TRANSPORT PROPERTIES OF MARKOVIAN ANDERSON MODEL

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ABSTRACT. We consider the Anderson model in $l^2(\mathbf{Z}^d)$, $d \geq 1$, with potentials whose values at any site of the lattice are Markovian independent random functions of time. The upper and lower bounds for the moments $|X|^p(t,\omega)$ with probability= 1 are obtained. We obtain also upper and lower bounds for the averaged diffusion constant and upper bounds for the correlation function. The results present diffusive behaviour in dimensions $d = 3D1, 2$ up to logarithmic factors.

1 Introduction

Since 50-th the Anderson model was one of the basic one for studying the transport phenomena in random environment:

$$i\frac{\partial\psi}{\partial t} = 3D - \Delta\psi + Q_\omega(n)\psi, \quad \psi|_{t=3D0} = 3D\psi_0(n), \quad \psi_0 \in l^2(\mathbf{Z}^d),$$

where the values of potential Q are random i.i.d. variables. Much less attention was devoted to the models with random potentials depending on time. There are physical motivations to consider such models (see [1] - [3]), but there are almost no mathematical results (see, however, [4] for referencies). In [4] the Markovian time-dependent version of Anderson model has been considered. Some result= s about the behaviour of solutions as t tends to infinity have been prove= n. In the present paper we shall study the transport properties of this mode= l.

Before describing the model and discussing the results we shall say some words about the mathematical problems related to the "classic" Anderson model. When considering the random Schrödinger operators, one can obtain in general two kind of results:

1. Spectral properties of random Schrödinger operators $H_\omega = 3D - \Delta + Q_\omega$ with probability 1. In particular, nature of the spectrum and properties of eigenfunctions are of interest.

2. Dynamical behaviour of solutions $\psi(t) = 3D \exp(-itH_\omega) psi_0$. Most interesting is the behaviour of the mean square displacement

$$X^2(t) = 3D \sum_{n\in\mathbf{Z}^d} |n|^2 |\psi(t,n)|^2$$

as $t \to \infty$, which determines the transport properties of the model. If in some sense $X^2(t) \sim t^{2\alpha}$, then $\alpha \in [0,1]$ is called diffusion exponent. If $X^2(t) \le C$, one speaks of dynamical localization.

As to the spectral properties of random Schrödinger operators, this is = the field which was most intensively explored by the mathematicians during la= st 20 years. The main result called mathematical localization is the pure po= int nature of the spectrum and the exponentional decay of eigenfunctions at infinity. This result was proved in great generality in dimension $d = 3D1$ and in dimension $d \ge 2$ for big disorder or low energy. One exp= ects that the continuous (absolutely continuous?) spectrum may exist in dimens= ion $d \ge 3$, however, there are still no mathematical prove of it.

The spectral properties of self-adjoint operator H are closely related = with the behaviour in time of solutions $\psi(t) = 3D \exp(-itH)\psi_0$. The = classic result is given by the so-called RAGE-Theorem [5]. If $\psi_0 \in \mathcal{H}_{pp}$, where $\mathcal{H}_{pp}$ is the subspace of pure point spect= rum of H, one has

$$\lim_{R\to+\infty} \sup_t \|F(|n| \ge R)\psi(t)\| = 3D0, \tag{1.1}$$

where F is a caracteristic function of the outside of the ball. The solutions $\psi(t)$ which satisfy (1.1) are called bound states.

If $\psi_0 \in \mathcal{H}_c$, where $\mathcal{H}_c$ is the continuous subspace of H, then for any $R > 0$ one has

$$\lim_{T\to+\infty} T^{-1} \int_0^T \|F(|n| \le R)\psi(t)\|^2 dt = 3D0. \tag{1.2}$$

The solutions $\psi(t)$ which verify (1.2) are called propagating states. The definitions (1.1), (1.2) are valid also in the case of time-dependent Hamiltonians [6].

The relations between spectral properties of H and the dynamical behavi= our of solutions $\psi(t)$ became less obvious when one considers the quantities like $X^2(t)$. When the spectrum of H is pure point, most that one can claim is the absence of ballistic motion [7]: $\lim_{t\to+\infty} X^2(t)/t^2 = 3D0$. The example of [8] shows = that even the exponential decay of eigenfunctions does not garantee the dynamical localization, so the mathematical localization does not imply dynamical one (the inverse being true). To prove the dynamical localizati= on, one needs some additional information on the spacial behaviour of eigenfunctions [8], [9] or some bounds for expectations of the resolvent [10], [8]. For Anderson model these conditions were verified in some situations. One can expect that in the near future the dynamical localization will be established in most cases when one has mathematical one.

There remains still two fundamental mathematical problems for the Anderson model in dimensions $d \ge 2$ which are not resolved.

1. The existence of the continuous spectrum and its properties.

2. The possible diffusive behaviour of solutions $\psi(t)$.

One should say that in the last 10 years one has proved many results on the relations between properties of continuous spectrum of abstract self-adjoint operators and the dynamical behaviour of solutions (see, for instance, [11] for the review of recent papers). These results, however, are rather of theoretical nature. To apply them in concrete cases (in particular, for Anderson model), one should first establish the existence and the properties of continuous spectrum, which is a very difficult problem. Generally speaking, there are only few quantum models of random media for which the existense of propagating states with nontrivial diffusion exponent $\alpha \in (0,1)$ has been mathematically proved. The Markovian Anderson model we present below is one of these examples, the diffusive behaviour in time of solutions being a generic one.

Before describing the model, let us make the following general observatio= n concerning Schrödinger equations with random time-dependent potentials.= For such models the behaviour of solutions $\psi(t)$ to the time-dependent equation $i\psi_t = 3DH(t)\psi$ is not related with the spectral propert= ies of operators $H(t)$ even in the sense of (1.1), (1.2). It follo= ws from the results of [4] for $d = 3D1$, where it was shown that with probability 1 all the states are propagating ones. At the same time it is well known that the spectrum of operators $H(t)$ for any t is pure point. Therefore, to study the dynamical behaviour of solutions $\psi(t)$, one should do it directly which differs considerably from the spectral methods used in the case of static potentials. On the other hand, the results we obtain for $\psi(t)$ do not give information about spectral properties of operators $H(t)$ for a fixed t. For instanse, we shall prove the existense of diffusion in dimension $d = 3D2$ (up to logarithmic factor= s), but this gives nothing about the spectrum of two-dimensional static Anderson model.

Let us describe now the model we shall consider. Let $V : E \to \mathbf{R}$ be some real bounded continuous function on the topological space E with Borel σ-algebra $\mathcal{B}$.

Let $u(t,\omega)$ be a stationary Markov process with values on E:

$$u \; : \; \mathbf{R} \times \Omega \to E,$$

where $(\Omega, \mathcal{A}, P)$ is the probability space with probability measure P. Let

$$\mathbf{\Omega} = 3D\Omega^{(\mathbf{Z}^d)}$$

with corresponding product probability measure $\mathbf{P}$ and expectation $\mathbf{E}$, the elements of $\mathbf{\Omega}$ being $\omega = 3D\{\omega_n : \omega_n \in \Omega, \; n \in \mathbf{Z}^d\}$.

One considers the time-dependent Schrödinger equation on the latti= ce $\mathbf{Z}^d$:

$$i\frac{\partial\psi(t,n)}{\partial t} = 3DH\psi(t,n) + V(u(t,\omega_n))\psi(t,n), \quad \psi|_{t=3D0} = 3D\psi_0(n), \quad t \geq 0. \qquad (1.3)$$

Here $\psi_0 \in l^2(\mathbf{Z}^d), \; H = 3D - \Delta + Q_0(n)$, Δ is the discrete Laplacian:

$$(-\Delta f)(n) = 3D \sum_{p:|p|=3D1} f(n+p),$$

$Q_0 : \mathbf{Z}^d \to \mathbf{R}$ is some nonrandom bounded function.

We shall assume that the Markov process $u(t,\omega)$ satisfies the following conditions:

1. The paths $u(\cdot,\omega)$ are P-a.s. right continuous and have a finite number of discontinuities on any compact time interval. =09

2. We assume the existence of a unique invariant measure μ on $(E,\mathcal{B})$ which gives the stationary distribution of the Markov process u. =09

3. Let B be the generator of the Markov process. One assumes that B is a self-adjoint nonnegative operator in $L^2(E,\mathcal{B},\mu)$ with discrete spectrum: $0 = 3D\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$, the only eigenfunction corresponding to $\lambda = 3D0$ being $e_0(u) \equiv 1, \quad u \in E$.

The last condition on the Markov process implies the exponentional decay = in time of correlations of potential [4]. Finally, we suppose that $V : E \to \mathbf{R}$ is bounded continuous function such that

$$\int_E V(u)d\mu(u) = 3D0, \quad \int_E V^2(u)d\mu(u) > 0.$$

One can note that for any fixed t the expression $V(u(t,\omega_n))$ g= ives some realisation of Anderson potential. Therefore we can consider our mod= el as dynamical generalisation of the usual Anderson model.

Let $\psi(t,n,\omega)$ be solutions to the equation (1.3). The main results showed in [4] is the following (Theorem 1): with probability 1 for any $\psi_0 \in l^2(\mathbf{Z}^d)$

$$\lim_{T\to+\infty} T^{-1}\int_0^T \|F(|n| \geq a(t))\psi(t,n)\|^2 dt = 3D0, \tag{1.4}$$

$$\lim_{T\to+\infty} T^{-1}\int_0^T \|F(|n| \leq b(t))\psi(t,n)\|^2 dt = 3D0, \tag{1.5}$$

where $b(t) = 3Dt^\beta, \ \beta < 1/2$ for $d = 3D1, 2$ and $\beta < 1/d$ for $= d \geq 3, \quad a(t) = 3Dt^\alpha, \ \alpha > 1/2$ for any d.

This means, in particular, that all the states are propagating ones (1.2). However, the equalities (1.4), (1.5) give much more information about the behaviour of solutions, namely, that the quantum particle spends most time (in average) in the region $n : a(t) < |n| < b(t)$. We shall call (1.4) and (1.5) RAGE-like results for the outside and inside occupation time respectively.

As it was stressed in [4], (1.4) and (1.5) do not determine directly the behaviour of $X^2(t)$ as $t \to \infty$. In the present paper we shall study the behaviour of the time averages moments

$$\langle |X|^p \rangle_{T,\omega} = 3DT^{-1}\int_0^T \sum_n |n|^p |\psi(t,n,\omega)|^= 2dt, \quad p > 0.$$

At the same time, we shall considerably improve (1.4) and (1.5)= , especially in dimensions $d \geq 3$.

Let us expose now the main results of the paper. First, we ameliorate the RAGE-like results (1.4), (1.5) for the outside and inside occupation time. Namely, we show (Theorems

3.10 and 5.5) that (1.4) and (1.5) are true with probability 1 for any $\psi_0 \in l^2(\mathbf{Z}^d)$ with the following choice of $a(t), b(t)$:

$$a(t) = 3Dt^{\frac{1}{2}} \log(t+1)^{\nu},$$

where $\nu > 0$ for $d = 3D1, 2$ and $\nu > 1$ for $d \geq 3$,

$$b(t) = 3Dt^{\frac{1}{2}} (\log(t+2))^{-r},$$

where $r > 1,\ d = 3D1;\quad r > 3/4,\ d = 3D2$, and

$$b(t) = 3Dt^{\frac{2}{d+2}} (\log(t+2))^{-r}, \quad r > \frac{2}{d+2},\ d \geq= 3.$$

The second group of results concerns the behaviour of the time averaged moments. We show (Theorem 5.8) that for any $\psi_0 \in l^2(\mathbf{Z}^d)$ with probability 1 for T big enough

$$\langle |X|^p \rangle_{T,\omega} \geq \frac{1}{2} \|\psi_0\|^2 b^p(T),$$

where $b(T)$ was defined above. We show also (Theorem 3.9) that fo= r any ψ_0 exponentially decreasing at infinity

$$\langle |X|^p \rangle_{T,\omega} \leq C_{r,p}(\psi_0, \omega =) T^{\frac{p}{2}} (\log T)^r,$$

where $r > 1$ for $d = 3D1, 2$ and $r > p+1$ for $d \geq 3$.

The third group of results gives the upper and lower bounds for the averaged diffusion constant

$$D(z) = 3D\mathbf{E}[z^2 \int_0^{+\infty} dt \exp(-tz) |X|^2(t, \cdot)], \quad \psi_0 \neq 0.$$

We show (Corollaries 3.5 and 5.4) the following uniform in $z \in (0, 1/3]$ estimates with some positive constants $C_1, C =_2$:

$$0 < C_1 \leq D(z) \leq C_2 < +\infty,\ d = 3D1,$$

$$C_{1,r} |\log z|^{-r} \leq D(z) \leq C_2 (\log|\log z|)^2, \quad r > 1/2,\ d = 3D2,$$

$$C_1 z^{\frac{d-2}{d+2}} \leq D(z) \leq C_2 (\log z)^2, \quad d \geq 3.$$

Finally, we obtain upper bounds for the correlation function

$$C_{\psi_0, \psi_1}(T, \omega) = 3DT^{-1} \int_0^T dt |(\psi(t, \omega), \psi_1)|^2.$$

We show (Theorem 5.9) that for any $\psi_0, \psi_1 \in l^2(\mathbf{Z}^d)$ w= ith probability 1 the uniform in $T \geq 3$ estimates hold:

$$C(T, \omega) \leq C_r T^{-\frac{1}{2}} (\log T)^r, \quad r > 1, \quad d = 3D1,$$

$$C(T,\omega) \le C_r T^{-1}(\log T)^r, \quad r > 1, \quad d = 3D2,$$
$$C(T,\omega) \le CT^{-1}, \quad d \ge 3.$$

That gives us the following result for the autocorrelation function $C_{\psi_0,\psi_0}(T,\omega)$:

$$\limsup_{T\to+\infty} \frac{\log C(T,\omega)}{\log T} \le -\frac{1}{2}, \quad d = 3D1;$$

$$\lim_{T\to+\infty} \frac{\log C(T,\omega)}{\log T} = 3D - 1, \quad d \ge 2, \ psi_0 \ne 0.$$

We obtain also (Theorem 5.6) bounds for the rate of decay of average occupation time

$$\langle W_R \rangle_{T,\omega} = 3DT^{-1} \int_0^T dt \sum_{n:|n|\le R} |\psi(t,n,\omega)|^2 =, \quad R > 0.$$

In particular (Corollary 5.7), in dimensions $d \ge 3$ the total occupation time of any ball is finite with probability 1:

$$\int_0^{+\infty} dt \sum_{n:|n|\le R} |\psi(t,n,\omega)|^2 < +\infty.$$

2 Bounds for the average density matrix in weighted spaces.

The result of [4] which will be the basis for our considerations concerns the Laplace transform of the average density matrix. It was obtained with the so-called generalised Feynman-Kac formula [12], [4]. Let $\psi(t,n,\omega)$ be the solution to the equation (1.3). We set

$$f(n,m,z) = 3D \int_0^{+\infty} dt \exp(-tz) \mathbf{E}[\psi(t,n,\cdot)\overline{\psi(t,m,\cdot)}\ ], \quad z > 0. \tag{2.1}$$

It was shown in [4] that for any $z > 0$ the function f satisfies the following equation in $l^2(\mathbf{Z}^{2d})$:

$$Y(z)f = 3Dg,$$

where $g(n,m) = 3D\psi_0(n)\overline{\psi_0(m)}$, $Y(z) = 3Di\Delta + iQ + z + G(z)$,

$$(\Delta f)(n,m) = 3D \sum_{p:|p|=3D1} f(n+p,m) - \sum_{p:|p|=3D1} f(n,m+p),$$

$$(Qf)(n,m) = 3D(Q_0(n) - Q_0(m))f(n,m)$$

and $G(z)$ is some family of bounded operators in $l^2(\mathbf{Z}^{2d})$. Consider for any $\alpha \ge 0$ the weighted space

$$l^2_\alpha(\mathbf{Z}^{2d}) = 3D\{\psi : \psi \in l^2(\mathbf{Z}^{2d}), \ \sum_{n,m} \exp(\alpha(|n|+|m|))|\psi(n,m)|^2 < +\infty\},$$

where $l^2_\alpha(\mathbf{Z}^{2d}) \subset l^2_0(\mathbf{Z}^{2d}) = 3Dl^2(\mathbf{Z}^{2d})$. We shall denote the inner product in $l^2_\alpha(\mathbf{Z}^{2d})$ as $\langle\cdot,\cdot\rangle_\alpha$ and the norm as $\|\cdot\|_\alpha$.

It has been proven in [4] (Lemma 3.1, Corollary 3.2, Lemma 4.2) that operators $\Delta, Q, G(z)$ have the following properties:

1. Operators Δ, Q are bounded in $l^2_\alpha(\mathbf{Z}^{2d})$ for any $\alpha \geq 0$ and $\|\Delta\|_\alpha \leq C,\ \|Q\|_\alpha \leq C$ with constants uniform in $\alpha \in [0,1]$.

2. The operator Q is self-adjoint in $l^2_\alpha(\mathbf{Z}^{2d})$ and for Δ one has $\|\Im\Delta\|_\alpha \leq C\alpha$ with C uniform in $\alpha \in [0,1]$.

3. Let T be the operator of multiplication by Kronecker symbol:

$$(Tf)(n,m) = 3D\delta_{n,m}f(n,m), \quad (T'f)(n,m) = 3D(1-\delta_{n,m})f(n,m).$$

Operators T, T' are self-adjoint in $l^2_\alpha(\mathbf{Z}^{2d})$, $T + T' = 3DI$, $\|T\|_\alpha \leq 1$, $\|T'\|_\alpha \leq 1$. The identities hold:

$$T^2 = 3DT, \quad (T')^2 = 3DT', \quad TQ = 3DQT = 3D0, \quad T\Delta T = 3D = 0.$$

4. Operators $G(z)$ are bounded in $l^2_\alpha(\mathbf{Z}^{2d})$ for any $\alpha \in [0,\nu]$ with some $\nu(d) > 0$: $\|G(z)\|_\alpha \leq C$ with C uniform in $z \in (0,1],\quad \alpha \in [0,\nu]$.

5. For any $z > 0$ the identities holds: $TG(z) = 3DG(z)T = 3D0$.

6. For any $z \in (0,1],\ \alpha \in [0,\nu]$ the estimate holds:

$$\Re\langle G(z)f, f\rangle_\alpha \geq C_1\|T'f\|^2_\alpha$$

with some uniform $C_1 > 0$.

Using the properties 1-6, one can prove necessary estimates for $f = 3DY^{-1}(z)g$ in $l^2_\alpha(\mathbf{Z}^{2d})$. The main interest for us will present the quantities like

$$\rho(z) = 3D\langle Y^{-1}(z)g, h\rangle_\alpha, \quad g(n,m) = 3D\psi_0(n)\overline{\psi_0(m)}.$$

We shall take $h(n,m) = 3D\delta_{n,m}F(|n| \geq A)\exp(-\alpha|n|)|n|^p$, $\alpha = 3DC\sqrt{z}=$ to get the upper bounds for the moments $|X|^p$ and bounds (1.4) for the outside occupation time. To get the lower bounds for $|X|^p$ and bounds (1.5) for the inside occupation time, we shall take $h(n,m) = 3D\delta_{n,m}F(|n| \leq A)$, $\alpha = 3D = 0$. Finally, we choose $h(n,m) = 3D\psi_1(n)\overline{\psi_1(m)}$ to get up= per bounds for the correlation function. Some of results we prove in this sec= tion were obtained in [4]. However, the proofs we present here are much mor= e simpler.

Lemma 2.1 *For any $C_2 > 0,\ \delta > 0$ we consider the set in $\mathbf{R}^2$:*

$$D(C_2,\delta) = 3D\{(z,\alpha) : z \in (0,\delta], \alpha \in [0, C_2\sqrt{z}]\}.$$

There exist $C_2 > 0,\ \delta \in (0,1/2)$ such that the bounded inverse $Y^{-1}(z)$ exist in $l^2_\alpha(\mathbf{Z}^{2d})$ for any $(z,\alpha) \in D(C_2,\delta)$. The uniform in $(z,\alpha) \in D(C_2,\delta)$ estimates hold:

$$\|Y^{-1}(z)\|_\alpha \leq Cz^{-1}, \quad \|T'Y^{-1}(z)\|_\alpha \leq Cz^{-\frac{1}{2}}, \tag{2.2}$$

$$\|\Delta TY^{-1}(z)\|_\alpha \leq Cz^{-\frac{1}{2}}, \quad \|\Delta^* TY^{-1}(z)\|_\alpha \leq Cz^{-\frac{1}{2}}, \tag{2.3}$$

where Δ^* is the adjoint of Δ in $l^2_\alpha(\mathbf{Z}^{2d})$.

Proof. Let $f \in l^2_\alpha(\mathbf{Z}^{2d})$, $H = 3D\Delta + Q$. One can write the equality:

$$\|Y(z)f\|^2_\alpha = 3D\langle (iH + z + G(z))f, (iH + z + G(z))f\rangle =_\alpha = 3D$$

$$\|(iH + G(z))f\|^2_\alpha + z^2\|f\|^2_\alpha + 2z\Re\langle (iH + G(z))f, f\rangle_\alpha \tag{2.4}$$

Properties 6 and 2 imply that for $\alpha \leq \nu$

$$\Re\langle G(z)f, f\rangle_\alpha \geq C_1\|T'f\|^2_\alpha, \tag{2.5}$$

$$\Re\langle iHf, f\rangle_\alpha = 3D - \langle (\Im\Delta)f, f\rangle_\alpha, \quad \Im\Delta = 3D\frac{1}{2i}(\Delta - \Delta^*). \tag{2.6}$$

As $T\Delta T = 3D0$ and $T^* = 3DT$ (property 3), we have also $T\Delta^* = T = 3D0$. Therefore one can rewrite (2.6) as follows:

$$2i\langle (\Im\Delta)f, f\rangle_\alpha = 3D\langle (T + T')(\Delta- = lta^*)(T + T')f, f\rangle_\alpha = 3D$$

$$\langle T(\Delta - \Delta^*)T'f, f\rangle_\alpha + \langle T'(\Delta - \Delta^*)f, f\rangle_\alpha.$$

Properties 2 and 3 imply the uniform in α estimate

$$|\langle (\Im\Delta)f, f\rangle_\alpha| \leq C\alpha\|T'f\|_\alpha\|f\|_\alpha \tag{2.7}$$

With (2.4)-(2.7) we estimate for $\alpha \in [0, \nu]$:

$$\|Y(z)f\|^2_\alpha \geq z(za^2 + 2C_1b^2 - 2C\alpha ab), \tag{2.8}$$

where $a = 3D\|f\|_\alpha$, $b = 3D\|T'f\|_\alpha$. Obviously, there exists $C_2 > 0$ such that if $\alpha \leq C_2\sqrt{z}$, th= en for any a, b

$$\frac{z}{2}a^2 + C_1b^2 - 2C\alpha ab \geq 0.$$

For such (z, α), (2.8) yields:

$$\|Y(z)f\|^2_\alpha \geq \frac{z^2}{2}\|f\|^2_\alpha + C_1z\|T'f\|^2_\alpha. \tag{2.9}$$

If we take δ sufficiently small such that $C_2\sqrt{\delta} \leq nu$, (2.9) will be valid for any $(z, \alpha) \in D(C_2, \delta)$. In the same manner one can show that for the adjoint in $l^2_\alpha(\mathbf{Z}^{2d})$ operator for any $(z, \alpha) \in D(C_2, \delta)$

$$\|Y^*(z)f\|^2_\alpha \geq \frac{z^2}{2}\|f\|^2_\alpha + C_1z\|T'f\|^2_\alpha \tag{2.10}$$

The operators $Y(z), Y^*(z)$ being bounded in $l^2_\alpha(\mathbf{Z}^{2d})$, we get from (**??**), (2.10) that $KerY(z) = 3DKerY^*(z) = 3D\{0\}$, $RanY(z) = 3DRanY^*(z) = 3Dl^2_\alpha(\mathbf{Z}^{2d})$ and

$$\|Y^{-1}(z)\|_\alpha \leq \sqrt{2}z^{-1}, \quad \|(Y^*)^{-1}(z)\|_\alpha \leq \sqrt{2}z^{-1}. \tag{2.11}$$

One gets also

$$\|T'Y^{-1}(z)\|_\alpha \leq Cz^{-\frac{1}{2}}, \quad \|T'(Y^*)^{-1}(z)\|_\alpha \leq Cz^{-\frac{1}{2}} \tag{2.12}$$

for any $(z,\alpha) \in D(C_2,\delta)$. Let us write now the equation for $f = 3DY^{-1}(z)$. As $QT = 3D0, \quad TG(z) = 3DG(z)T = 3D0$, one has

$$i\Delta f + iQT'f + zf + T'G(z)T'f = 3Dg.$$

The estimates (2.11), (2.12) together with the uniform bounded= ness of $G(z)$ imply

$$\|\Delta f\|_\alpha \le Cz^{-\frac{1}{2}}\|g\|_\alpha.$$

As $\Delta f = 3D\Delta Tf + \Delta T'f$, we get immediately

$$\|\Delta Tf\|_\alpha \le Cz^{-\frac{1}{2}}\|g\|_\alpha. \tag{2.13}$$

As $\|\Delta - \Delta^*\|_\alpha \le C'\alpha \le C\sqrt{=}z$, one gets also

$$\|\Delta^* Tf\|_\alpha \le Cz^{-\frac{1}{2}}\|g\|_\alpha. \tag{2.14}$$

The statement of the Lemma follows now from (2.11), (2.12), (2.13) and (2.14). The proof is completed.

For any $z > 0$ define the following functions $l_d(z) : l_d(z) = 3Dz^{-\frac{3}{4}}, \quad d = 3D1, \quad l_d(z) = 3Dz^{-\frac{1}{2}}|\log z|^{\frac{1}{2}}, \quad d = 3D2 =$ and $l_d(z) = 3Dz^{-\frac{1}{2}}, \quad d \ge 3$. We shall denote by D the set $D(C_2,\delta)$ from Lemma 2.1.

Lemma 2.2 *Let $(z,\alpha) \in D, \ g \in l^2_\alpha(\mathbf{Z}^{2d}), \ n \in \mathbf{Z}^d$. The uniform estimat= es hold:*

$$|(Y^{-1}(z)g)(n,n)| \le Cl_d(z)\exp(-\alpha|n|)\|g\|_\alpha, \tag{2.15}$$

$$|((Y^*)^{-1}(z)g)(n,n)| \le Cl_d(z)\exp(-\alpha|n\check{=}ert)\|g\|_\alpha. \tag{2.16}$$

Proof. One can easily see ([4], Lemma 5.1) that $(\Delta TY^{-1}(z)g)(n,m) = 3D0$ for $|n-m| \ne 1$ and $(\Delta TY^{-1}(z)g)(n,m) = 3D\rho(m) - \rho(n)$ for $|n-m| = 3D = 1$, where $\rho(n) = 3D(Y^{-1}(z)g)(n,n)$. Therefore

$$\|\Delta TY^{-1}(z)g\|^2_\alpha = 3D \sum_{n,p:|p|=3D1=} |\rho(n+p) - \rho(n)|^2 \exp(\alpha(|n| + |n+p|)). \tag{2.17}$$

Consider the function $h(n) = 3D\rho(n)\exp(\alpha|n|)$. The result of Lemma **??** lp1 implies

$$\|h\|^2_0 = 3D\sum_n |h(n)|^2 = 3D\|TY^{-1}(z)g\|^2_\alpha \le Cz^{-2}\|g\|^2_\alpha. \tag{2.18}$$

Let us fix $p : |p| = 3D1$. One can write the identity

$$h(n) - h(n+p) = 3D(\rho(n) - \rho(n+p))\exp(\alpha|n|) + \rho(n+p)(\exp(\alpha|n|) - \exp(\alpha|n+p|)) = .$$

As $|p| = 3D1$ and $0 \le \alpha \le C_2\sqrt{z} \le \nu$, one can easily see with (2.17), (2.18 and (2.3) that

$$\sum_n |h(n) - h(n+p)|^2 \le$$

$$C\sum_n |\rho(n)-\rho(n+p)|^2\exp(\alpha(|n|+|n+p|))+C\alpha^2\sum_n |\rho(n+p)|^2\exp(2\alpha|n+p|) = 3D$$

$$C\|\Delta TY^{-1}(z)g\|_\alpha^2 + C\alpha^2\|TY^{-1}(z)g\|_\alpha^2 \le Cz^{-1}\|g\|_\alpha^2.$$

Therefore,

$$\sum_{n,p:|p|=3D1} |h(n)-h(n+p)|^2 \le Cz^{-1}\|g\|_\alpha^2. \tag{2.19}$$

The bounds (2.18) and (2.19) allow us to estimate $\max_n |h(n)|$ exactly in the same manner as in [4], Lemmas 5.1-5.2. The result is

$$\max_n |h(n)| \le Cl_d(z)\|g\|_\alpha$$

with $l_d(z)$ defined above, which gives immediately (2.15). For the adjoint operator the proof is the same. The proof is completed.

Theorem 2.3 *Let $h \in l^2_\alpha(\mathbf{Z}^{2d})$, $g(n,m) = 3D\psi_0(n)\overline{\psi_0(m)}$, where $\sum_n |\psi_0(n)|^2\exp(a|n|) < +\infty$ for so= me $a > 0$. There exists $\delta > 0$ such that for $(z,\alpha) \in D(C_2,\delta)$ the uniform estimate holds:*

$$|\langle Y^{-1}(z)g,h\rangle_\alpha| \le C(\psi_0)l_d(z)\|h\|_\alpha. \tag{2.20}$$

Proof. One can write the identity:

$$\langle Y^{-1}(z)g,h\rangle_\alpha = 3D\langle Tg, T(Y^*)^{-1}(z)h\rangle_\alpha + \langle T'g, T'(Y^*)^{-1}(z)h\rangle_\alpha \equiv r_1(z)+r_2(z).$$

The estimate (2.16) of Lemma 2.2 implies

$$|r_1(z)| \le Cl_d(z)\sum_n \exp(2\alpha|n|)|\psi_0(n)|^2\exp(-\alpha|n|)\|h\|_\alpha \le C(\psi_0)l_d(z)\|h\|_\alpha \tag{2.21}$$

if α is sufficiently small ($\alpha \le a$). The estimate $\|T'(Y^*)^{-1}(z)\|_\alpha \le Cz^{-\frac{1}{2}}$ of Lemma 2.1 yields

$$|r_2(z)| \le Cz^{-\frac{1}{2}}\|h\|_\alpha \cdot \|T'g\|_\alpha \le C(\psi_0)l_d(z)\|h\|_\alpha, \tag{2.22}$$

again if $\alpha \le a$. The statement of the Theorem follows from (**??**) and (2.22). The proof is completed.

The result of this Theorem for the maximal allowed value of α will be used to get the upper bounds for $|X|^p$ and bounds for the outside occupation time (1.4). In principle, taking $\alpha = 3D0$, one can get the lower bounds for $|X|^p$ and bounds (1.5) for the inside occupation time. In fact, it was done so in [4] (Lemma 5.3, Corollary 5.4) for the inside occupation time. There exists, however, another method wich gives better results. We shall discuss it in section 4.

3 Upper bounds for the moments.

In this section we shall suppose that ψ_0 satisfies the condition of the Theorem 2.3, namely, that

$$\sum_n |\psi_0(n)|^2 \exp(a|n|) < +\infty \tag{3.1}$$

for some $a > 0$. Let $\psi(t,n,\omega)$ be the solution to the Schrödinger equation (1.3). For any $p > 0,\ N > 0$ define the positive functiona= ls

$$|X|_N^p(t,\omega) = 3D \sum_{n:|n|\le N} |n|^p |\psi(t,n,\omega)|^2.$$

Obviously,

$$|X|_N^p(t,\omega) \le N^p \|\psi_0\|^2 \tag{3.2}$$

for any t, ω. Define now

$$|X|^p(t,\omega) = 3D \sum_n |n|^p |\psi(t,n,\omega)|^2 = 3D \lim_{N\to+\infty} |X|_N^p(t,\omega),$$

where it is possible that $|X|^p = 3D + \infty$ for some $t, \o = m$. For $\mathbf{P} - a.e.\ \omega$ the functions $|X|_N^p(\cdot, \omega\ =)$ are continuous as $\psi(\cdot, n, \omega)$ are continuous (see [4]=). Therefore, with probability 1 the functions $|X|^p(\cdot, \o = m)$ are Lebesgue measurable and one can consider the integrals:

$$\eta_p(z,\omega) = 3D \int_0^{+\infty} \exp(-tz)|X|^p(t,\omega)dt, \quad z > 0.$$

Again, it is possible that $\eta_p(z,\omega) = 3D + \infty$.

Lemma 3.1 *For any $z > 0, N > 0$ the equality holds:*

$$\mathbf{E}[\int_0^{+\infty} dt \exp(-tz)|X|_N^p(t,\cdot)] = 3D \sum_{n:|n|\le N} |n|^p f(n,n,z), \tag{3.3}$$

where $f(z) = 3DY^{-1}(z)g, \quad g(n,m) = 3D\psi_0(n)\overline{\psi_0(m)}$.

Proof. First, for any t the functionals $|X|_N^p$ are $\mathbf{P}$-measurable as for any t, n the functionals $\psi(t,n,\cdot)$ are measurable [**?**]. Moreover, for $\mathbf{P} - a.e.\ \omega$ the functions $|X|_N^p(\cdot,\omega)$ are continuous. The result of Lemma 2.5 of [4] and (3.2) imply that the functionals

$$\eta_{p,N}(z,\omega) = 3D \int_0^{+\infty} dt \exp(-tz)|X|_N^p(t,\omega)$$

are $\mathbf{P}$-integrable for any positive p, N, z and

$$\mathbf{E}[\eta_{p,N}(z,\cdot)] = 3D \int_0^{+\infty} dt \exp(-tz)\mathbf{E}[|X|_N^p(t,\cdot)] = 3D$$

$$\int_0^{+\infty} dt \exp(-tz) \sum_{n:|n|\le N} |n|^p \mathbf{E}[|\psi(t,n,\cdot)|^2] = 3D \sum_{n:|n|\le N} |n|^p f(n,n,z),$$

where f is defined by (2.1). The proof is completed.

Lemma 3.2 *Let us fix some ψ_0, p. Consider the function*

$$\rho_p(z) = 3D\sum_n |n|^p f(n,n,z).$$

Suppose that $\rho_p(z) < +\infty$ for any $z > 0$. Then
1. $\mathbf{P}$*-almost surely for any $z > 0$ $\eta_p(z,\omega) < +\infty$.*
2. *For any $z : z > 0$ the functionals $\eta_p(z,\cdot)$ are $\mathbf{P}$-integrable and $\mathbf{E}[\eta_p(z,\cdot)] = 3D\rho_p(z)$.*

The proof is based on the monotone convergence theorem (the limit $N \to +\infty$) and is rather straightforward, so we shall omit it.

Corollary 3.3 *If $\rho_p(z) < +\infty$ for any $z > 0$, then*
1. $\mathbf{P}$*-almost surely $|X|^p(t,\omega) < +\infty$ for Lebesgue-almost every t.*
2. $\mathbf{P}$*-almost surely the functions $\eta_p(\cdot,\omega)$ are contin= uous in $z : z > 0$.*

Now we shall estimate the functions $\rho_p(z)$.

Theorem 3.4 *Let $p > 0$ and ψ_0 satisfy (3.1) for some $a > 0$. The uniform in $z \in (0, 1/3)$ estimates hold:*

$$\rho_p(z) \le Cz^{-\frac{p}{2}-1}, \quad d = 3D1,$$
$$\rho_p(z) \le Cz^{-\frac{p}{2}-1}(\log|\log z|)^p, \quad d = 3D = 2,$$
$$\rho_p(z) \le Cz^{-\frac{p}{2}-1}|\log z|^p, \quad d \ge 3.$$

Proof. Let $A > 0$. One can write the identity:

$$\rho_p(z) = 3D(\sum_{n:|n|\ge A} + \sum_{n:|n|<A})|n|^p f(n,n,z) \equiv a(z) + b(z).$$

One can write $a(z)$ as

$$a(z) = 3D\sum_n \exp(2\alpha|n|)F(|n| \ge A)\exp(-2\alpha|n|)|n|^p f(n,n,z) = 3D\langle f(z), h_A\rangle_\alpha,$$

where $\alpha \ge 0$, $h_A(n,m) = 3D\delta_{n,m}F(|n| \ge A)|n|^p \exp(-2\alpha|n|)$. For any $z \in (0,\delta]$, where δ is the number from Theorem 2.3, we take $\alpha = 3DC_2\sqrt{z}$. The estimate (2.20) of this Theorem implies

$$|a(z)| \le C(\psi_0)l_d(z)\|h_A\|_\alpha.$$

The calculation shows that

$$\|h_A\|^2_\alpha = 3D\sum_{n:|n|\ge A}|n|^{2p}\exp(-2\alpha|n|) \le C\alpha^{-2p-d}\exp(-A\alpha)$$

This gives us

$$|a(z)| \le C(\psi_0,p,d)z^{-p/2-d/4}l_d(z)\exp(-CA\sqrt{z} =) \tag{3.4}$$

with some $C > 0$. To estimate $b(z)$, we observe first that $f(n,n,z) \geq 0 =$ and

$$\sum_n f(n,n,z) = 3D \sum_n \int_0^{+\infty} dt \exp(-tz) \mathbf{E}[\psi(t,n,\cdot)|^2] = 3Dz^{-1}\|\psi_0\|^2.$$

Therefore,

$$|b(z)| \leq A^p \sum_{n:|n|<A} f(n,n,z) \leq A^p z^{-1} C(\psi_0). \tag{3.5}$$

To get the optimal bound for $\rho_p(z)$, we take $A = 3D0$ for $d = 3D1$, $A = 3DDz^{-1/2} \log|\log z|$ with D big enough for $d = 3D2$ and $A = 3DDz^{-1/2}|\log z|$ with D big enough for $d \geq 3$. As $\rho_p(z)$ are decreasing in z, the bounds (3.4), (3.5) give the statement of the Theorem. The proof is completed.

Corollary 3.5 *Let $D(z) = 3D\mathbf{E}[z^2 \int_0^{+\infty} dt \exp(-tz)|X|^2(t,\cdot)]$ be the average diffusion constant. The following uniform in $z \in (0, 1 = /3]$ estimates hold:*

$$D(z) \leq C, \quad d = 3D1; \quad D(z) \leq C(\log|\log z|)^2, \quad d = 3D = 2;$$

$$D(z) \leq C(\log z)^2, \quad d \geq 3.$$

Let $T > 0$, $p > 0$. Consider the time-averaged quantities

$$\langle |X|^p \rangle_{T,\omega} = 3DT^{-1} \int_0^T dt |X|^p(t,\omega).$$

To get the upper bounds with probability 1 for $\langle |X|^p \rangle_{T,\omega}$, we shall use the following elementary fact.

Lemma 3.6 *Let*

$$h(z,\omega) = 3Dz^\beta |\log z|^{-\gamma} \int_0^{+\infty} dt \exp(-tz) g(t,z,\omega),$$

where $\beta \geq 0, \gamma \geq 0, z \in (0, 1/3]$, $g(t,z,\omega) \geq 0$ and for $\mathbf{P}$-a.e. ω and any $t > 0$ the function $g(t,\cdot,\omega)$ is decreasing. Suppose that $h(z,\cdot)$ is $\mathbf{P}$-integrable for any $z > 0$.

1. If the uniform in $z \in (0, 1/3]$ estimate holds:

$$\mathbf{E}[h(z,\cdot)] \leq C|\log z|^{-1}(\log|\log z|)^{-r} \tag{3.6}$$

with some $r > 1$, then $\lim_{z \to +0} h(z,\omega) = 3D0$ $\mathbf{P}$-almost $=$ surely.

2. If $\beta = 3D0$ and the uniform estimate holds:

$$\mathbf{E}[h(z,\cdot)] \leq C(\log|\log z|)^{-r}$$

with some $r > 1$, then $\lim_{z \to +0} h(z,\omega) = 3D0$ $\mathbf{P}$-almost surely.

3. If $\beta = 3D\gamma = 3D0$, g does not depend on z and the uniform estimate holds:

$$\mathbf{E}[h(z,\cdot)] \leq C,$$

then $\mathbf{P}$-almost surely for any $z \in (0, 1/3]$

$$h(z,\omega) \leq \int_0^{+\infty} g(t,\omega) dt < +\infty.$$

Proof. We shall use the same idea as in [8], proof of Theorem 7.6 (one could use also the Borel-Cantelli Theorem, see [13], proof of the Theorem 4.4.2). If $\beta > 0$ and (3.6) holds, we set $z_k = 3D \exp(-k)$, $k = 3D1, 2, \ldots$ and define the functional

$$Q(\omega) = 3D \sum_{k=3D1}^{+\infty} h(z_k, \omega).$$

One sees easily that $\mathbf{E}[Q(\cdot)] < +\infty$. Therefore, for $\mathbf{P}$-a.e. ω we have $Q(\omega) < +\infty$ and

$$\lim_{k\to+\infty} h(z_k, \omega) = 3D0. \tag{3.7}$$

On the other hand, if $z \in [z_{k+1}, z_k]$, one can estimate:

$$h(z, \omega) \le z_k^\beta |\log z_k|^{-\gamma} \int_0^{+\infty} dt \exp(-tz_{k+1}) g(t, z_{k+1}, \omega) = 3D$$

$$h(z_{k+1}, \omega) \left(\frac{z_k}{z_{k+1}}\right)^\beta \left|\frac{\log z_{k+1}}{\log z_k}\right|^\gamma \le e^\beta 2^\gamma h(z_{k+1}, \omega). \tag{3.8}$$

It follows from (3.7) and (3.8) that $\lim_{z\to+0} h(z, \omega) = 3D0$.

In the case $\beta = 3D0$ one takes $z_k = 3D \exp(-\exp k)$, the proof bei= ng the same.

Finally, if

$$\mathbf{E}[\int_0^{+\infty} dt \exp(-tz) g(t, \cdot)] \le C,$$

the monotone convergence theorem implies $\int_0^{+\infty} g(t, \omega) dt < +\infty$ for $\mathbf{P}$-a.e. ω. The proof is completed.

Corollary 3.7 *Let $G(z, \omega) = 3D \int_0^{+\infty} dt \exp(-tz) g(t, z, \omega)$, where g is the same as in Lemma 3.6. Suppose that for any $z > 0$ the functionals $G(z, \cdot)$ are $\mathbf{P}$-integrable.*

1. If $\mathbf{E}[G(z, \cdot)] \le Cz^{-\beta} |\log z|^\gamma$, where $\beta > 0, \gamma \ge 0$, then with probability 1 the uniform in $z \in (0, 1/3]$ estimate holds:

$$G(z, \omega) \le C_r(\omega) z^{-\beta} |\log z|^r, \tag{3.9}$$

where $r > \gamma + 1$.

2. If $\mathbf{E}[G(z, \cdot)] \le C |\log z|^\gamma$ with so= me $\gamma > 0$, then with probability 1 the uniform estimate holds:

$$G(z, \omega) \le C_r(\omega) |\log z|^r, \tag{3.10}$$

where $r > \gamma$.

Proof. One takes $h(z, \omega) = 3Dz^\beta |\log z|^{-r} G(z, \omega)$, where $r > \gamma + 1$ in the first case and $h(z, \omega) = 3D |\log z|^{-r} G(z, \omega)$, where $r > \gamma$ in the= second case. Lemma 3.6 implies in both cases $\lim_{z\to+0} h(z, \omega) = 3D = 0$ with probability 1, which gives immediately (3.9), (3.10). The proof is completed.

Now we can estimate functionals $\eta_p(z, \omega)$ with probability 1.

Lemma 3.8 *Let $p > 0$ and ψ_0 satisfy (3.1) for some $a > 0$. For any $r > 1$ with probabilit= y 1 the following uniform in $z \in (0, 1/3]$ estimates hold:*

$$\eta_p(z, \omega) \le C(r, \omega) z^{-p/2-1} |\log z|^r, \quad d = 3D1, 2,$$
$$\eta_p(z, \omega) \le C(r, \omega) z^{-p/2-1} |\log z|^{p+r}, \quad d \ge 3.$$

Proof. We take $G(z, \omega) = 3D\eta_p(z, \omega)$. The result follows directly from Lemma 3.2, Theorem 3.4 and the first statement of Corollary 3.7.

Theorem 3.9 *Let ψ_0 be the same as above, $p > 0$, $r > 1$. With probability 1 the uniform in $T \ge 3$ estimates hold:*

$$\langle |X|^p \rangle_{T,\omega} \le C(r, \omega) T^{p/2} (\log T)^r, \quad d = 3D1, 2,$$

$$\langle |X|^p \rangle_{T,\omega} \le C(r, \omega) T^{p/2} (\log T)^{p+r}, \quad d \ge 3.$$

Proof. Again, we can estimate:

$$\langle |X|^p \rangle_{T,\omega} \le eT^{-1} \int_0^T dt \exp(-t/T) |X|^p(t, \omega) \le$$

$$eT^{-1} \int_0^{+\infty} dt \exp(-t/T) |X|^p(t, \omega) = 3Dez\eta_p(z, \omega), \tag{3.11}$$

where $z = 3DT^{-1}$. The result follows from (3.11) and Lemma 3.8.

With this result one can get bounds for the outside occupation time (1.4). Let us fix some $\psi_0 \in l^2(\mathbf{Z}^d)$ and consider

$$W(t, \omega) = 3D \sum_{n:|n| \ge a(t)} |\psi(t, n, \omega)|^=2,$$

where $a(t)$ is some increasing function. In [4] it was shown tha= t for $a(t) = 3Dt^\alpha, \ \alpha > 1/2$, in any dimension with probability 1

$$\lim_{T \to +\infty} \langle W \rangle_{T,\omega} = 3D0 \tag{3.12}$$

for any $\psi_0 \in l^2(\mathbf{Z}^d)$. We shall improve now this result.

Theorem 3.10 *Let $a(t) = 3Dt^{\frac{1}{2}} (\log(t+1))^\nu$, where $\nu > 0$ for $d = 3D = 1, 2$ and $\nu > 1$ for $d \ge 3$. With probability 1 for any $\psi_0 \in l^2(\mathbf{Z}^d) =$ (3.12) hold.*

Proof. Suppose that ψ_0 is finite, so the result of Theorem 3.9 hold= s. For any $p > 0$ we have

$$|X|^p(t, \omega) \ge a^p(t) \sum_{n:|n| \ge a(t)} |\psi(t, n, \omega)|^2.$$

Therefore,

$$W(t, \omega) \le a^{-p}(t) |X|^p(t, \omega). \tag{3.13}$$

As $\lim_{T\to+\infty} T^{-1}\int_0^3 dtW(t,\omega)=3D0$, it is suff= icient to consider

$$L(T,\omega)=3DT^{-1}\int_3^T dtW(t,\omega).$$

Let

$$F(T,\omega)=3D\int_3^T dt|X|^p(t,\omega).$$

Theorem 3.9 yields with probability 1

$$F(T,\omega)\leq CT^{1+p/2}(\log T)^\gamma, \tag{3.14}$$

where $\gamma>1$ for $d=3D1,2$ and $\gamma>p+1$ for $d\geq 3$. One can = estimate L using (3.13):

$$L(T,\omega)\leq T^{-1}\int_3^T a^{-p}(t)F'(t,\omega)dt=3DT^{-1}F(T,\omega)a^{-p}(T)-T^{-1}\int_3^T F(t,\omega)s(t)dt, \tag{3.15}$$

where $s(t)=3D(a^{-p}(t))'$. It is not difficult to show that (3.14) and (3.15) imply

$$\lim_{T\to+\infty} L(t,\omega)=3D0$$

if $p\nu>\gamma$. This condition is satisfied for p big enough in any dimension. So, we have proved (3.12) for any finite ψ_0 with probability 1. By standard density arguments (see [**?**], Lemma 2.4), (3.12) holds with probability 1 for any $\psi_0\in l^2(\mathbf{Z}^d)$. The proof is completed.

4 Bounds for the average density matrix in $l^2(\mathbf{Z}^{2d})$.

In this section we shall denote by (f,g) and $\|f\|$ the scalar product and the norm in $l^2(\mathbf{Z}^{2d})$. To get the upper bounds for $\rho(z)=3D(Y^{-1}(z)g,h)$, we shall consider the operators

$$X(z)=3Di\Delta+z+T',$$

where operators Δ, T' were defined in section 2. The idea consists to compare the inverse $Y^{-1}(z)$ with $X^{-1}(z)$ and to show that these two operators are close in some sense as $z\to+0$. The advantage of thi= s method is that the inverse operators $X^{-1}(z)$ can be explicitely calcu= lated (in Fourier-representation), which gives good estimates for $\rho(z)$, better than that of Theorem 2.3 for $\alpha=3D0$. One should make = here some comments about operators $X(z)$. Let us consider the potential $pV(u)$, and take λB as generator of Markov process, p,λ being positive parameters. If $p/\lambda=$ is small enough, one can show using the results of [4] that for t= he corresponding operators $G(z)$ one has:

$$G(z)=3DCp^2\lambda^{-1}T'+T'R(z)T',$$

where $\|R(z)\|\leq Cp^3\lambda^{-2}$. If one takes the limit $p^2=3D\lambda\to+\infty$, one gets $G(z)=3DCT'$ with some constant $=C$. As $\lambda\to+\infty$, the correlation time of Markov process tends to 0. Therefore the equation $X(z)f = 3Dg$ can be interpreted as equa= tion for the Laplace transform of the average density matrix in the case of δ-correlated potentials (see also [14]). One can explicitely calculate the inverse $X^{-1}(z)$ and show that the averaged solutions present exac= tly diffusive behaviour in any dimension.

Before calculating the inverse $X^{-1}(z)$, we shall first establish the relation between $X^{-1}(z)$ and $Y^{-1}(z)$.

Lemma 4.1 *The following uniform in $z \in (0, 1]$ estimates hold:*

$$\|T'X^{-1}(z)T'\| \leq C, \quad \|T'Y^{-1}(z)T'\| \leq C. \tag{4.1}$$

Proof. First, it is easy to see that operators $X(z)$ have all the properties 1-6 (section 2) of $Y(z)$. Therefore, all the results of section 2 are va= lid for $X(z)$. In particular, the inverse operators $X^{-1}(z)$ exist in $l^2(\mathbf{Z}^{2d})$. Let us write now the equation for $f = 3DY^{-1}(z)T'g$:

$$(i\Delta + iQ + z + G(z))f = 3DT'g.$$

As $TQ = 3DQT = 3D0$, $TG(z) = 3DG(z)T = 3D0$, one can write

$$i\Delta f + iT'QT'f + zf + T'G(z)T'f = 3DT'g. \tag{4.2}$$

Let us apply T to both sides of (4.2). As $T\Delta T = 3D0$, one gets

$$iT\Delta T'f + zTf = 3D0. \tag{4.3}$$

Now apply T' to both sides of (4.2):

$$iT'\Delta Tf + iT'\Delta T'f + iT'QT'f + zT'f + T'G(z)T'f = 3DT'g. \tag{4.4}$$

¿From (4.3) we get $Tf = 3D - iz^{-1}T\Delta T'f$ and (4.4) impli= es the following equation for $T'f$:

$$N(z)T'f \equiv z^{-1}T'\Delta T\Delta T'f + iT'\Delta T'f + iT'QT'f + zT'f + T'G(z)T'f = 3DT'g,$$

where the operators $N(z)$ act in $\mathcal{H}' = 3DT'l^2(\mathbf{Z}^{2d})$, the closed subspace of $l^2(\mathbf{Z}^{2d})$. One has

$$T'\Delta T\Delta T' = 3D(T\Delta T')^*(T\Delta T') \geq 0.$$

As operators $T'\Delta T'$, $T'QT'$ are self-adjoint in $l^2(\mathbf{Z}^{2=d})$, with the property 6 of $G(z)$ one estimates for any $h \in \mathcal{H}^{=me}$:

$$\Re(N(z)h, h) \geq \Re(T'G(z)T'h, h) \geq C_1\|h\|^2$$

with some positive uniform constant C_1. Therefore,

$$\|T'g\|^2 = 3D\|N(z)T'f\|^2 = 3D\|(N(z) - C_1)T'f + C_1T'f\|^2 \geq C_1^2\|T'f\|^2. \tag{4.5}$$

The estimate (4.5) gives the statement of Lemma for $Y^{-1}(z)$. For the inverse $X^{-1}(z)$ the proof is the same. The proof is completed.

Lemma 4.2 *The following equality holds:*

$$Y^{-1}(z) = 3DX^{-1}(z) + X^{-1}(z)T'M(z)T'X^{-1}(z),$$

where $\|M(z)\| \le C$ *uniformly in* $z \in (0,1]$.

Proof. Let K, D be some bounded operators such that bounded inverses $(K+D)^{-1}$ and K^{-1} exist. It is easy to check the identity:

$$(K+D)^{-1} = 3DK^{-1} - K^{-1}(D - D(K+D)^{-1}D)K^{-1}. \tag{4.6}$$

One takes $K = 3DX(z),\ D = 3DY(z) - X(z) = 3DiQ + G(z) - T'$. As $TQ = 3DQT = 3D0,\quad TG(z) = 3DG(z)T = 3D0,\quad TT' = 3DT'T = 3D0$, we have $= TD = 3DDT = 3D0$. Therefore, (4.6) implies

$$Y^{-1}(z) = 3DX^{-1}(z) - X^{-1}(z)T'(D - DT'Y^{-1}(z)T'D)T'X^{-1}(z). \tag{4.7}$$

It follows directly from the properties of $G(z), Q$ that

$$\|D\| \le C \tag{4.8}$$

uniformly in $z \in (0,1]$. The result of Lemma follows from (4.1), (4.7) and (4.8). The proof is completed.

Theorem 4.3 *Let* $\rho(z) = 3D(Y^{-1}(z)g, h)$, *where* $g(n,m) = 3D\psi_0(n)\overline{\psi_0(m)},\ \psi_0 \in l^2(\mathbf{Z}^d)$, $h \in l^2(\mathbf{Z}^{2d})$. *The uniform in* z, ψ_0, h *estimate holds:*

$$|\rho(z)| \le C\|\psi_0\|^2 \max_{n,m} |((X^*)^{-1}(z)Th)(n,m)| +$$

$$C(\|T'g\| + \|T'X^{-1}(z)Tg\|)(\|T'h\| + \|T'(X^*)^{-1}(z)Th\|).$$

Proof. Lemma 4.2 and $T + T' = 3DI$ imply

$$\rho(z) = 3D((T+T')X^{-1}(z)(T+T')g, h) + (X^{-1}(z)T'M(z)T'X^{-1}(z)g, h).$$

As $\|M(z)\| \le C$ (Lemma 4.2) and $\|T'X^{-1}(z)T'\| \le C$ (Lemma 4.1), one estimates:

$$|\rho(z)| \le |(TX^{-1}(z)Tg, h)| + |(T'X^{-1}(z)Tg, h)| +$$

$$|(TX^{-1}(z)T'g, h)| + C\|T'g\|\|T'h\| + C\|T'X^{-1}(z)g\|\|T'(X^*)^{-1}(z)h\|. \tag{4.9}$$

Further,

$$(TX^{-1}(z)Tg, h) = 3D(Tg, T(X^*)^{-1}(z)Th) = 3D\sum_n |\psi_0(n)|^2((X^*)^{-1}(z)Th)(n,n). \tag{4.10}$$

Using again the result of Lemma 4.1 for operators $X(z), X^*(z)$, one estimates:

$$\|T'X^{-1}(z)g\| \le C(\|T'g\| + \|T'X^{-1}(z)Tg\|), \tag{4.11}$$

$$\|T'(X^*)^{-1}(z)h\| \le C(\|T'h\| + \|T'(X^*)^{-1}(z)Th\|). \tag{4.12}$$

The equality (4.10) and estimates (4.9), (4.11), (4.12) give the statement of the Theorem. The proof is completed.

It follows from the Theorem that the key quantities to estimate are

$$J_1 = 3D \max_{n,m} |((X^*)^{-1}(z)Th)(n,m)|,$$

$$J_2 = 3D\|T'X^{-1}(z)Tg\|, \ \ J_3 = 3D\|T'(X^*)^{-1}(z)Th\|.$$

It turns out that the functions $X^{-1}(z)Tg, \ (X^*)^{-1}(z)Th$ can be calculated explicitely in Fourier-representation. Let $f \in l^2(\mathbf{Z}^{2d}), \ \widehat{f}$- its Fourier transform:

$$\widehat{f}(s,k) = 3D \sum_{n,m} \exp(-i(s,n) - i(k,m))f(n,m),$$

$$f(n,m) = 3D(2\pi)^{-2d} \int_{[-\pi,\pi]^d} ds \int_{[-\pi,\pi]^d} dk \exp(i(s,n) + i(k,m)\widehat{f}(s,k).$$

As the function $\exp(i(k,m)\widehat{f}(s,k)$ is 2π-periodic in $k_j, \ j = 3D1,...d,$ for any s, one can change the variables $r = 3Dk + s$ and write

$$\int_{[-\pi,\pi]^d} dk \exp(i(s,n)+i(k,m)\widehat{f}(s,k) = 3D \int_{[-\pi,\pi]^d+s} dr \exp(i(s,n-m)+i(r,m))\widehat{f}(s,r-s) = 3D$$

$$\exp(i(s,n-m)) \int_{[-\pi,\pi]^d} dr \exp(i(r,m))\widehat{f}(s,r-s).$$

We shall use the variables s, r and for any $f \in l^2(\mathbf{Z}^{2d})$ we shal= l note $\widetilde{f}(s,r) \equiv \widehat{f}(s,r-s)$, so

$$f(n,m) = 3D(2\pi)^{-2d} \int_{[-\pi,\pi]^{2d}} dsdr \exp(i(s,n-m) + i(r,=m)\widetilde{f}(s,r), \tag{4.13}$$

$$\|f\|^2 = 3D(2\pi)^{-2d} \int_{[-\pi,\pi]^{2d}} dsdr|\widetilde{f}(s,r)|^2. \tag{4.14}$$

If one takes $n = 3Dm$ in (4.13), one gets

$$\widetilde{Tf}(s,r) = 3D(2\pi)^{-d} \int_{[-\pi,\pi]^d} ds\widetilde{f}(s,r), \tag{4.15}$$

i.e. $\widetilde{Tf}(s,r)$ does not depend on s. We shall write therefore $\widetilde{Tf}(r)$. One can show also from (4.13) that

$$(\widetilde{\Delta f})(s,r) = 3DL(s,r)\widetilde{f}(s,r), \tag{4.16}$$

where

$$L(s,r) = 3D4 \sum_{j=3D1}^{d} \sin(r_j/2)\sin(s_j - r_j/2), \quad r = 3D(r_1,...,r_d), \quad s = 3D(s_1,...,s_d).$$

We can calculate now $X^{-1}(z)Tf, \ (X^*)^{-1}(z)Tf$ for any f.

Theorem 4.4 *Let $0 < z \leq 1,\ f \in l^2(\mathbf{Z}^{2d}),\ u =3DX^{-1}(z)Tf,\ v =3D(X^*)^{-1}(z)Tf$. The identities hold:*

$$\tilde{u}(s,r) =3D\frac{\widetilde{Tf}(r)}{1-\eta_z(r)}(iL(s,r)+z+1)^{-1}, \tag{4.17}$$

$$\tilde{v}(s,r) =3D\frac{\widetilde{Tf}(r)}{1-\eta_z(r)}(-iL(s,r)+z+1)^{-1}, \tag{4.18}$$

where $\eta_z(r)$ is some real function such that for any r, z

$$0 < \eta_z(r) \leq 1 - C(z+r^2) \tag{4.19}$$

with positive uniform constant C.

Proof. As $X(z)u =3DTf$, one can write with (4.15), (4.16) the equation for u in variables s, r:

$$(iL(s,r)+z+1)\tilde{u}(s,r) - (2\pi)^{-d}\int_{[-\pi,\pi]^d} ds\tilde{u}(s,r) =3D\widetilde{Tf} = (r). \tag{4.20}$$

Let $p(r) =3D\widetilde{Tu}(r)$. The equations (4.20) and (**??**) yield

$$p(r) =3D(\widetilde{Tf}(r)+p(r))\eta_z(r), \tag{4.21}$$

where

$$\eta_z(r) =3D(2\pi)^{-d}\int_{[-\pi,\pi]^d} ds(iL(s,r)+z+1)^{-1}.$$

Using the periodicity of $L(s,r)$ in s_j, one can write

$$\eta_z(r) =3D(2\pi)^{-d}\int_{[-\pi,\pi]^d} ds \left(4i\sum_{j=3D1}^{d}\sin(r_j/2)\sin(s_j)+z+1\right)^{-1}.$$

Let $K(s,r) =3D4\sum\sin(r_j/2)\sin(s_j)$. As $K(-s,r) =3D-K(s,r)$, we= get

$$\eta_z(r) =3D(z+1)(2\pi)^{-d}\int_{[-\pi,\pi]^d} ds\left((z+1)^2+K^2(s,r)\right)^{-1}. \tag{4.22}$$

One sees that $\eta_z(r)$ is real and $0 < \eta_z < (z+1)^{-1}$ for any r, z. Further, one can write

$$(z+1)^{-1}-\eta_z(r) =3D(2\pi)^{-d}(z+1)^{-1}\int_{[-\pi,\pi]^d} dsK^2(s,r)\left((z+1)^2+K^2(s,r)\right)^{-1}. \tag{4.23}$$

As $K^2(s,r) \leq 16d^2,\ z \in (0,1]$, (4.23) implies

$$(z+1)^{-1}-\eta_z(r) \geq C\int_{[-\pi,\pi]^d} dsK^2(s,r) \tag{4.24}$$

with some positive uniform constant C. The calculation gives

$$\int_{[-\pi,\pi]^d} dsK^2(s,r) = 3D16\pi \sum_{j=3D1}^{d} \sin^2(r_j/2) =\geq Cr^2, \quad r \in [-\pi,\pi]^d. \tag{4.25}$$

The estimates (4.24), (4.25) imply the estimate (4.19) for $\eta_z(r)$. Now we go back to the equation (4.21) and find

$$p(r) = 3D\widetilde{Tf}(r)\eta_z(r)(1-\eta_z(r))^{-1} \tag{4.26}$$

The identities (4.20) and (4.26) yield (4.17). For the adjoint operator the proof of (4.18) is the same. The proof is completed.

Corollary 4.5 *The uniform in n, m, z estimate holds:*

$$|((X^*)^{-1}(z)Th)(n,m)| \leq C \int_{[-\pi,\pi]^d} dr|\widetilde{Th}(r)|(r^2+z)^{-1}.$$

Proof. As $L(s,r)$ is real, the result follows directly from Theorem 4.4 and (4.13).

Corollary 4.6 *The uniform in z estimates hold:*

$$\|T'X^{-1}(z)Tg\|^2 \leq C \int_{[-\pi,\pi]^d} dr|\widetilde{Tg}(r)|^2 r^2 (r^2+z)^{-2}, \tag{4.27}$$

$$\|T'(X^*)^{-1}(z)Th\|^2 \leq C \int_{[-\pi,\pi]^d} dr|\widetilde{Th}(r)|^2 r^2 (r^2+z)^{-2}. \tag{4.28}$$

Proof. Let $u = 3DX^{-1}(z)Tg$. The equalities (4.17) and (4.26) imply

$$\widetilde{T'u}(s,r) = 3D\frac{\widetilde{Tg}(r)}{1-\eta_z(r)} \left((iL(s,r)+z+1)^{-1} - \eta_z(r)\right). \tag{4.29}$$

As $|L(s,r)| \leq C|r|$ uniformly in s, r, it is ea= sy to show that

$$|(iL(s,r)+z+1)^{-1} - \eta_z(r)| \leq C|r|. \tag{4.30}$$

The equality (4.29) and the estimates (4.30), (4.19) yield

$$|\widetilde{T'u}(s,r)| \leq C|\widetilde{Tg}(r)| \cdot |r|(r^2+z)^{-1},$$

which together with (4.14) gives (4.27). For the adjoint operator the proof of (4.28) is the same. The proof is completed.

5 Bounds for inside occupation time and lower bounds for moments.

First, we shall obtain lower bounds for the expectations

$$\rho_p(z) = 3D\mathbf{E}[\int_0^{+\infty} dt \exp(-tz)|X|^p(t, om)] = 3D\sum_n |n|^p f(n, n, z).$$

To do it, one should be able to estimate the quantities like

$$M_B(z) = 3D \sum_{n:|n|\le B} f(n, n, z).$$

Lemma 5.1 *Let $\psi_0 \in l^2(\mathbf{Z}^d)$. The uniform in $z \in (0, 1/3]$, $B \ge 2$, $zB^2 \le 1$ estimates hold:*

$$|M_B(z)| \le CBz^{-\frac{1}{2}},\ d = 3D1;\ |M_B(z)| \le CB^2\sqrt{|\log z|(1 + |\log zB^2|)},\quad d = 3D2;$$

$$|M_B(z)| \le CB^{\frac{d+2}{2}},\ d \ge 3.$$

Proof. First, as $f(n, n, z) \ge 0$, we have

$$M_B(z) \le \sum_{n:\forall j |n_j| \le B} f(n, n, z) \equiv N(z). \tag{5.1}$$

One can write $N(z)$ as $N(z) = 3D(TY^{-1}(z)g, h)$, where $g(n, m) = 3D\psi_0(n)\overline{\psi_0(m)}$,

$$h(n, m) = 3D\delta_{n,m} \prod_{j=3D1}^{d} F(|n_j| \le B).$$

As $T'h = 3D0$, Theorem 4.3 gives the estimate:

$$|N(z)| \le CJ_1(z) + (C(\psi_0) + J_2(z))J_3(z), \tag{5.2}$$

where

$$J_1(z) = 3D \max_{n,m} |((X^*)^{-1}(z)Th)(n, m)|,$$

$$J_2(z) = 3D\|T'X^{-1}(z)Tg\|,\ J_3(z) = 3D\|T'(X^*)^{-1}(z)Th\|.$$

Corollaries 4.5 and 4.6 imply

$$J_1(z) \le C \int_{[-\pi,\pi]^d} dr |\widetilde{Th}(r)|(r^2 + z)^{-1}, \tag{5.3}$$

$$J_2^2(z) \le C \int_{[-\pi,\pi]^d} dr |\widetilde{Tg}(r)| =^2 r^2(r^2 + z)^{-2}, \tag{5.4}$$

$$J_3^2(z) \le C \int_{[-\pi,\pi]^d} dr |\widetilde{Th}(r)| =^2 r^2 (r^2+z)^{-2}. \tag{5.5}$$

One can explicitly calculate the function $\widetilde{Th}(r)$, $r \ne= 0$:

$$\widetilde{Th}(r) = 3D \sum_n \exp(-i(r,n)) h(n,n) = 3D \prod_{j=3D1}^{d} \sum_{n_j : |n_j| \le B} \exp(-i n_j r_j) = 3D$$

$$\prod_{j=3D1}^{d} \frac{\sin(r_j B_1)}{\sin(r_j/2)},$$

where $B_1 \in [B-1/2, B+1/2]$. Using the uniform estimate $|\sin x| \ge C|x|$, $x \in [-\pi/2, \pi/2]$ and changing the variable $\rho = 3DB_1 r$, we obtain:

$$J_1(z) \le C \int_{[-\pi,\pi]^d} dr (r^2+z)^{-1} \prod_{j=3D1}^{d} \frac{|\sin(r_j B_1)|}{|r_j|} = 3D$$

$$CB_1^2 \int_{[-\pi B_1, \pi B_1]^d} d\rho (\rho^2 + zB_1^2)^{-1} \prod_{j=3D1}^{d} \frac{|\sin \rho_j|}{|\rho_j|}. \tag{5.6}$$

It is easy to estimate the integral in (5.6) and to show that

$$J_1(z) \le CBz^{-1/2}, \quad d = 3D1; \quad J_1(z) \le CB^2(|\log(zB^2)|+1), \quad d = 3D2;$$

$$J_1(z) \le CB^2, \quad d \ge 3. \tag{5.7}$$

Estimating $J_3^2(z)$ in the same manner, one gets

$$J_3^2(z) \le CB^2 z^{-1/2}, \quad d = 3D1; \quad J_3^2(z) \le CB^4(|\log(zB^2)|+1), \quad d = 3D2;$$

$$J_3^2(z) \le CB^{d+2}, \quad d \ge 3. \tag{5.8}$$

Let us estimate now $J_2(z)$. As $\widetilde{Tg}(r) = 3D \sum_n \exp(-i(n,r)) |\psi_0(n)|^2$, we have $|\widetilde{Tg}(r)| \le \|\psi_0\|^2$ uniformly in r. Therefore, one gets from (5.4):

$$J_2^2(z) \le Cz^{-1/2}, \quad d = 3D1; \quad J_2^2(z) \le C|\log z|, \quad d = 3D2;$$

$$J_2^2(z) \le C, \quad d \ge 3. \tag{5.9}$$

The result of Lemma follows now from (5.1), (5.2) and (5.7)-(5.9). The proof is completed.

Comment 5.2 *It follows from Theorem 2.3 for $\alpha = 3D0$ that $|M_B(z)| \le Cl_d(z)B^{d/2}$. One can verify that the estimates of Lemma 5.1 give better results with probability 1 in dimensions $d = 3D1, 2$ and significantly better results for the averaged and with probability 1 for $d \ge 3$.*

Theorem 5.3 *Let $\psi_0 \in l^2(\mathbf{Z}^d)$, $\psi_0 \neq 0$. The following estimates hold with positive constants uniform in $z \in (0, 1/3]$:*

$$\rho_p(z) \geq Cz^{-1-\frac{p}{2}}, \quad d = 3D1; \quad \rho_p(z) \geq Cz^{-1-\frac{p}{2}} |\log z|^{-p\alpha}, \ \alpha > \frac{1}{4}, \quad d = 3D2;$$

$$\rho_p(z) \geq Cz^{-1-\frac{2p}{d+2}}, \quad d \geq 3.$$

Proof. For any $B > 0$ one can estimate:

$$\rho_p(z) \geq \sum_{n:|n|>B} |n|^p f(n, n, z) \geq B^p \left(z^{-1} \|\psi_0\|^2 - \sum_{n:|n|\leq B} f(n, n, z) \right) = 3D$$

$$B^p(z^{-1}\|\psi_0\|^2 - M_B(z)). \tag{5.10}$$

We have again used the identity $\sum_n f(n, n, z) = 3Dz^{-1}\|\psi_0\|^2$. We choose B big but = such that

$$|M_B(z)| \leq \frac{1}{2} z^{-1} \|\psi_0\|^2. \tag{5.11}$$

To do it, we take

$$B = 3D\delta z^{-\frac{1}{2}}, \quad d = 3D1; \quad B = 3D\delta z^{-\frac{1}{2}} |\log z|^{-\alpha}, \ \alpha > \frac{1}{4}, \quad d = 3D2;$$

$$B = 3D\delta z^{-\frac{2}{d+2}}, \quad d \geq 3.$$

For δ sufficiently small and depending only on ψ_0, Lemma 5.1 yields (5.11). The result of the Theorem follows now from (5.10). The proof is completed.

Corollary 5.4 *The following uniform estimates for the averaged diffusion constant hold:*

$$D(z) \geq C, \ d = 3D1; \quad D(z) \geq C_\alpha |\log z|^{-2\alpha}, \ \alpha > \frac{1}{4}, \quad d = 3D2; \quad D(z) \geq Cz^{\frac{d-2}{d+2}}, \ d \geq 3.$$

The lower bounds for expectations $\rho_p(z) = 3D\mathbf{E}[\eta_p(z, \cdot)]$ do not give, however, lower bounds for $\eta_p(z, \omega)$ with probability 1. To obtain such bounds, one shall use the RAGE-like results for the inside occupation tim= e, namely,

$$\lim_{T\to+\infty} T^{-1} \int_0^T dt \sum_{n:|n|\leq b(t)} |\psi(t, n, \omega)|^2 = 3D0 \tag{5.12}$$

for some increasing $b(t)$ with probability 1. The result we prove below improves significantly the corresponding result of [4].

Theorem 5.5 *Let $b(t) = 3Dt^{\frac{1}{2}}(\log(t+2))^{-r}$, where $r > 1$, $d = 3D1$ and $r > \frac{3}{4}$, $d = 3D2$; $b(t) = 3Dt^{\frac{2}{d+2}}(\log(t+2))^{-r}$, $r > \frac{2}{d+2}$, $d \geq 3 =$. With probability 1 for any $\psi_0 \in l^2(\mathbf{Z}^d)$*

$$\lim_{T\to+\infty} L(T, \omega) \equiv \lim_{T\to+\infty} T^{-1} \int_0^T dt \sum_{n:|n|\leq b(T)} |\psi(t, n, \omega)|^2 = 3D0. \tag{5.13}$$

In particular, (5.12) holds.

Proof. Let us fix some ψ_0 and estimate the functional $L(T,\omega)$. As in (3.11), one has

$$L(T,\omega) \leq CzG(z,\omega), \tag{5.14}$$

where $z = 3DT^{-1}$,

$$G(z,\omega) = 3D \int_0^{+\infty} dt \exp(-tz) \sum_{n:|n|\leq b(z^{-1})} |\psi(t,n,\omega)|^2.$$

One can calculate expectations of $G(z,\cdot)$ in the same manner as it was made for $\eta_p(z,\cdot)$. One gets easily

$$\mathbf{E}[G(z,\cdot)] = 3Dz \sum_{n:|n|\leq b(z^{-1})} f(n,n,z).$$

With the choise of functions $b(t)$ in the statement of the Theorem, one can verify that the estimates of Lemma 5.1 yield in any dimensi= on:

$$\mathbf{E}[zG(z,\cdot)] \leq C|\log z|^{-r}$$

with some $r > 1$. As the function

$$g(t,z,\omega) = 3D \sum_{n:|n|\leq b(z^{-1})} \psi(t,n,\omega)|^2$$

is decreasing in z for z small enough, we can apply Lemma 3.6 to the functional $h(z,\omega) = 3DzG(z,\omega)$ and get

$$\lim_{z\to+0} zG(z,\omega) = 3D0 \tag{5.15}$$

with probability 1. It follows form (5.14) and (5.15) that (5.13) holds. As the function $b(t)$ is increasing for t big enough, (5.13) implies (5.12).

The equalities (5.12), (5.13) are proven for any $\psi_0 \in l^2(\mathbf{Z}^d)$ for $\mathbf{P}$-a.e. ω. By density arguments, (5.12) and (5.13) hold with probability 1 for any ψ_0. The proof is completed.

It follows from this Theorem that for any ball $K_R = 3D\{n : |n| \leq R\}$, $R > 0$, the average occupation tim= e tends to 0 (in the Cesaro sense) as $T \to +\infty$:

$$\lim_{T\to+\infty} \langle W_R\rangle_{T,\omega} = 3D0, \quad W_R(t,\omega) = 3D \sum_{n:|n|\leq R} |\psi(t,n,\omega)|^2.$$

One can estimate the rate of decay of $\langle W_R\rangle_{T,\omega}$ as $T \to +\infty$.

Theorem 5.6 *Let $\psi_0 \in l^2(\mathbf{Z}^d)$, $R > 0$, $r > 1$. With probability 1 the follow= ing uniform in $T \geq 3$ estimates hold:*

$$\langle W_R\rangle_{T,\omega} \leq CT^{-1/2}(\log T)^r,\ d = 3D1; \quad \langle W_R\rangle_{T,\omega} \leq CT^{-1}(\log T)^r,\ d = 3D2;$$

$$\langle W_R\rangle_{T,\omega} \leq CT^{-1},\ d \geq 3.$$

Proof. We can show exactly in the same manner as in the proof of Theorem **??** that

$$\langle W_R \rangle_{T,\omega} \leq CzG(z,\omega), \tag{5.16}$$

where $z = 3DT^{-1}$,

$$G(z,\omega) = 3D\int_0^{+\infty} dt \exp(-tz) \sum_{n|n|\leq R} |\psi(t,n,\omega)|^2,$$

$$\mathbf{E}[G(z,\cdot)] = 3D \sum_{n:|n|\leq R} f(n,n,z) = 3DM_R(z).$$

The statement of the Theorem follows directly from (5.16), Lemma 5.1 and Corollary 3.7 in dimensions $d = 3D1, 2$ and the third statement of Lemma 3.6 in dimensions $d \geq 3$. The proof is completed.

Corollary 5.7 *In dimensions $d \geq 3$ for any ψ_0, R with probability 1 the total occupation time of the ball K_R is finite:*

$$\int_0^{+\infty} dt \sum_{n:|n|\leq R} |\psi(t,n,\omega)|^2 < +\infty.$$

In particular, for any $n \in \mathbf{Z}^d$

$$\int_0^{+\infty} dt |\psi(t,n,\omega)|^2 < +\infty.$$

Now we can prove the lower bounds for the moments $|X|^p$ with probability 1.

Theorem 5.8 *Let $\psi_0 \in l^2(\mathbf{Z}^d)$, $\psi_0 \neq 0$. With probability 1 for T big enough the uniform estimate holds:*

$$\langle |X|^p \rangle_{T,\omega} \geq \frac{1}{2} \|\psi_0\|^2 \cdot b^p(T),$$

where $b(T)$ was defined above.

Proof. As in the proof of Theorem 5.3, one estimates:

$$\langle |X|^p \rangle_{T,\omega} \geq b^p(T) \left(\|\psi_0\|^2 - L(T,\omega) \right),$$

where $L(T,\omega)$ was defined in Theorem 5.5. The statement of the Theorem follows directly from (5.13). The proof is completed.

Finally, we shall discuss the behaviour of the correlation function:

$$C_{\psi_0,\psi_1}(T,\omega) = 3DT^{-1} \int_0^T |(\psi(t,\omega),\psi_1)|^2 dt,$$

where $\psi_0, \psi_1 \in l^2(\mathbf{Z}^d)$.

Theorem 5.9 *Let $\psi_0, \psi_1 \in l^2(\mathbf{Z}^d)$, $\quad r > 1$. With probability 1 the foll= owing uniform in $T \geq 3$ estimates hold:*

$$C(T,\omega) \leq CT^{-\frac{1}{2}}(\log T)^r,\ d = 3D1;\quad C(T,\omega) \leq T^{-1}(\log T)^r,\ d = 3D2;$$
$$C(T,\omega) \leq CT^{-1}, \quad d \geq 3.$$

Proof. Again,

$$C(T,\omega) \leq CzG(z,\omega), \tag{5.17}$$

where $z = 3DT^{-1}$,

$$G(z,\omega) = 3D \int_0^{+\infty} dt \exp(-tz) |(\psi(t,\omega), \psi_1)|^2,$$

and

$$\widehat{G}(z) = 3D\mathbf{E}[G(z,\cdot)] = 3D \left(Y^{-1}(z)g, h\right),$$

where $g(n,m) = 3D\psi_0(n)\overline{\psi_0(m)}$, $\quad h(n,m) = 3D\psi_1(n)\overline{\psi_1(m)}$. As in the proof of Lemma = fll1,

$$\widehat{G}(z) \leq CJ_1 + (C + J_2)(C + J_3),$$

where J_1, J_2, J_3 are estimated as (5.3)-(5.5) and C are some constant depending on ψ_0, ψ_1 but uniform in z. As

$$|\widetilde{Tg}(r)| \leq \|\psi_0\|^2, \quad |\widetilde{Th}(r)| \leq \|\psi_1\|^2,$$

one can show easily that

$$\widehat{G}(z) \leq Cz^{-\frac{1}{2}},\ d = 3D1;\quad \widehat{G}(z) \leq C|\log z|,\ d = 3D2;\quad \widehat{G}(z) \leq C,\ d \geq 3. \tag{5.18}$$

One can apply the Corollary 3.7 for $d = 3D1, 2$ and Lemma 3.6 for $d \geq 3$ and show that (5.17)-(5.18) imply the statement of the Theorem. The proof is completed.

Corollary 5.10 *For $d \geq 3$ with probability 1*

$$\int_0^{+\infty} |(\psi(t,\omega), \psi_1)|^2 dt < +\infty. \tag{5.19}$$

Comment 5.11 *For Schrödinger equation with Hamiltonian H wich does not depend on t= ime, where $\psi(t) = 3D\exp(-itH)\psi_0$, (5.19) implies the absolute continuity of spectral measures μ_{ψ_0,ψ_1}, μ_{ψ_0}.*

Corollary 5.12 *Let $\psi_1 = 3D\psi_0 \neq 0$. Consider the autocorrelation function*

$$C_{\psi_0}(T,\omega) = 3DT^{-1} \int_0^T |(\psi(t,\omega), \psi_0)| =^2.$$

For $d \geq 2$ with probability 1

$$\lim_{T\to+\infty} \frac{\log C(T,\omega)}{\log T} = 3D - 1.$$

For $d = 3D1$ with probability 1

$$\limsup_{T\to+\infty} \frac{\log C(T,\omega)}{\log T} \leq -1/2.$$

Proof. As $|(\psi(0,\omega),\psi_0)| = 3D\|\psi_0\|^2 > 0$, by continuity in time

$$C(T,\omega) \geq \delta T^{-1} \tag{5.20}$$

with some $\delta > 0$. The result of the Corollary follows directly from (5.20) and Theorem 5.9. The proof is completed.

One can conjecture that for $d = 3D1$

$$\limsup_{T\to+\infty} \frac{\log C(T,\omega)}{\log T} = 3D - 1/2.$$

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