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Transformations of Gibbs measures [1]

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Abstract

We study local transformations of Gibbs measures. We establish sufficient conditions for the quasilocality of the images and obtain results on the existence and continuity properties of their relative energies. General results are illustrated using simple examples.

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1 Introduction

Central in equilibrium statistical mechanics is a function, called interaction potential, from which relative energies can be computed. They determine the conditional probability distribution for a random field inside a finite region given the field outside. The random field is called a Gibbs measure with respect to this potential. In many problems, however, we have to investigate the probability law on transformed variables. We have in mind in particular renormalization group transformations or transformations induced by a (stochastic) dynamics on the configuration space.

Over the last decade many interesting examples have been produced in which non-Gibbsian states resulted by transforming Gibbs states (for a recent review see [11]). In the context of renormalization group theory or probabilistic cellular automata Gibbs measures depend on parameters such as the temperature, magnetic field, noise level etc. The discussion of the Gibbsian or non-Gibbsian character of transformed measures therefore has to be done with respect to the parameter values. Typically it seems that at very high temperatures or sufficiently strong magnetic fields the emerging state remains Gibbsian [20, 22, 23, 36, 15, 28, 29], whereas for low temperature regions and/or small external fields non-Gibbsian images commonly appear [19, 20, 22, 38, 14, 32, 33, 41, 12]. There are examples warning one, however, that this separation of behaviour in the parameter space cannot be taken for granted [13, 10]. One of the points of our paper is giving sufficient conditions excluding such high temperature pathologies.

More recently it was suggested and for certain cases proven [8, 35, 30, 3], that even for non-Gibbsian measures one can make sense of the notion of relative energy. A potential can be defined which is absolutely summable only for 'typical' configurations. In [30] we called the resulting probability measures weakly Gibbsian measures. One purpose for which weakly Gibbsian measures might be useful is the extending of a meaningful thermodynamic description also to those parameter regions in which the transformed measures are non-quasilocal (not almost Markovian).

This paper is organized as follows. In the preparatory section following this introduction we introduce notations, fix the classes of transformations we will consider further on, and establish some basic results on these transformations. The main section is the last one, divided into four subsections. The first subsection presents general results. There, by using disagreement percolation methods, we investigate sufficient conditions for the quasilocality of conditional distributions for transformed Gibbs measures. This is continued in the third subsection where we concentrate on relative energies for weakly Gibbsian states. Subsections 2 and 4 are devoted to more specific examples illustrating various points.

2 Preliminaries

Lattices and graphs For convenience and without serious restriction of generality we will mostly think of the regular d-dimensional cubic lattice $\mathbb{Z}^d$. The symbol $\mathcal{E}$ will denote the set of finite subsets of the lattice, and we write $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$, for every $\Lambda \in \mathbb{Z}^d$. The distance between two sets $\Lambda, \tilde{\Lambda} \in \mathcal{E}$ is given by $\mathrm{dist}(\Lambda, \tilde{\Lambda}) = \min\{|x-y| : \ x \in \Lambda, y \in \tilde{\Lambda}\}$.

The label $\partial\Lambda$ stands for the boundary of Λ, i.e. the set of sites in Λ^c at distance 1 from Λ. Whenever considering infinite volume limits $\Lambda \to \mathbb{Z}^d$, we take Λ along an increasing sequence of cubes.

Often in the following we will look at $\mathbb{Z}^d$ as being a graph. We turn $\mathbb{Z}^d$ into a graph by adding edges between nearest neighbours sites. The graph $\mathcal{G}$ then consists of the pair made up by the vertex set $\mathbb{Z}^d$ and the set of edges $\mathbb{E}^d$. Two vertices x and y are adjacent, denoted $x \sim y$, if they are connected by an edge. A path of length n is a sequence of distinct vertices $x_1, x_2, ..., x_n$ in which the consecutive vertices are adjacent. For two disjoint sets $\Lambda_1, \Lambda_2 \in \mathcal{E}$ a path from Λ_1 to Λ_2 is a path which has a vertex in Λ_1 and another in Λ_2.

Configurations At each site of the lattice a 'spin' is placed. These spins take their values from the same finite set S. The (spin) configuration space is thus $\Omega = S^{\mathbb{Z}^d}$, and its projection to a finite volume Λ is $\Omega_\Lambda = S^\Lambda$. If $\sigma, \sigma' \in \Omega$ are such that $\sigma_x = \sigma'_x$, for every $x \in \Lambda$, then we say that the two configurations agree on the set Λ and denote this event by $\sigma_\Lambda = \sigma'_\Lambda$. This notion of nearness of configurations induces the product topology on Ω. We further consider the measurable space $(\Omega, \mathcal{F})$ by associating Ω with its Borel σ-field $\mathcal{F}$. We also consider the sub-σ-fields $\mathcal{F}_\Lambda$ of events measurable with respect to spins sitting within Λ.

Functions A function $f : \Omega \to \mathbb{R}$ is continuous if for every $\varepsilon > 0$ there exists a $\Lambda \in \mathcal{E}$ such that $|f(\sigma) - f(\sigma')| < \varepsilon$ whenever $\sigma_\Lambda = \sigma'_\Lambda$. The set $\mathcal{C}(\Omega)$ of continuous functions is a Banach space with respect to the norm $||f||_\infty = \sup_{\sigma\in\Omega} |f(\sigma)|$. Continuous functions are approximated by local functions. Denote by $\mathcal{D}_f$ the dependence set of $f : \Omega \to \mathbb{R}$, i.e. the minimal set (in the sense of inclusion) Λ for which f is $\mathcal{F}_\Lambda$-measurable. The function f is local if $\mathcal{D}_f \in \mathcal{E}$.

A system of independent spins is described by a product measure. We can take such a measure (e.g., products of the normalized one-site counting measure) as a reference measure on which an interaction will be superimposed.

Markov fields A probability measure μ on $(\Omega, \mathcal{F})$ is a Markov field if for each $\Lambda \in \mathcal{E}$ and every $\eta \in \Omega_\Lambda$ we have μ-almost surely

$$\mu(\sigma_\Lambda = \eta_\Lambda \mid \sigma_x,\, x \in \Lambda^c) = \mu(\sigma_\Lambda = \eta_\Lambda \mid \sigma_x,\, x \in \partial\Lambda) \tag{2.1}$$

Gibbs measures More generally than in (2.1), whenever the local conditional probabilities $\mu(\sigma_\Lambda = \eta_\Lambda \mid \sigma_{\Lambda^c} = \bar{\eta}_{\Lambda^c})$ are all continuous functions of $\bar{\eta}$ for η and $\Lambda \in \mathcal{E}$ fixed, then we say that μ is quasilocal (or almost Markovian, see [40]). When, moreover, these conditional probabilities are strictly positive, we say that μ is a Gibbs measure. Equivalent definitions and more details can be found in [16, 14]. Typically in classical equilibrium statistical mechanics the conditional probablities in the left hand side of (2.1) are specified in terms of a potential and one is interested in characterizing the corresponding probability measures. A potential $U = \{U_B\}_{B\in\mathcal{E}}$ is a family of real valued functions on Ω such that each U_B is $\mathcal{F}_B$-measurable, with $U_\emptyset = 0$.

Relative energies This function gives the energy difference obtained under a change made in the configuration. For our purposes it suffices to compare the energy for a configuration σ with the energy for a reference configuration, say $\mathbb{1}$, $\mathbb{1}_x = 1$, $\forall x \in \mathbb{Z}^d$. Then we consider for each $\sigma \in \Omega$ the modified configuration ${}^\Lambda\sigma \in \Omega$, where $({}^\Lambda\sigma)_x = \sigma_x$, if $x \notin \Lambda$, and $({}^\Lambda\sigma)_x = +1$, if $x \in \Lambda$. We say that a potential U allows a relative Hamiltonian if the relative energies ($\Lambda \in \mathcal{E}$)

$$H_\Lambda(\sigma) = \sum_{B \cap \Lambda \neq \emptyset} [U_B(\sigma) - U_B({}^\Lambda\sigma)] \tag{2.2}$$

converge uniformly in σ. (Here and henceforth convergence means convergence along a net of finite subsets of the lattice, directed by inclusion.) A sufficient condition for uniform convergence is that the potential is absolutely summable, i.e. for any $x \in \mathbb{Z}^d$

$$\sum_{B \ni x} ||U_B||_\infty < \infty \tag{2.3}$$

It is not difficult to see [40, 16] that if a potential allows a relative Hamiltonian, then there exists a Gibbs measure μ such that for each $\Lambda \in \mathcal{E}$ and μ-almost every $\eta \in \Omega$

$$H_\Lambda(\eta) = -\log \frac{\mu(\sigma_\Lambda = \eta_\Lambda \mid \sigma_{\Lambda^c} = \eta_{\Lambda^c})}{\mu(\sigma_\Lambda = +1 \mid \sigma_{\Lambda^c} = \eta_{\Lambda^c})} \tag{2.4}$$

Sullivan has proved that the converse is also true [40]: Local conditional probabilities of a Gibbs measure can be expressed in terms of a potential allowing a relative Hamiltonian.

Denote by $\mathfrak{S}(S)$ the permutation group for the finite set S, and take $\pi \in \mathfrak{S}(S)$. Also, let us denote by

$$\Delta_x^\pi H(\sigma) = \sum_{B \ni x} [U_B(\sigma) - U_B(\pi_x\sigma)] \tag{2.5}$$

the relative energy associated with the change $\sigma \mapsto \pi_x\sigma$, where $(\pi_x\sigma)_y = \sigma_y$ if $y \neq x$ and $(\pi_x\sigma)_x = \pi(\sigma_x)$. This is well defined if (2.2) is. With some abuse of notation we also write $\pi_x f(\sigma) = f(\pi_x\sigma)$ for the action of the permutation group transposed to functions $f : \Omega \to \mathbb{R}$. If μ is a Gibbs measure for the potential $\{U_B\}_{B \in \mathcal{E}}$, then for all $\pi \in \mathfrak{S}(S), x \in \mathbb{Z}^d$ and all $f \in \mathcal{C}(\Omega)$ we have

$$\mu(\pi_x^{-1} f) = \mu(e^{\Delta_x^\pi H} f) \tag{2.6}$$

This means that $\mu \circ \pi_x$ is absolutely continuous with respect to μ, for every x, with a positive and continuous version of the Radon-Nikodým derivative given by the Boltzmann-factor $\exp(\Delta_x^\pi H)$.

Weakly Gibbsian measures A weaker notion of Gibbsianness can be obtained by relaxing the condition (2.2) on the relative energy. In [30] we called such measures weakly Gibbsian states. They have appeared in [3] and [35] and their study was inspired by a talk of R.L. Dobrushin in 1995, see [8] and [9]. A probability measure μ on $(\Omega, \mathcal{F})$ is weakly Gibbsian if there is a tail event $K \in \mathcal{T} = \cap_{\Lambda \in \mathcal{E}} \mathcal{F}_{\Lambda^c}$ with $\mu(K) = 1$, and a potential $\{U_B\}$ which is μ-almost surely summable, namely for all $\sigma \in K$

$$H_\Lambda(\sigma) = \sum_{B \cap \Lambda \neq \emptyset} [U_B(\sigma) - U_B({}^\Lambda\sigma)] \tag{2.7}$$

converges and is bounded so that (2.6) holds and (2.4) holds μ-almost surely. For a weakly Gibbsian state the local conditional probabilities need not be continuous; all that matters is that relative energies (2.7) can be well defined on a measure one subset so that (2.4) holds. In all of the examples we know of the configurations $\sigma \in K$ are characterized by the fact that in (2.7) $H_\Lambda({}^{V^c}\sigma) \to H_\Lambda(\sigma)$, as $V \to \mathbb{Z}^d$. This means that for the left hand side in (2.1) there is a version g of the conditional probabilities which, though not continuous, satisfies $\lim_{V \to \mathbb{Z}^d} g({}^{V^c}\sigma) = g(\sigma)$, for all $\sigma \in K$.

Transformations The last ingredient we need is a class of transformations of probability measures μ on $(\Omega, \mathcal{F})$. The image measure is a probability measure on $(\Omega', \mathcal{F}')$. We always take here $\Omega' = S'^{\mathcal{L}}$, where $|S'| \leq |S|^{L^d}$ and $\mathcal{L} = L\mathbb{Z}^d$ for some integer $L \geq 1$. (An exception from this choice will be discussed in the example in Subsection 3.3 where the transformation will be chosen as a projection to a hyperplane $\mathcal{L} = \mathbb{Z}^{d-1}$.) The transformation T is given by a probability kernel

$$T(d\xi|\sigma) = \prod_{x \in \mathcal{L}} t_x(\xi_x|\sigma) d\xi_x \tag{2.8}$$

where $t_x(\cdot|\sigma)d\xi_x$ is a probability measure on S' for each $\sigma \in \Omega$, $x \in \mathcal{L}$. Here $d\xi_x$ is the counting measure on S'. In this paper we consider the following two cases:

- (a) Stochastic parallel updating: In this case we take $\mathcal{L} = \mathbb{Z}^d$ (or $L = 1$), $S' = S$. The updating kernels t_x are bounded, positive and local, that is we assume that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

  $$\varepsilon_1 \leq t_x(\xi_x|\sigma) \leq \varepsilon_2 \tag{2.9}$$

  and $t_x(\xi_x|\cdot)$ is B_x-measurable with a $B_x \in \mathcal{E}$, $x \in B_x$, with uniformly bounded diameter, $\operatorname{diam} B_x < r$, for some $r > 0$. These transformations correspond to a discrete time dynamics, often called a probabilistic cellular automaton (see [27]).

- (b) Deterministic updating with no overlap: In this case we take the dependence region $\{B_x\}_{x \in \mathcal{L}}$ as a partition of the lattice $\mathbb{Z}^d$, i.e. for $x \in \mathcal{L} = L\mathbb{Z}^d$, B_x is a cube of size L^d on $\mathbb{Z}^d$ centered around x. For updating we take here

  $$t_x(\xi_x|\sigma) = I[\xi_x = T_x(\sigma)] \tag{2.10}$$

  that is the indicator function of the event $\xi_x = T_x(\sigma)$, where now $T_x : S^{B_x} \to S'$ is a copy of one and the same T_0. We assume, in order to avoid trivialities, that for every $\alpha \in S'$ there is a $\sigma \in S^{B_x}$ such that $\alpha = T_x(\sigma)$. An example of deterministic updating with overlap (as in cellular automata) will be presented in Subsection 3.3 below. At any rate, case (b) can always be taken as some (zero noise) limit of case (a).

Since our transformations are local, all local functions are mapped into local functions, i.e. if $\mathcal{D}_f = W$, then we have

$$Tf(\sigma) = \int_{\Omega'} f(\xi) \prod_{x \in W} t_x(\xi_x|\sigma) d\xi_x \tag{2.11}$$

In this paper we study some aspects of the image $\nu = \mu T$ of (Gibbs) probability measures μ. The image measure ν is the marginal on $\Omega' = S'^{\mathcal{L}}$ of the product measure $\mu \times T$ defined on the product space $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}')$:

$$\mu \times T(f) = \int_{\Omega} \mu(d\sigma) \int_{\Omega'} T(d\xi|\sigma) f(\sigma, \xi) \tag{2.12}$$

for all $\mathcal{F} \times \mathcal{F}'$-measurable real valued functions f. We list below some useful properties the proof of which is immediate:

Lemma 2.1 *The following properties hold:*

1. *If $f(\sigma, \xi) = g(\xi)$, then $\mu \times T(f) = \nu(g)$.*
2. *If $f(\sigma, \xi) = h(\sigma)g(\xi)$, then $\mu \times T(f) = \mu \times T(h \times g) = \mu(hTg)$.*
3. *If $g(\xi) = G_1(\xi)G_2(\xi)$ with G_1 and G_2 local functions on Ω' having disjoint dependence, i.e. $\mathcal{D}_{G_1} \cap \mathcal{D}_{G_2} = \emptyset$, then $Tg = TG_1 \cdot TG_2$.*

For $\alpha \in \Omega', \Lambda \in \mathcal{E}' = \{V \subset \mathcal{L}, V \in \mathcal{E}\}$, write $I_{\alpha,\Lambda}(\xi)$ for the indicator function of the event $\xi_\Lambda = \alpha_\Lambda$. Then $TI_{\alpha,\Lambda} = \prod_{x \in \Lambda} t_x(\alpha_x|\sigma)$, and if μ is non-null, then $\mu(TI_{\alpha,\Lambda}) > 0$. This enables us to define μ^{α_Λ} as the projection to Ω of the measure $\mu \times T$ conditional on having $\xi = \alpha$ on Λ. That is, for $h \in \mathcal{C}(\Omega)$,

$$\mu^{\alpha_\Lambda}(h) := \frac{\mu \times T(h \times I_{\alpha,\Lambda})}{\mu \times T(\mathbb{1} \times I_{\alpha,\Lambda}(\xi))} = \frac{\mu(hTI_{\alpha,\Lambda})}{\nu(I_{\alpha,\Lambda})} \tag{2.13}$$

Obviously, $\mu \times T = \nu \mu^{\xi_\Lambda}$, and in particular, for all $f \in \mathcal{C}(\Omega)$,

$$\mu(f) = \mu \times T(f \times \mathbb{1}) = \nu(\mu^{\xi}(f)) \tag{2.14}$$

where $\mu^\alpha = \mu \times T[\cdot|\xi = \alpha], \alpha \in \Omega'$ is obtained by conditioning on the second marginal of $\mu \times T$.

Notice that μ^{α_Λ} can be written as

$$\mu^{\alpha_\Lambda}(h) = \frac{\int \mu(d\sigma) h(\sigma) \prod_{x \in \Lambda} t_x(\alpha_x|\sigma)}{\int \mu(d\sigma) \prod_{x \in \Lambda} t_x(\alpha_x|\sigma)} \tag{2.15}$$

Therefore it is easy to prove (see also, e.g., [14], section 2.4.8.)

Lemma 2.2 *Suppose μ is a Gibbs measure on $(\Omega, \mathcal{F})$ for the potential $\{U_B\}$. Under condition (2.9), μ^{α_Λ} is also a Gibbs measure on $(\Omega, \mathcal{F})$ for the potential*

$$U_B^{\alpha_\Lambda}(\sigma) = \begin{cases} U_B(\sigma) - \log t_x(\alpha_x|\sigma) & \textit{if } B = B_x \quad \textit{for some } x \in \Lambda \\ U_B(\sigma) & \textit{otherwise} \end{cases} \tag{2.16}$$

For deterministic updating (case (b), (2.10)) the measure μ^{α_Λ} is equal to μ conditioned on having $\alpha_x = T_x(\sigma)$, for $x \in \Lambda$, i.e.,

$$\mu^{\alpha_\Lambda}(h) = \mu(h|\alpha_x = T_x(\sigma),\ x \in \Lambda) \tag{2.17}$$

Since the conditioning is made on disjoint blocks B_x, for every $x \in \Lambda$, it is convenient to think of μ^{α_Λ} as a measure on $\tilde{\Omega}^\Lambda = \otimes_{x\in\Lambda} S^{B_x} \otimes S^{\mathbb{Z}^d \setminus \cup_{x\in\Lambda} B_x}$, where the B_x are thought of as being a super-site with new single spin space S^{B_x}, $x \in \Lambda$. The conditioning $\alpha_x = T_x(\sigma)$ then determines a reference measure on each S^{B_x}, $x \in \Lambda$.

Lemma 2.3 *If μ is a Markov field on $(\Omega, \mathcal{F})$, then μ^{α_Λ} is a Markov field as well for the graph $\tilde{\mathcal{G}}$ obtained by adding edges to $(\mathbb{Z}^d, \mathbb{E}^d)$ between those $x, y \in \mathbb{Z}^d$ for which $x, y \in B_z$ for some $z \in \Lambda$.*

Proof: The statement is obvious once one realizes that μ^{α_Λ}-a.s.

$$\mu^{\alpha_\Lambda}(\sigma_V = \cdot|\sigma_{V^c} = \eta_{V^c}) = \mu^{\alpha_\Lambda}(\sigma_V = \cdot|\sigma_{\partial\tilde{V}} = \eta_{\partial\tilde{V}}) \tag{2.18}$$

with

$$\partial\tilde{V} = \bigcup_{\substack{z\in\Lambda \\ B_z \cap V \neq \emptyset}} B_z \bigcap V^c \tag{2.19}$$

□

This previous lemma becomes relevant for our purposes from the following simple fact.

Lemma 2.4 *For all local functions $f \in \mathcal{C}(\Omega')$ with $\mathcal{D}_f \cap \Lambda = \emptyset$ for $\Lambda \in \mathcal{E}', \alpha \in \Omega'$*

$$\nu(f|\xi_\Lambda = \alpha_\Lambda) = \mu^{\alpha_\Lambda}(Tf). \tag{2.20}$$

Proof: Notice that by Lemma 2.1 we have

$$\frac{\nu(fI_{\alpha,\Lambda})}{\nu(I_{\alpha,\Lambda})} = \frac{\mu(Tf \cdot TI_{\alpha,\Lambda})}{\mu(TI_{\alpha,\Lambda})} = \frac{\mu \times T(Tf \times I_{\alpha,\Lambda})}{\mu \times T(\mathbb{1} \times I_{\alpha,\Lambda})} = \mu^{\alpha_\Lambda}(Tf) \tag{2.21}$$

as required. □

3 Transformations of Gibbs measures

3.1 Quasilocality of image measures

The most recent reference on the concept of quasilocality is [15]. We also refer to [16, 14] for more details. The central issue in our context is to understand how $\nu(f|\xi_{\bar{\Lambda}} = \alpha_{\bar{\Lambda}})$ depends on $\alpha \in \Omega'$, for arbitrary $\mathcal{F}'_\Lambda$-measurable f and $\Lambda, \bar{\Lambda} \subset \mathcal{E}'$. For this purpose we fix a $\Lambda \in \mathcal{E}'$ and an $\mathcal{F}'_\Lambda$-measurable function f. We take $\Lambda \subset V \subset W \in \mathcal{E}'$, and write $V \setminus \Lambda = \Lambda_1$ and $W \setminus V = \Lambda_2$. Furthermore, we choose $\alpha, \bar{\alpha} \in \Omega'$ such that $\alpha_{\Lambda_1} = \bar{\alpha}_{\Lambda_1}$.

Lemma 2.4 allows us to estimate how much in the state ν events at the site of observation depend on regions far away (the distance being measured by the length of a path connecting Λ_1 with Λ_2). This amounts to estimating some truncated pair correlation function for which we can follow the methods in [2, 17].

For example, suppose that there is just one site $y \in \Lambda_2$ for which $\alpha_y \neq \bar{\alpha}_y$. Then, as is straightforward to see from Lemma 2.4 (with $\Lambda_1 \cup \Lambda_2$ replacing the Λ there), say under condition (2.9),

$$\nu(f|\xi_{\Lambda_1 \cup \Lambda_2} = \bar{\alpha}_{\Lambda_1 \cup \Lambda_2}) - \nu(f|\xi_{\Lambda_1 \cup \Lambda_2} = \alpha_{\Lambda_1 \cup \Lambda_2}) = \frac{\mu^{\alpha_{\Lambda_1 \cup \Lambda_2}}(Tf; p_y^{\alpha,\bar{\alpha}})}{\mu^{\alpha_{\Lambda_1 \cup \Lambda_2}}(p_y^{\alpha,\bar{\alpha}})} \tag{3.1}$$

where

$$p_y^{\alpha,\bar{\alpha}} = \frac{t_y(\bar{\alpha}_y|\sigma)}{t_y(\alpha_y|\sigma)} \tag{3.2}$$

and $\mu(f;g) = \mu(fg) - \mu(f)\mu(g)$ denotes the truncated pair correlation function for a probability measure μ and functions f, g. The conclusion is that all we need for qualilocality of ν is that the measure μ^ξ has good decay of correlations uniformly in $\xi \in \Omega'$. Sufficient conditions can be obtained in a variety of ways. For example, [21] use that the so called "constrained measures" μ^ξ obtained after a majority transformation from the two dimensional Ising model at the critical point fall into the regime of complete analyticity. We use a general disagreement percolation approach which has worked well in other disordered settings (where the ξ correspond to independent random variables), see [17].

In what follows we take μ to be a Markov field on $(\Omega, \mathcal{F})$ with single vertex conditional probabilities

$$\mu_x(\cdot|\eta) = \mu(\sigma_x = \cdot|\sigma_y = \eta_y,\ y \sim x) \tag{3.3}$$

with $\eta_y \in S$, $y \sim x$, so that $\mu(\sigma_y = \eta_y,\ y \sim x) > 0$. The maximal variation of this conditional probability defines the parameters

$$q_x = \max_{\eta,\bar{\eta}} \mathrm{var}[\mu_x(\cdot|\eta), \mu_x(\cdot|\bar{\eta})] \tag{3.4}$$

We distinguish between two types of transformations: transformations with or without overlapping. Transformations under case (a) can allow overlapping, while case (b) is always non-overlapping.

Proposition 3.1 *Suppose T is a transformation with non-overlapping blocks. There is a $q^* > 0$ such that if $q_x < q^*$ for every $x \in \mathbb{Z}^d$, then $\nu = \mu T$ is quasilocal. In general q^* depends on the transformation T, however if $d = 1$, we can always take $q^* = 1$.*

Corollary 3.1 *Suppose T is a transformation with non-overlapping blocks of type covered by case (a). If μ is a Gibbs measure on $(\Omega, \mathcal{F})$ for a nearest neighbour pair potential $\{U_B\}$, i.e. $U_B(\sigma) = \beta U(\sigma_x, \sigma_y)$, if $B = \{x, y\}$, $|x - y| = 1$, and $U_B(\sigma) = 0$ otherwise, then there is a $\beta^* > 0$ such that $\nu = \mu T$ is a Gibbs measure for all $\beta < \beta^*$. If $d = 1$, then $\nu = \mu T$ is a Gibbs measure for every β (i.e., $\beta^* = \infty$).*

Proposition 3.2 *Suppose T is a transformation with overlapping blocks belonging to case (a). Suppose, moreover, that for every x*

$$k_x = \max_{\sigma,\bar{\sigma}} \operatorname{var}[t_x(\cdot|\sigma), t_x(\cdot,\bar{\sigma})] < k^* \tag{3.5}$$

Then for $k^* > 0$ sufficiently small, Proposition 3.1 and Corollary 3.1 apply unchanged. If $d = 1$, we can take $k^* = 1$.

We now turn to proving the above statements. The general strategy is as follows: We prove uniform convergence of the conditional probabilities $\nu(f|\xi_{W\setminus\Lambda} = \alpha_{W\setminus\Lambda})$ as $W \to \mathbb{Z}^d$. For doing that we use (3.1) and the methods of [2, 17] for estimating truncated correlation functions and the influence of boundary conditions in terms of disagreement percolation.

Proof of Proposition 3.2: Fix $x \in \mathbb{Z}^d$ and consider the set $\mathcal{N}_x = \cup_{z:B_z \ni x} B_z \setminus \{x\}$. The single site conditional probability of $\mu^{\alpha_{\Lambda_1\cup\Lambda_2}}$ is given by

$$\mu_x^{\alpha_{\Lambda_1\cup\Lambda_2}}(\cdot|\eta) = \mu^{\alpha_{\Lambda_1\cup\Lambda_2}}(\sigma_x = \cdot|\sigma_{\mathcal{N}_x} = \eta_{\mathcal{N}_x}) \tag{3.6}$$

for such configurations η for which $\mu^{\alpha_{\Lambda_1\cup\Lambda_2}}(\sigma_{\mathcal{N}_x} = \eta_{\mathcal{N}_x}) > 0$. Its maximal variation parameters are

$$q_x^{\alpha_{\Lambda_1\cup\Lambda_2}} = \max_{\eta,\bar{\eta}} \operatorname{var}[\mu_x^{\alpha_{\Lambda_1\cup\Lambda_2}}(\cdot \mid \eta), \mu_x^{\alpha_{\Lambda_1\cup\Lambda_2}}(\cdot \mid \bar{\eta})] \tag{3.7}$$

Observe now that $q_x^{\alpha_{\Lambda_1\cup\Lambda_2}}$ can be made small uniformly in x and $\alpha_{\Lambda_1\cup\Lambda_2}$ if we take both q^* and k^* small. This is clear because $q_x^{\alpha_{\Lambda_1\cup\Lambda_2}} = 0$ if $q_x = 0$ and $k_y = 0$, for all $y \in \mathbb{Z}^d$ for which $x \in B_y$. So we choose q^* and k^* so small that $q_x^{\alpha_{\Lambda_1\cup\Lambda_2}} \leq q$, for a fixed q to be determined below. Let λ_q be the Bernoulli measure on $\{0,1\}^{\mathbb{Z}^d}$ with density q. Furthermore consider the graph $\tilde{\mathcal{G}}$ with vertex set $\mathbb{Z}^d$ where edges connect any two sites $x, y \in \mathbb{Z}^d$ for which there is a $z \in \mathbb{Z}^d$ with the property that $x, y \in B_z$, or for which $|x - y| = 1$. Let $p_c(\tilde{\mathcal{G}})$ be the threshold density for Bernoulli site percolation on the graph $\tilde{\mathcal{G}}$. We have then that if $q < p_c(\tilde{\mathcal{G}})$, there exist $C_1 < \infty$ and $m > 0$ such that

$$\lambda_q(A \rightsquigarrow B) \leq C_1 \exp(-m \operatorname{dist}(A,B)) \tag{3.8}$$

Here $A \rightsquigarrow B$ denotes the event that there is an open path (i.e., a path of occupied vertices) connecting the regions $A, B \in \mathcal{E}$. This is useful for applying the methods in [2, 17]. From there it follows that differences in conditional expectations can be estimated as

$$\begin{aligned} &|\nu(f|\xi_{\Lambda_1\cup\Lambda_2} = \bar{\alpha}_{\Lambda_1\cup\Lambda_2}) - \nu(f|\xi_{\Lambda_1\cup\Lambda_2} = \alpha_{\Lambda_1\cup\Lambda_2})| \leq \\ &\leq\ 2||Tf||_\infty\, \lambda_q(\bar{\Lambda} \rightsquigarrow \Lambda_2) \end{aligned} \tag{3.9}$$

where $\bar{\Lambda} = \Lambda \cup_{z\in\Lambda} B_z$. It remains to notice that $p_c(\tilde{\mathcal{G}}) > 0$ for every (fixed) local transformation T, and that $p_c(\tilde{\mathcal{G}}) = 1$ if $d = 1$. $\square$

Proof of Proposition 3.1: It is useful here to take the approach outlined above Lemma 2.3. We glue the sites $y \in B_x$, $x \in \mathcal{L}$, together into a single super-site, again denoted by x. Consider thus the random field $(s_x,\ x \in \mathcal{L})$ with $s_x \in S^{B_x}$ and joint distribution induced by $\mu^{\alpha_{\Lambda_1 \cup \Lambda_2}}$. For example, in case (b) this means that $s_x = (\sigma_y,\ y \in B_x)$ such that $\alpha_x = T_x(\sigma)$, if $x \in \Lambda_1 \cup \Lambda_2$; the rest of the dependence is between blocks and originates from correlations already present in μ.

We look again at the single-supersite conditional probabilities

$$\mu_x^{\alpha_{\Lambda_1 \cup \Lambda_2}}(\cdot|e) = \mu_x^{\alpha_{\Lambda_1 \cup \Lambda_2}}(s_x = \cdot|s_y = e_y,\ y \sim x) \tag{3.10}$$

where we put $y \sim x$ for those $x, y \in \mathcal{L}$ for which $\mathrm{dist}(B_x, B_y) = 1$. The maximal variations are now given by

$$Q_x^{\alpha_{\Lambda_1 \cup \Lambda_2}} = \max_{e,\bar{e}} \mathrm{var}[\mu_x^{\alpha_{\Lambda_1 \cup \Lambda_2}}(\cdot, e), \mu_x^{\alpha_{\Lambda_1 \cup \Lambda_2}}(\cdot, \bar{e})] \tag{3.11}$$

Clearly, $Q_x^{\alpha_{\Lambda_1 \cup \Lambda_2}}$ tends to zero as q goes to zero, for $y \in B_x$. The rest of the proof is then similar to that of Proposition 3.2. □

Proof of Corollary 3.1: The statement follows immediately from Proposition 3.1 by noticing that $q_x \to 0$ as $\beta \to 0$. Moreover, it is easily seen that $\nu = \mu T$ is non-null. Therefore we conclude that ν is a Gibbs measure. □

Remark: The above deals with the so called 'positive' results regarding the Gibbsian nature of the image measure. The results are overlapping with earlier work dealing with transformations of Gibbs measures in a well-understood region of the phase diagram. We believe that the approach given above unifies these results, at least for transformations of Gibbs measures for short range interactions. We refer to [20, 22, 23, 37, 7, 6, 14, 34, 31, 28, 29, 15] for the earlier work.

3.2 Counterexamples

In this subsection we discuss by two examples why some of the conditions on the transformations appearing in Propositions 3.1 and 3.2 are necessary and that the conclusion of these Propositions cannot be taken for granted even in the regime of so called complete analyticity for the original Gibbs measure μ.

First let us note that in Proposition 3.1 the temperature range where the statement is true depends on the transformation. It is important to keep account of this since it is easy to see that for arbitrarily high temperatures one can find a transformation for which the image is non-Gibbsian. This has been first discussed in [10]: Consider the standard nearest neighbour Ising model on $\mathbb{Z}^d$ with $d \geq 2$. (This is obtained from (3.29) below in the limit $n \to \infty$.) The transformation is of type (b), see (2.10), where

$$T_x(\sigma) = \begin{cases} L^d & \text{if } \sigma_y = -1 \text{ for all } y \in B_x \\ \sum_{y: y \in B_x} \sigma_y & \text{otherwise} \end{cases} \tag{3.12}$$

Then it is shown in [10] that for every temperature β there exists a side-length $L^*(\beta)$ such that for all $L > L^*(\beta)$ the measure obtained by transforming the Gibbs measure μ_β of the

Ising model under $T = T_L$ is non-Gibbsian. What happens here is that in the new graph $\tilde{\mathcal{G}}$ for any vertex the number of edges incident to the vertex is increasing with L. Therefore we have for the corresponding percolation threshold $p_c(\tilde{\mathcal{G}}) \to 0$ as $L \to \infty$, and hence we must take $\beta = \beta(L) \to 0$ as $L \to \infty$ in order to ensure that μT_L is Gibbsian.

Secondly, we turn to discussing another interesting example which shows that for deterministic transformations we have to require in Proposition 3.2 that no overlap between the blocks B_x occurs. The following example shows why (originally due to J. van den Berg). Let us consider independent binary spins σ_x, $x \in \mathbb{Z}$, distributed identically by $\mu(\sigma_x = +1) = p \neq 1/2$, $\mu(\sigma_x = -1) = 1 - p$, $0 < p < 1$. Define the transformation

$$\xi_x = \sigma_x \sigma_{x+1}, \qquad x \in \mathbb{Z} \tag{3.13}$$

This is a deterministic transformation with overlapping blocks. The $(\xi_x, x \in \mathbb{Z})$ form a one-dependent random field. In particular, the induced measure on the spins on the even (resp. on the odd) sites is a product measure. Still, the measure ν on the $(\xi_x, x \in \mathbb{Z})$ is in some sense strongly non-Gibbsian:

Proposition 3.3 *There is no version of conditional probabilities of the measure induced under the transformation (3.13) which is quasilocal for at least one configuration.*

Proof: First notice that for $m \geq 1$

$$\xi_0 = \sigma_{-m} \xi_{-m} \cdots \xi_{-1} \xi_1 \cdots \xi_m \sigma_{m+1} \tag{3.14}$$

Choose $\alpha_i \in \{-1, +1\}$, for $i = 1, 2, ..., m$, $m > 1$, and denote by $\mu^{\alpha_{1m}}(f)$ the expectation of a function f in the conditioned measure

$$\mu^{\alpha_{1m}}(\cdot) = \mu(\cdot \mid \sigma_1 \sigma_2 = \alpha_1, ..., \sigma_m \sigma_{m+1} = \alpha_m) \tag{3.15}$$

Take $f(\sigma) = \sigma_1$, and look at how its expectation changes if we make the change $\alpha_m \mapsto \bar{\alpha}_m = -\alpha_m$ while leaving the other α_i unchanged $(i = 1, ..., m-1)$. An easy computation gives

$$\mu^{\alpha_{1m}}(\sigma_1) - \mu^{\bar{\alpha}_{1m}}(\sigma_1) = [1 - \mu^{\alpha_{1m}}(\sigma_1)] \frac{A}{1 - A} \tag{3.16}$$

where

$$A = \tanh(2h \prod_{i=1}^{m} \alpha_i)$$
$$\tanh h = 2p - 1$$

This implies that there is an $\varepsilon(p) > 0$ such that no matter how large m is taken,

$$|\nu(\xi_0 = +1 \,|\, \xi_{-m} = \alpha_{-m}, ..., \xi_{-1} = \alpha_{-1}, \xi_1 = \alpha_1, ..., \xi_m = +1) - \\ \nu(\xi_0 = +1 \,|\, \xi_{-m} = \alpha_{-m}, ..., \xi_{-1} = \alpha_{-1}, \xi_1 = \alpha_1, ..., \xi_m = -1)| > \varepsilon(p)$$

for every configuration $\xi_{-m} = \alpha_{-m}$, ..., $\xi_{m-1} = \alpha_{m-1}$, whenever $p \neq 1/2$. In other words, the conditional probabilities are nowhere quasilocal. $\square$

The measure ν is therefore non-Gibbsian in a very strong sense. The reason why this phenomenon occurs is that the severe constraint (3.13) makes it possible that spins exert an influence on arbitrarily distant spins without that this effect is shielded by others in the intermediate region. There is no decay whatsoever in the corresponding measures (2.17). As a result (see Lemma 2.4) the conditional probability

$$\begin{aligned}
&\nu(\xi_0 = + \mid \xi_{-m}, ..., \xi_{-1}, \xi_1, ..., \xi_m) = \\
&\mu(\sigma_0 = + \mid \sigma_{-m}\sigma_{-m+1} = \xi_{-m}, ..., \sigma_{-1}\sigma_0 = \xi_{-1})\mu(\sigma_1 = + \mid \sigma_1\sigma_2 = \xi_1, ..., \sigma_m\sigma_{m+1} = \xi_m) + \\
&\mu(\sigma_0 = - \mid \sigma_{-m}\sigma_{-m+1} = \xi_{-m}, ..., \sigma_{-1}\sigma_0 = \xi_{-1})\mu(\sigma_1 = - \mid \sigma_1\sigma_2 = \xi_1, ..., \sigma_m\sigma_{m+1} = \xi_m)
\end{aligned}$$

oscillates between two values with non-zero difference uniformly in $m \to \infty$ upon varying $\xi_m \in \{-1, +1\}$, no matter how we fixed $\xi_{-m}, ..., \xi_{-1}, \xi_1, ..., \xi_{m-1}$.

If, however, the constraint is taken to be less severe, the properties of the induced measure improve. Consider therefore a modified transformation on the σ-spins so that $(\sigma_x, x \in \mathbb{Z})$ are mapped into a pair of variables $((\xi_x, \zeta_x), x \in \mathbb{Z})$, both taking the values ± 1:

$$\begin{aligned}
&\zeta_x = -1 \text{ with probability } w,\ 0 < w < 1, \text{ independently} \\
&\xi_x = \begin{cases} \sigma_x & \text{if } \zeta_x = -1 \\ \\ \sigma_x\sigma_{x+1} & \text{if } \zeta_x = +1 \end{cases}
\end{aligned}$$

One can now repeat the computation in the proof of Proposition 3.3 but one easily finds that the dependence of the spin value at the origin on spins far away does not extend beyond the spins in the largest interval containing the origin on which $\zeta = +1$. Formally (and in the sense of (2.4)–(2.7)), the measure ν on the variables (ξ, ζ) is of the form

$$\nu(d\xi, d\zeta) = \exp\left(-H_\zeta(\xi)\right)\lambda_{1/2}(d\xi)\lambda_w(d\zeta) \tag{3.17}$$

where λ_w is the product measure with density w. The Hamiltonian H_ζ is not behaving well uniformly in ζ. $H_\zeta(\xi^x) - H_\zeta(\xi)$ is a function of all those ξ_y for which $\zeta_z = +1$ for z between x and y, and it depends strongly on all these ξ_y. For $\zeta \not\equiv +1$, there is only a finite number of sites in the $+$ cluster of x, thus $H_\zeta(\xi^x) - H_\zeta(\xi)$ is a strictly local function of ξ. However, although the range of the interaction is finite almost surely, it is not bounded uniformly in ζ. The situation is better than in the previous example: although ν is still non-Gibbsian, it is weakly Gibbsian.

Let us conclude this section by pointing out that a stochastic version of this model is behaving even better. The stochastic variant is defined by putting (case (a))

$$t(\xi_x|\sigma_x, \sigma_{x+1}) = \frac{\exp(v\sigma_x\sigma_{x+1}\xi_x)}{2\cosh v} \tag{3.18}$$

with $v > 0$. Then, by following the argument cf. Proposition 3.1 we get

$$q_x^\xi(p) \leq \frac{1}{2}[\tanh(2v + h) + \tanh(2v - h)] < 1 \tag{3.19}$$

with $\tanh h = 2p - 1$, and hence the image measure is Gibbsian. For $v \to \infty$ the argument breaks down; then we have indeed the deterministic version and the resulting measure is not Gibbsian. The example shows how variations of the same transformation can yield genuinely non-Gibbsian, weakly Gibbsian or Gibbsian images.

3.3 Relative energies for image measures

We restrict attention here to the case where the $\{B_x\}$ of (2.9)–(2.10) are non-overlapping boxes. Consider the permutation groups $\mathfrak{S}(S)$ respectively $\mathfrak{S}(S')$. Suppose there is an element $\pi \in \mathfrak{S}(S')$ for which a conjugated permutation $\varphi_x \in \mathfrak{S}(S^{B_x})$ can be found such that the following invariance of updating kernels holds:

$$t_x(\pi\xi_x|\sigma) = t_x(\xi_x|\varphi_x^{-1}\sigma) \tag{3.20}$$

Let $\Delta_x^\pi H(\sigma)$ be the relative energy associated with the modification $\sigma \mapsto \pi_x\sigma$ (see (2.5)).

Proposition 3.4 *Suppose μ is a weakly Gibbsian measure (see (2.6)–(2.7)) having then in particular*

$$\int \mu(d\sigma)g(\varphi_x^{-1}\sigma) = \mu(g\exp(\Delta_x^\varphi H)) \tag{3.21}$$

Then $\nu\circ\pi_x^{-1}$ is absolutely continuous with respect to ν, and the Radon-Nikodým derivative equals

$$\frac{d(\nu\circ\pi_x^{-1})}{d\nu}(\xi) = \mu^\xi(\exp(\Delta_x^\varphi H)) \tag{3.22}$$

ν-almost surely.

Proof: We have

$$\nu(\pi_x^{-1}f) = \mu(T\pi_x^{-1}f) = \int\mu(d\sigma)\int T(d\xi|\sigma)f(\pi_x^{-1}\xi) \tag{3.23}$$

On changing variables by $\pi_x^{-1}\xi \mapsto \xi$ and $\sigma \mapsto \varphi_x\sigma$, we use the hypothesis and the invariance property of the kernels (3.20) to find that

$$\nu(\pi_x^{-1}f) = \mu\times T(\exp(\Delta_x^\varphi H)\times f) \tag{3.24}$$

We conclude the proof by applying the definition (2.13) and using that f is $\mathcal{F}'$-measurable. $\square$

From Proposition 3.4 we learn that ν will be a Gibbs measure if $\mu^\xi(\exp(\Delta_x^\varphi H))$ can be obtained as a continuous function in ξ. Indeed, this is what also appeared from Section 3.1 which was built on Lemma 2.4. We further illustrate that point via the following important example.

Example: One of the problems in non-equilibrium statistical mechanics is to find characterizations of time invariant measures going beyond the mere implicit stationarity condition $\nu T = \nu$, see also [32]. The transformation T (as in case (a)) refers to the transfer operator of e.g. a PCA. In [27, 18] it was investigated how to characterize this stationary ν as being the projection to a hyperplane of its Markov-extension μ_ν under T. μ_ν is then a Gibbs measure on the space-time configurations. The problem is to see what can be said of measures $\nu = \mu P$, resulting as restrictions of Gibbs measures to hyperplanes, i.e.

$$P(d\xi|\sigma) = \prod_{x\in\mathbb{Z}^{d-1}} I(\xi_x = \sigma_{(x,0)})d\xi_x \tag{3.25}$$

with $x \in \mathbb{Z}^{d-1}$, $(x,0) \in \mathbb{Z}^d$, and $d\xi_x$ being the counting measure on S. Although this transformation P is not covered by any one of the cases discussed so far, Proposition 3.4 still applies. Suppose $S = \{-1,+1\}$, and take π corresponding to the spin flip operation $\pi_x f(\sigma) = f(\sigma^x)$, defined by $(\sigma^x)_y = \sigma_y, y \neq x, = -\sigma_x, y = x$. Then it is true that

$$\frac{d(\nu \circ \pi_x)}{d\nu}(\xi) = \mu(\exp[\Delta_x H] \mid \sigma_{(x,0)} = \xi_x) \tag{3.26}$$

where $\Delta_x H(\sigma)$ is the relative Hamiltonian for the Gibbs measure μ under a flip at site x. For PCA this Hamiltonian is directly obtained from the transformation T. In general, $\Delta_x H(\sigma)$ can be decomposed into a sum

$$\Delta_x H = \Delta_x^{\text{upper}} H + \Delta_x^{\text{lower}} H \tag{3.27}$$

where $\Delta_x^{\text{upper}} H$ is the contribution into the relative Hamiltonian containing interactions between spins in the upper half of $\mathbb{Z}^d$, and $\Delta_x^{\text{lower}} H$ is similarly the contribution of spins from the lower half. In the language of probabilistic cellular automata, "upper" refers to the "past" and "lower" refers to the "future", while the conditioning (3.26) is being made on the "present". If μ has the global Markov property with respect to the hyperplane $\{(x,0),\, x \in \mathbb{Z}^d\}$ (which is the case for PCA), then

$$\frac{d(\nu \circ \pi_x)}{d\nu}(\xi) = \mu(\exp[\Delta_x^{\text{upper}} H] \mid \xi)\mu(\exp[\Delta_x^{\text{lower}} H] \mid \xi) \tag{3.28}$$

In other words, in order to prove that ν is Gibbsian, we have to understand the continuity properties of the transformation T (which is trivial for local PCA), and its time reversal with respect to ν (which is far from trivial). In that sense, (2.14) gives the formula $\nu(fTg) = \nu(\mu^\xi(f)f)$ for the time reversal of the PCA T with respect to the stationary measure ν. It is then clear that the continuity (Feller-property) of the time reversal is equivalent with the condition that ν is Gibbsian. This intimate relation between a stationary measure and time reversal is also discussed in [25] and [26]. A proof that the projection ν^+ of the two dimensional Ising phase μ^+ at low temperatures is weakly Gibbsian can be found in [35]. In the above language, this Ising-projection is stationary (and reversible) under the so called transfer matrix (which is however highly non local).

3.4 One-site Kadanoff transformation

Here we consider the plus phase of the standard Ising model. More specifically, for $n \geq 1$ we take the cubes $\Lambda_n = [-n,n]^d \cap \mathbb{Z}^d$ and consider on $\Omega = \{-1,+1\}^{\mathbb{Z}^d}$ the probability measure

$$\mu_n(\sigma) = \frac{1}{Z_n^+(\beta)} \exp(-\beta H_n^+(\sigma)) I(\sigma = +1 \text{ on } \Lambda_n^c) \tag{3.29}$$

for the Hamiltonian

$$H_n^+(\sigma) = -\sum_{\substack{x,y \in \Lambda_n \\ x \sim y}} (\sigma_x \sigma_y - 1) - \sum_{\substack{x \in \Lambda_n, y \in \Lambda_n^c \\ x \sim y}} (\sigma_x - 1) \tag{3.30}$$

and with partition function $Z_n^+(\beta)$. As $n \to \infty$, the weak limit $\mu_n \to \mu^+$ exists and is called the plus phase of the Ising model.

We study the action of the Kadanoff transformation with block size equal to 1. This is a stochastic transformation defined by the probability kernel (see type (a) transformations above)

$$t_x(\xi_x|\sigma) = \frac{\exp(p\xi_x\sigma_x)}{2\cosh p} \tag{3.31}$$

The transformation is often used in image processing analysis (see [4, 14]).

In [14], Section 4.3.3., it is proven that for a given $0 < p < \infty$ there exists a $\beta_0 > 0$ such that for all $\beta > \beta_0$ (small temperatures) the Gibbs measure μ^+ is transformed into a non-Gibbsian measure $\nu^+ = \mu^+ T$ under T. This holds for any dimension $d \geq 2$ (but not for $d = 1$ as we have seen before). Yet, by applying the methods of [3], one can show that ν^+ is weakly Gibbsian with respect to some set K, $\nu^+(K) = 1$, and a potential absolutely summable on K (recall Section 2 around (2.7)). A crucial moment in obtaining this result is to understand why the constrained measure $\mu^{+,\xi}$ (see below) is clustering for $\xi \in K$. More precisely, we will show that there is a set $K \subset \Omega$, $\nu^+(K) = 1$, for which the two-point correlation function $\mu^{+,\xi}(f;g) = \mu^{+,\xi}(fg) - \mu^{+,\xi}(f)\mu^{+,\xi}(g)$ decays exponentially fast with the distance between the dependence sets $\mathcal{D}_f$ and $\mathcal{D}_g$ of local functions f and g as this distance becomes larger than a specific length determined by $\xi \in K$. Here $\mu^{+,\xi}$ is the measure obtained from μ^+ by conditioning on having $\xi \in \Omega$ as image under the Kadanoff transformation; it is the Gibbs measure obtained as the infinite volume limit of

$$\mu_n^{+,\xi}(\sigma) = \frac{1}{Z_n^+(\beta,\xi,p)} \exp(-\beta H_n^+(\sigma) + p \sum_{x\in\Lambda_n} \xi_x\sigma_x) I(\sigma = +1 \text{ on } \Lambda_n^c) \tag{3.32}$$

(see Lemma 2.2). If $\mu^{+,\xi}(f)$ were continuous in ξ for every local function f on Ω, then ν^+ would be Gibbsian. Now suppose that $0 \in \mathcal{D}_f$ and consider the differences $\mu^{+,\xi}(f) - \mu^{+,\bar{\xi}}(f)$ where $\xi = \bar{\xi}$ on Λ_n. As is easy to understand (and as we learn from (3.1)), this difference is governed by correlations $\mu^{+,\xi}(f;g)$. Combining this with (3.22) we see that $\mu^{+,\xi}(f;g)$ with $f = \exp(\Delta_x^\varphi H)$ measures how the relative energy is influenced (with respect to a spin flip at the site x) on changing the configuration far away (monitored by g . The following proposition gives information on the control of these correlations.

Suppose there is a real number $m > 0$ such that

$$\mu^+(\sigma_0\sigma_x) - \mu^+(\sigma_0)\mu^+(\sigma_x) \leq \exp(-2m|x|) \tag{3.33}$$

This is always the case when β is sufficiently large or sufficiently small (and is expected to be true whenever $\beta \neq \beta_c$, i.e. the system is away from the critical point).

Proposition 3.5 *There is a set $K \subset \Omega$, $\nu^+(K) = 1$, such that for every $\xi \in K$ there is $l(\xi) < \infty$ for which*

$$|\mu^{+,\xi}(\sigma_0\sigma_x) - \mu^{+,\xi}(\sigma_0)\mu^{+,\xi}(\sigma_x)| \leq \exp(-m|x|) \tag{3.34}$$

whenever $|x| > l(\xi)$.

Proof: We notice the following facts:

(1) By the FKG inequality

$$\mu^{+,\xi}(\sigma_0\sigma_x) - \mu^{+,\xi}(\sigma_0)\mu^{+,\xi}(\sigma_x) \geq 0 \tag{3.35}$$

(2) Also, we have

$$\begin{aligned} &\nu^+(\mu^{+,\xi}(\sigma_0\sigma_x)) = \mu^+(\sigma_0\sigma_x) \\ &\nu^+(\mu^{+,\xi}(\sigma_0)) = \nu^+(\mu^{+,\xi}(\sigma_x)) = \mu^+(\sigma_0) = \mu^+(\sigma_x) \end{aligned} \tag{3.36}$$

by the definition of $\mu^{+,\xi}$ (see (2.14)).

(3) Since both $\mu^{+,\xi}(\sigma_0)$ and $\mu^{+,\xi}(\sigma_x)$ are non-decreasing functions, we have again by the FKG inequality that

$$\nu^+(\mu^{+,\xi}(\sigma_0)\mu^{+,\xi}(\sigma_x)) \geq \nu^+(\mu^{+,\xi}(\sigma_0))\nu^+(\mu^{+,\xi}(\sigma_x)) \tag{3.37}$$

Hence, by combining observations (1-3) and (3.33), we get

$$\begin{aligned} &\nu^+(\mu^{+,\xi}(\sigma_0\sigma_x) - \mu^{+,\xi}(\sigma_0)\mu^{+,\xi}(\sigma_x)) \leq \\ &\quad \mu^+(\sigma_0\sigma_x) - \mu^+(\sigma_0)\mu^+(\sigma_x) \leq \exp(-2m|x|) \end{aligned} \tag{3.38}$$

The proof is then concluded by a standard application of the Borel-Cantelli lemma. □

At this stage one may start wondering which configurations ξ actually belong to the set K. Certainly the + configuration (i.e., $\xi_x = +1, \forall x$) is an element of K, but we would expect that all configurations with magnetization $1/|V|\sum_{y\in V}\xi_y$ (with $V \in \mathcal{E}$ large) close enough to the spontaneous magnetization $\mu^+(\sigma_0)$ are also "good" configurations. In order to make this more specific and to link it more directly to the relative energies we consider

$$E^+(x,\eta) = \log \frac{\nu^+(\xi_x = \eta_x \mid \xi_y = \eta_y, y \neq x)}{\nu^+(\xi_x = +1 \mid \xi_y = \eta_y, y \neq x)} \tag{3.39}$$

We want to estimate differences like $E^+(x,\eta) - E^+(x,\bar\eta)$, where we assume that $\eta = \bar\eta$ on a large set centered around x.

For simplicity we restrict the discussion to $d = 2$. Take a sequence of volumes $D_k \subset \mathbb{Z}^2$ constructed iteratively by adding one of the sites x_k nearest to the origin but not yet contained in D_{k-1}, $D_k = D_{k-1} \cup x_k$, $x_k \in \mathbb{Z}^2$, with the properties that $|x_k| > |x_{k-1}|$, $x_1 = 0$, $D_0 = \emptyset$, $D_1 = \{0\}$. Also, we write $D_k(x) = D_k + x$. For a connected set V we denote by $\bar V$ the union of V with the finite components of its complement.

Proposition 3.6 *(Take β large) Fix $x \in \mathbb{Z}^2$, $l < \infty$ and suppose that $\xi \in \Omega$ is chosen in such a way that for all $k > l$ and all connected sets V, $\bar V \ni x$, $|V| \geq k$*

$$\frac{1}{|V|}\sum_{y\in V}\xi_y \geq \frac{1}{2} \tag{3.40}$$

holds. Then

$$|E^+(x,\xi_{D_k(x)}) - E^+(x,\xi)| \leq \mathit{const}(\beta,p)\exp(-m(p)\,\mathrm{diam}\, D_k) \tag{3.41}$$

whenever $\mathrm{diam}\, D_k > l$, *and $m(p) \to \infty$ as $p \to \infty$ (*$\mathrm{diam}\, D_k \sim \sqrt{k}$ *is the diameter of the set D_k, and $\xi_{D_k(x)} = {}^{D_k^c(x)}\xi$).*

Proof: Translation invariance allows us to restrict to $x=0$ and first study the object

$$h_n^+(\xi) = E_n^+(0,\xi) = \log \frac{\nu_n^+(\xi)}{\nu_n^+({}^0\xi)} \tag{3.42}$$

We have that

$$h_n^+(\xi) = \log \frac{Z_n^{+,\xi}}{Z_n^{+,{}^0\xi}} \tag{3.43}$$

$$Z_n^{+,\xi} := \sum_{\sigma\in\{-1,+1\}^{\Lambda_n}} \exp(-H_n^{+,\xi}(\sigma)) \tag{3.44}$$

$$H_n^{+,\xi}(\sigma) := -\beta \sum_{\substack{x,y\in\Lambda_n\\ x\sim y}} (\sigma_x\sigma_y - 1) - \beta \sum_{\substack{x\in\Lambda_n, y\in\Lambda_n^c\\ x\sim y}} (\sigma_x - 1)$$

$$-p\sum_{x\in\Lambda_n} \xi_x(\sigma_x - 1) - p\sum_{x\in\Lambda_n} \xi_x \tag{3.45}$$

For a shorthand we put $\xi_k = {}^{D_k^c(0)}\xi$ (this is not to be confused with the value of ξ at a site). We define for $k \geq 1$

$$\Phi_k^n(\xi) := h_n^+(\xi_k) - h_n^+(\xi_{k-1}) \tag{3.46}$$

More explicitly, for all $k > 1$

$$\Phi_k^n(\xi) = \log \frac{Z_n^{+,\xi_k}(\beta) Z_n^{+,{}^0\xi_{k-1}}(\beta)}{Z_n^{+,{}^0\xi_k}(\beta) Z_n^{+,\xi_{k-1}}(\beta)} \tag{3.47}$$

Note that this expression is zero whenever $\xi_0 = +1$ or $\xi_{x_k} = +1$ and that the ratio of partition functions can be rewritten in terms of correlation functions to get

$$\Phi_k^n(\xi) = \frac{1-\xi_0}{2}\frac{1-\xi_{x_k}}{2} \log \frac{\mu_n^{+,\xi_k}(\exp(2p\sigma_0)\exp(2p\sigma_{x_k}))}{\mu_n^{+,\xi_k}(\exp(2p\sigma_0)\mu_n^{+,\xi_k}(\exp(2p\sigma_{x_k})))} \tag{3.48}$$

where $\mu_n^{+,\xi}$ is the Gibbs state for $H_n^{+,\xi}$.

Now, by monotonicity (FKG inequality), $\lim_n \Phi_k^n(\xi) = \Phi_k(\xi)$ exists for all $\xi \in \Omega$. Moreover, as we will show, for p sufficiently large, we have uniformly as $n \to \infty$ and uniformly in β (large enough) that

$$|\Phi_k^n(\xi)| \leq C(\beta,p)\ \exp(-m(p)\,\mathrm{diam}\, D_k) \tag{3.49}$$

for ξ and k satisfying the hypotheses of the proposition. From (3.49) we obtain that

$$\sum_{k=1}^{(2n+1)^2} |\Phi_k^n(\xi)| \leq c(\xi) < \infty \tag{3.50}$$

and since

$$\sum_{k=0}^{2n+1} \Phi_k^n(\xi) = \sum_{k=0}^{\infty} \Phi_k^n(\xi) \tag{3.51}$$

(the extra terms on the right-hand side being identically zero), we can then use dominated convergence to get

$$\lim_n \sum_{k=0}^{2n+1} \Phi_k^n(\xi) = \sum_{k=0}^{\infty} \Phi_k(\xi) = \lim_n h_n^+(\xi) = h^+(\xi) \tag{3.52}$$

We can now conclude that

$$\begin{aligned} |E^+(0,\xi_k) - E^+(0,\xi)| &= |h^+(\xi_k) - h^+(\xi)| \\ &= |\sum_{i=k+1}^{\infty} \Phi_i(\xi)| \\ &\leq \sum_{i=k+1}^{\infty} |\Phi_i(\xi)| \\ &\leq \text{const}(\beta,p)\ \exp(-m(p)\,\text{diam}\, D_k) \end{aligned} \tag{3.53}$$

as required. □

For the rest it remains to prove (3.49). This is a consequence of the following

Proposition 3.7 *Take $f_x(\sigma) = \exp(2\beta\sigma(x))$. If p is large enough, then there exists $m = m(p) > 0$ ($m(p) \to \infty$ as $p \to \infty$), such that if ξ satisfies 3.40 for $x = 0$ and $l \leq \text{diam}\, D_k \leq 2n+1$, then*

$$|\mu_n^{+,\xi}(f_0 f_{x_k}) - \mu_n^{+,\xi}(f_0)\mu_n^{+,\xi}(f_{x_k})| \leq \text{const}(\beta,p)\ \exp(-m(p)\,\text{diam}\, D_k) \tag{3.54}$$

For proving Proposition 3.7 we use the strategy outlined in [5] and we need the following

Lemma 3.1 *Under the same conditions as before, there exists $u = u(p)$ ($u \to \infty$ as $p \to \infty$), such that for all $r \geq l$*

$$\mu_n^{+,\xi} \times \mu_n^{+,\xi}[0 \rightsquigarrow \Lambda_r^c] \leq \text{const}(p)\exp(-u(p)r) \tag{3.55}$$

Here the notation $A \rightsquigarrow B$ corresponds to the event of having a path of disagreement connecting the regions A and B. By a path of disagreement between A and B we mean a sequence $x_1 \in A$, $x_2, ..., x_n \in B$ of consecutive nearest neighbour sites for which $(\sigma_{x_j}, {\sigma'}_{x_j}) \neq (+,+)$, for all $j = 1, ..., n$.

Proof: Following [5] we call a finite set $A \subset \mathbb{Z}^2$ enclosing if A and A^c are connected sets. Recall that for a connected set B we denote $\bar{B}$ for the union of B with the finite components of its complement. $\bar{B}$ is always enclosing. There is a one-to-one correspondence between enclosing sets and contours as they are usually defined for the Ising model (see e.g. [39]). Each contour borders an enclosing set, i.e. for each contour γ there is an enclosing set A such that $A = \text{Int}\gamma$ (the interior of the contour). We use the notation $\partial_i\Lambda = \{x \in \Lambda : \exists y \in \Lambda^c,\ x \sim y\}$ for the inner boundary of Λ.

Denote by $C(\sigma, \sigma')$ the maximal connected set containing 0 on which $(\sigma, \sigma') \neq (+,+)$. Denote by $\mathcal{K}(0)$ all enclosing sets containing the origin. Let $C \in \mathcal{K}(0)$.

$$\begin{aligned}
&\mu_n^{+,\xi} \times \mu_n^{+,\xi}(\bar{C}(\sigma, \sigma') = C) \leq \\
&\sum_{R \subset \partial_i C} \mu_n^{+,\xi} \times \mu_n^{+,\xi}(\sigma \equiv - \text{on } R,\ \sigma \equiv + \text{on } \partial C,\ \sigma' \equiv - \text{on } \partial_i C \setminus R,\ \sigma' \equiv + \text{on } \partial C) \\
&\sum_{R \subset \partial_i C} \mu_n^{+,\xi}(\sigma \equiv - \text{on } R,\ \sigma \equiv + \text{on } \partial C)\ \mu_n^{+,\xi}(\sigma' \equiv - \text{on } \partial_i C \setminus R,\ \sigma' \equiv + \text{on } \partial C)
\end{aligned} \tag{3.56}$$

We now need the following lemma that we will prove afterwards.

Lemma 3.2 *Let $C \in \mathcal{K}(0)$ and $R \subset \partial_i C$. If $\xi \in \Omega_l^+$ and $\partial_i C$ contains at least $r \geq l$ sites, then there exists $\delta = \delta(p)$ ($\delta \to \infty$ as $p \to \infty$), such that*

$$\begin{aligned}
&\mu_n^{+,\xi}(\sigma \equiv - \textit{ on } R,\ \sigma \equiv + \textit{ on } \partial C)\ \mu_n^{+,\xi}(\sigma' \equiv - \textit{ on } \partial_i C \setminus R,\ \sigma' \equiv + \textit{ on } \partial C) \\
&\leq \textit{const}(p)\ \exp(-\delta(p)|\partial_i C|)
\end{aligned} \tag{3.57}$$

If we require that there is a path from 0 to Λ_r^c, then indeed $|\partial_i \bar{C}(\sigma, \sigma')| > r$. By Lemma 3.2 we have that for $\xi \in \Omega_l^+$

$$\begin{aligned}
\mu_n^{+,\xi} \times \mu_n^{+,\xi}(\bar{C}(\sigma, \sigma') = C) &\leq 2^{|\partial_i C|} \text{const}(p)\ \exp(-\delta(p)|\partial_i C|) \\
&\leq \text{const}(p)\ \exp(-m(p)|\partial_i C|)
\end{aligned} \tag{3.58}$$

and $m(p) \to \infty$ when $p \to \infty$.

We can now proceed as follows. Let W_s denote the number of enclosing sets bordered by a contour of length s and containing 0. By the isoperimetric inequality on the square lattice we have that $W_s \leq (s^2/16)4^s$. Hence

$$\begin{aligned}
\mu_n^{+,\xi} \times \mu_n^{+,\xi}[0 \rightsquigarrow \Lambda_r^c] &= \mu_n^{+,\xi} \times \mu_n^{+,\xi}[C(\sigma, \sigma') \cap \Lambda_r^c \neq \emptyset] \\
&\leq \sum_{s=r}^{\infty} \sum_{\substack{C \in \mathcal{K}(0) \\ |\partial_i C| = s}} \mu_n^{+,\xi} \times \mu_n^{+,\xi}[\bar{C}(\sigma, \sigma') = C] \\
&\leq \text{const}(p) \sum_{s=r}^{\infty} \exp(-m(p)s) \frac{s^2}{16} 4^s \\
&\leq \text{const}(p)\ \exp(-um(p))
\end{aligned} \tag{3.59}$$

for some $u > 0$. $\square$

Proof of Lemma 3.2: For any enclosing set $V \subset \mathbb{Z}^2$ we put

$$\begin{aligned}
\tilde{H_V}^{+,\xi}(\sigma) &:= -\beta \sum_{\substack{x,y\in V\\ x\sim y}} (\sigma_x\sigma_y - 1) - \beta \sum_{\substack{x\in V, y\in V^c\\ x\sim y}} (\sigma_x - 1) - p\sum_{x\in V} \xi_x(\sigma_x - 1)\\
\tilde{H_V}^{-,\xi}(\sigma) &:= -\beta \sum_{\substack{x,y\in V\\ x\sim y}} (\sigma_x\sigma_y - 1) - \beta \sum_{\substack{x\in V, y\in V^c\\ x\sim y}} (-\sigma_x - 1) - p\sum_{x\in V} \xi_x(\sigma_x + 1)\\
\tilde{Z}^{+,\xi}(V) &:= \sum_{\sigma_x; x\in V} \exp(-\tilde{H_V}^{+,\xi}(\sigma))\\
\tilde{Z}^{-,\xi}(V) &:= \sum_{\sigma_x; x\in V} \exp(-\tilde{H_V}^{-,\xi}(\sigma))
\end{aligned} \tag{3.60}$$

and observe that

$$\frac{\tilde{Z}^{-,\xi}(V)}{\tilde{Z}^{+,\xi}(V)} \leq \exp(2\beta|\partial V|)\exp(2p\sum_{x\in V}\xi_x) \tag{3.61}$$

We have that

$$\begin{aligned}
&\mu_n^{+,\xi}(\sigma \equiv - \text{ on } R,\ \sigma \equiv + \text{on } \partial C)\, \mu_n^{+,\xi}(\sigma' \equiv - \text{ on } \partial_i C \setminus R,\ \sigma' \equiv + \text{ on } \partial C) \leq\\
&\mu_n^{+,\xi}(\sigma \equiv - \text{ on } R \mid \sigma \equiv + \text{on } \partial C)\, \mu_n^{+,\xi}(\sigma' \equiv - \text{ on } \partial_i C \setminus R \mid \sigma' \equiv + \text{ on } \partial C) =\\
&\mu_C^{+,\xi}(\sigma \equiv - \text{ on } R)\, \mu_C^{+,\xi}(\sigma' \equiv - \text{ on } \partial_i C \setminus R) \leq\\
&\sum_{\substack{\{\gamma\},\{\gamma'\} \text{ inside } C\\ \{\gamma\} \text{ compatible with } R\\ \{\gamma'\} \text{ compatible with } \partial_i C\setminus R}} \prod_{\gamma} w_\xi(\gamma) \prod_{\gamma'} w_\xi(\gamma')
\end{aligned} \tag{3.62}$$

The set $\{\gamma\}$ of contours is compatible with R if it is compatible with $\sigma \equiv -$ on R, $R \subset \cup_\gamma \operatorname{Int}\gamma$ and $R \cap \operatorname{Int}\gamma \neq \emptyset$ for each γ. The weight $w_\xi(\gamma)$ of a contour γ is given by

$$w_\xi(\gamma) := \exp(-2\beta|\gamma| - 2p\sum_{x\in \operatorname{Int}\gamma} \xi_x)\frac{\tilde{Z}^{-,\xi}(\operatorname{Int}\gamma \setminus \partial_i \operatorname{Int}\gamma)}{\tilde{Z}^{+,\xi}(\operatorname{Int}\gamma \setminus \partial_i \operatorname{Int}\gamma)} \tag{3.63}$$

By using (3.61) we see that

$$w_\xi(\gamma) \leq \exp(-2p\sum_{x\in \partial_i \operatorname{Int}\gamma} \xi_x) \tag{3.64}$$

Now regard the sets $\{\gamma\}$, $\{\gamma'\}$ in (3.62) as parts of a single set of contours $\{\tilde\gamma\} = \{\gamma\}\cup\{\gamma'\}$. The contours $\tilde\gamma$ can overlap, but a site of $\partial_i C$ can be

inside of at most two contours. Moreover, every site of $\partial_i C$ is in the interior of at least one contour of $\{\tilde\gamma\}$ and $|\partial_i C| > r$.

Let $V := \cup_{\tilde\gamma}\partial_i \operatorname{Int}\tilde\gamma$. V is a connected set such that $\bar{V} \ni 0$ and $|V| > r$. Since $\xi \in \Omega_l^+$, $r \geq l$,

$$\sum_{\tilde\gamma}\sum_{x\in\partial_i \operatorname{Int}\tilde\gamma} \xi_x \geq \frac{1}{4}\sum_{\tilde\gamma} |\partial_i \operatorname{Int}\tilde\gamma| \tag{3.65}$$

This takes into account the worst situation where all $-$ spins are counted twice and all $+$ spins only once. Hence in (3.62)

$$\prod_{\gamma} w_\xi(\gamma) \prod_{\gamma'} w_\xi(\gamma') \le \prod_{\gamma} e^{-\frac{1}{2}p|\partial_i \operatorname{Int}\gamma|} \prod_{\gamma'} e^{-\frac{1}{2}p|\partial_i \operatorname{Int}\gamma'|} \tag{3.66}$$

Now we can proceed as in Lemma 2.5 of [5]. Enumerate the set R as $\{x_1, ..., x_q\}$. Each x_i, $i = 1, ..., q$, is enclosed by a contour of the set $\{\gamma\}$. The set $\{\gamma\}$ contains at most q contours. Suppose $\{\gamma\}$ has s contours; we denote them by $\gamma_{i_1}, ..., \gamma_{i_s}$, where x_{i_j} is the point of R with the smallest index, enclosed by γ_{i_j}. Put $L_{i_j} = |\gamma_{i_j}|$ for $j = 1, ..., s$, and $L_k = 0$ if $k \in \{1, ..., q\}$ but $k \ne L_{i_1}, ..., L_{i_s}$. Let $L = \sum_\gamma |\gamma|$. Then $L_1, ..., L_q$ gives a partition of L into q nonnegative numbers. It is well known that the number of such partitions is smaller than $K\, 4^L$ for some constant K. On the other hand, the number of contours crossing a fixed point x and having length l is bounded by $l\, 4^l \le 8^l$. Therefore

$$\sum_{\substack{\{\gamma\},\{\gamma'\}\text{ inside } C \\ \{\gamma\}\text{ compatible with } R \\ \{\gamma'\}\text{ compatible with } \partial_i C \setminus R}} \prod_{\gamma} w_\xi(\gamma) \prod_{\gamma'} w_\xi(\gamma') \le$$

$$\sum_{r \ge |R|} K\, 32^r e^{-\frac{1}{2}pr} \sum_{s \ge |\partial_i C \setminus R|} K\, 32^s e^{-\frac{1}{2}ps} \;\le\; \text{const}(p)\, \exp(-\delta(p)|\partial_i C|) \tag{3.67}$$

with a suitable $\delta(p) > 0$. $\square$

We can now turn to prove Proposition 3.7.

Proof of Proposition 3.7: For any configuration $\xi \in \Omega$ we have that

$$\begin{aligned} & |\mu_n^{+,\xi}(f_0 f_{x_k}) - \mu_n^{+,\xi}(f_0)\mu_n^{+,\xi}(f_{x_k})| \\ &= |\mu_n^{+,\xi}\left(\mu_n^{+,\xi}(f_{x_k} \mid \sigma = \eta \text{ on } \partial D_k)\, [\mu_{D_k}^{+,\xi,\eta}(f_0) - \mu_n^{+,\xi}(f_0)]\right)| \\ &\le e^{2\beta}\, \mu_n^{+,\xi}(|\mu_{D_k}^{+,\xi,\eta}(f_0) - \mu_n^{+,\xi}(f_0)|) \\ &= e^{2\beta}\mu_n^{+,\xi}(|\mu_{D_k}^{+,\xi,\eta} \times \mu_n^{+,\xi}(f_0 \times \mathbb{1} - \mathbb{1} \times f_0)|) \end{aligned} \tag{3.68}$$

The rest of the argument is similar to the ideas in [1, 2]. Pick a configuration (σ, σ') from the distribution $\mu_{D_k}^{+,\xi,\eta} \times \mu_n^{+,\xi}$. If there is no path from 0 to D_k^c on which $(\sigma, \sigma') \not\equiv (+,+)$, then there exists a $*$-chain around 0 separating it from D_k^c and on which $\sigma \equiv \sigma' \equiv +$. It follows that there exists a maximal $*$-chain Δ with this property inside D_k (maximal in the sense that it is contained in no other $*$-chain). Therefore

$$\mu_{D_k}^{+,\xi,\eta} \times \mu_n^{+,\xi}(f_0 \times \mathbb{1} - \mathbb{1} \times f_0) = \tag{3.69}$$

$$\sum_{\substack{*\text{-chains}\,\Delta \\ \text{around}\, 0}} \mu_{D_k}^{+,\xi,\eta} \times \mu_n^{+,\xi}(f_0 \times \mathbb{1} - \mathbb{1} \times f_0 \mid \Delta)\, \mu_{D_k}^{+,\xi,\eta} \times \mu_n^{+,\xi}(\Delta) +$$

$$\mu_{D_k}^{+,\xi,\eta} \times \mu_n^{+,\xi}(f_0 \times \mathbb{1} - \mathbb{1} \times f_0 \mid 0 \rightsquigarrow D_k^c)\, \mu_{D_k}^{+,\xi,\eta} \times \mu_n^{+,\xi}[0 \rightsquigarrow D_k^c]$$

Now, since $\sigma = \sigma'$ on Δ

$$\mu_{D_k}^{+,\xi,\eta} \times \mu_n^{+,\xi}(f_0 \times \mathbb{1} - \mathbb{1} \times f_0 \mid \Delta) = 0 \tag{3.70}$$

for every Δ, we obtain

$$|\mu_{D_k}^{+,\xi,\eta} \times \mu_n^{+,\xi}(f_0 \times \mathbb{1} - \mathbb{1} \times f_0)| \leq 2e^{2\beta}\mu_{D_k}^{+,\xi,\eta} \times \mu_n^{+,\xi}[0 \rightsquigarrow D_k^c] \tag{3.71}$$

Lemma 3.1 and (3.68), (3.71) then yield

$$\begin{aligned} |\mu_n^{+,\xi_k}(f_0 f_{x_k}) - \mu_n^{+,\xi_k}(f_0)\mu_n^{+,\xi_k}(f_{x_k})| &\leq e^{2\beta}\mu_n^{+,\xi_k}[|\mu_{D_k}^{+,\xi_k,\eta} \times \mu_n^{+,\xi_k}(f_0 \times \mathbb{1} - \mathbb{1} \times f_0)|] \\ &\leq 2e^{4\beta}\mu_n^{+,\xi_k} \times \mu_n^{+,\xi_k}[0 \rightsquigarrow \Lambda_k^c] \\ &\leq \mathrm{const}(\beta, p)\exp(-m(p)\,\mathrm{diam}\, D_k) \end{aligned} \tag{3.72}$$

□

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