On Eigenfunction Decay for Two Dimensional Magnetic Schrödinger Operators

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Abstract

For two dimensional Schrödinger operators with a nonzero constant magnetic field perturbed by a magnetic field and a scalar potential, both vanishing arbitrarily slow at infinity, it is proved that eigenfunctions corresponding to the discrete spectrum decay faster than any exponential. Under more restrictive conditions on the perturbations, even quicker decay is obtained.

1 Introduction

The aim of this paper is to study the decay properties of the eigenfunctions for the two dimensional magnetic Schrödinger operators

$$H = (\mathbf{p} - \mathbf{a}(\mathbf{x}))^2 + V(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{R}^2 \tag{1.1}$$

The two dimensional case is interesting both from physical (e.g. the theory of quantum Hall effect) and mathematical point of view; in particular, since for $V = 0$ and constant magnetic field B_0 the spectrum of H is the well known "Landau spectrum"

$$\sigma_L(B_0) = \{(2n+1)B_0 \,|\, n = 0, 1, \ldots\} \tag{1.2}$$

by adding a perturbation one obtains a very rich spectral structure: eigenvalues in the gaps of the essential spectrum, "bands" created by an infinite number of magnetic and / or scalar "wells", etc. In this paper we shall consider a particle in a constant magnetic field perturbed by a magnetic field and

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a scalar potential vanishing at infinity, i.e. the "one well" problem (we stress the fact that we impose conditions on the magnetic field and not on the corresponding vector potential); in this case it is known [I, H] that the essential spectrum of H coincide with the Landau spectrum. More precisely, we shall obtain a strong decay at infinity of the eigenfunctions $\psi_E(\mathbf{x})$, corresponding to discrete eigenvalues, E, irrespective of the fact that they lie below the essential spectrum or in between the Landau levels. In this context, a fairly straightforward application of Combes-Thomas-Agmon theory gives an exponential decay, $|\psi_E(\mathbf{x})| \leq const. \exp(-\mu(E)|\mathbf{x}|)$, where $\mu(E)$ depends upon the distance between E and the essential spectrum of H and moreover $\mu(E) \to 0$ as E approaches the essential spectrum (see however the remark at the end of the proof of Theorem 4.1). For a thorough review in this direction, including the many-body problem, see [C-F-K-S, A-H-S 1, A-H-S 2, H, Hu] and references therein.

On the other hand, if the scalar potential and the perturbing magnetic field are compactly supported or have rotational symmetry, it is not hard to see that the eigenfunctions have a gaussian decay irrespective of the position of E. In this context, T. Hoffmann-Ostenhof posed the problem of the decay of eigenfunctions under less stringent conditions on the perturbation.

Results concerning this question were obtained by Erdös [E] for energies below the essential spectrum. He proved that if V decays at infinity and is analytic in the angle variable then the gaussian decay of the eigenfunctions is preserved and more important, he gave an example of a potential decaying at infinity for which the decay of the ground state is *slower* than a gaussian.

Our main result (see Theorem 4.1 for precise formulation) is that if the perturbations

$$B'(\mathbf{x}), V(\mathbf{x}) \to 0 \quad \text{when} \quad |\mathbf{x}| \to \infty \tag{1.3}$$

then the eigenfunctions corresponding to $E \notin \sigma_L(B_0)$ decay *faster* than any exponential. We believe our result to be optimal; additional conditions have to be added in order to ensure a quicker decay of eigenfunctions and we give some results in this direction (see Theorem 4.2): if the perturbing magnetic field is compactly supported and the scalar potential has a decay of the form $\exp(-\delta|\mathbf{x}|^\beta)$ with $0 < \beta \leq 2$ then the eigenfunctions decay like $\exp(-\mu|\mathbf{x}|^{1+\beta/2})$.

These results can be applied to the "magnetic multiple wells" problem (see [H-S, H-H, Na 1, Na 2]) in order to have a better control on the width of the mini-bands which appear in the tight-binding limit (see [B-C-D]); this will be done in a companion paper.

The content of the paper is as follows. Section 2 has a preparatory character; we write down some simple properties of the family of "local" transversal gauges we shall use as well as the closed formula for the integral kernel $K_0(\mathbf{x}, \mathbf{x}'; z)$ of the resolvent of H in the case of constant magnetic field B_0 and $z \notin \sigma_L(B_0)$.

Section 3 contains one of the main results of the paper: in the case when

$V=0$ and the magnetic field is of the form ($b>0$):

$$B(\mathbf{x}) = B_0 + b\,B'(\mathbf{x}) \tag{1.4}$$

with $B_0>0$ and B' uniformly bounded and smooth, we prove that for $z \notin \sigma_L(B_0)$ fixed and $b \to 0$ then $z \in \rho(H)$ and we obtain an upper bound on the integral kernel $K(\mathbf{x},\mathbf{x}';z)$ of $(H-z)^{-1}$ (see Theorem 4.1 for the precise statement), which essentially says that as $|\mathbf{x}-\mathbf{x}'| \to \infty$, we have

$$|K(\mathbf{x},\mathbf{x}';z)| \approx \exp\left[-\mu(b,z)|\mathbf{x}-\mathbf{x}'|\right], \quad \lim_{b\to 0}\mu(b,z) = \infty \tag{1.5}$$

Our method, which is interesting in itself, used for proving the above result is a "regularised" magnetic perturbation theory: the main idea is the fact that most of the "singularity" of the perturbation is given by a (nonvanishing at infinity) "gauge phase"; by an appropriate "factorisation" of this phase one obtains a much more regular perturbation theory (see [N] for another use of the same idea).

Section 4 contains the main results of the paper concerning the decay of eigenfunctions. Theorem 4.1. follows (with the technical complications due to the magnetic field) from Theorem 3.1. by substantiating the general idea that the decay at infinity of the eigenfunctions is governed by the "asymptotic" hamiltonian and its resolvent's integral kernel behaviour.

The proof of Theorem 4.2. is based on the fact that while the gaussian decay of the "free kernel" K_0 may not "propagate" itself in the Neumann series when adding a perturbing scalar potential, one can use the decay properties of the potential in order to obtain a decay which is stronger than that of the potential.

2 Preliminaries

In this paper we shall consider only the bidimensional case (i.e. the particle is confined in the plane $x_3=0$ and the magnetic field is orthogonal to that plane). For $b \in \mathbf{R}$ and $\mathbf{d}=(d_1,d_2) \in \mathbf{R}^2$, define

$$\mathbf{b} \wedge \mathbf{d} \equiv (-b\,d_2, b\,d_1) \in \mathbf{R}^2 \tag{2.1}$$

Let $B(\mathbf{x}) \in C^1(\mathbf{R}^2)$. We shall use the following family of vector potentials corresponding to $B(\mathbf{x})$:

$$\mathbf{a}(\mathbf{x},\mathbf{x}') = \int_0^1 ds\, s\,\mathbf{B}(\mathbf{x}'+s(\mathbf{x}-\mathbf{x}')) \wedge (\mathbf{x}-\mathbf{x}') \tag{2.2}$$

For $\mathbf{x}'=0$, this is nothing but the usual transversal gauge (see e.g. [T]):

$$\mathbf{a}(\mathbf{x},0) \equiv \mathbf{a}(\mathbf{x}) = \int_0^1 ds\, s\,\mathbf{B}(s\,\mathbf{x}) \wedge \mathbf{x} \tag{2.3}$$

If we define

$$\mathbf{f}(\mathbf{x},\mathbf{x}') = \mathbf{a}(\mathbf{x}) - \mathbf{a}(\mathbf{x},\mathbf{x}') \tag{2.4}$$

then there exists $\varphi(\mathbf{x},\mathbf{x}')$ such that

$$\nabla_{\mathbf{x}}\varphi(\mathbf{x},\mathbf{x}') = \mathbf{f}(\mathbf{x},\mathbf{x}') \tag{2.5}$$

The additional requirement

$$\varphi(\mathbf{x}',\mathbf{x}') = 0 \tag{2.6}$$

gives

$$\varphi(\mathbf{x},\mathbf{x}') \;=\; \int_{x_1'}^{x_1} dt\; f_1(t,x_2;\mathbf{x}') + \int_{x_2'}^{x_2} dt\; f_2(x_1',t;\mathbf{x}') \tag{2.7}$$

where x_i, x_i', f_i are the cartesian components of $\mathbf{x},\mathbf{x}'$ and $\mathbf{f}$ respectively. From the explicit formulae (2.2), (2.4) and (2.7), it follows at once that

Lemma 2.1 *If $|B(\mathbf{x})| \leq M < \infty$, then*
i)

$$|f_j(\mathbf{x},\mathbf{x}')| \leq \; M\,(|\mathbf{x}| + |\mathbf{x}-\mathbf{x}'|) \tag{2.8}$$

$$|\varphi(\mathbf{x},\mathbf{x}')| \leq \; 2\,M\,|\mathbf{x}-\mathbf{x}'|(|\mathbf{x}| + |\mathbf{x}-\mathbf{x}'|) \tag{2.9}$$

ii)

$$\mathbf{a}(\mathbf{x},\mathbf{x}')\cdot(\mathbf{x}-\mathbf{x}') = 0 \tag{2.10}$$

iii)

$$\begin{aligned}\nabla_{\mathbf{x}}\mathbf{a}(\mathbf{x},\mathbf{x}') = \int_0^1 ds\; s\;\times \\ \times\left(\frac{\partial}{\partial x_2}B(\mathbf{x}'+s(\mathbf{x}-\mathbf{x}')),-\frac{\partial}{\partial x_1}B(\mathbf{x}'+s(\mathbf{x}-\mathbf{x}'))\right)\cdot(\mathbf{x}-\mathbf{x}')\end{aligned} \tag{2.11}$$

iv) If $B(\mathbf{x}) = B_0$ is constant, then

$$\begin{aligned}\varphi_0(\mathbf{x},\mathbf{x}') &= -\frac{1}{2}B_0\,(x_1\,x_2' - x_1'\,x_2) \\ \mathbf{a}_0(\mathbf{x},\mathbf{x}') &= \frac{1}{2}\mathbf{B}_0 \wedge (\mathbf{x}-\mathbf{x}')\end{aligned} \tag{2.12}$$

The hamiltonian of a particle in the presence of the magnetic field and a scalar potential V is (in the transversal gauge):

$$\begin{aligned} H &= (\mathbf{p}-\mathbf{a}(\mathbf{x}))^2 + V(\mathbf{x}) \\ \mathbf{p} &= \left(-i\frac{\partial}{\partial x_1}, -i\frac{\partial}{\partial x_2}\right) \\ \mathbf{a}(\mathbf{x}) &= \left(-x_2\int_0^1 ds\; s\,B(s\,\mathbf{x}), x_1\int_0^1 ds\; s\,B(s\,\mathbf{x})\right)\end{aligned} \tag{2.13}$$

In the case of constant magnetic field, one has the hamiltonian

$$\begin{aligned} H_0 &= (\mathbf{p} - \mathbf{a}_0(\mathbf{x}))^2 \text{ where} && (2.14)\\ \mathbf{p} &= -i\nabla_{\mathbf{x}} \quad \text{and} \quad \mathbf{a}_0(\mathbf{x}) = \left(-\frac{1}{2}B_0\, x_2, \frac{1}{2}B_0\, x_1\right) && (2.15)\end{aligned}$$

which is essentially self-adjoint on $C_0^\infty\left(\mathbf{R}^2\right)$ and its spectrum is the well known Landau spectrum

$$\sigma(H_0) = \sigma_{ess}(H_0) = \{(2n+1)B_0 \mid n = 0, 1, 2, \ldots\} \tag{2.16}$$

For $z \notin \sigma(H_0)$ and $g \in L^2\left(\mathbf{R}^2\right)$, we write

$$\begin{aligned} &\left[(H_0 - z)^{-1} g\right](\mathbf{x}) = \int d\mathbf{x}'\, K_0(\mathbf{x}, \mathbf{x}'; z) g(\mathbf{x}') \\ &(H_0 - z) K_0(\mathbf{x}, \mathbf{x}'; z) = \delta(\mathbf{x} - \mathbf{x}') \end{aligned} \tag{2.17}$$

Then takes place (see e.g. [J-P]):

Lemma 2.2 *Let*

$$\begin{aligned} \varphi_0(\mathbf{x}, \mathbf{x}') &= -\frac{B_0}{2}\,(x_1\, x_2' - x_2\, x_1') \\ \psi(\mathbf{x}, \mathbf{x}') &= \frac{B_0}{4}\, |\mathbf{x} - \mathbf{x}'|^2 \\ \alpha &= -\frac{1}{2}\left(\frac{z}{B_0} - 1\right) \neq -1, -2, \ldots \end{aligned}$$

Then

$$\begin{aligned} K_0(\mathbf{x}, \mathbf{x}'; z) &= e^{i\,\varphi_0(\mathbf{x}, \mathbf{x}')}\, G_0(\mathbf{x}, \mathbf{x}'; z) = \\ &= \frac{\Gamma(\alpha)}{4\,\pi} e^{i\,\varphi_0(\mathbf{x}, \mathbf{x}')}\, e^{-\psi(\mathbf{x}, \mathbf{x}')} U(\alpha, 1; 2\,\psi(\mathbf{x}, \mathbf{x}')) \end{aligned} \tag{2.18}$$

where Γ is the Euler function and $U(\alpha, \gamma; \xi)$ is the confluent hypergeometric function [A-S].

For a proof using the eigenfunctions of H_0, see [J-P]. Alternatively, for $\Re(z) < B_0$ one can obtain (2.18) by taking the Laplace transform of the formula for the "heat" kernel (see e.g. [F-H]) and using an appropriate integral representation for the confluent hypergeometric function (see formula (13.2.5) in [A-S]). The formula (2.18), for arbitrary $z \in \rho(H_0)$ follows by analytic continuation.

The holomorphy and asymptotic behaviour of $U(\alpha, \gamma; \xi)$ (see e.g. [A-S]) imply the following inequality which we need later on: if $\beta, m > 0$, $[c, d] \subset \rho(H_0)$ and $z \in [c, d]$, then:

$$\sup_{t \in \mathbf{R}^+} t^\beta\, e^{-m\,t} |U(\alpha, 1, \frac{B_0}{2}\, t)| \leq M(\beta, m, c, d) < \infty \tag{2.19}$$

¿From Lemma 2.2 it follows that $K_0(\mathbf{x},\mathbf{x}';z)$ has a gaussian decay as $|\mathbf{x}-\mathbf{x}'|\to\infty$. We shall use this in the following form:

Corollary 2.1 *Let $\mu>0$ and $T(\mu,\mathbf{x}_0)$ be the operator of multiplication with $e^{\mu|\mathbf{x}-\mathbf{x}_0|^2}$, $\mathbf{x}_0\in\mathbf{R}^2$. Then for all $0<\delta<\frac{B_0}{8}$ and $z\in\rho(H_0)$, one has that the operators*

$$T(\delta,\mathbf{x}_0)\,(H_0-z)^{-1}\,T(-2\delta,\mathbf{x}_0)\;:\;L^2\left(\mathbf{R}^2\right)\to L^2\left(\mathbf{R}^2\right)\left(L^\infty\left(\mathbf{R}^2\right)\right) \tag{2.20}$$

are bounded.

Proof:
Since for all $\mathbf{x},\mathbf{x}'\in\mathbf{R}^2$, $\delta|\mathbf{x}-\mathbf{x}_0|^2\le 2\delta|\mathbf{x}-\mathbf{x}'|^2+2\delta|\mathbf{x}'-\mathbf{x}_0|^2$ one has

$$\begin{aligned}&|e^{\delta|\mathbf{x}-\mathbf{x}_0|^2}K_0(\mathbf{x},\mathbf{x}';z)e^{-2\delta|\mathbf{x}'-\mathbf{x}_0|^2}f(\mathbf{x}')|\le\\ \le\;& e^{-2(\frac{B_0}{8}-\delta)|\mathbf{x}-\mathbf{x}'|^2}|U(\alpha,1;2\psi(\mathbf{x},\mathbf{x}'))|\frac{|\Gamma(\alpha)|}{4\,\pi}|f(\mathbf{x}')|\end{aligned} \tag{2.21}$$

Since (2.19) takes place, one has that:

$$e^{-2(\frac{B_0}{8}-\delta)y^2}U(\alpha,1;\frac{B_0y^2}{2})\in L^1\left(\mathbf{R}^2\right)\cap L^2\left(\mathbf{R}^2\right) \tag{2.22}$$

and the use of Young inequality ([R-S 2]) finishes the proof.
Remark:
Since under a gauge transformation

$$\begin{aligned}(U_\chi f)(\mathbf{x}) &= e^{i\,\chi(\mathbf{x})}f(\mathbf{x})\quad\text{and} &(2.23)\\ \left(U_\chi^*(H_0-z)^{-1}U_\chi f\right)(\mathbf{x}) &= \int_{\mathbf{R}^2}d\mathbf{x}'\;K_\chi(\mathbf{x},\mathbf{x}';z)=\\ &= \int_{\mathbf{R}^2}d\mathbf{x}'\;e^{-i\,\chi(\mathbf{x})}K_0(\mathbf{x},\mathbf{x}';z)e^{i\,\chi(\mathbf{x}')}f(\mathbf{x}')\end{aligned}$$

one has

$$|K_\chi(\mathbf{x},\mathbf{x}';z)|=|K_0(\mathbf{x},\mathbf{x}';z)| \tag{2.24}$$

i.e. the gaussian decay is valid for an arbitrary gauge.

3 Schrödinger operators with nonconstant magnetic field

Suppose that we are given a magnetic field

$$\begin{aligned}\mathbf{B}(\mathbf{x}) &= (0,0,B_0+B'(\mathbf{x}))\quad,\quad B_0>0\\ ||B'||_{C^1(\mathbf{R}^2)} &\equiv \max_{|\alpha|\le 1}\{||D^\alpha B'||_\infty\}\le 1\end{aligned} \tag{3.1}$$

A vector potential corresponding to B' is (see Section 2):

$$\begin{aligned}\mathbf{a}'(\mathbf{x},\mathbf{x}') &= \Big(-(x_2-x_2')\int_0^1 ds\, s\, B'(\mathbf{x}'+s(\mathbf{x}-\mathbf{x}')),\\ &\quad (x_1-x_1')\int_0^1 ds\, s\, B'(\mathbf{x}'+s(\mathbf{x}-\mathbf{x}'))\Big)\end{aligned} \tag{3.2}$$

Then

$$\begin{aligned}\mathbf{a}(\mathbf{x},\mathbf{x}') &= \mathbf{a}_0(\mathbf{x},\mathbf{x}')+\mathbf{a}'(\mathbf{x},\mathbf{x}') \quad \text{and}\\ |\mathbf{a}'(\mathbf{x},\mathbf{x}')| &< |\mathbf{x}-\mathbf{x}'| \quad , \quad |\nabla_{\mathbf{x}}\mathbf{a}'(\mathbf{x},\mathbf{x}')| < |\mathbf{x}-\mathbf{x}'|\end{aligned} \tag{3.3}$$

We showed that there exist $\varphi(\mathbf{x},\mathbf{x}')$ and $\varphi'(\mathbf{x},\mathbf{x}')$ such that:

$$\begin{aligned}\mathbf{a}(\mathbf{x}) &= \mathbf{a}(\mathbf{x},\mathbf{x}')+\nabla_{\mathbf{x}}\varphi(\mathbf{x},\mathbf{x}') =\\ &= \mathbf{a}_0(\mathbf{x})+\mathbf{a}'(\mathbf{x},\mathbf{x}')+\nabla_{\mathbf{x}}\varphi'(\mathbf{x},\mathbf{x}')\end{aligned} \tag{3.4}$$

Where

$$\varphi'(\mathbf{x},\mathbf{x}') = \varphi(\mathbf{x},\mathbf{x}') - \varphi_0(\mathbf{x},\mathbf{x}') \tag{3.5}$$

We consider hamiltonians of the form:

$$H_b = (\mathbf{p}-\mathbf{a}_0-b\mathbf{a}')^2 \quad , \quad b>0 \tag{3.6}$$

which are essentially self-adjoint on $C_0^\infty\left(\mathbf{R}^2\right)$ (see e.g. [R-S]).

The main result of this section is the following perturbative expansion of the integral kernel $K_b(\mathbf{x},\mathbf{x}';z)$ of the resolvent of H_b:

Theorem 3.1 *Let $[c,d]\subset\rho(H_0)$. Then there exists $b_0>0$, depending on $[c,d]$ such that for any $0\le b\le b_0$ we have that $[c,d]\subset\rho(H_b)$ and for any $z\in[c,d]$, $(H_b-z)^{-1}$ has an integral kernel given by the following perturbative expansion:*

$$K_b(\mathbf{x},\mathbf{x}';z) = e^{i\,\varphi_b'(\mathbf{x},\mathbf{x}')}K_0(\mathbf{x},\mathbf{x}';z)+\sum_{n\ge 0}\mathcal{F}_n(\mathbf{x},\mathbf{x}';z) \tag{3.7}$$

where $\mathcal{F}_n(\mathbf{x},\mathbf{x}';z)$ are integral kernels corresponding to the operators

$$F_n\in\mathcal{B}\left(L^2\left(\mathbf{R}^2\right),L^2\left(\mathbf{R}^2\right)\right)$$

with

$$||F_n||_{\mathcal{B}(L^2(\mathbf{R}^2),L^2(\mathbf{R}^2))} \le [const\,([c,d])\;b]^{n+1} \tag{3.8}$$

Proof:

Denote with $H_{b,\mathbf{x}'} = (\mathbf{p}-\mathbf{a}_0(\mathbf{x})-b\mathbf{a}'(\mathbf{x},\mathbf{x}'))^2$. Then by gauge covariance:

$$H_b e^{i\,\varphi_b'(\mathbf{x},\mathbf{x}')} = e^{i\,\varphi_b'(\mathbf{x},\mathbf{x}')}H_{b,\mathbf{x}'} \tag{3.9}$$

The equation satisfied by K_b takes the form:

$$\begin{aligned}\delta(\mathbf{x}-\mathbf{x}') &= (H_b - z)K_b(\mathbf{x},\mathbf{x}';z) = \\ &= e^{i\,\varphi_b'(\mathbf{x},\mathbf{x}')}(H_{b,\mathbf{x}'} - z)e^{-i\,\varphi_b'(\mathbf{x},\mathbf{x}')}K_b(\mathbf{x},\mathbf{x}';z)\end{aligned} \tag{3.10}$$

The main idea of the proof is to factorise the "gauge phases". This is achieved by making the following ansatz:

$$K_b(\mathbf{x},\mathbf{x}';z) = e^{i\,\varphi_b'(\mathbf{x},\mathbf{x}')}K_0(\mathbf{x},\mathbf{x}';z) + \mathcal{F}(\mathbf{x},\mathbf{x}';z) \tag{3.11}$$

with $\mathcal{F}$ to be found. One can rewrite (3.10) as:

$$\begin{aligned}\delta(\mathbf{x}-\mathbf{x}') = (H_b - z)\left[e^{i\,\varphi_b'(\mathbf{x},\mathbf{x}')}K_0(\mathbf{x},\mathbf{x}';z) + \mathcal{F}(\mathbf{x},\mathbf{x}';z)\right] = \\ = (H_b - z)\mathcal{F}(\mathbf{x},\mathbf{x}';z) + e^{i\,\varphi_b'(\mathbf{x},\mathbf{x}')}(H_{b,\mathbf{x}'} - z)K_0(\mathbf{x},\mathbf{x}';z)\end{aligned} \tag{3.12}$$

But $H_{b,\mathbf{x}'} = H_0 - R_b(\mathbf{x},\mathbf{x}')$, where

$$R_b(\mathbf{x},\mathbf{x}') = 2b\mathbf{a}'(\mathbf{x},\mathbf{x}')\cdot(\mathbf{p} - \mathbf{a}_0(\mathbf{x})) - ib\nabla_{\mathbf{x}}\mathbf{a}'(\mathbf{x},\mathbf{x}') - b^2{\mathbf{a}'}^2(\mathbf{x},\mathbf{x}') \tag{3.13}$$

Since $\varphi_b'(\mathbf{x}',\mathbf{x}') = 0$, (3.12) becomes (using (2.17)):

$$(H_b - z)\mathcal{F}(\mathbf{x},\mathbf{x}';z) = e^{i\,\varphi_b'(\mathbf{x},\mathbf{x}')}R_b(\mathbf{x},\mathbf{x}')K_0(\mathbf{x},\mathbf{x}';z) \tag{3.14}$$

Using the explicit form of $K_0(\mathbf{x},\mathbf{x}';z)$ given in Lemma 2.2, the equality in (3.3) and the equality (2.10), one can derive that

$$\mathbf{a}'(\mathbf{x},\mathbf{x}')\cdot(\mathbf{p} - \mathbf{a}_0(\mathbf{x}))K_0(\mathbf{x},\mathbf{x}';z) = -\mathbf{a}'(\mathbf{x},\mathbf{x}')\cdot\mathbf{a}_0(\mathbf{x},\mathbf{x}')K_0(\mathbf{x},\mathbf{x}';z) \tag{3.15}$$

Moreover, using (2.19) with $m = B_0/8$, one can obtain the following estimate:

$$|R_b(\mathbf{x},\mathbf{x}')K_0(\mathbf{x},\mathbf{x}';z)| \leq\; b\,\mathcal{C}(c,d)\exp\left(-\frac{B_0|\mathbf{x}-\mathbf{x}'|^2}{8}\right) \tag{3.16}$$

Write as $S_b(z)$ the operator corresponding to $e^{i\,\varphi_b'(\mathbf{x},\mathbf{x}')}K_0(\mathbf{x},\mathbf{x}';z)$ and as $T_b(z)$ the operator linked to $e^{i\,\varphi_b(\mathbf{x},\mathbf{x}')}R_b(\mathbf{x},\mathbf{x}')K_0(\mathbf{x},\mathbf{x}';z)$.

Using Young inequalities as in Corollary 2.1, one can see that

$$T_b(z),\; S_b(z) \in \mathcal{B}\left(L^2\left(\mathbf{R}^2\right), L^2\left(\mathbf{R}^2\right)\right) \cap \mathcal{B}\left(L^2\left(\mathbf{R}^2\right), L^\infty\left(\mathbf{R}^2\right)\right) \tag{3.17}$$

and

$$\begin{aligned}||S_b(z)||_{\mathcal{B}(L^2(\mathbf{R}^2),L^2(\mathbf{R}^2))} &\leq ||K_0(y;z)||_1 \leq \mathcal{C}_1(c,d) < \infty \\ ||T_b(z)||_{\mathcal{B}(L^2(\mathbf{R}^2),L^2(\mathbf{R}^2))} &\leq \mathcal{C}_2(c,d)\,b\end{aligned} \tag{3.18}$$

Define

$$F_n(z) = S_b(z)\left[T_b(z)\right]^{n+1} \quad , \quad n \geq 0 \tag{3.19}$$

¿From (3.14) and (3.11) one has:

$$(H_b - z)^{-1}(1 - T_b(z)) = S_b(z) \tag{3.20}$$

and if

$$b < \frac{1}{\mathcal{C}_2(c,d)} \tag{3.21}$$

then (3.20) becomes:

$$(H_b - z)^{-1} = S_b(z) + \sum_{n \geq 0} F_n(z) \tag{3.22}$$

Corollary 3.1 *Let K be a compact in $\mathbf{R}$, such that $K \cap \sigma(H_0) = \emptyset$. Define:*

$$\begin{array}{rcl} B_n(\mathbf{x}) & = & B_0 + B'_n(\mathbf{x}) \quad , \quad B_0 > 0 \\ b_n & = & \|B'_n\|_{C^1(\mathbf{R}^2)} \quad , \quad \lim_{n\to\infty} b_n = 0 \end{array} \tag{3.23}$$

Let $\hat{\mathbf{v}}$ be a unit vector in $\mathbf{R}^2$ and fix $\mathbf{y} \in \mathbf{R}^2$. Denote with $T(u,\mathbf{y})$ and $D(u,\mathbf{y})$ the multiplication operators with $e^{u|\hat{\mathbf{v}}\cdot(\mathbf{x}-\mathbf{y})|}$ and $e^{u|\mathbf{x}-\mathbf{y}|}$.

Fix $u > 0$. Define

$$H_n = (\mathbf{p} - \mathbf{a}_0(\mathbf{x}) - \mathbf{a}'_n(\mathbf{x}))^2 \tag{3.24}$$

Then there exists $N(u) \geq 1$ s.t. for any $n \geq N(u)$ we have $K \cap \sigma(H_n) = \emptyset$ and

$$i) \quad T(u,\mathbf{y})(H_n - z)^{-1}T(-u,\mathbf{y}) \in \mathcal{B}\left(L^2\left(\mathbf{R}^2\right), L^2\left(\mathbf{R}^2\right)\right) \tag{3.25}$$

$$ii) \quad T(u,\mathbf{y})(H_n - z)^{-1}T(-u,\mathbf{y}) \in \mathcal{B}\left(L^2\left(\mathbf{R}^2\right), L^\infty\left(\mathbf{R}^2\right)\right) \tag{3.26}$$

$$iii) \quad D(u,\mathbf{y})(H_n - z)^{-1}D(-u,\mathbf{y}) \in \mathcal{B}\left(L^2\left(\mathbf{R}^2\right), L^2\left(\mathbf{R}^2\right)\right) \tag{3.27}$$

Moreover, the norms are uniformly bounded in $\mathbf{y}$, $\hat{\mathbf{v}}$, in $n \geq N(u)$ and $z \in K$.

Proof:

We treat only the case $i)$, the other ones being analogous. We can apply then Theorem 3.1 if n is sufficiently large. One obtains:

$$(H_n - z)^{-1} = S_n(z) \sum_{m \geq 0} T_n^m(z) \tag{3.28}$$

Because both $S_n(z)$ and $T_n(z)$ are integral operators, using (3.16) and the explicit form of $K_0(\mathbf{x}, \mathbf{x}'; z)$, one can prove (as in Corollary 2.1) that for any $\psi \in L^2\left(\mathbf{R}^2\right)$ and $m \geq 0$

$$\begin{aligned} & \|T(u,\mathbf{y})S_n(z)T_n^m(z)T(-u,\mathbf{y})\,\psi\|_{2\,(\infty)} \leq \\ \leq \; & \frac{|\Gamma(\alpha)|}{4\,\pi} \sup_{t\in\mathbf{R}^+} \left\{ \exp\left[u\,t - \frac{B_0\,t^2}{8}\right]\right\} \|U(\alpha, 1, (B_0\,\mathbf{x}^2)/2)e^{-\frac{B_0\mathbf{x}^2}{8}}\|_{1\,(2)} \times \\ \times \; & \left\{\mathcal{C}_2(K)\,b_n \sup_{t\in\mathbf{R}^+} \exp\left[u\,t - \frac{B_0\,t^2}{16}\right]\right\}^m \|\psi\|_2 \end{aligned} \tag{3.29}$$

Take now $n \geq N(u)$ sufficiently large such that

$$b_n < \left\{ \frac{e^{-\frac{4\,u^2}{B_0}}}{\mathcal{C}_2(K)} \right\} \tag{3.30}$$

and the proof follows.

4 Decay of eigenfunctions

In this section we consider the decay of the eigenfunctions of

$$H = (\mathbf{p} - \mathbf{a})^2 + V \tag{4.1}$$

where the scalar potential V and the magnetic field which corresponds to $\mathbf{a}$

$$\nabla \wedge \mathbf{a}(\mathbf{x}) = \mathbf{B}_0 + \mathbf{B}'(\mathbf{x}) \quad , \quad B_0 > 0 \tag{4.2}$$

satisfy the following conditions:

$$\begin{aligned} & B' \in C^1\left(\mathbf{R}^2\right) \; ; \; \lim_{n\to\infty} ||B'||_{C^1(\mathbf{R}^2 \setminus \{|\mathbf{x}| \leq n\})} = 0 \\ & V = V_1 + V_2 \; ; \; V_1 \in L^2\left(\mathbf{R}^2\right), \; V_2 \in L^\infty\left(\mathbf{R}^2\right) \\ & \lim_{n\to\infty} \sup_{|\mathbf{x}| \geq n} |V_2(\mathbf{x})| = 0 \end{aligned} \tag{4.3}$$

Under these conditions, H is essentially self-adjoint on $C_0^\infty\left(\mathbf{R}^2\right)$ (see e.g. [C-F-K-S]). Moreover, V is relatively compact with respect to $(\mathbf{p}-\mathbf{a})^2$ [C-F-K-S] which together with the results in [I, H] it implies that

$$\sigma_{ess}(H) = \sigma(H_0) = \{(2n+1)B_0 \mid n = 0, 1, 2, \ldots\} \tag{4.4}$$

Let $E \in \sigma_{disc}(H)$ (the discrete spectrum of H) and let ψ be a normalised eigenfunction corresponding to E. The main result of this section is that ψ decays faster than any exponential:

Theorem 4.1 *Suppose $B_0 > 0$ and B', V satisfy (4.3). Let $K \subset \mathbf{R} \setminus \sigma(H_0)$, K compact, $E \in K$, $(H - E)\psi = 0$, $\psi \in L^2\left(\mathbf{R}^2\right)$.*

Then for any $0 \leq u < \infty$ there exists a constant $C(u, K) < \infty$ such that:

$$|\psi(\mathbf{x})| \leq C(u, K) e^{-u|\mathbf{x}|} \tag{4.5}$$

Remark:

As proved by Erdös [E] (see also Theorem 4.2 below), under more restrictive conditions one can prove the existence of $\mu > 0$ such as

$$|\psi(\mathbf{x})| \leq const \; e^{-\mu|\mathbf{x}|^r} \quad , \quad 1 < r \leq 2. \tag{4.6}$$

The problem whether under the conditions (4.3) the decay given by Theorem 4.1 is optimal, remains open.

In what follows,

$$g \in C^{\infty}\left(\mathbf{R}^2\right) \; ; \; g = \overline{g} \, ; \; ||g||_{C^2(\mathbf{R}^2)} = M < \infty \tag{4.7}$$

Under the conditions (4.3), one has $D(H) = D((\mathbf{p}-\mathbf{a})^2)$ and $(p_j - a_j)(H+i)^{-1}$ is bounded, $j \in \{1, 2\}$. Moreover, because $[H, g] = -i\{(\mathbf{p}-\mathbf{a}) \cdot \nabla g + \nabla g \cdot (\mathbf{p}-\mathbf{a})\}$, it follows that

$$||[H, g](H+i)^{-1}|| \leq const(M)$$

.

Denote with $\Phi = [H, g]\psi = (E+i)[H, g](H+i)^{-1}\psi$. Then

$$||\Phi||_2 \leq const([c, d], M). \tag{4.8}$$

In what follows, we shall need a L^{∞} estimate on ψ.

Lemma 4.1

$$|\psi(\mathbf{x})| \leq \mathcal{C}(E) \tag{4.9}$$

where $\mathcal{C}(E)$ is bounded on compacts.

Proof:

As known, (4.9) follows from the general results (Harnack type inequalities); actually in our case, for sufficiently large λ, one has $(H+\lambda)^{-1} \in \mathcal{B}\left(L^2, L^{\infty}\right)$ and then the result follows from

$$\psi = (E+\lambda)(H+\lambda)^{-1}\psi$$

Proof of Theorem 4.1:

Let now $\hat{\mathbf{v}}$ be a unit vector in $\mathbf{R}^2$ and

$$g \in C^{\infty}(\mathbf{R}) \, , \; 0 \leq g(x) \leq 1 \, , \; g(x) = \begin{cases} 0 & \text{if} \quad x \leq \frac{1}{2} \\ 1 & \text{if} \quad x \geq 1 \end{cases}$$

$$g_{n,v} \equiv g\left(\frac{\hat{\mathbf{v}} \cdot \mathbf{x}}{n}\right) \; ; \; \Omega_n \equiv \left\{\mathbf{x} \in \mathbf{R}^2 \mid \hat{\mathbf{v}} \cdot \mathbf{x} \geq n\right\} \tag{4.10}$$

¿From now on, we shall denote $g_{n,v}$ with g_n. Let

$$\begin{aligned} B_n(\mathbf{x}) &= B_0 + g_n B'(\mathbf{x}) \quad , \quad V_n = g_n V(\mathbf{x}) \\ \mathbf{a}_n(\mathbf{x}) &= \int_0^1 ds\, s\, B_n(s\,\mathbf{x}) \wedge \mathbf{x} \end{aligned} \tag{4.11}$$

and

$$H_n = (\mathbf{p} - \mathbf{a}_n(\mathbf{x}))^2 + V_n(\mathbf{x}) \tag{4.12}$$

For $\mathbf{x}^{(n)} \in \Omega_n$, we have for all $\mathbf{x} \in \Omega_n$ (remember that $g_n(\mathbf{x}) = 1$ on Ω_n):

$$\begin{aligned} \mathbf{a}_n(\mathbf{x}) &= \mathbf{a}_n(\mathbf{x}, \mathbf{x}^{(n)}) + \nabla_{\mathbf{x}} \varphi_n(\mathbf{x}, \mathbf{x}^{(n)}) = \\ &= \mathbf{a}(\mathbf{x}) + \nabla_{\mathbf{x}} \delta\varphi_n(\mathbf{x}, \mathbf{x}^{(n)}) \end{aligned} \tag{4.13}$$

where

$$\delta\varphi_n(\mathbf{x}, \mathbf{x}^{(n)}) = \varphi_n(\mathbf{x}, \mathbf{x}^{(n)}) - \varphi(\mathbf{x}, \mathbf{x}^{(n)}) \tag{4.14}$$

As a consequence (see (4.12) and (4.13)):

$$\begin{aligned} H g_{2n} &= \left[(\mathbf{p} - \mathbf{a})^2 + V\right] g_{2n} = \\ &= e^{i\,\delta\varphi_n(.,\mathbf{x}^{(n)})} H_n e^{-i\,\delta\varphi_n(.,\mathbf{x}^{(n)})} g_{2n} \end{aligned} \tag{4.15}$$

¿From (4.3) and (4.10) it follows that, uniformly in $\hat{\mathbf{v}}$,

$$\begin{aligned} &\lim_{n\to\infty} ||{B_n}'||_{C^1(\mathbf{R}^2)} = 0 \quad ; \quad \lim_{n\to\infty} ||g_n V_1||_2 = 0 \\ &\lim_{n\to\infty} ||g_n V_2||_\infty = 0 \end{aligned} \tag{4.16}$$

and then from Corollary 3.1 it follows that if $E \in K$ then $E \in \rho\left((\mathbf{p} - \mathbf{a}_n)^2\right)$ and in addition

$$\begin{aligned} &||V_n \left[(\mathbf{p} - \mathbf{a}_n)^2 - E\right]^{-1} ||_{\mathcal{B}(L^2,L^2)} \leq \\ \leq\ &||g_n V_1||_2 || \left[(\mathbf{p} - \mathbf{a}_n)^2 - E\right]^{-1} ||_{\mathcal{B}(L^2,L^\infty)} + \\ +\ &||g_n V_2||_\infty || \left[(\mathbf{p} - \mathbf{a}_n)^2 - E\right]^{-1} ||_{\mathcal{B}(L^2,L^2)} \to 0\ , \text{ when } n \to \infty \end{aligned} \tag{4.17}$$

It follows that there exists $n(K)$ such that for $n \geq n(K)$,

$$\begin{aligned} &(H_n - E)^{-1} = \\ =\ &\left[(\mathbf{p} - \mathbf{a}_n)^2 - E\right]^{-1} \sum_{k\geq 0} \left\{V_n \left[(\mathbf{p} - \mathbf{a}_n)^2 - E\right]^{-1}\right\}^k \end{aligned} \tag{4.18}$$

where the series in the R.H.S. of (4.18) is $\mathcal{B}\left(L^2, L^2\right)$ and $\mathcal{B}\left(L^2, L^\infty\right)$ convergent.

Fix now $u > 0$. Corollary 3.1 implies the existence of $N_1(u, K)$ such that

$$\begin{aligned} ||T(u,0) \left[(\mathbf{p} - \mathbf{a}_n)^2 - E\right]^{-1} T(-u,0)||_{\mathcal{B}(L^2,L^2)} &\leq C_1(u, K) \\ ||T(u,0) \left[(\mathbf{p} - \mathbf{a}_n)^2 - E\right]^{-1} T(-u,0)||_{\mathcal{B}(L^2,L^\infty)} &\leq C_2(u, K) \end{aligned} \tag{4.19}$$

as soon as $n \geq N_1(u, K)$ and $E \in K$.

¿From (4.15) it follows that:

$$\begin{aligned} (H - E) g_{2n} \psi &= [H, g_{2n}]\psi \equiv \Phi_{2n} = \\ &= e^{i\,\delta\varphi_n} (H_n - E) e^{-i\,\delta\varphi_n} g_{2n} \psi \end{aligned} \tag{4.20}$$

Together with (4.18), one has (n large enough):

$$\begin{aligned} g_{2n}\psi &= \\ &= e^{i\,\delta\varphi_n}\left[(\mathbf{p}-\mathbf{a}_n)^2-E\right]^{-1}\times \\ &\times \sum_{k\geq 0}\left\{V_n\left[(\mathbf{p}-\mathbf{a}_n)^2-E\right]^{-1}\right\}^k e^{-i\,\delta\varphi_n}\Phi_{2n} \end{aligned} \tag{4.21}$$

Because $T(u,0)$ comutes with V, (4.21) becomes:

$$\begin{aligned} &||T(u,0)g_{2n}\psi||_\infty \leq \\ \leq\ & C_2(u,K)\sum_{k\geq 0}||V_nT(u,0)\left[(\mathbf{p}-\mathbf{a}_n)^2-E\right]^{-1}T(-u,0)||^k_{\mathcal{B}(L^2,L^2)}\times \\ \times\ & ||T(u,0)\Phi_{2n}||_2 \end{aligned} \tag{4.22}$$

Since $supp\,\Phi_{2n}\subset\overline{\Omega_n\setminus\Omega_{2n}}$, one has:

$$||T(u,0)\Phi_{2n}||_2 \leq e^{2\,n\,u}||\Phi_{2n}||_2 \tag{4.23}$$

Using (4.19), there exists $N(u,K)>0$ such that:

$$\begin{aligned} &||V_nT(u,0)\left[(\mathbf{p}-\mathbf{a}_n)^2-E\right]^{-1}T(-u,0)||_{\mathcal{B}(L^2,L^2)} \leq \\ \leq\ & ||V_1g_n||_2C_2(u,K)+||V_2g_n||_\infty C_1(u,K)<1 \quad \text{if } n\geq N(u,K) \end{aligned} \tag{4.24}$$

Because $||g_{2n}||_{C^2(\mathbf{R}^2)}\leq const\frac{1}{n}$ and using (4.24), (4.23), (4.22) and (4.8) one obtains:

$$e^{u\,\hat{\mathbf{v}}\cdot\mathbf{x}}|\psi(\mathbf{x})|g_{2N(u,K)}\leq\mathcal{C}(u,K)e^{2uN(u,K)} \tag{4.25}$$

and since $\hat{\mathbf{v}}$ is arbitrary, the proof of Theorem 4.1 is completed.

Remark:

To obtain a proof of Theorem 4.1 in the Combes-Thomas scheme, in order to apply the O'Connor's Lemma, one needs to show that the essential spectrum of $\exp(\alpha(1+\mathbf{x}^2)^{1/2})H\exp(-\alpha(1+\mathbf{x}^2)^{1/2})\equiv H(\alpha)$ is (at least) included in $\sigma_L(B_0)$ independently of α. One can try to prove this by showing that

$$\sigma_{ess}(H(\alpha))\subset\sigma_{ess}(H_0(\alpha))\subset\sigma_L(B_0).$$

The second inclusion follows at once from the gaussian decay of the kernel of $(H_0-z)^{-1}$. If $B'(x)=0$ the first inclusion (in fact equality) follows from a standard compactness argument. The hard case (at least for us) is $B'(x)\neq 0$ and actually Theorem 3.1 can be used to prove the inclusion in this case.

In the rest of this section,

$$B\in C^\infty \quad \text{and} \quad B(\mathbf{x})=B_0 \quad \text{if} \quad |\mathbf{x}|\geq 1 \tag{4.26}$$

The scalar potential V belongs to $L^2 + (L^\infty)_\epsilon$ and we suppose that there exist $\delta > 0$ and $0 < \beta \le 2$ such that:

$$|V(\mathbf{x})| \le Ce^{-\delta|\mathbf{x}|^\beta}, \quad \text{for} \quad |\mathbf{x}| \ge 1, \quad C > 0 \tag{4.27}$$

Then takes place

Theorem 4.2 *Let $r = 1 + \frac{\beta}{2}$, $(1 < r \le 2)$. Then there exists $\mu > 0$ such that*

$$|\psi(x)| \le const(E)\, e^{-\mu\, |x|^r}. \tag{4.28}$$

One can write (see (4.11)):

$$|V_n(\mathbf{x})| \le C_n e^{-\frac{\delta}{2}|\mathbf{x}|^\beta}, \quad \lim_{n\to\infty} C_n = 0 \tag{4.29}$$

Take N_1 sufficiently large such that if $n \ge N_1$, then:

$$C_n < dist(E, \sigma(H_0)), \tag{4.30}$$

and by a similar argument with that made in Theorem 4.1, one has:

$$\begin{aligned} g_{2n}\psi &= \\ &= e^{i\,\delta\varphi_n}\,(H_0 - E)^{-1} \times \\ &\times \sum_{k\ge 0} \left[V_n\,(H_0 - E)^{-1}\right]^k e^{-i\,\delta\varphi_n}\Phi_{2n} \end{aligned} \tag{4.31}$$

Denote with $x_v = \hat{\mathbf{v}} \cdot \mathbf{x}$, with $T(u,r)$ the multiplication operator with $e^{u|x_v|^r}$ (where $u > 0$ and $1 \le r \le 2$) and with $\chi = \chi_{\Omega_{\frac{n}{2}}}$.

Because $suppV_n \subset \Omega_{\frac{n}{2}}$, one has

$$\begin{aligned} g_{2n}\psi &= \\ &= e^{i\,\delta\varphi_n}\chi\,(H_0 - E)^{-1}\,\chi \times \\ &\times \sum_{k\ge 0} \left[\chi V_n\,(H_0 - E)^{-1}\,\chi\right]^k e^{-i\,\delta\varphi_n}\Phi_{2n} \end{aligned} \tag{4.32}$$

Lemma 4.2 *Let $z \in \rho(H_0)$ and $0 \le u < \frac{B_0}{2^{r+1}}$. Then*

$$T(u,r)\chi\,(H_0 - z)^{-1}\,\chi T(-2^{r-1}u, r) \in \mathcal{B}(L^2, L^2) \quad \left[\mathcal{B}(L^2, L^\infty)\right] \tag{4.33}$$

Proof:

One has:

$$\frac{B_0}{4}|\mathbf{x} - \mathbf{x}'|^2 \ge \frac{B_0}{4}|x_v - x'_v|^r - \frac{B_0}{4} \tag{4.34}$$

If $\mathbf{x}, \mathbf{x}' \in \Omega_{\frac{n}{2}}$, then x_v and x'_v are positive, therefore

$$ux_v^r \le 2^{r-1}u|x_v - x'_v|^r + 2^{r-1}ux_v'^r. \tag{4.35}$$

Using Young inequalities (as in Corollary 3.1), the result is straightforward.

The technical core of the proof of Theorem 4.2 is contained in the following

Lemma 4.3 *Let $r = 1 + \frac{\beta}{2}$, ($1 < r \leq 2$). Then there exists $u > 0$ such that:*

$$T(u,r)\chi V_n(H_0 - E)^{-1}\chi T(-u,r) \in \mathcal{B}(L^2, L^2) \tag{4.36}$$

and has a norm which goes to zero when n goes to infinity.

Proof:
We intend to use Young inequalities again, therefore we shall estimate for an arbitrary $\psi \in L^2$ the term:

$$\begin{aligned} & |T(u,r)\chi V_n(H_0 - E)^{-1}\chi T(-u,r)\psi|(\mathbf{x}) \leq \\ \leq \quad & e^{ux_v^r} C_n \chi(\mathbf{x}) \frac{|\Gamma(\alpha)|}{4\pi} \int d\mathbf{x}'\, e^{-\frac{\delta}{2}|\mathbf{x}|^\beta} e^{-\frac{B_0|\mathbf{x}-\mathbf{x}'|^2}{8}} e^{-\frac{B_0|\mathbf{x}-\mathbf{x}'|^2}{8}} \times \\ \times \quad & |U(|\mathbf{x}-\mathbf{x}'|^2)|\chi(\mathbf{x}')e^{-ux_v'^r}|\psi(\mathbf{x}')|. \end{aligned} \tag{4.37}$$

We want to prove the existence of $u > 0$ such that for any $\mathbf{x}, \mathbf{x}' \in \Omega_{\frac{n}{2}}$ to take place

$$\frac{B_0|\mathbf{x}-\mathbf{x}'|^2}{8} - ux_v^r + \frac{\delta}{2}x_v^\beta + ux_v'^r \geq 0 \tag{4.38}$$

First of all, $|\mathbf{x}-\mathbf{x}'|^2 \geq (x_v - x_v')^2$, so if $x_v \leq x_v'$, the inequality is trivial. Therefore, the only interesting zone is $0 \leq x_v' \leq x_v$.
Define

$$f(y) = \frac{B_0}{8u}(x_v - y)^2 + y^r - x_v^r, \quad 0 \leq y \leq x_v, \quad \frac{n}{2} \leq x_v. \tag{4.39}$$

Then

$$f'(y) = -\frac{B_0}{4u}(x_v - y) + ry^{r-1}; \quad f''(y) = \frac{B_0}{4u} + r(r-1)y^{r-2}. \tag{4.40}$$

Because $f''(y)$ is positive on the considered domain and $f'(y)$ is changing sign in the extremities of the interval, there will be only one point $z \in (0, x_v)$ in which f reaches its minimum.
Let $t = \frac{x_v - y}{x_v} \in [0,1]$; then $y = (1-t)x_v$ and

$$f(t) = \frac{B_0 x_v^2}{8u}\left\{t^2 - \frac{8ux_v^{r-2}}{B_0}\left[1 - (1-t)^r\right]\right\}. \tag{4.41}$$

The equation $f'(t_m) = 0$ can be easily rewritten as:

$$t_m = \lambda(1 - t_m)^{r-1}; \quad \lambda = \frac{4ur}{B_0 x_v^{2-r}}. \tag{4.42}$$

If $\frac{8u}{B_0} < 1$, an approximate solution for t_m is

$$t_m = \lambda - (r-1)\lambda^2 + o(\lambda^3). \tag{4.43}$$

Introducing t_m in $f(t)$ one obtains:

$$f(t_m) = -\frac{2ur^2}{B_0}x_v^\beta\left[1 + o\left(\frac{8u}{B_0}\right)\right]. \tag{4.44}$$

Choosing u sufficiently small, one has $uf(y) + \frac{\delta}{2}x_v^\beta \geq 0$. This implies:

$$\begin{aligned} & ||T(u,r)\chi V_n(H_0 - E)^{-1}\chi T(-u,r)\psi||_2 \leq \\ \leq \quad & \frac{|\Gamma(\alpha)|}{4\pi}C_n||e^{-\frac{B_0\mathbf{x}^2}{8}}|U(\mathbf{x}^2)|||_1||\psi||_2 \end{aligned} \tag{4.45}$$

and the proof of Lemma 4.3 is complete.

Proof of Theorem 4.2:

With the results of Lemma 4.2 and Lemma 4.3, (4.32) becomes:

$$\begin{aligned} & ||T\left(\frac{u}{2^{r-1}},r\right)g_{2n}\psi||_\infty \leq \\ \leq \quad & ||T\left(\frac{u}{2^{r-1}},r\right)\chi(H_0 - E)^{-1}\chi T(-u,r)||_{\mathcal{B}(L^2,L^\infty)} \times \\ \times \quad & \sum_{k\geq 0}||T(u,r)\chi V_n(H_0 - E)^{-1}\chi T(-u,r)||^k_{\mathcal{B}(L^2,L^2)} \times \\ \times \quad & ||T(u,r)\Phi_{2n}||_2 \end{aligned} \tag{4.46}$$

Choose now N_0 large enough, such that the above series to converge. Then:

$$|\psi(\mathbf{x})| \leq const\ \exp\left(-\frac{u}{2^{r-1}}x_v^r\right), \quad x_v \geq 2N_0, \tag{4.47}$$

where the constant is independent of $\hat{\mathbf{v}}$.

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