|^2)+\sum_{y\in R} e^{-m|x-y|}\mu_R^\tau(|\nabla_y f|^2)\big\}^{\frac12}\Big\}^2\\ &\le 3\bar k e^{-CL}\mu_R^\tau(|\nabla_R f|^2)+(1+\bar k e^{-CL})k'_m(L)\big\{\mu_R^\tau(|\nabla_x f|^2)\\ &+\sum_{y\in R} e^{-m|x-y|}\mu_R^\tau(|\nabla_y f|^2)\big\}\end{aligned} \tag{4.36}$$

where, without any harm, we have extended the summation over $y\in R_1$ to the whole rectangle R.

Observe that

$$3\bar k e^{-CL}\mu_R^\tau(|\nabla_R f|^2)\ \le\ e^{-(C/2)L}\sum_{y\in R} e^{-m|x-y|}\mu_R^\tau(|\nabla_y f|^2)$$

if m is small enough compared to the constant C and CL is large enough. Thus the r.h.s of (4.36) is smaller than

$$k'_m(L)(1+e^{-(C/3)L})\big\{\mu_R^\tau(|\nabla_x f|^2)+\sum_{y\in R} e^{-m|x-y|}\mu_R^\tau(|\nabla_y f|^2)\big\}$$

i.e.

$$k_m(R)\ \le\ k'_m(L)(1+e^{-(C/3)L}) \tag{4.37}$$

If we now reapply the whole procedure to an arbitrary rectangle in $\mathcal{R}_{2L}$ but with the shortest side smaller than L, we get from (4.37) that

$$k'_m(L)\ \le\ (1+e^{-(C/3)L})\,k_m(\frac32 L)$$

In conclusion

$$k_m(R)\ \le\ (1+e^{-(C/3)L})^2 k_m(\frac32 L)\qquad \forall R\in\mathcal{R}_{2L} \tag{4.38}$$

provided that L is large enough and m small enough.
Two iterations of (4.38) suffices to conclude the proof. □

We conclude by stating one final result, whose proof is a minor modification of the previous ones, that will turn out to be quite useful when discussing systems with random interactions.

Proposition 4.10. *Fix L_0 and let Λ be a multiple of Q_{L_0}. Assume that there exists $m > 0$ such that $SMT(V, L_0/2, m)$ holds for all subsets V of Λ that are multiples of Q_{L_0}. Then there exists a constant $\varepsilon = \varepsilon(m)$ such that, if $L_0^d e^{-m\sqrt{L_0}} \leq \varepsilon$, then*

$$\begin{aligned}\inf_\tau \operatorname{gap}(L_\Lambda^\tau) &\geq \frac{1}{2} \inf_\tau \operatorname{gap}(L_{Q_{L_0}}^\tau)\\ \sup_\tau c_s(\mu_\Lambda^\tau) &\leq 2 \sup_\tau c_s(\mu_{Q_{L_0}}^\tau)\end{aligned}$$

5. Boundary Phase Transitions

In this section we analyze more closely the problem of the exponential ergodicity in the uniform norm of the dynamics for three dimensional systems in the one phase region.

Already in [MO1] (see also [MOS]) it was discussed the possibility that, even if the interaction is such that there is no phase transition and weak mixing holds, in a finite large enough cube Q_L one could have some sort of "boundary phase transition" which could slow down dramatically the approach to equilibrium.

We will provide here a substantial evidence that this phenomenon happens in the three dimensional Ising model at low temperature and nonzero magnetic field h (see (2.4)) for which weak mixing holds. More precisely if β is large enough we will argue that it is possible to tune the magnetic field as a function of β in such a way that the gap in the spectrum of the generator of the dynamics in a finite cube with suitably chosen boundary conditions becomes *exponentially small* in the side of the cube. Notice that, for the same value of the thermodynamic parameters weak mixing holds *i.e.* theorem 4.1 applies and the gap of any attractive *infinite volume* Glauber dynamics is bounded away from zero. Actually the same arguments that proved theorem 4.1 could be used to show that the thermodynamic limit of the gap in finite cubes with $+$ boundary conditions is bounded away from zero [SchYo] (see also section 5.3 below).

In order to clarify the discussion, let us consider the three dimensional Ising model in a cube Q_L of side L, at low temperature and small, positive external field h. As boundary conditions we take $-$ on the bottom face V of the cube and free (*i.e.* absent) on the other faces. Since the magnetic field is positive and the temperature is very low, the typical configurations of the systems will be mostly plus spins (plus phase) away from the bottom face. Thus there will be, with large probability, a unique Peierls contour (*i.e.* a connected union of dual plaquettes separating nearest neighbor spins of opposite sign) Γ separating the plus bulk phase from the minus boundary conditions on the bottom face, and it is quite clear that the statistical properties of such a contour will play an important role in the mixing properties of the Gibbs measure of the system and on the rate of approach to equilibrium of an associated Glauber dynamics. Unfortunately a detailed analysis of the probability distribution of the contour Γ and its dynamical propertie requires quite deep ideas and technology like the Pirogov-Sinai theory (see e.g. [Zha]). Such analysis becomes simpler in of uses the so called solid-on-solid (SOS) approximation for Γ.

5.1 The Solid-on-Solid Approximation

We approximate the contour Γ with a two dimensional surface $\varphi = \{\varphi(x)\}_{x\in V}$, where the random variable $\varphi(x) \in \mathbb{N}$ represents the height of the surface at x above the bottom face of the cube, and we assume that the probability distribution of the surface with ψ boundary condition outside V is that of the solid–on–solid model (SOS)

$$\mu_V^{h,\psi}(\varphi) = \frac{1}{Z_V^{h,\psi}} \exp\left[-H_V^{h,\psi}\right]$$

where

$$H_V^{h,\psi}(\varphi) = \frac{J}{2} \sum_{\substack{x,y \in V; \\ |x-y|=1}} |\varphi(x) - \varphi(y)| \; + \; h \sum_{x \in V} \varphi(x) \; + \; J \sum_{\substack{x \in V \; y \in V^c \\ |x-y|=1}} |\varphi(x) - \psi(y)| \tag{5.1}$$

The SOS approximation corresponds to taking the original Gibbs measure and conditioning on the event that for any $x = (x_1, x_2, -1)$ in V there exists a non–negative integer $\varphi(x)$ such that

$$\sigma(x_1, x_2, n) = \begin{cases} -1 & \text{if } n \le \varphi(x) \\ +1 & \text{if } n > \varphi(x) \end{cases}$$

Although strictly speaking the above event has a very small probability w.r.t the original Ising Gibbs measure, the SOS model is considered a reasonable approximation at very low temperatures (large values of J).

A kinetic version of the model is readily obtained by considering a Glauber dynamics for it, namely a single spin Markov process on the configuration space $\Omega = \mathbb{N}^V$, reversible with respect to $\mu_V^\psi(\varphi)$ and such that each move consists in replacing at some site x, $\varphi(x)$ with $\varphi(x) \pm 1$ (but the new φ has to be non–negative) with rates that satisfy H1, H3, H4 and are reversible w.r.t the Gibbs measure $\mu_V^{h,\psi}$.

Two cases one may want to keep in mind are (χ is the characteristic function, $s = \pm$ and $\varphi^{x,s} = \varphi(x) + s$)

$$c(x, \varphi, s) = \min\bigl\{e^{-J\Delta_{x,s} H_V^{h,\psi}(\varphi)}\,, 1\bigr\}\, \chi\{\varphi^{x,s} \in \Omega\}$$

and

$$c(x, \varphi, s) = \Bigl[1 + \exp\bigl(J\Delta_{x,s} H_V^{h,\psi}(\varphi)\bigr)\Bigr]^{-1} \chi\{\varphi^{x,s} \in \Omega\}$$

where

$$\Delta_{x,s} H_V^{h,\psi}(\varphi) = H_{\{x\}}^{\varphi}(\varphi(x) + s) - H_{\{x\}}^{\varphi}(\varphi(x))\,.$$

As before we will denote by $L_V^{h,\psi}$ the corresponding generator in the region V with boundary conditions ψ. It is worthwhile to mention that, with above choice of the jump rates, the dynamics becomes attractive.

For the SOS model described by (5.1), and variants of it, it has long been realized (see e.g. [BEF], [FP1], [FP2] and more recently [LM], [BDZ], [DM]) that the relevant statistical properties of the surface φ are determined by the competition between the attraction to the wall due to the external field h and the “entropic repulsion” from the wall. Although the techniques to rigorously study these competing effects are quite involved, nevertheless the basic ideas can be understood in simple terms.

Imagine that the relevant configurations of the SOS Gibbs measure are “flat” rigid surfaces at certain height, with isolated small fluctuations that we assume to be only up or down parallelepipeds of unit base and certain length. Let us denote by $\Phi(k)$ the set of such surfaces which sit at height $k \in \mathbb{N}$. Clearly, if $\varphi \in \Phi(k)$, the downward fluctuations cannot be longer than k itself because of the presence of the wall under the surface. Thus, if $k < k'$, the cardinality of $\Phi(k')$ is larger than that of $\Phi(k)$. Having this in mind one can then try to compute the value of k such that $\mu_V^{h,\psi}\big(\Phi(k)\big)$ takes its maximum, or, what is the same, compute the ratio

$$\lambda(k) \equiv \frac{\mu_V^{h,\emptyset}\big(\Phi(k)\big)}{\mu_V^{h,\emptyset}\big(\Phi(k-1)\big)}$$

where $\emptyset$ means free (*i.e.* absent) boundary conditions. One easily gets from (5.1) that $\lambda(k)$ is approximately (neglecting the boundary conditions) given by

$$\lambda(k) \approx \exp(-hL^2 + L^2 e^{-4Jk})$$

where the term $-hL^2$ takes into account the energy one has to pay in order to lift up by one unit a rigid flat interface, while the term $L^2 e^{-4Jk}$ represents the entropy gain obtained by increasing by one the height. Notice that the factor e^{-4Jk} is the energy one has to pay in order to create a downward fluctuation of length k, namely a fluctuation which can be present in a surface $\varphi \in \Phi(k)$ and not in a surface $\varphi \in \Phi(k-1)$.

The above computation suggests that, when L is very large and the magnetic field h is decreased from value $e^{-4J(k-1)}$ to the new value e^{-4Jk}, the average height of the surface should jump from $k-1$ to k. Thus there should exist a critical value $h^*_{k-1}(J)$ between $e^{-4J(k-1)}$ and e^{-4Jk} such that, for $h = h^*_{k-1}(J)$, the surface is undecided whether to stay at height $k-1$ or k. It is such a situation that we call "layering phase transition".

In [DM] the phenomenon of "layering phase transition" was proved for the first time. Their result was then extended and improved in [CM1]. Let us recall the main result of [CM1], since it plays an essential role in our analysis of the non equilibrium case.

Theorem 5.1. *There exists J_0 such that for all $J \geq J_0$ there are positive numbers $\{h^*_k(J)\}_{k=1}^{k_{max}}$, with $k_{max} = \lfloor e^{\frac{J}{20000}} \rfloor$, such that the following holds for $k = 1, \dots, k_{max}$*

i) $\frac{1}{4} e^{-4Jk} \leq h^*_k(J) \leq 4e^{-4Jk}$

ii) if $h^*_k(J) < h < h^*_{k-1}(J)$ (define $h^*_0(J) = +\infty$) then

a) there exists a unique Gibbs measure

b) there exist $m(J,h) > 0$, $C(J,h) > 0$ such that for any $L \geq \lfloor 8/h + 1 \rfloor$

$$\sup_{\psi,\psi' \in \Omega} |\mu_{Q_L}^{h,\psi}(\varphi(x_0)) - \mu_{Q_L}^{h,\psi'}(\varphi(x_0))| \leq C(J,h) e^{-m(J,h)\, L}$$

where $\mu_{Q_L}^{h,\psi}(\varphi(x_0))$ denotes the expected value of the height of the surface at the center x_0 of the square Q_L with boundary conditions ψ.

iii) if $h = h^*_k(J)$ there are at least two distinct extreme Gibbs measures.

Remark.

a) Notice that, although the spins $\varphi(x)$ are unbounded, ii.b) above is nothing else but weak mixing in the attractive case (see section 3.1).

b) In [LMaz] the restriction $k \le k_{max}$ was removed and the proof of ii.b) was considerably simplified.

A byproduct of the results of [CM1] is that if $h^*_k(J) < h < h^*_{k-1}(J)$ then the typical configurations of the Gibbs measure $\mu^{h,\psi}_{Q_L}$ are, in a sense that can be made precise, "flat" surfaces at height k. On the other hand, at $h = h^*_k(J)$ and with free boundary condition, in [CM1] it was proved that the ± 1 random variables

$$\sigma(x) = \mathrm{sign}(\varphi(x) - k - \frac{1}{2})$$

behave roughly as a *two dimensional* Ising model at low temperature with *zero* external field and free boundary conditions, i.e a system in the phase coexistence region.

The associated Glauber dynamics in a square Q_L of side L, with boundary conditions ψ and magnetic field h, was then studied in [CM2]. The main result of [CM2] is

Theorem 5.2. *In the same setting as in Theorem [CM1] we have, for all $k = 1, \dots, k_{max}$,*

i) if $h^*_k(J) < h < h^*_{k-1}(J)$ then there exist $L_0(J, h)$, $\kappa(J, h) > 0$ such that

$$\inf_{L \ge L_0} \inf_{\psi \in \Omega} \mathrm{gap}(L^{h,\psi}_{Q_L}) \ge \kappa(J, h)$$

ii) if $h = h^*_k(J)$, then there exist positive constants $C_1(J, h)$, $C_2(J, h)$ such that for all $L > 10/h$

$$C_1(J, h)\, e^{-100JkL} \le \mathrm{gap}(L^{h,\emptyset}_{Q_L}) \le C_2(J, h)\, e^{-\frac{1}{40}JL}$$

where $\emptyset$ means free boundary conditions.

Proof. The proof of i) goes as follows. Since the SOS system is two dimensional one can adapt the arguments of [MOS] to transform the weak mixing condition ii.b) of theorem 5.1 into the strong mixing condition of section 2.3 and then use the recursive analysis of the proof of theorem 4.5 to get i).

To prove part ii), namely to find an upper bound on $\mathrm{gap}(L^{h^*_k(J),\emptyset}_{Q_L})$, one uses the "look for the bottleneck" approach, *i.e.* one takes advantage of the variational characterization for the gap (3.12) and choose an appropriate test function f which illustrates how the system, in order to relax to equilibrium, has to make an excursion to a very unlikely region of the phase space.
Given $\varphi \in \Omega$ we set

$$\sigma(x) = \mathrm{sign}(\varphi(x) - k - 1/2)$$

and, for $U \subset\subset \mathbb{Z}^2$

$$M_U(\sigma) = M_U(\sigma(\varphi)) = \sum\nolimits_{x \in U} \sigma(x)$$
$$m_U(\sigma) = m_U(\sigma(\varphi)) = |U|^{-1} M_U(\sigma)\,.$$

In analogy with the solution of the similar problem for the two dimensional Ising model in the phase coexistence region, we take as a test function

$$f(\varphi) = \chi\{M_{Q_L}(\varphi) > 0\} - \chi\{M_{Q_L}(\varphi) < 0\}\,.$$

Using (3.3), we find

$$\begin{aligned}\mathcal{E}^{h_k^*,\emptyset}_{Q_L}(f,f) &\le \sup_{\varphi\in\Omega}\sum_{s=\pm} c(x,\varphi,s)\,\mu^{h_k^*,\emptyset}_{Q_L}\{|M_{Q_L}(\varphi)| \le 2\}\\ &\le 2c_M(J,h_k^*)\,\mu^{h_k^*,\emptyset}_{Q_L}\{|M_{Q_L}(\varphi)| \le 2\},\end{aligned} \tag{5.2}$$

while

$$\mathrm{Var}^{h_k^*,\emptyset}_{Q_L}(f) = \mu^{h_k^*,\emptyset}_{Q_L}\{M_{Q_L}(\varphi) \neq 0\} - \big(\mu^{h_k^*,\emptyset}_{Q_L}\{M_{Q_L}(\varphi) > 0\} - \mu^{h_k^*,\emptyset}_{Q_L}\{M_{Q_L}(\varphi) < 0\}\big)^2 \tag{5.3}$$

The sougth upper bound now follows from the following two estimates valid for any L large enough. These bounds should be compared with the analogous ones for the two dimensional Ising model with free boundary conditions in the phase coexistence region (see (6.24) and (6.30)).

$$\mu^{h_k^*,\emptyset}_{Q_L}\{|M_{Q_L}(\varphi)| \le 2\} \le e^{-\frac{1}{12}JL}.$$

and

$$\begin{aligned}&\mu^{h_k^*,\emptyset}_{Q_L}\{\varphi\in\Omega_{Q_L} : M_{Q_L}(\sigma(\varphi)) < 0\} \ge e^{-\frac{1}{20}JL}\\ &\mu^{h_k^*(\beta),\emptyset}_{\Lambda}\{\varphi\in\Omega_{Q_L} : M_{Q_L}(\sigma(\varphi)) > 0\} \ge e^{-\frac{1}{20}JL}\end{aligned}$$

5.2 Back to the Ising Model

Going back to the 3D Ising model discussed above, we can conclude that there is a very good evidence that for three dimensional systems satisfying the weak mixing condition it is possible to have at the same time

$$\inf_{\Lambda,\tau}\mathrm{gap}(L^\tau_\Lambda) = 0 \qquad \text{and} \qquad \mathrm{gap}(L^J_{\mathbb{Z}^d}) \,>\, 0$$

due to the occurrence of some sort of "boundary phase transition". Theorem 4.5 or theorem 4.6 imply that in this situation the system cannot satisfy strong mixing.

It is interesting to better understand how the expected value of some local observable over the dynamics in a finite cube Q_L at time t is affected by the phenomenon of boundary phase transition. As before we consider this problem for the three dimensional Ising model (2.4) in Q_L with the boundary conditions described at the beginning of this paragraph, $\beta >> 1$ and $h \equiv h_k^*(\beta)$. Here, in analogy with the SOS approximation, the critical value $h_k^*(\beta)$ should be such that the height of the Peierls contour Γ above the bottom face V is undecided between level k and $k+1$. Without loss of generality we will consider the average magnetization in the center x_0 of the cube.

Our first claim is that up to time $T = k_0^{-1} L$, k_0 being the constant appearing in lemma 3.2

$$\|T^{\tau}_{Q_L}(t)\, \sigma(x_0) - \mu^{\tau}_{Q_L}(\sigma(x_0))\|_\infty \le Ce^{-t} \tag{5.4}$$

with C independent of L.
To prove (5.4) we write

$$\begin{aligned} \|T^{\tau}_{Q_L}(t)\sigma(x_0) - \mu^{\tau}_{Q_L}(\sigma(x_0))\|_\infty \le \quad & \|T^{\tau}_{Q_L}(t)\sigma(x_0) - T^J(t)\sigma(x_0)\|_\infty \\ & + \|T^J(t)\sigma(x_0) - \mu^J(\sigma(x_0))\|_\infty \\ & + |\mu^J(\sigma(x_0)) - \mu^{\tau}_{Q_L}(\sigma(x_0))| \end{aligned} \tag{5.5}$$

where μ^J is, as before, the unique infinite volume Gibbs state. Because of lemma 3.2

$$\|T^{\tau}_{Q_L}(t)\sigma(x_0) - T^J(t)\sigma(x_0)\|_\infty \le e^{-t} \qquad \forall\, t \le k_0^{-1} L$$

Using weak mixing together with theorem 3.3, the second term in the r.h.s of (5.5) is bounded from above by $C_1 e^{-m_1 t}$ for all $t \ge 0$ and suitable constants C_1 and m_1. Finally, thanks again to weak mixing, the third term in the r.h.s of (5.5) is bounded from above by $C_2 e^{-m_2 L} \le C_2 e^{-m_2 k_0 t}$ and (5.4) follows.

For times t larger than $k_0^{-1} L$ but smaller than $e^{\varepsilon L}$, where ε is a small positive constant, we conjecture that

$$T^{\tau}_{Q_L}(t)\sigma(0)(+) - T^{\tau}_{Q_L}(t)\sigma(0)(-) \ge e^{-m' L} \tag{5.6}$$

for a suitable $m' > 0$ independent of L, where $+$ and $-$ denote the two configurations identically equal to $+1$ and -1 respectively. In order to justify (5.6), we observe that the analysis of the SOS model (see [CM2]) for $h = h^*_k(J)$ shows that, with very high probability (larger than $1 - e^{-cL}$, $c > 0$), any interface Γ starting below (above) height k does not jump to height $k+1$ (k) before a time which is exponentially large in L. Therefore, the difference in the l.h.s of (5.6) should be bounded from below, using monotonicity, by the difference of magnetizations at $x = x_0$ computed with the Gibbs states in the parallepiped $V \times [k, L-1]$ with plus and minus b.c. on the bottom face and free b.c. on the remaining faces. Such a difference can be shown to be exactly of the order of the r.h.s of (5.6).

5.3 Recent Progresses

We conclude by discussing some recent progresses on the relaxation properties of the d-dimensional Ising model at low temperature and very small positive magnetic field h. Although our discussion indicates that one may have

$$\inf_{\tau} \mathrm{gap}(L^{\tau}_{Q_L}) \approx e^{-CL}$$

for the three dimensional Ising model at low enough temperature and suitably chosen small magnetic field h, it is also clear that for many natural boundary conditions, e.g plus b.c. of free b.c., such a patology should not occur, since no large contour is present in the system. The next theorem (see [Yo2] and also [SchYo]) goes exactly in this direction.

Theorem 5.3. *Consider the standard d-dimensional Ising model given by (2.4). There exists $\beta_0(d)$ such that for any $\beta \geq \beta_0$ and any $h_0 > 0$ there exists a $c > 0$ such that if $h > 0$ and*

$$\min_{x \in Q_L} \Big\{ h + \sum_{\substack{y \notin Q_L \\ d_2(x,y)=1}} \tau(y) \Big\} \geq h_0$$

then $\mathrm{gap}(L^{\tau}_{Q_L}) \geq c$. *Moreover, if the Glauber dynamics is attractive, there exist two finite positive constants C_1, C_2 independent of L such that*

$$\|T^{\tau}_{Q_L}(t)f - \mu^{\tau}_{Q_L}(f)\|_\infty \leq C_1 |||f||| e^{-C_2 t}$$

The reader may at this point be quite confused about what is known and what it remains to prove in the absence of a phase transition even for the simplest model like the Ising model. We thus summarize the results for the latter in dimension greater than one in the one phase region, *i.e.* when either the magnetic field h is different from zero or the inverse temperature β is strictly smaller than the critical value β_c.

i) The simplest case is two dimensions. Here weak mixing holds and therefore, thanks to theorem 4.1, we have infinite volume exponential ergodicity in the $\|\cdot\|_\infty$ norm. Moreover, thanks to theorems 2.5 and 2.7, the assumptions of theorem 4.6 hold and therefore, thanks to theorem 3.3, exponential ergodicity in any finite cube holds with constants independent of the size of the cube. One can say that in two dimensions the problem of the exponential ergodicity in the one phase region is fully solved.
ii) In higher dimension one still knows that weak mixing, and thus exponential ergodicity in infinite volume, holds for any $\beta < \beta_c$ and any h (see [Hi]) or for any large enough β (much larger than the critical value β_c) and any $h \neq 0$ (see [MO1]). However there is no result concerning weak mixing in the region of non-zero h and β just above β_c. Analogously strong mixing and thus the strong form of exponential ergodicity given by theorems 4.6 and 3.3 for cubes or "fat" parallelepipeds, has been proved only deep inside the one phase region *i.e.* at high temperature ($\beta << \beta_c$) or for $h\beta$ large enough. Moreover, contrary to the two dimensional case, we conjecture (but not prove) that strong mixing fails for β large enough and suitably small h. For these special values of the thermodinamic parameters, the relaxation time in a large cube should drastically depend on the boundary conditions.

6. Phase Coexistence

So far we have discussed the relaxation to equilibrium of the Glauber dynamics in the one phase region. A natural question arises as to what happens when the thermodynamic parameters (e.g inverse temperature and the external magnetic field in the Ising model) are such that we do have a phase transition in the thermodynamic limit.

To be more concrete, let us consider the usual Ising model (2.4) in d-dimensions $d \geq 2$ without external field and let us suppose that the inverse temperature β is larger than the critical value β_c. Then, thanks to theorem 3.1, any associated infinite volume Glauber dynamics is not ergodic and it is rather natural to ask how this absence of ergodicity is reflected if we look at the dynamics in a finite, but large cube Q_L of side L, where ergodicity is never broken.

A first partial answer was provided in [Th] few years ago for very low temperatures. In [Th] it was proved that, if the boundary conditions are free, *i.e.* absent, then the relaxation time to equilibrium, that in a first approximation can be taken equal to the inverse of the gap in the spectrum of the generator $L^{J,\tau}_{V_L}$ of the dynamics, diverges, as $L \to \infty$, at least as an exponential of the *surface* L^{d-1}.

The reason for such a result is the presence of a rather tight "bottleneck" in the phase space. When in fact the boundary conditions are either free or periodic, the energy landscape determined by the function $H^{J,\tau}_{Q_L}(\sigma)$ has only two absolute minima, corresponding to the two configurations identically equal to either $+1$ or to -1. Thus the dynamics started e.g. from all minuses, in order to relax to equilibrium, has to reach the neighborood of the opposite minimum by necessarily crossing the set of configurations of zero magnetization. Since the Gibbs measure gives to the latter a very small weight, of the order of a negative exponential of the *surface* of Q_L (see e.g. [P]), a kind of bottleneck is present and the result follows by rather simple arguments (see section 6.5 below).

The same reasoning also suggests that if the double well structure of the Gibbs measure is completely removed by the boundary conditions, e.g. by fixing equal to $+1$ all spins outside Q_L, or if we measure the relaxation to equilibrium of a function f which is *even* w.r.t a global spin flip $\sigma \to -\sigma$, then the relaxation time should be much shorter than in the previous case since there are no bottlenecks to cross. Actually in this case the interesting but unproven conjecture, is that, at least in two dimensions with plus boundary conditions, the relaxation time will diverge, as $L \to \infty$, like L^2. The proof of the above conjecture would have some very nice consequences on the analysis of the speed of relaxation to equilibrium for the infinite volume dynamics started in one of the pure phases of the system, e.g the plus phase, for which it has been predicted (see [FH]) a stretched exponential decay of the form $e^{-\sqrt{t}}$ in $d = 2$ and a pure exponential law in $d = 3$.

In this section, following [M], [CGMS] and [MM], we consider the above and other related questions for the two dimensional Ising model at inverse temperature β above the critical one β_c and without external field. We will first prove a *lower* bound on the gap in the spectrum of the generator $L^{J,+}_{Q_L}$ of the Glauber dynamics with + boundary conditions of the form:

$$\text{gap}(L^{J,+}_{Q_L}) \geq e^{-\delta L} \tag{6.1}$$

where $\delta = \delta(L)$ is such that $\lim_{L\to\infty} \delta(L) = 0$. Such a bound, although quite far from the conjectured L^2 law, is in any case much larger than the *upper bound* obtained in [Th] with free boundary conditions. We will also show how to use (6.1) to derive an upper bound on the time autocorrelation functions of the infinite volume dynamics started in the plus phase.

Then we will compute exactly the asymptotics of the gap with open boundary conditions and show that

$$\lim_{L\to\infty} -\frac{1}{\beta L}\log(\mathrm{gap}(L^{J,\emptyset}_{Q_L})) = \tau_\beta \tag{6.2}$$

where τ_β denotes the surface tension in the direction of e.g. the horizontal axis (see e.g [Pf] for a precise definition) and $\emptyset$ denotes free (*i.e.* absent) boundary conditions.

In this case, the picture of the relaxation behaviour to the Gibbs equilibrium measure that comes out of our analysis (see also [MM] for an important improvement), is the following: the system first relaxes rather rapidly to one of the two phases and then it creates, via a large fluctuation, a thin layer of the opposite phase along one of the sides of Λ. Such a process requires already a time of the order of $e^{\beta\tau_\beta L}$. After that, the opposite phase invades the whole system by moving, in a much shorter time scale the interface to the side opposite to the initial one and equilibrium is finally reached. The time required for this final process can be computed to be of the order of L^3 at least in the SOS approximation described in section 5 (see [Po]).

Once this picture is established it is not difficult to show that, under a suitable stretching of the time by a factor $a(L) \approx e^{\beta\tau_\beta(0)L}$, the magnetization in the square Q_L behaves in time as a continuous Markov chain with state space $\{-m^*(\beta), +m^*(\beta)\}$ and unitary jump rates , where $m^*(\beta)$ is the spontaneous magnetization (see Theorem 6.1 of [M]).

The key ingredients of our analysis (sse [M] and [CGMS]) are

- (*i*) a geometric bound on the gap in the spectrum of the generator of the dynamics in a rectangle
- (*ii*) some detailed equilibrium estimates related to the large fluctuations of an horizontal interface of length L
- (*iii*) a precise estimate of the equilibrium probability of having anomalous magnetization $m_\Lambda(\sigma) \in (-m^*(\beta), m^*(\beta))$.

The first one (*i*) is borrowed from a clever technique to bound from below the gap of symmetric Markov chains on complicated graphs introduced in [JS1], [JS2] in the framework of hard computational problems. It is important for us that its validity is independent of β. The second, at least for the Ising model, is nowdays available for any temperature below the critical one (see [CGMS]) thanks to the powerful methods of duality (see [Pf]). The third, after the extension of [Io1] and [Io2] of the basic results of [DKS], is also available for any $\beta > \beta_c$ (see [CGMS]).

6.1. Some Preliminary Key Equilibrium Results

Given the two dimensional lattice $\mathbb{Z}^2$ let $\mathbb{Z}^2_* = \mathbb{Z}^2 + (1/2, 1/2)$ denotes its dual. For $x, y \in \mathbb{R}^2$ $[x, y]$ is the *closed segment* with x, y as its endpoints. The *edges* of $\mathbb{Z}^2$ ($\mathbb{Z}^2_*$) are those $e = [x, y]$ with x, y nearest neighbors in $\mathbb{Z}^2$ ($\mathbb{Z}^2_*$). Given an edge e of $\mathbb{Z}^2$, e^* is the unique edge in $\mathbb{Z}^2_*$ that intersects e. We denote by $\mathcal{E}_\Lambda$ the set of all edges such that both endpoints are in Λ and by $\bar{\mathcal{E}}_\Lambda$ the set of all edges with at least one endpoint in Λ.

Given $\Lambda \subset \mathbb{Z}^2$ we define Λ^* as the set of all $x \in \mathbb{Z}^2_*$ such that $d(x,\Lambda) = 1$. The set of the dual edges is defined as

$$\mathcal{E}^*_\Lambda = \{e^* : e \in \bar{\mathcal{E}}_\Lambda\}$$

Notice that, in general, $\mathcal{E}^*_\Lambda \subset \mathcal{E}_{\Lambda^*}$ (the equality holds, for instance, in the case of rectangles). We also define the *boundary* $\delta\Lambda = \{e^* : e \in \bar{\mathcal{E}}_\Lambda \backslash \mathcal{E}_\Lambda\}$

We will often consider our model on a $(2L+1) \times (2M+1)$ rectangle

$$R_{L,M} = \{\, (x_1, x_2) \in \mathbb{Z}^2 : -L \leq x_1 \leq L \,, \; -M \leq x_2 \leq M \,\}$$

and thus $R_{L,L}$ coincides with B_L.

Given $V \subset\subset \mathbb{Z}^2$ and some *boundary condition* (b.c.) $\tau \in \Omega$, we consider the generalized Ising hamiltonian

$$H^{J,\tau}_V(\sigma) = -\frac{1}{2} \sum_{\substack{x,y \\ [x,y] \in \mathcal{E}_V}} J(x,y)\big(\sigma(x)\sigma(y)\big) - \sum_{\substack{x,y \\ [x,y]^* \in \delta V}} J(x,y)\big(\sigma(x)\tau(y)\big) \tag{6.3}$$

The coupling J have been introduced for technical reasons, but our main result is for $J = \beta$, $\beta > \beta_c$ (see section 2). We always assume $0 \leq J(x,y)$ for all x, y.

For further convenience we define the hamiltonian with *free boundary conditions*

$$H^{J,\emptyset}_V(\sigma) = -\frac{1}{2} \sum_{\substack{x,y \\ [x,y] \in \mathcal{E}_V}} J(x,y)\,\sigma(x)\sigma(y) \tag{6.4}$$

When $J(x,y) = \beta$ for all x, y, we substitute the superscript J with β.

When V is a rectangle $V = R_{L,M}$ and $\eta \in \Omega$, we define the $[k]_\eta$ boundary condition by (let $x = (x_1, x_2)$)

$$[k]_\eta(x) = \begin{cases} \eta(x) & \text{if } x_2 \geq M - k + 1 \\ +1 & \text{if } x_2 \leq M - k \end{cases} \tag{6.5}$$

so, in particular, $[0]_\eta$ b.c. means η on the top side of the rectangle and $+1$ on the remaining three sides. A rectangle V has a δ–boundary δV consisting of a top, bottom, left and right side, that we denote respectively with $\delta_t V$, $\delta_b V$, $\delta_l V$ and $\delta_r V$. So, for instance

$$\delta_t R_{L,M} = \big\{\, e = [x,y]^* : [x,y] = [(j,M),(j,M+1)] \;\, j = -L, \ldots, L \,\big\}$$

In the following we will often choose $J = \beta$ everywhere with the exception of one or more sides of a certain rectangle where we take $J = \varepsilon\beta$, $\varepsilon\langle 1$. So, we introduce the notation

$$\begin{aligned}
J^{\square}_\varepsilon(V; x,y) &= \beta \begin{cases} \varepsilon & \text{if } [x,y]^* \in \delta\Lambda \\ 1 & \text{otherwise} \end{cases} \\
J^{\sqcup}_\varepsilon(V; x,y) &= \beta \begin{cases} \varepsilon & \text{if } [x,y]^* \in \delta\Lambda \backslash \delta_t\Lambda \\ 1 & \text{otherwise} \end{cases} \\
J^{\|}_\varepsilon(V; x,y) &= \beta \begin{cases} \varepsilon & \text{if } [x,y]^* \in \delta_l\Lambda \cup \delta_r\Lambda \\ 1 & \text{otherwise} \end{cases} \\
J^{\sqcap}_\varepsilon(V; x,y) &= \beta \begin{cases} \varepsilon & \text{if } [x,y]^* \in \delta\Lambda \backslash \delta_b\Lambda \\ 1 & \text{otherwise} \end{cases}
\end{aligned} \tag{6.6}$$

We are now in a position to state the three main equilibrium estimates that are crucial for our analysis of the spectral gap of the generator with plus or free boundary conditions. The corresponding proofs can be found in [CGMS].

Proposition 6.1. *Let $\beta > \beta_c$ and let $\varepsilon, \alpha \in (0,1]$. Given a positive integer L we set $M = \lfloor \varepsilon L \rfloor$, $k = \lfloor \varepsilon L/10 \rfloor$. Let $\Lambda = R_{L,M} + (0,h)$ be a vertical translate of $R_{L,M}$ contained in B_L and let $\bar\Lambda$ be a vertical translate of $R_{L,N}$ (with $M \le N < L$) such that the bottom sides of Λ and $\bar\Lambda$ coincide. We also let*

$$\Lambda_{\text{bot}} = (x = (x_1, x_2) \in \Lambda : x_2 \le M - 3k + h)$$

Take $J_\alpha = J_\alpha^{\square}(B_L)$. Then there exists $L_0 = L_0(\beta, \alpha, \varepsilon)$ and $m = m(\beta, \alpha, \varepsilon)$ such that if $L \ge L_0$

(i) If the horizontal sides of Λ do not touch the horizontal sides of B_L, then

$$\sup_\eta |\mu_\Lambda^{J_\alpha,+}(A) - \mu_\Lambda^{J_\alpha,[0]_\eta}(A)| \le e^{-mL} \qquad \forall\, A \in \mathcal{F}_{\Lambda_{\text{bot}}} \tag{6.7}$$

(ii)

$$|\mu_\Lambda^{J_\alpha,+}(A) - \mu_{\bar\Lambda}^{J_\alpha,+}(A)| \le e^{-mL} \qquad \forall\, A \in \mathcal{F}_{\Lambda_{\text{bot}}} \tag{6.8}$$

To state the next result we have to introduce the notion of *plus *-chain* for a given configuration σ.

We say that $\{x^1, \dots, x^n\}$ is a *plus *-chain* for σ if $d(x^{i+1}, x^i) = 1$ and $\sigma(x^i) = +1 \qquad \forall\, i$.

Proposition 6.2. *In the same assumptions as in Proposition 6.1, let*

$$\Lambda_{\text{top}} = \Lambda \setminus \Lambda_{\text{bot}}$$
$$\Lambda_{\text{middle}} = \{x = (x_1, x_2) \in \Lambda : |x_2 - h| \le 2k\}$$

and consider the event

$$A = \{\sigma \in \Omega_\Lambda : \exists \text{ a plus *-chain } \{x^1, \dots, x^n\} \subset \Lambda_{\text{top}} \text{ such that } x_1^1 = -L,\ x_1^n = L\}$$

There exists $L_0 = L_0(\beta, \alpha, \varepsilon)$ such that if $L \ge L_0$ then

$$\inf_\eta \mu_\Lambda^{J_\alpha^{\sqcup},[0]_\eta}(A) \ge e^{-\beta(\tau_\beta + 14\varepsilon)(2L+1)} \tag{6.9}$$

Moreover, if $\nu_\Lambda^{J_\alpha^{\sqcup},[0]_\eta}$ denotes the conditional measure $\mu_\Lambda^{J_\alpha^{\sqcup},[0]_\eta}(\cdot\,|A)$ then there exists $m = m(\beta, \alpha, \varepsilon)$ such that, for any $L \ge L_0$

$$\begin{aligned}
&\sup_\eta |\nu_\Lambda^{J_\alpha^{\sqcup},[0]_\eta}(B) - \mu_{\bar\Lambda}^{J_\alpha^{\sqcup},+}(B)\,| \le e^{-mL} \qquad \forall\, B \in \mathcal{F}_{\Lambda_{\text{bot}}} \\
&\sup_\eta |\nu_\Lambda^{J_\alpha^{\sqcup},[0]_\eta}(B) - \mu_{\bar\Lambda}^{J_\alpha^{\|},[0]_\eta}(B)\,| \le e^{-mL} \qquad \forall\, B \in \mathcal{F}_{\Lambda_{\text{middle}}}
\end{aligned} \tag{6.10}$$

In order to formulate our last equilibrium bound we define $m_\Lambda(\sigma) = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sigma(x)$. Then we have (see [CGMS])

Proposition 6.3. *Let $\beta > \beta_c$ and $m \in (-m^*(\beta), m^*(\beta))$. Then*

$$\liminf_{L\to\infty} -\frac{1}{\beta(2L+1)} \log\left[\mu^{\beta,\emptyset}_{Q_L}\{\, m_{Q_L}(\sigma) = m\,\}\right] \geq \varphi(m)$$

where

$$\varphi(m) = \frac{1}{2}w\sqrt{\frac{m^*(\beta) - (|m| \vee m_1)}{2m^*(\beta)}}$$

and the constant w is the value of the Wulff functional W_τ on the Wulff curve (see e.g. [Pf] or [DKS]). The singularity point m_1 satisfies the equation

$$\frac{1}{2}w\sqrt{\frac{m^*(\beta) - m_1}{2m^*(\beta)}} = \tau_\beta$$

6.2 A Geometric Bound on the Spectral Gap

In this part, following [M], we establish via some geometric ideas (see e.g. [SC] for a review) a basic estimate on the spectral gap in a rectangle $R(l_1, l_2)$ which, besides being of independent interest, will play an important role in the determination of the exact asymptotics in the thermodynamic limit of the spectral gap $\text{gap}(L^{\beta,\emptyset}_{Q_L})$. Although we believe that our bound can be derived by other means we think that our derivation is a good illustration of how these geometric techniques work in our context. In what follows we will omit for simplicity in all the notation the boundary condition τ, the interaction J and the set Λ.

We start by introducing the set of *canonical paths* in Ω_Λ between configurations σ and η with $\sigma \neq \eta$. Let us first order the sites in Λ as follows

$$x < y \quad \text{if} \quad x_2 < y_2 \quad \text{or} \quad x_1 < y_1 \quad \text{and} \quad x_2 = y_2$$

Given now $\sigma, \eta \in \Omega_V$ we define the path $\gamma(\sigma, \eta)$ as the sequence of configurations obtained from σ by adjusting one by one, in increasing order, the values of its spins to those of of η. More precisely, if $x^1, \ldots, x^n$ are the sites in Λ ordered as above and such that $\sigma(x^i) \neq \eta(x^i)$, then $\gamma(\sigma, \eta) = (\sigma^0 \ldots \sigma^n)$ where $\sigma^0 = \sigma$ and for $i = 1, \ldots, n$,

$$\sigma^i(x) = \begin{cases} \eta(x) & \text{if } x \leq x^i \\ \sigma(x) & \text{if } x > x^i \end{cases} \tag{6.11}$$

Next, for any allowed transition of the Glauber dynamics $\sigma \to \sigma^x$ and any f we set

$$e = (\sigma, \sigma^x)\,, \qquad Q(e) = \mu(\sigma)c(x, \sigma)\,, \qquad f(e) = f(\sigma^x) - f(\sigma) \tag{6.12}$$

and we say that the transition e belongs to the canonical path γ if, for some i, $e = (\sigma^i, \sigma^{i+1})$. Finally we define the constant ρ_Λ as

$$\rho_\Lambda = \sup_e \sum_{\substack{\sigma,\eta: \\ e\in\gamma(\sigma,\eta)}} \frac{\mu(\sigma)\mu(\eta)}{Q(e)} \tag{6.13}$$

Then we have the following inequality

$$\text{gap}(L_\Lambda) \geq \frac{1}{|\Lambda|}\frac{1}{\rho_\Lambda} \tag{6.14}$$

Although the proof of (6.14) can be found in [SC], we reproduce it below because of its simplicity. Write

$$\begin{aligned}
\text{Var}(f) &= \frac{1}{2}\sum_{\sigma,\eta} \mu(\sigma)\mu(\eta)[f(\sigma) - f(\eta)]^2 \\
&= \frac{1}{2}\sum_{\sigma,\eta} \mu(\sigma)\mu(\eta)[\sum_{e\in\gamma(\sigma,\eta)} f(e)]^2 \\
&\leq \frac{1}{2}|\Lambda|\rho_\Lambda \sum_e Q(e)f(e)^2 \\
&= |\Lambda|\rho_\Lambda \mathcal{E}(f,f)
\end{aligned}$$

and (6.14) follows from the variational characterization of the spectral gap (3.12).

With the above notation our result for the the two dimensional Ising model with Hamiltonian (6.3) reads as follows

Theorem 6.4. *Let $\Lambda = R_{L,M}$ with $L \leq M$ and assume that for any $[x,y] \in \bar{\mathcal{E}}_\Lambda$ $J(x,y) \leq \beta$. Then, uniformly in the boundary conditions τ,*

$$\rho_\Lambda \leq c_m^{-1} e^{4\beta(2L+1)}$$

$$\inf_\tau \text{gap}(L_\Lambda^{J,\tau}) \geq \frac{1}{|\Lambda|} c_m\, e^{-4\beta(2L+1)}$$

where the constant c_m has been defined in (3.3).

Remark. The above estimate on the spectral gap is a very bad one for temperatures above the critical one, *i.e.* $\beta < \beta_c$, since in this case strong mixing holds (see the discussion at the end of section 2.3) and theorem 4.5 applies. At low temperature, instead, when the infinite volume dynamics is not ergodic, it gives the right dependence on the size of the set Λ, namely a negative exponential of the surface and not of the volume. However the constant appearing in the exponential is wrong by a factor 2 even in the limit $\beta \to \infty$. A more precise bound will be discussed in section 6.4.

Proof. The proof is based on a nice idea (see [JS1]) to bound the number of canonical paths that use a given transition e. Given a transition $e = (\xi, \xi^x)$ we define an *injective* mapping Φ from the set of all the canonical paths that use the transition e, $\Gamma(e)$, to Ω_Λ as follows:

$$\begin{aligned} \Phi(\gamma)(y) &= \sigma(y) \qquad \text{if} \qquad y < x \\ \Phi(\gamma)(y) &= \eta(y) \qquad \text{if} \qquad y \geq x \end{aligned}$$

where σ and η denote the starting and end point of γ. It is clear that Φ is iniective. In fact the knowledege of the transition $e = (\xi, \xi^x)$ together with $\Phi(\gamma)$ allow us to reconstruct completely the initial and final configurations σ and η and thus the path itself, simply because

$$\begin{aligned} \sigma(y) &= \xi(y) \qquad \forall\, y \geq x \\ \sigma(y) &= \Phi(\gamma)(y) \qquad \forall\, y < x \end{aligned}$$

and similarly for η.

Let now c_0 be the smallest constant such that for any transition e and any canonical path $\gamma \in \Gamma(e)$, $\gamma \equiv \gamma(\sigma, \eta)$,

$$\mu(\Phi(\gamma))\, Q(e) \;\geq\; \frac{1}{c_0} \mu(\sigma)\mu(\eta) \tag{6.15}$$

Then we have

$$\rho_\Lambda \;\leq\; c_0 \tag{6.16}$$

Using (6.15) we can in fact estimate the r.h.s. of (6.13) by:

$$c_0 \sup_e \sum_{\gamma \in \Gamma(e)} \mu(\Phi(\gamma))$$

Since the map Φ is injective and μ is a probability measure, the above sum is not greater than one and (6.16) follows.

In order to estimate the constant c_0, let $e = (\xi, \xi^x)$ and $\gamma = \gamma(\sigma, \eta) \in \Gamma(e)$ be given. Then, by direct inspection

$$|H(\sigma) + H(\eta) - H(\xi) - H(\Phi(\gamma))| \leq \sup_{[x,y] \in \mathcal{E}_\Lambda} J(x,y)\, 4(2L+1) \leq 4\beta(2L+1)$$

so that

$$\frac{\mu(\sigma)\mu(\eta)}{\mu(\Phi(\gamma))Q(e)} = \frac{\mu(\sigma)\mu(\eta)}{\mu(\Phi(\gamma))\mu(\xi)c(\xi,x)} \leq c_m^{-1}\, e^{4\beta(2L+1)}$$

Thus the constant c_0 can be taken equal to

$$c_0 \;=\; c_m^{-1}\, e^{4\beta(2L+1)}$$

and the theorem follows. $\square$

Remark. It is amusing to observe that, if one applies the above construction to the one dimensional case, one gets $\sup_\Lambda \rho_\Lambda < \infty$ even if the energy $H(\sigma)$ is replaced by a more general expression like

$$H(\sigma) = -\frac{1}{2}\sum_{x,y\in\Lambda} J(|x-y|)\sigma(x)\sigma(y)$$

provided that $\sum_{\substack{x<0\\ y>0}} |J(|x-y|)| < \infty$. Therefore in this case the spectral gap in a segment of length L in $\mathbf{Z}$ has a lower bound which is only proportional to L^{-1} without any negative exponential of L.

On the other hand it is known that a long range potential $J(|x-y|)$ satisfying the above condition is not able to induce any phase transition, the reason being that the energy between two semi-infinite lines is finite uniformly in the spin configuration. Thus, in some sense, the geometric approach is able to capture, at least at the level of the exponential, some (but certainly not all) of the physical aspects of the presence (absence) of a phase transition in the Ising model.

6.3 A Lower Bound on the Spectral Gap with + B.C.

Here we prove the bound (6.1) for any $\beta > \beta_c$ in a slightly more general case than the one discussed at the beginning of the section, namely when the b.c. are plus and the boundary coupling is weak or zero on three sides of the square B_L and strong on one. More precisely

Theorem 6.5. *Let $\beta > \beta_c$, $\delta \in [0,1]$ and let $J_\delta \equiv J_\delta^{\sqcap}(B_L)$. Then*

$$\lim_{L\to\infty} -\frac{1}{L}\log\bigl(\mathrm{gap}(L_\Lambda^{J_\delta,+})\bigr) = 0$$

Proof. Given $\varepsilon \in (0,1)$ and $\delta \in [0,1]$ let $\alpha = \varepsilon \vee \delta$. It easily follows from a rough estimate on the relative density of the two Gibbs measures $\mu_\Lambda^{J_\delta,+}$ and $\mu_\Lambda^{J_\alpha,+}$ that

$$\mathrm{gap}(L_\Lambda^{J_\delta,+}) \geq \frac{c_m}{c_M} e^{-C\varepsilon L}\,\mathrm{gap}(L_\Lambda^{J_\alpha,+})$$

for a suitable constant C.

Let now $\mathcal{D}_\varepsilon$ be the covering of the square $\Lambda \equiv B_L$ with thin horizontal rectangles $R_1, \dots, R_n$, where

$$R_1 = \{\, x \in \Lambda;\ -L \leq x_2 \leq -L + \lfloor \varepsilon L \rfloor \,\}$$

and each R^i, $i = 2 \dots n$ is a vertical translate of the previous one by an amount $\frac{\lfloor \varepsilon L\rfloor}{2}$. Thanks to (3.30) and theorem 6.4, we have

$$\begin{aligned}\mathrm{gap}(L_\Lambda^{J_\alpha,+}) &\geq \frac{1}{2}\inf_{i,\tau}\mathrm{gap}(L_{R_i}^{J_\alpha,\tau})\,\mathrm{gap}(L_{\mathcal{D}_\varepsilon}^{J_\alpha,+})\\ &\geq \frac{1}{4}L^{-2}c_m\, e^{-4\beta\varepsilon L}\,\mathrm{gap}(L_{\mathcal{D}_\varepsilon}^{J_\alpha,+})\end{aligned}$$

and therefore, due to the arbitrariness of ε, in order to prove (6.1), it suffices the following key result

Lemma 6.6. *For any $\varepsilon, \alpha \in (0,1]$ and any $\beta > \beta_c$*

$$\inf_L \operatorname{gap}(L_{\mathcal{D}_\varepsilon}^{J_\alpha,+}) > 0 \tag{6.17}$$

Proof. The lemma follows immediately if we can show that, in the above range of parameters, there exists a finite time t_0 and a number $r \in (0,1)$ such that for any large enough L the semigroup generated by $L_{\mathcal{D}_\varepsilon}^{J_\alpha,+}$ at time $t = t_0$ is a contraction in the sup norm, with norm less than $1 - r$. In more probabilistic terms if

$$\sup_\eta |\,\mathbb{E}^{\mathcal{D}_\varepsilon} f(\sigma_{t=t_0}^\eta)| \le (1-r)\|f\|_\infty \tag{6.18}$$

for any f such that $\mu_\Lambda^{J_\alpha,+}(f) = 0$. Here $\mathbb{E}^{\mathcal{D}_\varepsilon}\big(f(\sigma_t^\eta)\big)$ denotes the average of f over the block dynamics at time t starting from η.

In the present case, namely full plus b.c., we choose $t_0 = 1$. Let then $\{t_i\}$ be the random times at which the initial configuration σ is updated. Since the number n of rectangles depends only on ε and not on L, (6.18) follows if we show that there exists a number $\delta \in (0,1)$ such that, for any L large enough:

$$\sup_\eta |\,\mathbb{E}^{\mathcal{D}_\varepsilon}\big(f(\sigma_{t=1}^\eta)|t_n \le 1 < t_{n+1}\big)| \le (1-\delta)\|f\|_\infty \tag{6.19}$$

for any f such that $\mu_\Lambda^{J_\alpha,+}(f) = 0$. We will now concentrate on the proof of (6.19). Notice that, because of the definition of the block-dynamics, the following "multiple integral" formula holds

$$\mathbb{E}^{\mathcal{D}_\varepsilon}\big(f(\sigma_{t=1}^\eta)|t_n \le 1 < t_{n+1}\big) =$$

$$\sum_{i_1 \dots i_n \in \{1\dots n\}} \frac{1}{n^n} \int d\mu_{R_{i_1}}^{J_\alpha,\eta}(\sigma_1) \int d\mu_{R_{i_2}}^{J_\alpha,\sigma_1}(\sigma_2) \dots \int d\mu_{R_{i_n}}^{J_\alpha,\sigma_{n-1}}(\sigma_n) f(\sigma_n) \tag{6.20}$$

where the factor $\frac{1}{n^n}$ stands for the probability that during the first n updatings the rectangles $R_{i_1} \dots R_{i_n}$ are chosen in the given order. Notice that in the above formula we did not specify explicitly that each rectangle R_i has plus boundary conditions on those sites that are not in Λ. In other words each configuration σ_i is identically equal to $+1$ outside Λ.
In turn, in order to prove (6.19), it is sufficient to show that there exist $\bar\delta$ such that, for all large enough L,

$$\sup_\eta |\int d\mu_{R_1}^{J_\alpha,\eta}(\sigma_1) \int d\mu_{R_2}^{J_\alpha,\sigma_1}(\sigma_2) \dots \int d\mu_{R_{i_n}}^{J_\alpha,\sigma_{n-1}} f(\sigma_n)| \le (1-\bar\delta)\|f\|_\infty \tag{6.21}$$

Notice that in the above formula each configuration σ_i, $i = 1 \dots n$, is equal to the initial condition η above the top side of R_i.

For later purposes it is useful at this point to write down the expression of the constant r of (6.18) in terms of $\bar\delta$. A simple computation gives

$$r = \mathbb{P}(t_n \le 1 < t_{n+1}) n^{-n} \bar\delta \tag{6.22}$$

Let us finally prove (6.21). Define recursively

$$g_j(\sigma) = \int d\mu^{J_\alpha,\sigma}_{R_{j+1}}(\sigma_{j+1}) \dots \int d\mu^{J_\alpha,\sigma_{n-1}}_{R_n} f(\sigma_n) \qquad j = 1 \dots n-1$$

and observe that

i) $\|g_j\|_\infty \le \|f\|_\infty$.
ii) g_j depends only on the spins $\sigma(x)$ with $x \in \left(\cup_{k=1}^j R_k\right) \setminus R_{j+1}$

Because of *i), ii)* above, we can write, using (6.7) and (6.8) (remember that on the bottom and lateral sides of R_1 the b.c. are $+$)

$$\begin{aligned}
\sup_\eta \Big| \int d\mu^{J_\alpha,[0]_\eta}_{R_1}(\sigma_1) \int d\mu^{J_\alpha,\sigma_1}_{R_2}(\sigma_2) \dots \int d\mu^{J_\alpha,\sigma_{n-1}}_{R_n} f(\sigma_n)\Big| &= \\
&= \sup_\eta |\mu^{J_\alpha,[0]_\eta}_{R_1}(g_1)| \\
&\le \sup_\eta \big\{ |\mu^{J_\alpha,[0]_\eta}_{R_1}(g_1) - \mu^{J_\alpha,[0]_\eta}_{R_1\cup R_2}(g_1)| + |\mu^{J_\alpha,[0]_\eta}_{R_1\cup R_2}(g_1)| \big\} \\
&\le \|f\|_\infty e^{-m\varepsilon L} + \sup_\eta |\mu^{J_\alpha,[0]_\eta}_{R_1\cup R_2}(g_2)|
\end{aligned} \tag{6.23}$$

for a suitable $m = m(\varepsilon)$ and any L large enough. In the last step we used the DLR equation (2.2) to write

$$\mu^{J_\alpha,[0]_\eta}_{R_1\cup R_2}(g_1) = \mu^{J_\alpha,[0]_\eta}_{R_1\cup R_2}(g_2)$$

If we iterate $n-1$ times the above reasoning and use the fact that $\mu^{J_\alpha,+}_\Lambda(f) = 0$, we get

$$\sup_\eta \Big| \int d\mu^{J_\alpha,\eta}_{R_1}(\sigma_1) \int d\mu^{J_\alpha,\sigma_1}_{R_2}(\sigma_2) \dots \int d\mu^{J_\alpha,\sigma_{n-1}}_{R_n} f(\sigma_n)\Big| \le n e^{-m\varepsilon L} \|f\|_\infty$$

Thus (6.21) follows with $\bar\delta = 1 - n e^{-m\varepsilon L}$ and the lemma is proved. $\square$

Remark. It is interesting to notice that our lower bound on the spectral gap with plus boundary conditions on a square $\Lambda = Q_L$ allows one to estimate from below the probability of events like $|m_\Lambda(\sigma) - m| \le 2/|\Lambda|$, $m < m^*(\beta)$, in terms of the probability of the event $m_\Lambda(s) \le m$, with a negligeable error in the leading exponential behaviour.
Let us in fact define $f(\sigma) = \chi(m_\Lambda(\sigma) > m)$. Then we have

$$\mathrm{Var}^+_\Lambda(f) = \mu^+_\Lambda(m_\Lambda(\sigma) > m)\, \mu^+_\Lambda(m_\Lambda(\sigma) \le m)$$

$$\mathcal{E}^+_\Lambda(f,f) \le c_M |\Lambda| \mu^+_\Lambda(\,|\, m_\Lambda(\sigma) - m\,| \le \frac{2}{|\Lambda|})$$

and therefore, by the Poincaré inequality $\mathcal{E}^+_\Lambda(f,f) \geq \mathrm{gap}(L^+_\Lambda)\,\mathrm{Var}^+_\Lambda(f)$, we get

$$\mu^+_\Lambda(\,|\,m_\Lambda(\sigma) - m\,| \leq \frac{2}{|\Lambda|}) \geq \frac{1}{c_M|\Lambda|}\,\mathrm{gap}(L^+_\Lambda)\mu^+_\Lambda(m_\Lambda(\sigma) > m)\,\mu^+_\Lambda(m_\Lambda(\sigma) \leq m)$$

If we finally use our lower bound on $\mathrm{gap}(L^+_\Lambda)$ together with the observation that $\lim_{L\to\infty}\mu^+_\Lambda(m_\Lambda(\sigma) > m) = 1$ for any $m < m^*(\beta)$ and any $\beta > \beta_c$ (see e.g. [Io2]), we get that for any $\delta > 0$ and any L large enough

$$\mu^+_\Lambda(\,|\,m_\Lambda(\sigma) - m\,| \leq \frac{2}{|\Lambda|}) \geq e^{-\delta L}\mu^+_\Lambda(m_\Lambda(\sigma) \leq m)$$

6.4 A Lower Bound on the Spectral Gap with Free B.C.

Here we show that, for any $\beta > \beta_c$,

$$\lim_{L\to\infty} -\frac{1}{\beta(2L+1)}\log\bigl(\mathrm{gap}(L^{\beta,\emptyset}_{B_L})\bigr) \;\leq\; \tau_\beta \tag{6.24}$$

To prove (6.24) we first replace the free b.c. with weak plus b.c. More precisely we fix a small number $\varepsilon \in (0,1)$, that will be put equal to zero *after* the limit $L \to \infty$, and bound from below $\mathrm{gap}(L^{\beta,\emptyset}_{B_L})$ by

$$\mathrm{gap}(L^{\beta,\emptyset}_{B_L}) \;\geq\; \frac{c_m}{c_M}e^{-C\varepsilon L}\,\mathrm{gap}(L^{J^\square_\varepsilon,+}_{B_L})$$

where C is a numerical constant. Such a bound follows in a straightforward way from the very definition of the spectral gap. Then we proceed exactly as in the proof of the lower bound with full plus b.c. and reduce the problem to bound from below the gap of the block dynamics associated with the covering $\mathcal{D}_\varepsilon$ and coupling J_ε. Notice that, for simplicity, the small constant α giving the boundary couplings is the same as that fixing the height of the rectangles of the covering $\mathcal{D}_\varepsilon$. As in the case of full plus b.c. this amount to choose a finite time t_0 and bound from above the contraction constant r in (6.18). However, contrary to the case of plus b.c., if we choose t_0 independent of L here we cannot expect that the constant r stays bounded away from zero uniformly in the side L, since we know (see section 6.5 below) that $\mathrm{gap}(L^{\beta,\emptyset}_{B_L})$ shrinks to zero as $L \to \infty$ much more rapidly than $e^{-C\varepsilon L}$, at least if ε is small. Thus we have to proceed slightly more carefully than before. First of all we fix $t_0 = 2$ and observe that

$$\mathrm{gap}(L^{\mathcal{D}_\varepsilon}) \geq -\frac{1}{2}\log(1-r) \geq \frac{1}{2}r$$

Next we define $\mathcal{S}$ to be the event that up to time $t = 1$ only the first rectangle R_1 has been updated and, for any function f, we set

$$f_1(\eta) \equiv p\,\mathbb{E}^{\mathcal{D}_\varepsilon}\bigl(f(\sigma^\eta_{t=1})|\mathcal{S}\bigr) = p\mu^{J_\varepsilon,[0]_\eta}_{R_1}(f)$$

where $p = \frac{1}{n}\,\mathbb{P}\big(t_1 < 1 \le t_2\big)$ and notice that

i) f_1 does not depend on the spins in the first rectangle R_1

ii) $\|f_1\|_\infty \le p\|f\|_\infty$

iii) $\mu_\Lambda^{J_\varepsilon,+}(f_1) = 0$ iff $\mu_\Lambda^{J_\varepsilon,+}(f) = 0$

Moreover

$$\begin{aligned}\sup_\eta |\,\mathbb{E}^{\mathcal{D}_\varepsilon}\, f(\sigma^\eta_{t=2})| &\le (1-p)\|f\|_\infty + \sup_\eta |\,\mathbb{E}^{\mathcal{D}_\varepsilon}\, f_1(\sigma^\eta_{t=1})| \\ &\le (1 - p r_1)\|f\|_\infty\end{aligned}$$

where r_1 is the smallest constant such that

$$\sup_\eta |\,\mathbb{E}^{\mathcal{D}_\varepsilon}\, f(\sigma^\eta_{t=1})| \le (1-r_1)\|f\|_\infty \tag{6.25}$$

for all f that do not depend on the spins in R_1 and such that $\mu_\Lambda^{J_\varepsilon,+}(f) = 0$. Thus $r \ge p\,r_1$ and it is enough to prove

$$\lim_{L\to\infty} -\frac{1}{\beta(2L+1)}\log(r_1) \;\le\; \tau_\beta \tag{6.26}$$

In order to establish (6.26) we proceed exactly as in the case of full plus b.c. and we bound from below the constant $\bar\delta$ defined in (6.21). Using the same notation of the proof of (6.21) and remembering that on the bottom and lateral sides of the first rectangle we have plus b.c., we write, thanks to proposition 6.1, 6.2

$$\begin{aligned}&\sup_\eta |\,\mu_{R_1}^{J_\varepsilon^\sqcup,[0]_\eta}(g_1)| \\ &\le \sup_\eta \Big\{\mu_{R_1}^{J_\varepsilon^\sqcup,[0]_\eta}(A)|\,\mu_{R_1}^{J_\varepsilon^\sqcup,[0]_\eta}(g_1\,|A)| + \mu_{R_1}^{J_\varepsilon^\sqcup,[0]_\eta}(A^c)\|f\|_\infty\Big\} \\ &\le \sup_\eta \Big\{\mu_{R_1}^{J_\varepsilon^\sqcup,[0]_\eta}(A)\big[\,|\,\mu_{R_1}^{J_\varepsilon^\sqcup,[0]_\eta}(g_1\,|A) - \mu_{R_1\cup R_2}^{J_\varepsilon^\parallel,[0]_\eta}(g_1)| \;+\; |\,\mu_{R_1\cup R_2}^{J_\varepsilon^\parallel,[0]_\eta}(g_1)|\,\big]\Big\} \\ &\quad + \sup_\eta \mu_{R_1}^{J_\varepsilon^\sqcup,[0]_\eta}(A^c)\|f\|_\infty \\ &\le \sup_\eta \Big\{\mu_{R_1}^{J_\varepsilon^\sqcup,[0]_\eta}(A)\big[\,e^{-m\varepsilon L}\|f\|_\infty \;+\; |\,\mu_{R_1\cup R_2}^{J_\varepsilon^\parallel,[0]_\eta}(g_1)|\,\big] + \mu_{R_1}^{J_\varepsilon^\sqcup,[0]_\eta}(A^c)\|f\|_\infty\Big\}\end{aligned} \tag{6.27}$$

where A is the event appearing in proposition 6.2. It is important to observe that we have been able to use proposition 6.2 to bound from above the error term $|\,\mu_{R_1}^{J_\varepsilon^\sqcup,[0]_\eta}(g_1\,|A) - \mu_{R_1\cup R_2}^{J_\varepsilon^\parallel,[0]_\eta}(g_1)|$ because, by construction, the function g_1 does depend only on the spins $\sigma_1(x)$ with $x \in (R_1)_{\text{middle}}$

It is important to understand the reasoning that led us to introduce the event A. Let us suppose that the initial configuration η is identically eqaul to -1. In this case the typical configurations of the Gibbs measure $\mu_{R_1}^{J_\varepsilon^\sqcup,[0]_\eta}$ consists of a sea of minus spins with a thin layer of plus spins attached to the lateral and bottom boundary and small islands of pluses in the bulk. Thus the first updating will not modify substantially the initial configuration and similarly for the successive updatings. One possibility to drastically change the initial configuration and make a transition to configurations where now the majority of the spins are +1 is via a large deviation in the first updating. The event A not only does the job but, apparently, is also an efficient way to make the transition in the sense that the lower bound one gets on the spectral gap is of the same order of the upper bound of the next section.

Let us now bound the remaining terms in the r.h.s. of (6.27). Notice also that the Gibbs measure $\mu_{R_1\cup R_2}^{J_\varepsilon^{\|},[0]_\eta}$ has η b.c. on his top side, plus b.c. with boundary coupling β on its bottom side and weak plus b.c. on its lateral sides. Thus we can write

$$\mu_{R_1\cup R_2}^{J_\varepsilon^{\|},[0]_\eta}(g_1) = \mu_{R_1\cup R_2}^{J_\varepsilon^{\|},[0]_\eta}(g_2)$$

To estimate the remaining term $\mu_{R_1\cup R_2}^{J_\varepsilon^{\|},[0]_\eta}(g_2)$ we can at this point proceed exactly as in the case with full b.c. since we have them on the bottom side. The final result is

$$|\,\mu_{R_1\cup R_2}^{J_\varepsilon^{\|},[0]_\eta}(g_2)| \leq ne^{-m\varepsilon L}\|f\|_\infty + |\,\mu_\Lambda^{J_\varepsilon^{\sqcap},+}(f) - \mu_\Lambda^{J_\varepsilon^{\square},+}(f)| \tag{6.28}$$

To estimate the last term in the r.h.s. of (6.28) we observe that, since f does not depend on the spins in R_1, we can apply once more proposition 6.1 and get

$$|\,\mu_\Lambda^{J_\varepsilon^{\sqcap},+}(f) - \mu_\Lambda^{J_\varepsilon^{\square},+}(f)| \leq e^{-m\varepsilon L}\|f\|_\infty \tag{6.29}$$

In conclusion we have obtained that

$$\begin{aligned}\sup_\eta |\,\mu_{R_1}^{J_\varepsilon^{\sqcup},[0]_\eta}(g_1)| &\leq \sup_\eta\Big\{\mu_{R_1}^{J_\varepsilon^{\sqcup},[0]_\eta}(A)\; 2^n\; e^{-m\varepsilon L}\|f\|_\infty \;+\; \mu_{R_1}^{J_\varepsilon^{\sqcup},[0]_\eta}(A^c)\|f\|_\infty\Big\}\\ &\leq \big(1 - e^{-\beta\tau_\beta(1+14\varepsilon)(2L+1)}(1 - 2^n e^{-m(\varepsilon)\varepsilon L})\big)\|f\|_\infty\end{aligned}$$

for any L large enough, because of proposition 6.2. Thus the constant $\bar\delta$ in this new case can be bounded from below by $\frac{1}{2}e^{-\beta\tau_\beta(1+14\varepsilon)(2L+1)}$. In turn, using (6.22) and the arbitrariness of the constant ε, the above lower bound on $\bar\delta$ immediately implies (6.26). $\square$

6.5. Upper Bound on the Spectral Gap with Free B.C.

Here we show that, for the two dimensional Ising model

$$\lim_{L\to\infty} -\frac{1}{\beta(2L+1)}\log\big(\mathrm{gap}(L_{B_L}^{\beta,\emptyset})\big) \;\geq\; \tau_\beta \tag{6.30}$$

At the end of this paragraph we will also discuss an extension to higher dimensions.

Following [M], let $\Lambda = B_L$ and let $f_\Lambda(\sigma)$ be the trial function

$$f_\Lambda(\sigma) = \chi\{m_\Lambda(\sigma) > 0\} - \chi\{m_\Lambda(\sigma) < 0\} \tag{6.31}$$

If we plug f_Λ in the variational characterization of the gap (3.12) we get

$$\operatorname{gap}(L_\Lambda^{\beta,\emptyset}) \le 4\, c_M\, |\Lambda|\, \frac{\mu_\Lambda^{\beta,\emptyset}\{\, |m_\Lambda(\sigma)| \le \frac{2}{|\Lambda|}\,\}}{1-\mu_\Lambda^{\beta,\emptyset}\{\, m_\Lambda(\sigma)=0\,\}} \tag{6.32}$$

Therefore

$$\begin{aligned}\lim_{L\to\infty} -\frac{1}{\beta(2L+1)}\log\bigl(\operatorname{gap}(L_\Lambda^{\beta,\emptyset})\bigr) &\ge \lim_{L\to\infty} -\frac{1}{\beta(2L+1)}\log\Bigl(\mu_\Lambda^{\beta,\emptyset}\{\, |m_\Lambda(\sigma)| \le \frac{2}{|\Lambda|}\,\}\Bigr)\\ &= \tau_\beta\end{aligned}$$

because of proposition 6.3. □

Remark Notice that up to (6.32) our argument was completely dimension independent. Unfortunately, in dimensions higher than two there is yet no precise bound of the probability that the magnetization is close to zero. However there is a bound (see [P]) that says that, for any $d \ge 2$, there exists $\beta_0 < +\infty$ such that for any $\beta > \beta_0$ the r.h.s of (6.32) is exponentially small in L^{d-1}, *i.e.* the surface of the cube Λ. Moreover there are good chances that β_0 coincides with the critical temperature. Alternatively one could use the result of [Th]. If we combine such bounds with the very general lower bound of theorem 3.8 we get that for any $d \ge 2$ and low enough temperature

$$\lim_{L\to\infty} \frac{1}{\log L}\log\bigl(-\log(\operatorname{gap}(L_{Q_L}^{\beta,\emptyset}))\bigr) = d-1$$

6.6 Mixed B.C.

We would like to conclude our analysis of the spectral gap of the two dimensional stochastic Ising model at low temperature by discussing the case of "mixed" b.c., *i.e.* b.c. that do not have a definite sign.

Remember that, when the b.c. where free *i.e.* absent, the spectral gap was very small because of the presence of a very narrow "bottleneck" in the phase space, while for plus b.c. the spectral gap was much larger because the bottleneck is either absent or much wider. This phenomenon can be associated to the fact that in the case of free b.c. the two phases are equally likely while in the case of plus b.c. the plus phase is chosen by the b.c.

It is then natural to ask whether it is possible to have intermediate situations in which a bottleneck for the Glauber dynamics is still present but nevertheless the b.c. are able to choose the phase. The main result in this direction is the following one (see [HiYo]).

Given the square Q_L and a configuration $\tau \in \Omega$, define

$$v_L(\tau) = \max\bigl\{|\sum_{x\in A}\tau(x)|;\ A \text{ is a connected subset of } \partial_{\text{ex}}Q_L\bigr\}$$

where $\partial_{\text{ex}}Q_L = \{x \in Q_L^c;\ d_2(x,Q_L)=1\}$. Then we have

Theorem 6.7. *Let $d = 2$ and define a class $\Omega_0(\vartheta)$ $(0 \le \vartheta \le 1)$ of boundary conditions by*

$$\Omega_0(\vartheta) = \left\{\tau \in \Omega; \limsup_{L\to\infty} \frac{V_L(\tau)}{4L} \le \vartheta\right\}$$

Then, for every $\vartheta < \bar\vartheta < \frac{1}{8}$ and $\alpha \in]0, 1 - 8\bar\vartheta[$, there exist $\beta_0 > 0$ such that for every $\tau \in \Omega_0(\vartheta)$ and $\beta \ge \beta_0$

$$\operatorname{gap}(L^{\beta,\tau}_{Q_L}) \le c(\beta) e^{-\beta\alpha L}$$

for a suitable constant $c(\beta)$ independent of τ.

6.7 Applications

Here we discuss some additional results and applications that follow from the spectral gap estimates established so far.

Let us first consider the standard two dimensional Ising model at inverse temperature $\beta > \beta_c$ and zero external field h. We denote by μ^β_+ denote the so called "plus phase", *i.e.* the infinite volume Gibbs measure obtained as thermodynamic limit of the finite volume Ising Gibbs measure with plus b.c., and by $\| \; \|_{2,\mu^\beta_+}$ the L^2–norm w.r.t μ^β_+. Let also $T^\beta(t)$ be the Markov semigroup of an attractive Glauber dynamics (e.g. the heat-bath dynamics) for the above model. Then we have

Theorem 6.8. *Let $\alpha \in [0, \infty)$ be given. Then there exists $c < +\infty$ such that for any local function f*

$$\|T^\beta(t) f - \mu^\beta_+(f)\|_{2,\mu^\beta_+} \le \frac{c}{t^\alpha} |||f||| \quad \forall\, t \ge 1$$

Proof. Using the reversibility of the semigroup $T^\beta(t)$ w.r.t the Gibbs measure μ^β_+ we write

$$\begin{aligned}
&\|T^\beta(t) f - \mu^\beta_+(f)\|^2_{2,\mu^\beta_+} \\
&\quad = \int d\mu^\beta_+(\eta) \Big\{ \int d\mu^\beta_+(\eta') \big[T^\beta(t) f(\eta) - T^\beta(t) f(\eta')\big] \Big\}^2 \\
&\quad \le \int d\mu^\beta_+(\eta) \int d\mu^\beta_+(\eta') \big[T^\beta(t) f(\eta) - T^\beta(t) f(\eta')\big]^2 \\
&\quad = \int d\mu^\beta_+(\eta) \int d\mu^\beta_+(\eta') \big[\mathbb{E}\big(f(\sigma^{\beta,\eta}_t) - f(\sigma^{\beta,\eta'}_t)\big)\big]^2 \\
&\quad \le |||f||| \sum_x \|\nabla_x f\|_\infty \int d\mu^\beta_+(\eta) \int d\mu^\beta_+(\eta')\, \mathbb{P}\big(\sigma^{\beta,\eta}_t(x) \ne \sigma^{\beta,\eta'}_t(x)\big) \\
&\quad = |||f|||^2 \int d\mu^\beta_+(\eta) \int d\mu^\beta_+(\eta')\, \mathbb{P}\big(\sigma^{\beta,\eta}_t(0) \ne \sigma^{\beta,\eta'}_t(0)\big) \\
&\quad \le 2|||f|||^2 \int d\mu^\beta_+(\eta)\, \mathbb{P}\big(\sigma^{\beta,\eta}_t(0) \ne \sigma^{\beta,+}_t(0)\big)
\end{aligned} \tag{6.33}$$

where we have used the standard inequality

$$|f(\sigma) - f(\eta)| \le \sum_x \|\nabla_x f\|_\infty \chi(\sigma(x) \ne \eta(x))$$

together with translation invariance.
Let us now bound the factor $\int d\mu_+^\beta(\eta)\, \mathbb{P}\big(\sigma_t^{\beta,\eta}(0) \ne \sigma_t^{\beta,+}(0)\big)$ in the r.h.s of (6.33). Using monotonicity together with reversibility and denoting by $m^*(\beta)$ the spontaneous magnetization, $m^*(\beta) = \mu_+^\beta(\sigma(0))$, we have

$$\begin{aligned}
&\int d\mu_+^\beta(\eta)\, \mathbb{P}\big(\sigma_t^{\beta,\eta}(0) \ne \sigma_t^{\beta,+}(0)\big) = \\
&\qquad = \int d\mu_+^\beta(\eta) \Big[\mathbb{P}\big(\sigma_t^{\beta,+}(0) = +1\big) - \mathbb{P}\big(\sigma_t^{\beta,\eta}(0) = +1\big)\Big] \\
&\qquad \le \mathbb{P}\big(\sigma_t^{B_L,\beta,+,+}(0) = +1\big) - \frac{1}{2}(m^*(\beta) + 1) \\
&\qquad = \frac{1}{2}\Big\{\mathbb{E}\big(\sigma_t^{B_L,\beta,+,+}(0)\big) - m^*(\beta)\Big\}
\end{aligned} \tag{6.34}$$

where $\sigma_t^{B_L,\beta,+,+}$ denotes the dynamics in B_L with plus boundary conditions and starting from the configuration identically equal to plus one. Thanks to (3.15),

$$\mathbb{E}(\sigma_t^{B_L,\beta,+,+}(0)) \;\le\; \mu_{B_L}^{\beta,+}(\eta(0)) + e^{2\beta L^2 \,-\, t\,\mathrm{gap}(L_{B_L}^{\beta,+})} \tag{6.35}$$

If we plug (6.35) into (6.34) we obtain that the r.h.s. of (6.34) is bounded from above by:

$$\frac{1}{2}\big(\mu_{B_L}^{\beta,+}(\eta(0)) \,-\, m^*(\beta)\big) + e^{2\beta L^2 \,-\, t\,\mathrm{gap}(L_{B_L}^{\beta,+})} \le \frac{1}{2} C_1 e^{-mL} + e^{2\beta L^2 \,-\, t\,\mathrm{gap}(L_{B_L}^{\beta,+})} \tag{6.36}$$

because of the well known bound

$$0 \;\le\; \mu_{B_L}^{\beta,+}(\eta(0)) \,-\, m^*(\beta) \;\le\; C_1 \exp(-mL) \tag{6.37}$$

valid for any $\beta > \beta_c$ and suitable constants C_1, m. We now choose the size L depending on t as:

$$L = 2\frac{\alpha \log(t)}{m} \tag{6.38}$$

and apply (6.1) to (6.36). We get that, for any large enough t, the r.h.s of (6.36) is bounded from above by $3C_1 t^{-2\alpha}$. If we finally plug this bound in the r.h.s of (6.33) we get the sought polynomial decay. $\square$

The above result as an interesting consequence in terms of coercive inequality for the Glauber dynamics in the plus phase. Denote by $\mathcal{E}_+^\beta(f,f)$ and $\mathrm{Var}_+^\beta(f)$ the Dirichlet form and variance of a local function f w.r.t the Gibbs measure μ_+^β. Then

Corollary 6.9. *In the same hypotheses of theorem 6.8, for any $\delta > 0$ there exists a finite constant c such that for any local function f*

$$\mathrm{Var}^{\beta}_{+}(f)^{1+\delta} \le c\,\mathcal{E}^{\beta}_{+}(f,f)\,|||f|||^{2\delta}$$

Proof. Given a local function f with $\mu^{\beta}_{+}(f) = 0$, let $d\rho_f(\lambda)$ be its spectral measure w.r.t the selfadjoint operator $-L^{\beta}$ in $L^2(\Omega, \mu^{\beta}_{+})$. Then, for any $\delta > 0$ and $t > 0$, we have

$$\begin{aligned}\mathrm{Var}^{\beta}_{+}(f) &= \int_0^\infty d\rho_f(\lambda)\\ &\le e\int_0^\infty d\rho_f(\lambda)e^{-\lambda t} + t\int_0^\infty d\rho_f(\lambda)\lambda\\ &= e\|T^{\beta}(t)f\|^2_{2,\mu^{\beta}_{+}} + t\mathcal{E}^{\beta}_{+}(f,f)\\ &\le c(\delta)\frac{|||f|||^2}{t^{1/\delta}} + t\mathcal{E}^{\beta}_{+}(f,f)\end{aligned} \tag{6.39}$$

because of theorem 6.8. It suffices now to optimize the choice of t to get the final result. □

As a next topic we consider in more details the dynamics in a finite square Q_L with free b.c. and large β. For notation convenience, in what follows σ^{η}_t will always denote $\sigma^{Q_L,\beta,\eta,\emptyset}_t$.

As we said at the beginning of the section, the slow relaxation process to equilibrium is due to the presence of a bottleneck in the configuration space that somehow separates configuration with positive magnetization from those with negative magnetization. It is however clear that if the flip rates are invariant under a global spin flip $\sigma \to -\sigma$ and f is function that is even w.r.t such a sign inversion, then the relaxational behaviour of $f(\sigma^{\eta}_t)$ should not be affected by the presence of the bottleneck. In particular one would expect that functions that are *even* under global spin flip, *i.e.* they do not distinguish between the two phases, will relax to their equilibrium value in a time much shorter than the global relaxation time $\mathrm{T}_{\mathrm{spin}}$ given by the inverse of the spectral gap. More precisely, if we denote by $\mathrm{T}_{\mathrm{contour}}$ the inverse of the spectral gap in the spectrum of the generator $L^{\beta,\emptyset}_{Q_L}$ restricted to the invariant subspace of functions that are even with respect to a global spin flip, we expect that $\mathrm{T}_{\mathrm{contour}} << \mathrm{T}_{\mathrm{spin}}$. Since even functions can be thougth of as functions of the broken lines in the dual lattice separating regions of plus spins from regions of minus spins (Peierls contours), one could say that $\mathrm{T}_{\mathrm{contour}}$ characterizes the relaxation process as $t \to \infty$ of the probability distribution of the Peierls contours generated by the dynamics at time t.
The main result of [MM] concerning this new time scale goes as follows

Theorem 6.10. *There exists a positive constant β_o such that for any $\beta \ge \beta_o$*

$$\lim_{L\to\infty}\frac{1}{L}\log\Big(\frac{\mathrm{T}_{\mathrm{spin}}}{\mathrm{T}_{\mathrm{contour}}}\Big) > 0$$

Using the above result one can draw some interesting consequences that make the picture found in [M] more precise. The first one says that, under the dynamics, any initial configuration relaxes to one of the two phases within a time scale of the order of $L^2\mathrm{T}_{\mathrm{contour}}$ much shorter than $\mathrm{T}_{\mathrm{spin}}$. More precisely we have (see theorem 3.1 in [MM])

Theorem 6.11. *Let $t_o \equiv 10\beta L^2 \mathrm{T}_{\text{contour}}$. There exist positive constants β_o, L_o such that for any $\beta \geq \beta_o$ and any $L \geq L_o$*

$$\sup_\eta \mathbb{P}\big(\sigma_t^+ \neq \sigma_t^\eta \neq \sigma_t^-\big) \leq e^{-[t/t_o]} \quad \forall\, t \geq t_o$$

The second one (see theorem 3.2 in [MM]) says that, once the system decides to jump from one phase to the opposite one, then, with large probability, it does it on a time scale not larger than $L^3 \mathrm{T}_{\text{contour}}$, again much shorter than the average time one has to wait in order to see the jump. One could say that in our case the Glauber dynamics has a behavior similar, in some sense, to that of a finite dimensional reversible Markov processes with invariant measure having a symmetric double well structure in the low noise regime.

In order to formulate the result, let us define recursively, for a fixed small δ, the following sequence of stopping times:

$$\begin{aligned}
s_0 &\equiv 0 \\
t_i &\equiv \inf\{t > s_{i-1};\ \big| |m(\sigma_t^\eta)| - m^*(\beta) \big| \geq 2\delta\} \\
s_i &\equiv \inf\{t > t_i;\ \big| |m(\sigma_t^\eta)| - m^*(\beta) \big| < \delta\}
\end{aligned}$$

where $m(\sigma)$ denotes the (normalized) magnetization of σ, *i.e.* $m(\sigma) = \frac{1}{L^2}\sum_{x \in Q_L} \sigma(x)$. We also define the random variable $\nu(\eta)$ as

$$\nu(\eta) \equiv \min\{i;\ |m(\sigma_{s_i}^\eta) + m^*(\beta)| < \delta\}$$

Then we have:

Theorem 6.12. *Let t_0 be as in theorem 6.11. Then there exist positive constants β_o, L_o and c such that for any $\beta \geq \beta_o$ and any $L \geq L_o$*

i)

$$\sup_\eta \mathbb{P}\big(s_{\nu(\eta)} - t_{\nu(\eta)} \geq L t_0\big) \leq e^{-cL}$$

ii)

$$\lim_{L \to \infty} \frac{1}{L} \log\Big(\frac{\mathbb{E}(s_{\nu(+)})}{\mathrm{T}_{\text{spin}}}\Big) = 0$$

Remark. If η is such that $m(\eta) > m^*(\beta) - 2\delta$, then we may call $s_{\nu(\eta)} - t_{\nu(\eta)}$ and $s_{\nu(\eta)}$ the time scale of the last excursion before leaving the set $\{\sigma; m(\sigma) \geq -m^*(\beta) + \delta\}$ and the transition time for η respectively. Thus we may conclude that the last excursion occurs on a time scale much shorter than the average transition time, at least if the dynamics starts from the plus configuration.

We conclude this part by stating two last results on the time–dependent magnetization $m(\sigma_t^\eta)$ (see [M]). The first one concerns the large deviations of $m(\sigma_t^\eta)$ and makes a precise connection with analogous results for the equilibrium magnetization obtained in the framework of a rigorous

justification on the Wulff construction (see [Sh], [Io1], [Io2], [Pf] and [CGMS]).
Let $\rho \in (-m^*(\beta), m^*(\beta))$ and let ρ_L be a sequence of integers such that

$$\lim_{L\to\infty} \frac{\rho_L}{L^2} = \rho$$
$$\rho_L - L^2 = 0 \mod 2$$

Let also $\tau_{\rho_L}(\eta)$ be the stopping time

$$\tau_{\rho_L}(\eta) = \inf\{t \geq 0;\quad m(\sigma^\eta_t) \leq \rho_L\}$$

Then we have (see theorem 5.1 in [M])

Theorem 6.13. *For any β large enough and any ρ_L as above:*

$$\lim_{L\to\infty} \frac{1}{\beta L} \log \mathbb{E}(\tau_{\rho_L}(+)) = \lim_{L\to\infty} -\frac{1}{\beta L} \log \mu^{\beta,\emptyset}_{Q_L}(\rho \vee 0) \equiv \psi(\rho \vee 0)$$

Remark The rate function $\psi(\rho)$ can be computed easily in terms of the Wulff functional.

The second result makes more precise the picture of the Glauber dynamics as a symmetric double well Markov chain jumping from one phase to the opposite one in a time scale $\mathrm{T}_{\mathrm{spin}}$. Let M be the two state space $\{-m^*(\beta), m^*(\beta)\}$ and let Y_t be a continuous time Markov chain on M with unitary jump rate for both states. Then (see theorem 6.1 in [M]) we have

Theorem 6.14. *For any β large enough and for any $\varepsilon \in (0, \frac{m^*(\beta)}{2})$ there exists $\alpha(L) \equiv \alpha(L, \beta, \varepsilon)$ such that*

i) for any choice of times $t_1 < t_2 < \ldots < t_k$ and numbers $m_i \in M$, $i = 1 \ldots k$

$$\lim_{L\to\infty} \mathbb{P}^\mu\left(|m(\sigma^\eta_{\alpha(L)t_1}) - m_1| < \varepsilon, \ldots, |m(\sigma^\eta_{\alpha(L)t_k}) - m_k| < \varepsilon\right)$$
$$= \mathrm{P}^\nu(Y_{t_1} = m_1, \ldots, Y(t_k) = m_k)$$

where $\mathbb{P}^\mu$ (P^ν) denotes the probability over the Glauber dynamics (the chain Y_t) with initial distribution the invariant measure $\mu^{\beta,\emptyset}_{Q_L}$ (ν).

ii)

$$\lim_{L\to\infty} \frac{1}{\beta L} \log(\alpha(L)) = \tau_\beta$$

7. Glauber Dynamics for Random Systems

In this final section we discuss what happens to the relaxational properties of the Glauber dynamics when we remove the assumption of *translation invariance* of the interaction, see definition 2.1, and consider in particular *random interactions.* The simplest example of such a system is the (bond) dilute Ising ferromagnet. In this case the couplings between nearest neighbor spins, that in (2.4) were assumed to be constant, become a collection of i.i.d random variables $\{J_{xy}\}$ that take only two values, $J_{xy} = 0$ and $J_{xy} = \beta$ with probability $1-p$ and p respectively, independently for each pair of nearest neighbors $x, y \in \mathbb{Z}^d$. In a more pictorial form one starts from the standard Ising model and removes, independently for each bond $[x, y]$, the coupling J_{xy} with probability $1-p$.

Since the $\{J_{xy}\}$ are uniformly bounded, at sufficiently high temperatures (*i.e.* sufficiently small values of β) Dobrushin's uniqueness theory applies and detailed information about the unique Gibbs measure and the relaxation to equilibrium of an associated Glauber dynamics are available using the concept of complete analyticity [DS], [SZ], [MO1] and [MO2]. This regime is usually referred to as the *paramagnetic* phase and, at least for the two dimensional dilute Ising model, it is known to cover the whole interval $\beta < \beta_c$ where β_c is the critical value for the "pure" Ising system.

There is then a range of temperatures, below the paramagnetic phase, where, even if the Gibbs state is unique, certain characteristics of the paramegnetic phase like the analyticity of the free energy as a function of the external field disappear. This is the so called *Griffiths' regime* [G] (see also [F] for additional discussion on this and many other related topics). This "anomalous behavior" is caused by the presence of arbitrarily large clusters of bonds associated with "strong" couplings J_{xy}, which can produce a long–range order inside the cluster. Even above the percolation threshold, *i.e.* when one of such clusters is infinite with probability one, there may be a Griffiths phase for values of $\beta \in (\beta_c, \beta_c(p))$, where β_c is the critical value for the Ising model on $\mathbb{Z}^d$ and $\beta_c(p)$ the critical value of the dilute model above which there is a phase transition (see [F]). What happens is that for almost all realizations of the disorder J and for all site x there is a finite length scale $l(J, x)$, such that correlations between $\sigma(x)$ and $\sigma(y)$ start decaying exponentially at distances greater than $l(J, x)$.

In [BD] an "elementary" approach was given to the problem of uniqueness of the equilibrium state of disordered systems in the Griffiths regime (see also [FI]). In another paper [D] Dobrushin prepared the mathematical background for the study of (arbitrary order) truncated correlation functions for spin glasses. Bounds on $T_c(p)$ have been obtained in [ACCN] and [OPG]. More recent references where, at least for the statics, the situation has been considerably cleared up are [DKP], [GM1], [GM2], [GM3] and [Be]. In particular, under suitable conditions on the couplings distribution, one proves that the infinite volume Gibbs state is unique with probability one and the static correlation functions decay exponentially fast uniformly in the size of the system and its boundary conditions.

The effect of the Griffiths' singularities on the dynamical properties are much more serious since, as we will see, the long time behaviour of any associated Glauber dynamics is dominated by the islands of strongly coupled spins produced by large statistical fluctuations in the disorder (see [B1], [B2] for a non rigorous treatment and [GZ1], [GZ2], [CMM1], [CMM2]). In what follows we will analyze in some detail the simple case of the dilute Ising model. Although at first sight such a model is a very special one, it is important to say that most of the results, particularily those concerning the dynamical behaviour inside the Griffiths phase, apply with minor changes to a much wider class of models (see [CMM1], [CMM2]).

7.1 The dynamics in the paramagnetic phase

In this first part we assume $\beta < \beta_c$, where β_c is the critical value for the "pure" Ising model (2.4), while the parameter p is an arbitrary value in the interval $[0,1]$. It is not difficult to see that in this case, for *any* couplings configuration $\{J_{xy}\}$ there exists a unique infinite volume Gibbs measure with exponentially decaying correlations functions. Let us in fact consider, for an arbitrary configuration $J = \{J_{xy}\}$, the quantity

$$\mu_{B_L}^{J,+}(\sigma(0)) - \mu_{B_L}^{J,-}(\sigma(0)) \tag{7.1}$$

It is a well known fact that (7.1) is an increasing function of each coupling J_{xy}. Therefore, since $J_{xy} \le \beta$, (7.1) is bounded from above by the analogous quantity computed for the pure Ising model with constant coupling β. Thus, since $\beta < \beta_c$, (7.1) decays exponentially fast in L uniformly in the configuration J. In particular weak mixing holds (see (4.1)) uniformly in J. Uniqueness of the infinite volume Gibbs measure then follows from a simple argument (see for details in a more general case [Be]).

Let us now consider an attractive Glauber dynamics for the dilute Ising model like the heat bath dynamics or Metropolis. All the general results of section 3 apply without change also to this random case. Moreover, using the fact that weak mixing holds uniformly in J, we could repeat word by word the proof of theorem 4.1 and get exponential ergodicity in the $\|\cdot\|_\infty$ norm uniformly in J.

We can conclude that, as long as $\beta < \beta_c$, the dynamical dilute Ising model does not behave differently from the corresponding pure model at inverse temperature β.

7.2 The Dynamics in the Griffiths Phase: $\mathbf{P} < \mathbf{P_c}$

Here we consider the more interesting case of $\beta > \beta_c$ but we assume that p is below the percolation threshold p_c for the d-dimensional independent bond percolation on $\mathbb{Z}^d$ (see e.g. [Gri]). It is not difficult to convince oneself that in this case there still exists with probability one a unique infinite volume Gibbs measure. Let us in fact denote by $\{W_i\}$ the connected components of the set

$$W = \{x \in \mathbb{Z}^d : \exists\, y \text{ such that } d_2(x,y) = 1 \text{ and } J_{xy} = \beta\} \tag{7.2}$$

and let us call "regular" any site x that does not belong to the set W. It is well known that if $p < p_c$ all W_i are finite with probability one. Therefore, with probability one, the infinite volume Gibbs measure is simply the product of the Ising Gibbs measure at inverse temperature β and free boundary conditions for each component W_i and the Bernoulli measure of parameter $\frac{1}{2}$ for each site $x \in W^c$.

Although the situation might appear, and indeed is, quite simple, it is nevertheless very instructive to analyze the relaxational properties of the dynamics in this case. We will see in fact later on that many (but not all) of the features of the case $p < p_c$ remain true also for values of p slightly above the percolation threshold p_c.

Let us first observe that, with probability one, the Glauber dynamics in our case is a product dynamics for each of the clusters $\{W_i\}$ and for each of the "regular" sites $x \in W^c$. Thus, if we consider a local function f that for simplicity we can take as the spin at the origin, $\pi_0(\sigma) = \sigma(0)$, we get that

$$\|T^J(t)\pi_0\|_{L^2(\mu^J)} \leq e^{-\lambda_0 t} \tag{7.3}$$

where, for any $x \in \mathbb{Z}^d$,

$$\lambda_x = \begin{cases} 1 & \text{if } x \text{ is regular} \\ \text{gap}(L_{W_i}^{J,\emptyset}) & \text{if } x \in W_i \end{cases}$$

Notice that in (7.3) we used the obvious property $\mu^J(\pi_0) = 0$ valid because of the simmetry $\sigma \to -\sigma$. Since the clusters W_i are finite with probability one, we can immediately conclude that $\|T^J(t)\pi_0\|_{L^2(\mu^J)}$ converges exponentially fast to its equilibrium value but with an exponential rate, λ_0 in our case, that *depends* on the chosen local function, π_0, through its support. It is important to outline here two important features of the dynamics in the present case

i) in $d = 2$ for any $\beta > \beta_c$ or in $d \geq 3$ and any β large enough, $\inf_x \lambda_x = 0$ with probability one. By ergodicity we have in fact that, with probability one, for any $L \geq 1$ we can find $x(L)$ such that $Q_L(x(L)) = W_i$ for some i. Thanks to the results of section 6 the spectral gap of the cluster W_i is thus exponentially small in L^{d-1}. In particular the spectral gap of the infinite volume dynamics is zero. We can say that such non–uniformity of the rates λ_x is a first signal of the Griffiths phase.

ii) the fact that local functions relax exponentially fast although with a non–uniform rate is a specific feature of the dilute model and it does not extend to more general systems in which the interaction between clusters of strongly interacting spins is weak but non zero (see next section for more details)

Although the analysis of the relaxation to equilibrium for a fixed realization of the random couplings $\{J_{xy}\}$ is certainly interesting, much more relevant from the physical point of view is the same analysis when one takes average over the disorder. It is here that the differences between the dynamics in the paramagnetic phase and in the Griffiths phase appear more pronounced.

Let us denote by $\mathbb{E}(\cdot)$ the average w.r.t the disorder and let us compute upper and lower bounds on $\mathbb{E}\big(\|T^J(t)\pi_0\|_{L^2(\mu^J)}\big)$. For this purpose, for any $x \in \mathbb{Z}^d$, it is convenient to denote with C_x the set

$$C_x = \begin{cases} x & \text{if } x \text{ is regular} \\ W_i & \text{if } x \in W_i \end{cases} \tag{7.4}$$

Then, remembering theorem 3.8, we can write

$$\begin{aligned}\mathbb{E}\big(\|T^J(t)\pi_0\|_{L^2(\mu^J)}\big) &\le \sum_{V\subset\subset\mathbb{Z}^d} \mathbb{P}(C_0=V)e^{-\,\mathrm{gap}(L_V^{\beta,\emptyset})t} \\ &\le \sum_{V\subset\subset\mathbb{Z}^d} \mathbb{P}(C_0=V)e^{-t\,e^{-c(\beta)|V|^{\frac{d-1}{d}}}} \\ &\le \sum_{n\ge 1} C(p)e^{-m(p)n-te^{-c(\beta)n^{\frac{d-1}{d}}}} \\ &\le C_1(p,\beta)e^{-C_2(p,\beta)\log(t)^{\frac{d}{d-1}}}\end{aligned} \tag{7.5}$$

for suitable constants C_1, C_2. In the third inequality above we used the well known fact (see e.g. [Gri]) that

$$\mathbb{P}(|C_0|=n)\le C(p)\,e^{-m(p)n} \qquad \forall\, p<p_c$$

Notice that the same computation carried out at $\beta<\beta_c$ would have led to a pure exponential decay since (see theorem 4.1 a) of [SchYo])

$$\inf_{V\subset Z^d}\mathrm{gap}(L_V^{\beta,\emptyset})>0 \qquad \forall\,\beta<\beta_c$$

Let us now turn to the proof of a lower bound on the same quantity. We assume $\beta>\beta_c$ if $d=2$ or β large enough if $d\ge 3$ but, contrary to the proof of the upper bound, we don't need to restrict ourselves to values of p below the percolation threshold; we just need $p<1$.

Let $\Lambda=Q_L$ and let $\bar\Theta$ be the set of all interactions $J\in\Theta$ such that

(a) $J_{xy}=\beta$ for all $\{x,y\}$ such that $\{x,y\}\subset\Lambda$

(b) $J_{xy}=0$ for all $\{x,y\}$ which intersect both Λ and Λ^c (the boundary edges)

If we denote by $m_\Lambda=|\Lambda|^{-1}\sum_{x\in\Lambda}\sigma(x)$ the normalized magnetization in Λ, we can write (remember that $\mu^J(\pi_0)=0$)

$$\mathbb{E}\|T^J(t)\pi_0\|_{L^2(\mu^J)}\ge\mathbb{E}\|T^J(t)\,m_\Lambda\|_{L^2(\mu^J)}\ge\mathbb{P}(\bar\Theta)\inf_{J\in\bar\Theta}\|T^J(t)\,m_\Lambda\|_{L^2(\mu^J)} \tag{7.6}$$

Choose $J\in\bar\Theta$ and let

$$F_\Lambda=\{\sigma\in\Omega:\, m_\Lambda(\sigma)>\tfrac12\}$$

Then we have

$$\|T^J(t)\,m_\Lambda\|_{L^2(\mu^J)}\ge\sqrt{\mu(F_\Lambda)}\;\|T^J(t)\,m_\Lambda\|_{L^2(\mu^J(\cdot\,|\,F_\Lambda))}$$

and

$$\|T^J(t)\,m_\Lambda\|_{L^2(\mu^J(\cdot\,|\,F_\Lambda))}\ge\|T^J(t)\,m_\Lambda\|_{L^1(\mu^J(\cdot\,|\,F_\Lambda))}\ge\mu^J(T^J(t)\,m_\Lambda\,|\,F_\Lambda) \tag{7.7}$$

For $\sigma\in\Omega$, let $\{\eta_t^\sigma\}_{t\ge0}$ be the process associated with $T^J(t)$ with initial condition $\eta_0^\sigma=\sigma$, and let $\{\eta_t^\mu\}_{t\ge0}$ be the stationary process (the one with initial distribution μ^J). Consider the events

$$G_{\Lambda,t}^\sigma\equiv\{\,\exists s\in[0,t]:\,|m_\Lambda(\eta_s^\sigma)-1/2|\le 1/(100)\,\} \qquad \sigma\in\Omega$$

For each $\sigma \in F_\Lambda$, if $|\Lambda| > 100$, we have

$$m_\Lambda(\eta_t^\sigma) \geq \frac{1}{2}\mathbb{I}_{(G^\sigma_{\Lambda,t})^c} - \mathbb{I}_{G^\sigma_{\Lambda,t}} = \frac{1}{2} - \frac{3}{2}\mathbb{I}_{G^\sigma_{\Lambda,t}}$$

which implies

$$\mu^J(T^J(t)\, m_\Lambda \,|\, F_\Lambda) \geq \frac{1}{2} - \frac{3}{2}\int_\Omega \mu^J(d\sigma \,|\, F_\Lambda)\,\mathrm{Prob}(G^\sigma_{\Lambda,t}) \geq \frac{1}{2} - \frac{3}{2}\mu^J(F_\Lambda)^{-1}\,\mathrm{Prob}(G^\mu_{\Lambda,t}) \tag{7.8}$$

If $t_1, t_2, \ldots$ are the (random) times at which the stationary process η_t^μ is updated inside Λ and n_t is the number of updates up to time t, we have, for all $j \in \mathbb{Z}_+$,

$$P(G^\mu_{\Lambda,t}) \leq j\mu^J\{|m_\Lambda(\sigma) - 1/2| \leq 1/(100)\} + \mathrm{Prob}\{n_t > j\} \tag{7.9}$$

which, taking $j = k|\Lambda|t$ with $k = 2c_M$, can be bounded by

$$\begin{aligned} &k\,|\Lambda|\,t\;\mu^J\{|m_\Lambda(\sigma) - 1/2| \leq 1/(100)\} + e^{-k'|\Lambda|t} \\ &\leq k\,|\Lambda|\,t\,e^{-c(\beta)L^{d-1}} + e^{-k'|\Lambda|t} \end{aligned} \tag{7.10}$$

for suitable positive constants $C(\beta)$, k' thanks to the results of section 6.5. Moreover, because of the same results,

$$\mu^J(F_\Lambda) \geq \frac{1}{3}$$

for any L large enough.
Take now $L = L_t$ as the smallest integer for which $C(\beta)L^{d-1} \geq 4\log t$. In this way we find

$$\mu(T^J(t)\, m_\Lambda \,|\, F_\Lambda) \geq \frac{1}{3} \tag{7.11}$$

for all t large enough. From (7.6) $\ldots$ (7.11) it follows

$$\mathbb{E}\|T^J(t)\pi_0\|_{L^2(\mu^J)} \geq \frac{1}{3}\,\mathbb{P}(\bar{\Theta}) \geq \exp\left[\,-k''\,(\log t)^{\frac{d}{d-1}}\,\right]$$

for a suitable positive constant k''.

Thus we can conclude that, for any $\beta > \beta_c$ in $d = 2$ or for any β large enough in higher dimensions there exists two positive constants C_1, C_2 such that, for any large enough time t

$$e^{-C_1\log(t)^{\frac{d}{d-1}}} \leq \mathbb{E}\,\|T^J(t)\pi_0\|_{L^2(\mu^J)} \leq e^{-C_2\log(t)^{\frac{d}{d-1}}}$$

We would like to conclude this part with a short discussion of the almost sure scaling law of $\mathrm{gap}(L^{J,\tau}_{Q_L})$ as $L \to \infty$. Such a discussion is instructive since it represents a main guideline for the coming analysis of the dynamical dilute Ising model *above* the percolation threshold.

The starting observation is the following. For any $L \geq 1$ and any J let

$$v(J,L) \equiv \max_{x \in Q_L} |C_x|$$

Since the probability distribution of the volume of the cluster C_0 containing the origin has an exponential tail for any $p < p_c$, it follows from standard arguments that there exists a positive constant $k = k(p)$ and a set $\bar{\Theta}$ of measure one such that for any $J \in \bar{\Theta}$ there exists $L_0(J)$ such that for any $L \geq L_0(J)$

i) $v(J,L) \leq k \log(L)$

ii) there exists $x \equiv x(L,J)$, with $|x| \leq L/2$, such that $C_x = Q_l(x)$ and *all* couplings inside the cube $Q_l(x)$, $l = (k^{-1} \log L)^{1/d}$, are equal to β and *all* couplings connecting a point inside $Q_l(x)$ with one of its nearest neighbors outside it are zero. By construction $Q_l(x) \subset Q_L$ if L is large enough.

It follows immediately from i) above, theorem 3.8, proposition 3.9 and the simple observation that

$$\begin{aligned} \operatorname{gap}(L^{J,\tau}_{Q_L}) &= \min_{x \in Q_L} \operatorname{gap}(L^{J,\tau}_{C_x}) \\ c_s(L^{J,\tau}_{Q_L}) &= \max_{x \in Q_L} c_s(L^{J,\tau}_{C_x}) \end{aligned} \tag{7.12}$$

since the couplings among disjoint clusters are zero, that for any $J \in \bar{\Theta}$ and any $L \geq L_0(J)$

$$\begin{aligned} \operatorname{gap}(L^{J,\tau}_{Q_L}) &\geq e^{-kv(J,L)^{\frac{d-1}{d}}} \geq e^{-k' \log(L)^{\frac{d-1}{d}}} \\ c_s(L^{J,\tau}_{Q_L}) &\leq kv(J,L) e^{kv(J,L)^{\frac{d-1}{d}}} \leq e^{k'' \log(L)^{\frac{d-1}{d}}} \end{aligned} \tag{7.13}$$

for suitable constants k, k', k''. Again, using the results of section 6.5 and ii) above, one also obtain an upper bound on the spectral gap of the same order but with a different constant $\bar{k}$ in the usual range of inverse temperature β. Similarily for the logarithmic Sobolev constant. We can thus conclude that, almost surely, the spectral gap in the box Q_L shrinks, as $L \to \infty$, roughly as $e^{-k \log(L)^{\frac{d-1}{d}}}$. We can actually go further and estimate also the probability of a large deviation from the "typical" behaviour. Using (7.13) we can write for any $\varepsilon \in (0,1]$

$$\begin{aligned} \mathbb{P}\big(c_s(L^{J,\tau}_{Q_L}) \geq L^{\varepsilon}\big) &\leq \mathbb{P}\big(kv(J,L) e^{kv(J,L)^{\frac{d-1}{d}}} \geq L^{\varepsilon}\big) \\ &\leq L^d \, \mathbb{P}\big(|C_0| \geq k'_{\varepsilon} \log(L)^{\frac{d}{d-1}}\big) \\ &\leq e^{-k''_{\varepsilon} \log(L)^{\frac{d}{d-1}}} \end{aligned} \tag{7.14}$$

for suitable constants $k'_{\varepsilon}, k''_{\varepsilon}$ and any L large enough.

In the next final section we will see how to extend the above simple analysis to a much more complicate situation in which one connected component of the set of non–regular sites W is infinite.

7.3 The Dynamics in the Griffiths Phase: $\mathbf{P} \geq \mathbf{P_c}$

Here we analyze the more complex case when $\beta > \beta_c$ and $p \geq p_c$ but there still exists a unique infinite volume Gibbs state. More precisely, following [CMM1], we make the following assumption (recall definition 2.6 in section 2) that easily implies (see [Be]) almost sure uniqueness of the infinite volume Gibbs measure and exponential decay of the average (over the disorder) of the absolute value of covariances.

(H) There exist $L_0 \in \mathbb{Z}_+$, $\alpha > 0$, $\vartheta > 0$ such that for all $L \geq L_0$

$$\mathbb{P}\{\, SMT(Q_L, L/4, \alpha)\,\} \geq 1 - e^{-\vartheta L}$$

Notice that the above assumption is quite general and flexible in that it does not assume anything specific on the couplings (e.g. that strong couplings do not percolate).

It is rather elementary to show that hypothesis (H) holds as long as $p < p_c$. Much more interesting and less trivial is the problem of showing that it holds also for some value of p above the percolation threshold. The following result (see [CMM2]) provides a partial answer in two dimensions. For an extension to higher dimensions we refer the reader to [ACCMM]. Rember that in two dimensions $p_c = \frac{1}{2}$.

Theorem 7.1. *If*

$$\beta < \begin{cases} \frac{\beta_c}{p} \vee \log(\frac{p}{2p-1}) & \textit{if } p > 1/2 \\ \infty & \textit{if } p \leq 1/2 \end{cases}$$

then hypothesis (H) holds.

The picture below illustrates the region described in theorem 7.1.

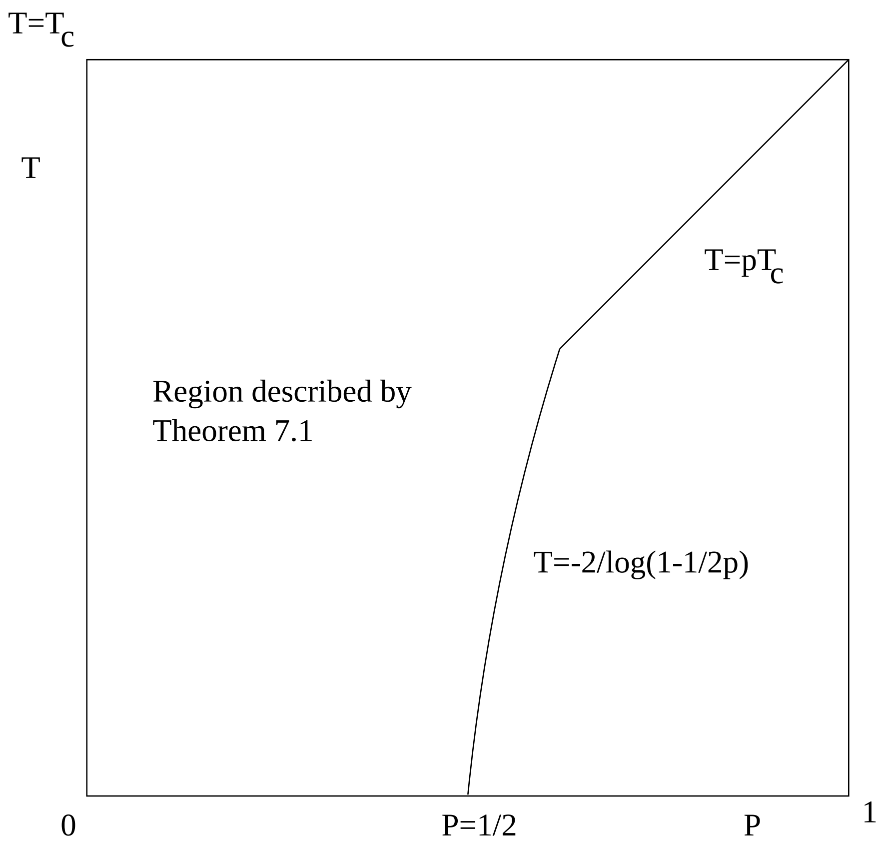

Fig. 1

Using hypothesis (H) we will now state the main results (see [CMM1]) concerning the relaxational properties of the dynamics, almost surely and in average, and the growth of the logarithmic Sobolev constant in the cube Q_L as $L \to \infty$.

Theorem 7.2. *Assume (H). Then*

(a) *If $d \geq 1$ there exists a set $\bar{\Theta} \subset \Theta$ of full measure such that for each $J \in \bar{\Theta}$ there exists a unique infinite volume Gibbs measure μ^J. Moreover there exists a constant k and, for each $J \in \bar{\Theta}$ and for any local function f there exists $0 < t_0(J, f) < \infty$ such that for all $t \geq t_0$*

$$\|T^J(t)f - \mu^J(f)\|_\infty \leq \exp\Big[-t\,\exp\big[-k\,(\log t)^{\frac{d-1}{d}}\,(\log\log t)^{d-1}\big]\Big] \tag{7.15}$$

(b) Assume $d \geq 2$. Then there exists a constant k and for any local function f there exists $0 < t_0(f) < \infty$ such that, if $t \geq t_0(f)$ then

$$\mathbb{E}\,\|T^J(t)f - \mu^J(f)\|_\infty \leq \exp\big[-k\,(\log t)^{\frac{d}{d-1}}\,(\log\log t)^{-d}\big] \tag{7.16}$$

Remark. The almost sure speed of relaxation to equilibrium is faster than any stretched exponential, *i.e.* a decay like e^{-Ct^δ} with $\delta < 1$, and , as the next theorem shows, it cannot be improved in general.

Also the bound (b) on the relaxation for the *averaged* dynamics in the bounded case is, apart from the technical factor $(\log\log t)^d$, optimal (see theorem 7.3 below).

Recall now that $\pi_0(\sigma) = \sigma(0)$ and that, by simmetry, $\mu^J(\pi_0) = 0$. Then we have

Theorem 7.3. *For each $d \geq 2$ there is $\beta_1(d) > 0$ such that the following holds: if $\beta > \beta_1(d)$ is such that for almost all J there exists a unique Gibbs measure μ^J, then*

(a) *for all large enough t,*

$$\mathbb{E}\|T^J(t)\pi_0\|_{L^2(\mu^J)} \geq \exp\left[-k\,(\log t)^{\frac{d}{d-1}}\right] \tag{7.17}$$

for some k which depends on d, p_1 and p_2.

(b) *Assume $p > p_c$ and choose the transition rates of the heat-bath dynamics given in (3.5). Then there exists $k > 0$ and a set Θ_0 of positive probability such that for all $J \in \Theta_0$ there exists $0 < t_0(J) < \infty$ such that for all $t \geq t_0$ we have*

$$\|T^J(t)\pi_0\|_\infty \geq \exp\Big[-t\,\exp\left[-k\,(\log t)^{\frac{d-1}{d}}\right]\Big] \tag{7.18}$$

Theorem 7.4.

(i) *Assume (H). Then there exist C_1, C_2 and L_1 depending on d, α and ϑ such that for all $L \geq L_1$,*

$$\mathbb{P}\left\{\,c_s(L^J_{Q_L}) > C_1\exp\left[C_2\beta\,(\log\log L)^{d-1}(\log L)^{\frac{d-1}{d}}\right]\right\} < L^{-2d} \tag{7.19}$$

(ii) *Assume (H) and $d \geq 2$. Then for any $\varepsilon \in (0,1]$ there exist positive constants C_3 and L_2 depending on d, r, α, ϑ, J_0 and ε such that for all $L \geq L_2$*

$$\mathbb{P}\left\{\,c_s(L^J_{Q_L}) > L^\varepsilon\,\right\} < \exp\left[-C_3\,(\log\log L)^{-d}(\log L)^{\frac{d}{d-1}}\right] \tag{7.20}$$

Remark. Using (i) together with the Borel–Cantelli lemma, it follows that, with probability one, $c_s(L^J_{Q_L})$ does not grow faster than the exponential appearing in (7.19). Such a growth is, apart from the $\log\log L$ factor, optimal if one compares it with the almost sure growth obtained for $p < p_c$. Similar comment apply also to the bound on the probability of a large deviations of $c_s(L^J_{Q_L})$.

7.4 A Coarse Graining Description Above $\mathbf{p_c}$

This paragraph represents really the core of the analysis of the dynamical dilute Ising model in the Griffiths phase above the percolation threshold. We give a deterministic upper bound on the logarithmic Sobolev constant $c_s(\mu^{J,\tau}_\Lambda)$ in the cube $\Lambda \equiv Q_L$.

For this purpose, recall first the result at the end of section 7.2. There we proved that if $\sup_i |W_i \cap Q_L| \leq v$ where $\{W_i\}$ are the connected components of the set of non–regular sites W, then

$$c_s(\mu_\Lambda^{J,\tau}) \le C_1 v\, e^{C_2 v^{\frac{d-1}{d}}} \tag{7.21}$$

for suitable constants C_1 and C_2 independent of L and v. A key ingredient for the proof of the above bound was the observation that the connected components of W are non–interacting since they are separated from each other by a "safety belt" of regular sites, *i.e.* sites completely decoupled from their complement. Unfortunately, when p is larger than p_c, the same approach becomes useless since the non–regular set W has exactly one infinite connected component (the percolating cluster) with a positive density so that v will typically be of the same order of magnitude as the original volume $|Q_L|$.

It is however very important to observe that, thanks to some of the results of section 4.2, the above conclusions in the non percolating regime remains true, modulo some irrelevant constant factors, even if the value $J_{xy} = 0$ is replaced by a very small number J_{min}, provided that $|J_{min}||W \cap Q_L| \ll 1$.

This remark suggests how to transpose to the dilute model above the percolation threshold the previous ideas. In a certain sense, if the coupling J are such that truncated correlations functions decay exponentially fast on a length scale $l_0 \ll L$, then our original model behaves, after a suitable "coarse-graining", quite closely to the same dilute model *below* the percolation threshold. Let us in fact make a coarse–grained description of the model on a new scale $l_0 \ll L$, by replacing sites with disjoint cubes C_i of side l_0 and declare "regular" those cubes C_i in which truncated correlations decay exponentially fast with rate $\alpha > 0$. In this way, if B is a collection of "non–regular" cubes C_i surrounded by a safety–belt of regular cubes, then the effective interaction of B with any other region outside the safety–belt will be not larger than $|B|\exp(-\alpha l_0)$. Thus, if l_0 is chosen so large that the effective interaction among the connected components of the set W_{l_0} of non–regular cubes C_i is much smaller than one, e.g. if $|W_{l_0}|\exp(-\alpha l_0) \ll 1$, then our system, on scale l_0, will behave like a diluted Ising model even if, strictly speaking, the regular cubes are connected by non–zero couplings. In particular we will be able to apply the results of Section 4.2 and, as a consequence, we will get the bound (7.13) on the spectral gap, with v equal to the volume of the largest connected component of the set W_{l_0}. Moreover, if the probability p_{l_0} that a cube C_i of side l_0 is non–regular is very small, *i.e.* if assumption (H) holds, then the net effect of the coarse–graining will be that of replacing the original parameter $p > p_c$ with a new, renormalized one, $p_{l_0} \ll p_c$.

Although the above reasoning looks quite appealing from a physical point of view, it is still unsatisfactory for the following reason. In a typical configuration of J, the volume of the set $W_{l_0} \cap Q_L$ is roughly $p(l_0)L^d$, where $p(l_0)$ is the probability that a cube C_i is not regular. Using our basic assumption (H), $p(l_0) \approx \exp(-\vartheta l_0)$ so that the minimal scale l_0 satisfying $|W_{l_0}|\exp(-\alpha l_0) \ll 1$ becomes of order $\log L$. This unfortunately is a too large scale: since v is at least l_0^d, the corresponding bound (7.21) on the logarithmic Sobolev constant, becomes at least of the order of a power of L.

In order to overcome this difficulty, we appeal to Proposition 4.10. More precisely we introduce an intermediate length scale $l_1 \ll L$ and we assume that the J in Λ are such that the hypotheses of Proposition 4.10 apply for l_1. If this is the case, then Proposition 4.10 basically allows us to replace the initial cube $\Lambda = Q_L$ with a smaller cube $Q_{l_1}(x)$, for a suitable $x \in \Lambda$. Once we have reduced the initial scale L to the new scale l_1, we make the coarse–grained analysis on scale $l_0 \ll l_1$ on the new cube $Q_{l_1}(x)$ and proceed as explained before.

The advantage of the above two–scale analysis is twofold. First of all the shortest scale l_0 is now at most of the order of $\log l_1$ instead of $\log L$. If we assume (H) it follows from the Borel–Cantelli lemma that in a typical configuration the intermediate scale l_1 can be taken already of the order of $\log L$. Thus we see that, with probability one, the smallest scale becomes $l_0 \approx \log\log L$ with an enormous gain in precision. We conclude this short heuristic discussion by observing that it is precisely the coarse–grained analysis on scale $\log\log L$ that is responsible for the various $\log\log L$ factors in Theorem 7.4.

We are now ready for a precise formulation of our results. Remember that $\mathcal{F}_l$ is the class of subsets of $\mathbb{Z}^d$ which are multiple of Q_l.

Definition 7.5. *A cube $C = Q_l(x)$ is said to be α–regular if, letting $n(l) = \lfloor l/(2\gamma_1)\rfloor$, $SMT(Q_n(y), n/2, \alpha)$ holds for all $y \in Q_l(x)$ where the constants γ_1 and $\widehat{m}(\alpha)$ are those appearing in Proposition 2.9.*

We immediately observe that, thanks to proposition 2.9 and the second remark after its proof, if $V \in \mathcal{F}_{n(l)}$ (*i.e.* if l is a multiple of the integer $2\gamma_1$) is the union of α–regular cubes of side length l then, if l is large enough depending on β and d, $SMT(V, l/2, \widehat{m}(\alpha))$ holds.

Definition 7.6. *Let $l \in \mathbb{Z}_+$, let $\alpha > 0$, let Λ be a multiple of Q_l and write $\Lambda = \cup_{i=1}^n Q_l(x_i)$ for some $n \in \mathbb{Z}_+$ and $x_i \in l\mathbb{Z}^d$. Let K be the set of all $i \in \{1, \dots, n\}$ such that $Q_l(x_i)$ is not α–regular and define*

$$W(\Lambda, l, \alpha) = \{\, x \in \Lambda :\ d_2\big(x, \cup_{i\in K} Q_l(x_i)\big) \le 2l \,\}$$
$$v(\Lambda, l, \alpha) = \text{the cardinality of the largest connected component of } W(\Lambda, l, \alpha)$$

Proposition 7.7. *Choose the transition rates c_J as in (3.6). Then, for each $\alpha > 0$ there exists $\bar{l}(d, \alpha, \beta)$ and $k(d, \alpha)$ such that the following holds for all integers $l_0 \ge \bar{l}$ which are multiples of $2\gamma_1$: let V be a multiple of Q_{l_0} and let $v = v(V, l_0, \alpha)$ (see Definition 7.6) and let $m = \widehat{m}(d, \alpha)$ be the decay rate given by Proposition 2.9. Assume that $ml_0 \ge \max\{200d\beta, C, 10\,(1 + \log|W(V, l_0, \alpha)|)\}$ where C is the constant appearing in Corollary 3.6. Then,*

$$\inf_{\tau\in\Omega} \mathrm{gap}(L_V^\tau) \ge \exp\big[\, -k(\, 2d\beta v^{\frac{d-1}{d}} + l_0^{d-1}\,)\,\big] \tag{7.22}$$

Proof. Write $V = \bigcup_{i=1}^s C_i$, where $C_i = Q_{l_0}(y_i)$ for some $y_i \in l_0\mathbb{Z}^d$. Let $B = W(V, l_0, \alpha)$ and let A be the union of all those (α–regular) cubes C_i such that $d(C_i, C_j) > l_0$ for all C_j which are not α–regular. Let also $A_0 = A \cap \partial_{l_0}^+ B$ and $B_0 = B \cap \partial_r^+ A$, $r = 1$. By Proposition 3.4 we have

$$\inf_{\tau\in\Omega} \mathrm{gap}(L_V^{J,\tau}) \ge \frac{1}{2}\Big[\inf_{\tau\in\Omega}\ \inf_{D\in\{A,B\}} \mathrm{gap}(L_D^{J,\tau})\,\Big] \inf_{\tau\in\Omega} \mathrm{gap}(L_{\{A,B\}}^{J,\tau}) \tag{7.23}$$

The proof of the Proposition can then be organized in the following steps:

(a) We can use Corollary 3.6 to show that the gap of the block dynamics generator $L^{J,\tau}_{\{A,B\}}$ is at least $1/2$. In order to show that 3.6 does indeed apply to our case, we first notice that $d(A_0, B_0) \geq l_0$, which, together with the trivial inequality $|B_0| \leq |W(V, l_0, \alpha)|$, implies the hypothesis (i) of 3.6. Then we observe that $A \backslash A_0$ can be expressed as a union of $\alpha-$regular cubes C_i. So, by Proposition 2.9, the property $SMT(A\backslash A_0, l_0/2, m)$ holds.

(b) Since the set A is a union of $\alpha-$regular cubes of side lenght l_0, property $SMT(V, l_0/2, \widehat{m}(\alpha))$ holds for any subset V of A which is a multiple of Q_{l_0}. Thus, using Proposition 4.10, one proves that $\mathrm{gap}(L^{J,\tau}_A)$ is bounded from below by a quantity which does not depend on the size of A:

$$\mathrm{gap}(L^{J,\tau}_A) \geq \frac{1}{2} \inf_\tau \inf_{C_i \subset A} \mathrm{gap}(L^{J,\tau}_{C_i}) \geq C_1 e^{-C_2 l_0^{d-1}} \tag{7.24}$$

(c) For what concerns the gap of $L^{J,\tau}_B$, we write B as the disjoint union of its connected components $\widetilde{B}_1, \ldots, \widetilde{B}_n$. Since $L^{J,\tau}_{\tilde{B}_i}$ commutes with $L^{J,\tau}_{\tilde{B}_j}$ for all $i \neq j$, it follows that

$$\mathrm{gap}(L^{J,\tau}_B) = \inf_{i \in \{1,\ldots,n\}} \mathrm{gap}(L^{J,\tau}_{\tilde{B}_i})$$

From (7.23), (a), (b) and (c), together with Theorem 3.8 (for the dynamics (3.6) we can take $c_m = 1/2$ in (3.3)) and the fact that trivially $\|J^{(\lambda)}\| \leq 2d\beta$, we get

$$\inf_{\tau \in \Omega} \mathrm{gap}(L^{J,\tau}_V) \geq \frac{1}{4} \min\{\frac{1}{2} \inf_i e^{-(\,k\, 2d\beta\, |\tilde{B}_i|^{\frac{d-1}{d}})}, \, C_1 e^{-C_2 l_0^{d-1}}\}$$

In order to obtain (7.22) we now observe that by definition of v, we have $|\widetilde{B}_i| \leq v$, and that the minimum of the two quantities in braces is greater than their product if l_0 is such that both terms are less than 1. $\square$

Theorem 7.8. *If the transition rates are given by (3.6), then for each $\alpha > 0$ there exist $\bar{l}$, C_1 and C_2 depending on d, β and α such that the following holds for all positive integers $l_0 \geq \bar{l}$ multiples of $2\gamma_1$: let l_1 be a multiple of l_0 and let Λ be a multiple of Q_{l_1} so that we can write*

$$\Lambda = \textstyle\bigcup_{i=1}^n B_i = \bigcup_{i=1}^s C_i \tag{7.25}$$

where $B_i = Q_{l_1}(x_i)$ and $C_i = Q_{l_0}(y_i)$ for some $x_i \in l_1\mathbb{Z}^d$ and $y_i \in l_0\mathbb{Z}^d$ Let $v = v(\Lambda, l_0, \alpha)$ (see Definition 7.6), and let $m = \widehat{m}(d, r, \alpha)$. Assume that

(i) For each $i \in \{1, \ldots n\}$ the cube B_i is $\alpha-$regular
(ii) $kv^{\frac{d-1}{d}} \leq ml_1/4$, where $k(d, \alpha, \beta)$ is the quantity defined in Theorem 3.8
(iii) $30d \log l_1 \leq ml_0 \leq (l_1)^{1/(2d)}$

Then we have

$$\sup_{\tau \in \Omega} c_s(\mu^{J,\tau}_\Lambda) \leq C_1 \exp\left[\, C_2\, (v^{\frac{d-1}{d}} + l_0^{d-1})\,\right]$$

Proof. Since Λ is the union of α−regular cubes B_i, for any $I \subset \{1,\dots,n\}$ property $SMT(V, l_1/2, \widehat{m}(\alpha))$ holds for $V = \bigcup_{i\in I} B_i$. Therefore, thanks hypothesis (ii) and to proposition 4.10

$$\begin{aligned}\sup_{\tau\in\Omega} c_s(L_\Lambda^{J,\tau}) &\le 2 \sup_{\tau\in\Omega} \sup_{B_i\in\Lambda} c_s(L_{B_i}^{J,\tau}) \\ &\le C' \left[4 + 4 l_1^d (2d\beta + 2\log 2) \right] \sup_{B_i\in\Lambda} \mathrm{gap}(L_{B_i}^{J,\tau})^{-1}\end{aligned}$$

We now observe that hypothesis iii) allow us to apply proposition 7.7 above to any cube B_i in Λ. Thus

$$\sup_{B_i\in\Lambda} \mathrm{gap}(L_{B_i}^{J,\tau})^{-1} \le \exp\left[+k(2d\beta v^{\frac{d-1}{d}} + l_0^{d-1}) \right]$$

and the result follows. □

7.5 Proof of the Main Results Above $\mathbf{P_c}$

In this final part we prove our main results above the percolation threshold, namely theorem 7.2, 7.3 and 7.4. First we need the following simple estimate.

Lemma 7.9. *Assume (H). Then there exist $L_0 \in \mathbb{Z}_+$, $\vartheta' > 0$ such that for all $L \ge L_0$,*

$$\mathbb{P}\{\, Q_L \text{ is } \alpha\text{−regular} \,\} \ge 1 - e^{-\vartheta' L}$$

Proof. The probability that Q_L is not α−regular is bounded by

$$\begin{aligned}&\mathbb{P}\{SMT(Q_n(y), n/4, \alpha) \text{ does not hold for some } y \in Q_L\} \le \\ &\le L^d\, e^{-\vartheta L/(3\gamma_1)} \\ &\le e^{-\vartheta' L}\end{aligned}$$

if L is greater than some L_0. □

Proof of Theorem 7.4. We give the proof in the special case of L which is a power of 2, which is enough to prove Theorem 7.2. We also assume, without loss of generality, that the constant γ_1 of proposition 2.9 is also a power of 2. A proof which works for all L requires a modification of Theorem 7.8 where one considers more general coverings of Λ with cubes and cuboids with slightly different sidelengths. This generalization is straightforward.

Part (i). We are going to use the key deterministic estimate of $c_s(\mu_{Q_L}^{J,\tau})$ given in Theorem 7.8. The idea is to prove that with probability greater than $1 - 3L^{-2}$, it is possible to choose the two parameters in Theorem 7.8, l_0, l_1 in such a way that the deterministic upper bound on $c_s(\mu_{Q_L}^{J,\tau})$ given in that proposition is not greater than

$$\exp\left[C(\log\log L)^d (\log L) \right]$$

for a suitable constant C. More precisely we define l_0 and l_1 as those powers of 2 (they are uniquely defined) such that

$$\frac{60d}{m}\log\log L \le l_0 < \frac{120d}{m}\log\log L \qquad \frac{3d}{\vartheta'}\log L \le l_1 < \frac{6d}{\vartheta'}\log L \tag{7.26}$$

where $m \equiv \widehat{m}(\alpha)$ (see Proposition 2.9) and ϑ' is given in lemma 7.9. We then take

$$v_* = l_0^d \log L \tag{7.27}$$

Since l_0 divides l_1 and l_1 divides L, we can write Q_L as in (7.25). We now observe that, if L is large enough, the hypotheses (i) – (iii) of Theorem 7.8 are satisfied for all $J \in \tilde{\Theta} \equiv \cap_{i=1}^2 \Theta_i$, where

$\Theta_1 = \{J : \text{each } B_i \text{ is } \alpha\text{−regular}\}$

$\Theta_2 = \{J : v(Q_L, l_0, \alpha) \le v_*\}$

and $v(Q_L, l_0, \alpha)$ has been defined in 7.6. By Theorem 7.8, (7.26) and (7.27), for any $J \in \tilde{\Theta}$, we have

$$c_s(\mu_{Q_L}^{J,\tau}) \le \exp\big[\, C(\log\log L)^d \log L\, \big]$$

for a suitable constant C. In order to prove the theorem it is therefore sufficient to estimate from above $\mathbb{P}\big(\tilde{\Theta}^c\big)$.

From Proposition 7.9, it follows

$$\mathbb{P}\big(\Theta_1^c\big) \le L^d e^{-\vartheta' l_1} \le L^{-2d} \tag{7.28}$$

for all L large enough. Let $p(l)$ be the probability that a cube Q_l is not α−regular. Then $p(l)$ goes to zero as $l \to \infty$, and a standard estimate for 2−dependent site percolation implies

$$\mathbb{P}\big(\Theta_2^c\big) \le L^d \left(k_1\, p(l_0)\right)^{k_2 v_* l_0^{-d}} \le L^{-3d} \tag{7.29}$$

for L large enough, where k_1 and k_2 are two suitable geometrical constants. This completes the proof.

Proof of part (ii). The proof is the same as in part (i), with a different choice of the three basic parameters l_0, l_1 and v_*. More precisely we define l_0 and l_1 as those powers of 2 (they are uniquely defined) such that (let again $m = \widehat{m}(\alpha)$)

$$\tfrac{60d^2}{(d-1)m}\log\log L \le l_0 < \tfrac{120d^2}{(d-1)m}\log\log L \qquad (\log L)^{\frac{d}{d-1}} \le l_1 < 2(\log L)^{\frac{d}{d-1}}$$

Given $\varepsilon \in (0,1)$ we then let

$$v_* = \left(\frac{\varepsilon \log L}{2J_0 C_2}\right)^{\frac{d}{d-1}} \tag{7.30}$$

where C_2 appears in Theorem 7.8. Write Q_L as in (7.25) and define the events $\tilde{\Theta}$, Θ_i as in the proof of part (i). Thanks to Theorem 7.8, we get

$$c_s(\mu_{Q_L}^{J,\tau}) \le C_1 L^{\frac{\varepsilon}{2}} \exp(C_2 l_0^d) \le L^{\varepsilon} \qquad \forall J \in \tilde{\Theta} \tag{7.31}$$

for all L sufficiently large. In order to prove (7.20) it is therefore sufficient to bound from above $\mathbb{P}(\tilde{\Theta}^c)$. As before, we find

$$\mathbb{P}(\Theta_1^c) \le L^d e^{-\vartheta' l_1} = L^d \exp\bigl[-\vartheta' (\log L)^{\frac{d}{d-1}}\bigr] \tag{7.32}$$

and

$$\mathbb{P}(\Theta_2^c) \le L^d\,(k_1 p(l_0))^{k_2 v_* l_0^{-d}} \le \exp\bigl[-C_3\,(\log\log L)^{-d} (\log L)^{\frac{d}{d-1}}\bigr] \tag{7.33}$$

for a suitable constant C_3 and all L large enough. Clearly (7.32) and (7.33) complete the proof of (ii). $\square$

Proof of Theorem 7.2. The proof of the almost sure bounds (part (a)) is a simple consequence of Theorem 7.4. Let $\bar{\Theta}$ be the set of interactions J such that for each $J \in \bar{\Theta}$ there exists $L_1(J)$ such that for all $L \ge L_1(J)$ (C is given in Proposition 7.4)

(i) $c_s(L^J_{B_L}) < \exp\bigl[C\,(\log L)\,(\log\log L)^d\bigr]$
(ii) $SMT(B_L, \gamma_1(2L+1), \alpha)$ holds

Using Theorem 7.4, (H) and the Borel–Cantelli lemma, one can check that $\mathbb{P}(\bar{\Theta}) = 1$. Moreover, thanks to (ii) and (iii), for all $J \in \bar{\Theta}$ there exists a unique infinite volume Gibbs measure that in the sequel will be denoted by μ^J. Let, in fact, f be any local function on Ω, and take L large enough such that $B_L \supset \Lambda_f$. Then, given two arbitrary boundary conditions τ and η, and using a telescopic interpolation between them, we get

$$\begin{aligned}\sup_{\tau,\eta\in\Omega}\bigl|\,\mu^{J,\tau}_{B_L}(f) - \mu^{J,\eta}_{B_L}(f)\,\bigr| &\le |\partial_r^+ B_L| \sup_{x\in\partial_r^+ B_L} \|\nabla_x[\mu^J_{B_L}(f)]\|_\infty = \\ &= |\partial_r^+ B_L| \sup_{x\in\partial_r^+ B_L}\sup_{\tau\in\Omega}\left|\frac{\mu^{J,\tau}_{B_L}(h_x, f)}{\mu^{J,\tau}_{B_L}(h_x)}\right|\end{aligned} \tag{7.34}$$

where $h_x \equiv \exp[-\nabla_x H_{B_L}]$.

Therefore, if L is larger than $L_1(J)$ and if $d(\Lambda_f, (B_L)^c) > \gamma_1(2L+1) + 1$, we can use $SMT(B_L, \gamma_1(2L+1), \alpha)$, and write

$$\sup_{\tau,\eta\in\Omega}\bigl|\,\mu^{J,\tau}_{B_L}(f) - \mu^{J,\eta}_{B_L}(f)\,\bigr| \le k L^{d+4} |\Lambda_f| \|f\|_\infty e^{-\alpha d(\Lambda_{h_x}, \Lambda_f)} \tag{7.35}$$

for a suitable constant k, and the uniqueness follows. At this point, in order to prove inequality (7.15), we simply appeal to (3.23) .

Proof of part (b). Let $L_t = \lfloor k_1 t\rfloor$ for some $k_1 > k_0$ such that L_t is a power of 2 (k_0 is given in Lemma 3.2) and, for simplicity, let $\Lambda_t = B_{L_t}$. For any $\varepsilon \in (0,1)$, let $\Theta(t,\varepsilon)$ be the set of interactions J such that

(i) $c_s(L^J_{\Lambda_t}) \le L_t^\varepsilon$
(ii) $SMT(\Lambda_t, (\log L_t)^{\frac{d}{d-1}}, \widehat{m}(\alpha))$.

We can write, for any $\tau \in \Omega$,

$$\begin{aligned}\mathbb{E}\|T^J(t)f - \mu^J(f)\|_\infty \leq |||f|||\, \mathbb{P}\big(\Theta(t,\varepsilon)^c\big) + \sup_{J\in\Theta(t,\varepsilon)} \|T^{J,\tau}_{\Lambda_t}(t)f - \mu^{J,\tau}_{\Lambda_t}(f)\|_\infty + \\ + \|T^J(t)f - T^{J,\tau}_{\Lambda_t}(t)f\|_\infty + \sup_{J\in\Theta(t,\varepsilon)} |\mu^{J,\tau}_{\Lambda_t}(f) - \mu^J(f)|\end{aligned} \tag{7.36}$$

We denote by X_1, X_2, X_3 and X_4 the four terms on the RHS of (7.36). The last two terms, using lemma 3.2 and property $SMT(\Lambda_t, (\log L_t)^{\frac{d}{d-1}}, \widehat{m}(\alpha))$ are exponentially small in t:

$$X_3 + X_4 \leq C_f\, e^{-2t} \tag{7.37}$$

Furthermore, we have

$$\begin{aligned}\mathbb{P}\big(\Theta(t,\varepsilon)^c\big) \leq \mathbb{P}\big\{\, c_s(L^J_{\Lambda_t}) \geq L^\varepsilon \,\big\} + \\ + \mathbb{P}\big\{\, SMT(\Lambda_t, (\log L)^{\frac{d}{d-1}}, \widehat{m}(\alpha)) \text{ does not hold } \big\}\end{aligned} \tag{7.38}$$

Of the above two terms the first one is estimated via (ii) of Theorem 7.4, which implies

$$\mathbb{P}\big\{\, c_s(L^J_{\Lambda_t}) \geq L^\varepsilon \,\big\} \leq \exp\big[\, -C_3\, (\log\log L_t)^{-d} (\log L_t)^{\frac{d}{d-1}}\big] \tag{7.39}$$

provided that t is large enough. The second term in the RHS of (7.38) can be bounded from above, using Proposition 2.9, by the probability that there exists a cube $Q_l(x)$ in Λ_t, with $l = \lceil (\log L)^{\frac{d}{d-1}} \rceil$, which is not $\alpha-$regular. Using Proposition 7.9 such a probability is bounded from above by

$$L_t^d \exp[-\vartheta' \,(\log L_t)^{\frac{d}{d-1}}] \tag{7.40}$$

provided that t is so large that $L_t \geq L_0$. In this way we have obtained

$$X_1 \leq |||f|||\, \Big[\exp[\, -C_3\, (\log\log L_t)^{-d} (\log L_t)^{\frac{d}{d-1}}] + L_t^d \exp[-\vartheta (\log L_t)^{\frac{d}{d-1}}]\Big] \tag{7.41}$$

As for X_2, we use (3.23) and the fact that now $c_s(L^J_{\Lambda_t}) \leq L_t^\varepsilon$, and we get

$$X_2 \leq 2\, |||f|||\, \exp\big[\, -k'\, t^{1-\varepsilon}\,\big] \tag{7.42}$$

for any t sufficiently large. From (7.37), (7.41) and (7.42) we get that for large t the dominant term in (7.36) is the first one and, by consequence (7.16) follows. □

Proof of the lower bound, Theorem 7.3
It is enough to prove part b) since part a) has already been proved in section 7.2.

The main idea for the lower bound on the a.s. relaxation of the spin at the origin can be divided into two distinct parts. The first part consists in showing that, with positive probability, for any L large enough, there exists a local function f_L, with $\Lambda_{f_L} \subset B_L \cap C_0$, C_0 being the cluster of the origin, whose relaxational behaviour is not faster than $\exp\Bigl[-t\,\exp\bigl[-k\,(\log L)^{\frac{d-1}{d}}\bigr]\Bigr]$.

The second part amounts to proving that the influence of the slow relaxation of f_L on the spin at the origin is not smaller than a negative exponential of L. This implies a lower bound on $\|T^J(t)\pi_0\|_\infty$ of the order of $\exp\Bigl[-mL\,-\,t\,\exp\bigl[-k\,(\log L)^{\frac{d-1}{d}}\bigr]\Bigr]$ and the result (remember that $\mu^J(\pi_0)=0$ by symmetry) follows by optimizing over $L \le t$. Let us now implement these sketchy ideas.

We say that a path $\gamma=(x_0,\dots,x_n)$ connects x to y if

$$x_0 = x,\; x_n = y \qquad d_2(x_i,x_{i+1}) = 1 \qquad \text{and} \qquad J_{x_i,x_{i+1}} = \beta \quad \forall\, i = 0,\dots,l-1$$

We will then write $\gamma : x \to y$ and set $|\gamma| = n$. Given a configuration J of the random couplings, a local function f and a finite set $\Lambda \subset C_0$, where C_0 is the cluster containing the origin, we set

$$d^J(\Lambda) = \sup_{x\in\Lambda}\, \inf_{\substack{\gamma:\\ \gamma:\,0\to x}} |\gamma|\,; \quad e^J(f) = \frac{\mathcal{E}^J(f,f)}{\operatorname{Var}^J(f)}\,; \quad e^J(\Lambda) = \inf_{\substack{f\in L^2(\Omega,d\mu^J)\\ \Lambda_f\subset\Lambda}} e^J(f) \tag{7.43}$$

(both the Dirichlet form and the variance are with respect to the unique infinite volume Gibbs measure). With the above definition we have the following two key results.

Lemma 7.10. *Under the same assumptions of part (b) of Theorem 7.3 there exists a set $\bar\Theta \subset \Theta$ of positive measure and two positive constants k and L_0 such that for each $J \in \bar\Theta$ and any $L \ge L_0$ there exists f such that*

$$\begin{aligned} \Lambda_f &\subset C_0\\ d^J(\Lambda_f) &\le kL\\ e^J(f) &\le \exp\Bigl[-k(\log L)^{\frac{d-1}{d}}\Bigr] \end{aligned}$$

Lemma 7.11. *There exists $m > 0$ such that for any $t \ge 1$ and any local function f such that $\Lambda_f \subset C_0$*

$$\|T^J(t)\pi_0\|_\infty \ge (8|\Lambda_f|)^{-1}\exp\Bigl[-md^J(\Lambda_f)\,-\,2e^J(f)t\Bigr]$$

Before proving the two lemmas we complete the proof of part (b) of the theorem. For this purpose choose J in the set of positive measure given by Lemma 7.10, define $L(t) = t\exp\Bigl[-(\log t)^{\frac{d-1}{d}}\Bigr]$ and assume that t is so large that $L(t) \ge L_0$. If we apply Lemma 7.11 to the function f given by lemma 7.10 for $L = L_t$ and use the upper bound on $e^J(f)$ given in Lemma 7.10 we immediately get the sought lower bound on $\|T^J(t)\pi_0\|_\infty$.

Sketch of the proof of Lemma 7.10. A detailed proof of the lemma is given in [ACCMM]. Here we only sketch the main ideas. Fix ε small enough and E large enough. Let $I(\varepsilon,E,L)$ be the event that inside the cube B_L there exists a smaller cube $Q_l(x)$, $l \ge \varepsilon\log(L)$, such that all the couplings inside $Q_l(x)$ are equal to β, all but one the boundary couplings are zero and $Q_l(x)$ is connected to the origin by a path γ of length smaller than EL. Using the fact that $p > p_c$ one can prove that the probability of $I(\varepsilon, E, L)$ is strictly bounded away from zero uniformly in L.

Let now $f(\sigma) \equiv \mathbb{1}\{m_{Q_l(x)}(\sigma) \geq 0\}$ where $m_{Q_l(x)}(\sigma)$ denotes the (normalized) magnetization in $Q_l(x)$. Notice that $\Lambda_f \subset Q_l(x) \subset B_L$. If we compute $e^J(f)$ and use the estimates of section 6.5 together with the symmetry under global spin flip, we get

$$e^J(f) \leq \frac{c_M\,|Q_l(x)|\,\mu^J\{\,|m_{Q_l(x)} - 1/2| \leq 1/100\,\}}{\mathrm{Var}^J(f)} \leq \exp\Big[-k(\log L)^{\frac{d-1}{d}}\Big]$$

for a suitable constant k depending on ε and any L large enough. $\square$

Proof of Lemma 7.11. For any given J we set $F^J(x,t) \equiv T^J(t)\pi_x(\underline{1})$. Notice that, since the nearest neighbor coupling J_{xy} are non–negative, the heat-bath dynamics is attractive (see section 3.4) so that $F^J(x,t)$ is a non–increasing function of t and $\|T^J(t)\pi_0\|_\infty = F^J(0,t)$. Next we define $m = m(\beta)$ by

$$\inf_{\substack{d_2(x,y)=1 \\ J_{xy}=\beta}} \quad \inf_{\substack{\sigma,\eta\in\Omega \\ \sigma(y)=1,\eta(y)=-1 \\ \sigma(z)\geq\eta(z)\ \forall z\neq y}} T^J(1)\pi_x(\sigma) - T^J(1)\pi_x(\eta) \equiv 2e^{-m} \tag{7.44}$$

Thanks to attractivity the quantity in (7.44) is non–negative, and, in particular, it is strictly positive with our choice of the transition rates.

Fix now a local function f and a path $\gamma = (x_0, \ldots, x_l) \subset C_0$. The result of the lemma is then a direct consequence of the following three simple inequalities valid for any $t \geq 1$ (see [CMM1] for a proof).

$$F^J(0,t) \geq e^{-ml} F^J(x_l, t) \tag{7.45}$$

$$\|T^J(t)f - \mu^J(f)\|^2_{L^2(\mu^J)} \leq 4\,\mathrm{Var}^J(f) \sum_{x\in\Lambda_f} F^J(x,t) \tag{7.46}$$

$$\|T^J(t)f - \mu^J(f)\|^2_{L^2(\mu^J)} \geq \frac{\mathrm{Var}^J(f)}{2} \exp[-2e^J(f)t] \tag{7.47}$$

In fact, for any local function f such that $\Lambda_f \subset C_0$, if we sum(7.45) over $x \in \Lambda_f$ and use (7.46) and (7.47), we get

$$F^J(0,t) \geq \frac{1}{8|\Lambda_f|} \exp[-md^J(\Lambda_f) - 2e^J(f)t] \geq \frac{1}{8|\Lambda|} \exp[-md^J(\Lambda) - 2e^J(f)t] \tag{7.48}$$

$\square$

8. Open Problems

We conclude these notes with a partial list of open problems.

Pb. 1 Prove (or disprove) the exponential ergodicity of the infinite volume dynamics for a general non–attractive system satisfying only the weak mixing condition of section 2 in dimensions $d \geq 3$.

Pb. 2 Discuss the validity of the strong mixing condition of section 2 for the three dimensional Ising model for $\beta < \beta_c$ or $\beta > \beta_c$ and $h \neq 0$.

Pb. 3 Prove the analogous of theorem 5.2 for the true three dimensional Ising model without making the SOS approximation.

Pb. 4 Analyze the relaxational properties under a Glauber dynamics for the Ising model at the critical point ($h = 0$, $\beta = \beta_c$). Such an interesting and difficult point was only marginally touched in these notes in the example at the end of section 3

Pb. 5 Consider a Glauber dynamics for the two dimensional Ising model in a box Q_L at low temperature, zero external field and plus boundary conditions. Prove that the spectral gap, as a function of L, shrinks as L^{-2}. Use then such a bound to prove (or disprove) the conjectured (see [FH] and [OG]) $e^{-\sqrt{t}}$ decay of the time autocorrelation of the spin at the origin for the infinite volume Glauber dynamics in the plus phase (see section 6.7). Discuss the same problem in three dimension. Here the conjectured decay is no longer a stretched exponential $e^{-\sqrt{t}}$ but, rather, a pure exponential. In particular the Glauber dynamics in the plus phase should have zero spectral gap in two dimensions and positive spectral gap in three dimensions. We refer the reader to [CSS] for some related results in a simplified context (zero temperature dynamics).

Pb. 6 This question has been posed to me by R.Dobrushin. Consider a Glauber dynamics for the two dimensional Ising model in the phase coexistence region and run the dynamics starting from a configuration which is all pluses except a large (e.g. spherical) bubble of minuses centered at the origin. As time goes on the bubble will shrink under the influence of the external plus spins and the question is whether, under a suitable time–dependent rescaling, it will reach asymptotically in time a limiting shape and whether the limiting shape has anything to do with the Wulff shape (see [DKS]).

Pb. 7 Improve theorem 6.10 by showing that in the phase coexistence region the spectral gap of the generator in a large box B_L with free boundary conditions restricted to the subspace of even functions shrinks not faster than an inverse power of L.

Pb. 8 Consider the two dimensional spin glass for which the random couplings take value $\pm J$ with equal probability. Extend to such model part a) of theorem 7.3 when $J > \beta_c$, β_c being the critical value of the two dimensional Ising model (see [RSP] for an heuristic approach).

References

[ACCN] M. Aizenman, J. T. Chayes, L. Chayes and C. M. Newman: *The phase boundary in dilute and random Ising and Potts ferromagnets.* J. Phys. A: Math. Gen. **20**, L313 (1987)

[ACCMM] K.Alexander, L. Chayes, F. Cesi, C. Maes and F. Martinelli: *Relaxation of Dilute Magnets in the Griffiths' Regime* in preparation

[B1] A. J. Bray: *Upper and lower bounds on dynamic correlations in the Griffiths phase.* J. Phys. A: Math. Gen. **22**, L81 (1989)

[B2] A. J. Bray: *Dynamics of dilute magnets above T_c.* Phys. Rev. Lett. **60**, No 8, 720 (1988)

[BD] L. A. Bassalygo and R. L. Dobrushin: *Uniqueness of a Gibbs field with random potential–an elementary approach.* Theory Prob. Appl. **31**, 572 (1986)

[Be] J. van den Berg: *A constructive mixing condition for 2-D Gibbs measures with random interactions.* Preprint 1996

[BEF] J. Bricmont, A. El Mellouki and J. Frölich: *Random surfaces in statistical mechanics: Roughening, Rounding, Wetting, …* J. Stat. Phys **42**, Nos 5/6, 743 (1986)

[BM] J. van den Berg and C. Maes: *Disagreement percolation in the study of Markov fields.* Ann. Prob. **22**, 749 (1994)

[BZ] L. Bertini and B. Zegarlinski: *Coercive inequalities for Gibbs measures* Preprint 1997

[BDZ] E. Bolthausen, J. Deuschel and O. Zeitouni: *Entropy repulsion of the lattice free field.* Commun. Math. Phys. **170**, No 2, 417, (1995)

[CGMS] F. Cesi, G. Guadagni, F. Martinelli and R. Schonmann: *On the 2D Stochastic Ising Model in the Phase Coexistence Region Near the Critical Point.* Journal. Stat. Phys. **85**, No 1–2, 55, (1996)

[CM1] F. Cesi and F. Martinelli: *On the layering transition of an SOS surface interacting with a wall I. Equilibrium results.* Journ. Stat. Phys. **82**, No 3–4, 823, (1996)

[CM2] F. Cesi and F. Martinelli: *On the layering transition of an SOS surface interacting with a wall II. The Glauber dynamics.* Commun. Math. Phys. **177**, No 1, 173, (1996)

[CMM1] F. Cesi, C. Maes and F. Martinelli: *Relaxation of Disordered Magnets in the Griffiths' Regime* Comm. Math. Phys.

[CMM2] F. Cesi, C. Maes and F. Martinelli: *Relaxation to equilibrium for two dimensional disordered Ising models in the Griffiths phase.* Comm. Math. Phys. to appear.

[CSS] L. Chayes, R. H. Schonmann and G. Swindle: *Lifshitz law for the volume of a two-dimensional droplet at zero temperature.* Journal of Stat. Phys. **79**, 821 (1995).

[D] R. L. Dobrushin: *A formula of full Semiinvariants.* In 'Cellular Automata and Cooperative Systems', N. Boccara, E. Goles, S. Martinez and P. Picco (eds.), Kluwer Acad. Publ., Dordrecht-Boston-London, pp. 135-140 (1993).

[DS] R. L. Dobrushin and S. Shlosman: *Constructive criterion for the uniqueness of Gibbs fields.* Statistical Physics and Dynamical Systems. Fritz, Jaffe and Szász eds. Birkhauser 347 (1985)

[DeSt] J. D. Deuschel and D. W. Stroock: *Large deviations.* Academic Press, Series in Pure and Applied Mathematics, **137**, (1989)

[DM] E. I. Dinaburg and A. E. Mazel: *Layering transition in SOS model with external magnetic field.* Journal. Stat. Phys. **74** Nos 3/4, 533 (1994)

[DKP] H. Von Dreyfus, A. Klein and J. F. Perez: *Taming Griffiths singularities: infinite differentiability of quenched correlations functions.* Commun. Math. Phys. **170**, 21 (1995)

[DKS] R.Dobrushin, R.Kotecký, S.Shlosman: *Wulff Construction. A Global Shape From Local Interaction* Translation of Math. Monographs, AMS **104** (1992)

[DRS] D. Dhar, M. Randeria and J. P. Sethna: *Griffiths singularities in the dynamics of disordered Ising models.* Europhys. Lett., **5**, No 6, 485 (1988)

[EFS] A. C. .D. van Enter, R. Fernandez and A. D. Sokal: *Regularity properties and pathologies of position–space renormalization–group transformations: scope and limitations of Gibbsian theory.* Journal of Stat. Phys. **72**, Nos 5/6, 879, (1993)

[F] J. Fröhlich: *Mathematical aspects of the physics of disordered systems.* in "Critical Phenomena, Random Systems, Gauge Theories" Eds. K.Osterwalder and R. Stora, Elsevier (1986)

[FI] J. Fröhlich and J. Z .Imbrie: *Improved perturbation expansion for disordered systems: beating Griffiths singularities.* Commun. Math. Phys. **96**, 145 (1984)

[FH] D. Fisher and D. Huse: Phys.Rev.B **35**, 6841 (1987)

[FP1] J. Frölich and C. Pfister: *Semi–infinite Ising model. I.* Commun. Math. Phys. **109**, 493 (1987)

[FP2] J. Frölich and C. Pfister: *Semi–infinite Ising model. II.* Commun. Math. Phys. **112**, 51 (1987)

[G] R. Griffiths: *Non–analytic behaviour above the critical point in a random Ising ferromagnet.* Phys. Rev. Lett. **23**, 17 (1969)

[GM1] G. Gielis and C. Maes: *Percolation Techniques in Disordered Spin Flip Dynamics: Relaxation to the Unique Invariant Measure.* Commun. Math. Phys. **177**, 83 (1996).

[GM2] G. Gielis and C. Maes: *Local analyticity and bounds on the truncated correlation functions in disordered systems.* Markov Proc. Relat. Fields **1**, 459 (1995)

[GM3] G. Gielis and C. Maes: *The Uniqueness regime of Gibbs Fields with Unbounded Disorder.* Journal Stat. Phys. **81**, 829 (1995)

[Gri] G. Grimmett: *Percolation.* Springer–Verlag, (1989)

[GZ1] A. Guionnet and B. Zegarlinski: *Decay to equilibrium in random spin systems on a lattice.* Commun. Math. Phys. **181**, No 3, 703 (1996)

[GZ2] A. Guionnet and B. Zegarlinski: *Decay to equilibrium in random spin systems on a lattice II.* Journal Stat. Phys. **86**, No 3–4, 899 (1997)

[Hi] Y. Higuchi: *Coexistence of infinite (*)–clusters II:–Ising percolation in two dimension.* Prob. Th. Rel. Fields, **97**, 1, (1993)

[HiYo] Y. Higuchi, N. Yoshida: *Slow relaxation of stochastic Ising models with random and non–random boundary conditions.* New trends in stochastic analysis, ed. by K. D. Elworthy, S. Kusuoka, I. Shigekawa, World Scientific Publishing, 153–167, (1997).

[Ho1] R.Holley: *Possible Rates of Convergence in Finite Range Attractive Spin Systems.* Contemp.Math. **41**, 215-234 (1985)

[Ho2] R.Holley: *Rapid convergence to equilibrium in ferromagnetic stochastic Ising models.* Reseñas IME–USP, **1**, No 2/3, 131 (1993)

[HS] R. A. Holley and D. W. Strook: *Uniform and L^2 convergence in one dimensional stochastic Ising models.* Commun. Math. Phys. **123**, 85 (1989)

[Io1] D. Ioffe: *Large Deviation for the 2D Ising Model: a Lower Bound without Cluster Expansion.* Journ. Stat. Phys. **74**, 411, (1994)

[Io2] D. Ioffe: *Exact Large Deviation Bounds up to T_c for the Ising Model in Two Dimensions.* Probab.Theory Related Fields, **102**, No 3, 313, (1995)

[J] S. Jain: *Anomalously slow relaxation in the diluted Ising model below the percolation threshold.* Physica A, **218**, 279 (1995)

[JS1] M.Jerrum, A.Sinclair: *Approximating the Permanent* Siam Journ.Comput. **18**, 1149, (1989)

[JS2] M.Jerrum, A.Sinclair: *Polynomial-Time Approximation Algorithms for the Ising Model* SIAM Journ. Comput. **22**, 1087, (1993)

[L] T. M. Ligget: *Interacting particles systems.* Springer–Verlag (1985)

[La] E. Laroche: *Hypercontractivité pour de systèmes des spin de portée infinie* Probab.Theor.Relat.Fields **101**, 89 (1995)

[LMaz] J. Lebowitz and A. E. Mazel: *A remark on the low temperature behavior of an sos interface in half space* Journal Stat. Phys. **84**, no 3/4, 379 (1996)

[LM] C. Maes and J. Lebowitz: *The effect of an external field on an interface, entropic repulsion.* Journal Stat. Phys. **46** No 1/2, 39 (1987)

[LY] ShengLin Lu, H.T.Yau: *Spectral Gap and Logarithmic Sobolev Inequality for Kawasaki and Glauber Dynamics.* Comm.Math.Phys. **156**, 399 (1993)

[M] F. Martinelli: *On the two dimensional dynamical Ising model in the phase coexistence region.* Journal Stat. Phys. **76**, No. 5/6, 1179 (1994)

[MM] E. Marcelli and F. Martinelli: *Some new results on the 2D kinetic Ising model in the phase coexistence region.* Journal Stat. Phys. **84**, No 3/4, 655, (1996)

[MO] F. Martinelli and E. Olivieri: *Finite volume mixing conditions for lattice spin systems and exponential approach to equilibrium of Glauber Dynamics* in "Cellular Automata and Cooperative Systems" ed. N.Boccara, E.Goles, S.Martinez and P.Picco, Nato Asi series, **396**, 473, (1993)

[MO1] F. Martinelli and E. Olivieri: *Approach to equilibrium of Glauber dynamics in the one phase region I: The attractive case.* Commun. Math. Phys. **161**, 447 (1994)

[MO2] F. Martinelli and E. Olivieri: *Approach to equilibrium of Glauber dynamics in the one phase region II: The general case.* Commun. Math. Phys. **161**, 487 (1994)

[MO3] F.Martinelli, E.Olivieri: *Some remarks on pathologies of renormalization group transformations for the Ising model* Journ. Stat. Phys. **72**, 1169 (1994)

[MO4 F.Martinelli, E.Olivieri: *Instability of Renormalization Group Pathologies under Decimation* Journ. Stat. Phys. **79**, 25 (1995)

[MOS] F. Martinelli, E. Olivieri and R.H.Schonmann: *For Gibbs state of 2D lattice spin systems weak mixing implies strong mixing.* Commun. Math. Phys. **165**, 33 (1994)

[O] E. Olivieri: *On a cluster expansion for lattice spin systems and finite size condition for the convergence.* Journal Stat. Phys. **50**, 1179 (1988)

[OG] Ogielski: *Dynamics of fluctuations in the order phase of kinetic Ising model.* Phys. Rev. Lett. **36**, No 13, 7315 (1987)

[OP] E. Olivieri and P. Picco: *Cluster expansion for D–dimensional lattice systems and finite volume factorization properties.* Journal Stat. Phys. **59**, 221 (1990)

[OPG] E. Olivieri, F. Perez and S. Goulart–Rosa–Jr.: *Some rigorous results on the phase diagram of the dilute Ising model.* Phys. Lett. **94A**, No 6,7, 309 (1983)

[P] A. Pisztora: *Surface order large deviations for Ising, Potts and percolation models.* Probab. Th. Rel. Fields **104**, 427 (1996)

[Pf] C.Pfister: *Large deviations and phase separation in the two dimensional Ising model.* Elv. Phys. Acta **64**, 953 (1991)

[Po] G. Posta: *Spectral gap for an unrestricted Kawasaki type dynamics* ESAIM, Probab. Statist. **1**, 145, (1995/97)

[RSP] M. Randeria, J. P. Sethna and R. G. Palmer: *Low–frequency relaxation in Ising spin–glasses.* Phys. Rev. Lett. **54**, No. 12, 1321 (1985)

[SC] L. Saloff-Coste: *Markov Chains* preprint 1996, to appear in the proceedings of the Saint Flour summer school in probability theory 1996

[Sch] R. H. Schonmann: *Slow droplet–driven relaxation of stochastic Ising Models in the vicinity of the phase coexistence region.* Commun. Math. Phys. **170**, 453 (1995)

[SchYo] R. H. Schonmann and N. Yoshida: *Exponential relaxation of Glauber dynamics with some special boundary conditions* Comm. Math.Phys. in press

[ShSch] R. H. Schonmann and S. B. Shlosman: *Complete analiticity for 2D Ising model completed.* Comm.Math.Phys. **170**, 453 (1995)

[SZ] D. W. Stroock and B. Zegarlinski: *The logarithmic Sobolev inequality for discrete spin systems on a lattice.* Commun. Math. Phys. **149**, 175 (1992)

[Th] L.Thomas: *Bounds on the Mass Gap for the Finite Volume Stochastic Ising Models at Low Temperatures.* Comm.Math.Phys. **126**, 1 (1989)

[Yo1] N. Yoshida: *Relaxed criteria of the Dobrushin-Shlosman mixing condition* Journal Stat. Phys. **87**, No 1–2, 293 (1997)

[Yo2] N. Yoshida: *Finite volume Glauber dynamics in a small magnetic field* to appear in Journal Stat.Phys.

[Z] B. Zegarlinski: *Strong decay to equilibrium in one dimensional random spin systems.* Journal Stat. Phys. **77**, 717 (1994)

[Zha] M.Zahradník: *An alternate versione of Pirogov–Sinai theory* Comm.Math.Phys. **93**, 559 (1984)