A new inductive approach to the lace expansion for self-avoiding walks

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Abstract

We introduce a new inductive approach to the lace expansion, and apply it to prove Gaussian behaviour for the weakly self-avoiding walk on $\mathbb{Z}^d$ where loops of length m are penalised by a factor $e^{-\beta/m^p}$ $(0 < \beta \ll 1)$ when:
(1) $d > 4$, $p \geq 0$;
(2) $d \leq 4$, $p > \frac{4-d}{2}$.
In particular, we derive results first obtained by Brydges and Spencer (and revisited by other authors) for the case $d > 4$, $p = 0$. In addition, we prove a local central limit theorem, with the exception of the case $d > 4$, $p = 0$.

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Contents

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1 Introduction and main theorems

Since its introduction by Brydges and Spencer [1] in 1985, the lace expansion has been developed into a powerful tool for the analysis of mean-field behaviour for self-avoiding walks, lattice trees and lattice animals, and percolation. Our purpose in this paper is to describe a new inductive approach to the lace expansion, which is simple and direct. We believe this approach to be sufficiently flexible so as to allow for a simplification and extension of the results that have been obtained with the lace expansion so far.

We develop the method in the setting of a model of weakly self-avoiding walks on $\mathbb{Z}^d$ where loops of length m are penalised by a factor $e^{-\beta/m^p}$ ($0 < \beta \ll 1$). For $d > 4$, $p \geq 0$ we recover the results proved by Brydges and Spencer [1] for $d > 4$, $p = 0$, namely, that the mean-square displacement is asymptotically linear in the number of steps and that the scaling limit of the endpoint is Gaussian. For $d \leq 4$, $p > \frac{4-d}{2}$ we prove similar results, thereby showing that diffusive behaviour persists for lower dimensions at the cost of sufficiently lowering the penalty of long loops. In addition, we prove a local central limit theorem for $d > 4$, $p > 0$ and for $d \leq 4$, $p > \frac{4-d}{2}$. This leaves open the important case $d > 4$, $p = 0$. Other aspects of the model have been studied by Caracciolo *et al.* [2] and Kennedy [9].

Several approaches to the lace expansion for self-avoiding walks have appeared previously in the literature, the principal difference between the approaches being the methods used to obtain convergence of the expansion. Brydges and Spencer [1] used induction on finite memory and advanced the induction with the help of generating functions (Laplace transforms) and complex analysis (to invert generating functions). Slade [14] proved convergence via generating functions with no induction argument, while using a finite memory cutoff. Hara and Slade [7] (see also Madras and Slade [11]) proved convergence via generating functions, but avoided the use of finite memory. Golowich and Imbrie [5] used induction on finite memory together with a cluster expansion (also called a polymer expansion). Khanin *et al.* [10] used induction on the length of the walk together with a cluster expansion.

Our method involves induction on the length of the walk, but does *not* use generating functions, complex analysis, finite memory, or a cluster expansion. The induction step is direct and relatively simple.

To indicate the nature of the induction, we begin by introducing the fundamental object of study. For $x \in \mathbb{Z}^d$, we set $c_0(x) = \delta_{0,x}$ and, for $n \geq 1$, $p \in \mathbb{R}$, $\beta \geq 0$, we define

$$\begin{aligned} c_n(x) &= \sum_{\substack{\omega : 0 \to x \\ |\omega| = n}} e^{-\beta \sum_{0 \leq s < t \leq n} \frac{U_{st}(\omega)}{|s-t|^p}} \\ &= \sum_{\substack{\omega : 0 \to x \\ |\omega| = n}} \prod_{0 \leq s < t \leq n} (1 - \lambda_{st} U_{st}(\omega)), \end{aligned} \tag{1.1}$$

where

$$U_{st}(\omega) = \begin{cases} 1 & \text{if } \omega(s) = \omega(t) \\ 0 & \text{if } \omega(s) \neq \omega(t) \end{cases}, \tag{1.2}$$

$$\lambda_{st} = \lambda_{st}(\beta, p) = 1 - e^{-\frac{\beta}{|s-t|^p}}, \tag{1.3}$$

and the sum in (1.1) is over all n-step simple random walk paths from 0 to x. The Fourier transform of (1.1) is written

$$\hat{c}_n(k) = \sum_{x \in \mathbb{Z}^d} c_n(x) e^{ik \cdot x}, \qquad k \in (-\pi, \pi]^d, \tag{1.4}$$

and we use the abbreviation

$$c_n = \hat{c}_n(0) = \sum_{x \in \mathbb{Z}^d} c_n(x). \tag{1.5}$$

We also need the characteristic function of the step distribution of simple random walk, which is

$$\hat{D}(k) = \frac{1}{d} \sum_{l=1}^{d} \cos k_l, \qquad k = (k_1, \ldots, k_d). \tag{1.6}$$

The lace expansion is a combinatorial identity in terms of a function $\hat{\pi}_m(k)$, defined in (A.6), stating that

$$\hat{c}_{n+1}(k) = 2d\hat{D}(k)\hat{c}_n(k) + \sum_{m=2}^{n+1} \hat{\pi}_m(k)\hat{c}_{n+1-m}(k). \tag{1.7}$$

A basic step in any lace expansion analysis is the observation that $\hat{\pi}_m$ can be bounded in terms of $(\hat{c}_j)_{0 \leq j < m}$. We emphasise that here only $0 \leq j < m$ appear, not $j = m$.

This means that the right-hand side of (1.7) can be analysed solely in terms of $(\hat{c}_j)_{0\leq j\leq n}$, which opens up the possibility of an *inductive analysis*, with the induction on n. This is precisely what we shall do.

Our approach should be contrasted with the inductive approaches in Golowich and Imbrie [5] (induction on finite memory) and Khanin *et al.* [10] (induction on n). In these papers the authors expand the right-hand side of (1.7) by iteration, until all $(\hat{c}_j(k))_{0\leq j\leq n}$ have been replaced by $(\hat{\pi}_i(k))_{0\leq i\leq n}$, and then use a cluster expansion to handle the myriad factors of $\hat{\pi}_i(k)$. Our approach, however, uses (1.7) in its current form, without further iteration or expansion. In this way we avoid a significant level of technical difficulty.

The following two theorems are our main results. We define

$$\epsilon = p + \frac{d-4}{2} > 0, \tag{1.8}$$

which turns out to be the key parameter in the model. The first theorem extends the results of Brydges and Spencer [1] for $d>4$, $p=0$.

Theorem 1.1 *Suppose that either $d>4$, $p\geq 0$ or $d\leq 4$, $p>\frac{4-d}{2}$. Then there is a $\beta_0=\beta_0(d,p)>0$ such that for $\beta<\beta_0$,*
(a)

$$c_n = A\mu^n[1+\mathcal{O}(n^{-\epsilon})], \tag{1.9}$$

(b)

$$\frac{1}{c_n}\sum_x x^2 c_n(x) = \begin{cases} Dn[1+\mathcal{O}(n^{-1\wedge\epsilon})] & \epsilon\neq 1 \\ Dn[1+\mathcal{O}(n^{-1}\log n)] & \epsilon = 1 \end{cases}, \tag{1.10}$$

(c)

$$\frac{1}{c_n}\hat{c}_n\Big(\frac{k}{\sqrt{Dn}}\Big) = e^{-\frac{k^2}{2d}[1+\mathcal{O}(n^{-\delta'})]}, \tag{1.11}$$

where $\mu, A, D>0$ are constants (depending on d,p,β), ϵ is given by (1.8), $\delta'\in(0,1\wedge\epsilon)$ is arbitrary, and the error estimate in (c) is uniform in $k\in\mathbb{R}^d$ provided $\|k\|_1(\log n)^{-1/2}$ is sufficiently small.

The second theorem is a local central limit theorem, but leaves open the important case $d>4$, $p=0$. (At the end of Section 4.2, we comment on what goes wrong in our proof of of the local central limit theorem for this case.)

Theorem 1.2 *Suppose that either $d>4$, $p>0$ or $d\leq 4$, $p>\frac{4-d}{2}$. Then there is a $\beta_0=\beta_0(d,p)>0$ such that for $\beta<\beta_0$,*

$$\frac{c_n(x)}{c_n} \sim 2\left(\frac{d}{2\pi Dn}\right)^{\frac{d}{2}} e^{-\frac{dx^2}{2Dn}} \quad \text{as } n\to\infty, \tag{1.12}$$

where n is taken to have the same parity as $\|x\|_1$, and the error estimate is uniform in $x\in\mathbb{Z}^d$ provided $\|x\|_1(n\log n)^{-1/2}$ is sufficiently small. For $d>4$, $p=0$, the following weaker result holds:

$$\sup_{x\in\mathbb{Z}^d}\frac{c_n(x)}{c_n} \leq \mathcal{O}(n^{-\frac{d}{2}}). \tag{1.13}$$

Our paper is organised as follows. The lace expansion is discussed in Appendix A, where $\hat{\pi}_m(k)$ is defined, and (1.7) is proved in Lemma A.1. The induction hypotheses used in the proof of Theorems 1.1 and 1.2 are stated in Section 2. In Section 2.4, we show how the induction hypotheses involving $1 \leq m \leq n$ can be used to bound $\hat{\pi}_{n+1}(k)$. This will be the primary driving force of the induction argument. The only fact that we will subsequently need about $\hat{\pi}_m(k)$ is that, under our induction hypotheses, it satisfies the bounds in Lemma 2.3. This lemma requires standard lace expansion methods, described in the Appendix in Section A.2, that are present in one form or another in all previous work on the problem. In Section 3, the induction is advanced. Finally, in Section 4, the completed induction is used to prove Theorems 1.1 and 1.2, the constants μ, A, D are identified, and some discussion is provided of potential extensions of our method.

2 The induction hypotheses (H1–H6)

2.1 Definitions and statement of induction hypotheses

Let $z_0 = \frac{1}{2d}$, and define z_n recursively by

$$z_{n+1} = \frac{1}{2d}\left[1 - \sum_{m=2}^{n+1} \hat{\pi}_m(0) z_n^m\right], \qquad n \geq 0, \tag{2.1}$$

with $\hat{\pi}_m(0)$ given by the Fourier transform of (A.6) at $k = 0$. For $z > 0$ and $n \geq 0$, define

$$A_n(k) = z^n \hat{c}_n(k), \tag{2.2}$$

$$B_n = 2dz - \sum_{m=2}^{n} \nabla^2 \hat{\pi}_m(0) z^m, \tag{2.3}$$

$$C_n = \sum_{m=2}^{n} (m-1)\hat{\pi}_m(0) z^m \tag{2.4}$$

(∇ is the gradient with respect to k) and

$$D_n = \frac{B_n}{1 + C_n}. \tag{2.5}$$

The z–dependence of $A_n(k), B_n, C_n, D_n$ will be left implicit in the notation.

Let $\vec{\pi}$ denote the vector in $\mathbb{R}^d$ whose components are all equal to the number π. Since $c_n(x)$ is nonzero only when $\|x\|_1$ and n have the same parity, we have $A_n(k + \vec{\pi}) = (-1)^n A_n(k)$. Thus it is sufficient to consider $k \in (-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\pi, \pi]^{d-1}$.

The induction hypotheses below involve a number of constants. For ϵ as defined in (1.8), we fix $\gamma, \delta, \delta' > 0$ according to

$$0 < \epsilon - \tfrac{1}{2}\delta < \gamma < \gamma + \delta' < 1 \wedge \epsilon. \tag{2.6}$$

We also fix $K_1, \ldots, K_6$ according to

$$K_3, K_6 \gg K_1, K_2, K_5 \gg K_4 \gg 1. \tag{2.7}$$

The amount by which, for instance, K_2 must exceed K_4 is independent of β and will be determined in the course of the advancement of the induction (see Sections 2.3 and 3.1–3.5).

We make separate induction hypotheses for small n and large n, using a k–dependent time scale $m(k) \geq 1$ to separate 'small' from 'large'. For $k \in (-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\pi, \pi]^{d-1}$ such that $1 - \hat{D}(k) \leq \frac{\gamma}{2}$ (recall (1.6)), we define

$$m(k) = \gamma \frac{1}{1 - \hat{D}(k)} \log \frac{1}{1 - \hat{D}(k)} \geq 2 \log \Big(\frac{2}{\gamma}\Big) > 1, \tag{2.8}$$

while for k such that $1 - \hat{D}(k) > \frac{\gamma}{2}$, we define $m(k) = 1$.

For $n \geq 1$, we define intervals

$$I_n = [z_n - K_1 \beta n^{-1-\epsilon}, z_n + K_1 \beta n^{-1-\epsilon}]. \tag{2.9}$$

Throughout the rest of this paper we require that either $d > 4$, $p \geq 0$ or $d \leq 4$, $p > \frac{4-d}{2}$, and we fix $\beta < \beta_0$ with $\beta_0 = \beta_0(d,p) > 0$ sufficiently small. Our induction hypotheses are that the following six statements hold, for all $k \in (-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\pi, \pi]^{d-1}$ and all $z \in I_n$:

$(H1)$ For $1 \leq j \leq n$: $\quad |z_j - z_{j-1}| \leq K_1 \beta j^{-2-\epsilon}$.

$(H2)$ For $1 \leq j \leq n$: $\quad |D_j - D_{j-1}| \leq K_2 \beta j^{-1-\epsilon}$.

$(H3)$ For $1 \leq j \leq n \leq m(k)$: $\quad A_j(k) = \prod_{i=1}^{j} [1 - D_i[1 - \hat{D}(k)] + E_i(k)][1 + F_i]$ with $|E_i(k)| \leq K_3 \beta k^2 i^{-\delta'}$, $|F_i| \leq K_3 \beta i^{-1-\epsilon}$.

$(H4)$ For $m(k) \leq j \leq n$: $\quad |A_j(k)| \leq K_4 k^{-4-\delta} j^{-2-\epsilon}$.

$(H5)$ For $m(k) \leq j \leq n$: $\quad |A_j(k) - A_{j-1}(k)| \leq K_5 k^{-2-\delta} j^{-2-\epsilon}$.

$(H6)$ For $1 \leq j \leq n$: $\quad |\nabla^2 A_j(0) - \nabla^2 A_{j-1}(0) + D_j A_{j-1}(0)| \leq K_6 \beta j^{-\epsilon}$.

Hypothesis (H3) is vacuous when $n > m(k)$, while (H4–H5) are vacuous when $n < m(k)$. For $k = 0$, (H3) reduces to $A_j(0) = \prod_{i=1}^{j} [1 + F_i]$.

We begin the induction by verifying (H1–H6) for $n = 1$. Since $z_0 = z_1 = \frac{1}{2d}$, (H1) holds. Since $D_0 = D_1 = 1$, (H2) holds. Since $A_1(k) = 2dz\hat{D}(k)$, with $z \in I_1$, (H3) holds provided $K_3 \geq 2dK_1$ (take $E_1(k) = 0$ and $F_1 = 2dz - 1 = 2d(z - z_1)$). For β small enough, we have $|A_1(k)| \leq 2dz \leq 2$ for all $z \in I_1$. Hence we can choose K_4 large enough that (H4) holds, and since $A_0(k) = 1$, we can also choose K_5 large enough that (H5) holds. Finally, since $A_0(k) = 1$, $\nabla^2 \hat{D}(0) = -1$ and $z \in I_1$, (H6) holds provided $K_6 \geq 2dK_1$.

2.2 Motivation

In this section, we motivate the induction hypotheses.

1. Because of the sub-multiplicativity bound $c_{m+n} \leq c_m c_n$ (which is a simple consequence of the definition of c_n in (1.5)), the limit $\mu = \lim_{n\to\infty} c_n^{1/n}$ exists. The factor z^n in (2.2), with $z \in I_n$ defined by (2.9), is intended to approximate μ^{-n}, and hence to cancel the exponential growth of c_n. Our initial lack of a convenient expression for μ prompts us to formulate the induction hypotheses for a small interval of z–values. The sequence z_n will ultimately converge to μ^{-1}. Hypothesis (H1) drives this convergence and gives some control on the rate. Moreover, as we will see in Section 2.4, (H1) guarantees that the intervals I_j are decreasing: $I_1 \supset I_2 \supset \cdots \supset I_n$. Because the length of these intervals is shrinking to zero, their intersection $\cap_{j=1}^{\infty} I_j$ is a single point, namely μ^{-1}. For large n, if $z \in I_n$, then z^n is close to μ^{-n}. Consequently, as we will see in Section 4.3, $\lim_{n\to\infty} A_n(0) = A$, where A is the amplitude in Theorem 1.1(a).

2. To motivate the recursion (2.1), we begin by substituting (2.2) into (1.7), obtaining

$$A_{n+1}(k) = 2dz\hat{D}(k)A_n(k) + \sum_{m=2}^{n+1} \hat{\pi}_m(k) z^m A_{n+1-m}(k). \tag{2.10}$$

Setting $k = 0$, taking $n \to \infty$, and replacing $A_{n+1-m}(0)$ by its limiting value A, we get

$$1 = 2dz + \sum_{m=2}^{\infty} \hat{\pi}_m(0) z^m \tag{2.11}$$

(with the series not yet shown to be convergent). The recursion (2.1) approximates this relation, namely, by discarding the $\hat{\pi}_m(0)$ for $m > n+1$ that cannot be handled at the n^{th} stage of the induction argument. In Section 4.3, we will show that (2.11) holds for $z = \mu^{-1}$.

3. The quantity D_n defined in (2.5) is an approximation to the diffusion constant D of Theorem 1.1(b). Hypothesis (H2) expresses the convergence of D_n to D and gives some control on the rate. Ignoring the error terms in (H3), replacing D_i by D for $1 \leq i \leq j$, and using the fact that $1 - \hat{D}(k) \sim \frac{k^2}{2d}$ as $k \to 0$, we see that the right-hand side of (H3) is an approximation to the exponential behaviour

$$A_j(k) \approx A_j(0) \exp\Big[- D\frac{k^2 j}{2d}\Big] \tag{2.12}$$

consistent with Theorem 1.2. Note that for $z = \frac{1}{2d}$ and $\beta = 0$, we have $D_i = 1$ for all i, so that (H3) reduces to $A_j(k) = \hat{D}(k)^j$, which is the correct simple random walk behaviour for all j and k.

4. For large j, we require less detailed control of $A_j(k)$, as expressed in (H4–H5). The overlap of (H3) with (H4–H5) for $j = n = m(k)$ places restrictions on the values of K_4 and K_5 (see Section 2.3). Hypothesis (H5) is needed only to advance (H4).

5. We will use (H3–H4) to obtain an estimate for $\|A_j\|_1$ for $1 \leq j \leq n$. This will provide us with a bound on $\|c_j\|_\infty$ for $1 \leq j \leq n$ and, by Lemma A.2, on $\hat{\pi}_m(k)$ for $1 \leq m \leq n+1$. This mechanism drives the induction argument.

6. For simple random walk with $z = \frac{1}{2d}$ and $\beta = 0$, we have $A_j(0) = 1$, $\nabla^2 A_j(0) = -j$, and the diffusion constant is 1. The left-hand side of (H6) is therefore zero for simple random walk, and (H6) is an appropriate generalisation for the interacting model. The form (2.5) of D_{n+1} can be motivated by the following rough argument. Differentiating (2.10) twice with respect to k, setting $k = 0$, and using the fact that odd derivatives vanish, we obtain

$$\begin{aligned} \nabla^2 A_{n+1}(0) &= 2dz[\nabla^2 A_n(0) - A_n(0)] \\ &\quad + \sum_{m=2}^{n+1} \Big[\hat{\pi}_m(0) z^m \nabla^2 A_{n+1-m}(0) + \nabla^2 \hat{\pi}_m(0) z^m A_{n+1-m}(0)\Big]. \end{aligned} \tag{2.13}$$

Approximating $2dz\nabla^2 A_n(0)$ by $[1 - \sum_{m=2}^{n+1} \hat{\pi}_m(0)z^m]\nabla^2 A_n(0)$ (recall (2.1)), $A_{n+1-m}(0)$ by $A_n(0)$ in the last term, and recalling (2.3), we get

$$\nabla^2 A_{n+1}(0) - \nabla^2 A_n(0) \approx -B_{n+1} A_n(0) + \sum_{m=2}^{n+1} \hat{\pi}_m(0) z^m \Big[\nabla^2 A_{n+1-m}(0) - \nabla^2 A_n(0)\Big]. \tag{2.14}$$

Next, approximating $\nabla^2 A_{n+1-m}(0) - \nabla^2 A_n(0)$ by $(m-1)D_{n+1}A_n(0)$, in accordance with (H6), we get (recall (2.4))

$$\nabla^2 A_{n+1}(0) - \nabla^2 A_n(0) \approx -B_{n+1} A_n(0) + D_{n+1} C_{n+1} A_n(0). \tag{2.15}$$

Putting the right-hand side equal to $D_{n+1}A_n(0)$, we find (2.5).

2.3 Extension of (H4–H5)

In this section we show that (H2–H3) imply (H4–H5) for $rm(k) \leq j \leq m(k)$, provided r is sufficiently close to 1. This will be used in Section 3.4. Here, and throughout the rest of this paper,

> C denotes a strictly positive constant that may depend on $d, p, \gamma, \delta, \delta', r$, but *not* on $K_1, \ldots, K_6$, *not* on k, and *not* on β (provided β is sufficiently small, possibly depending on $K_1, \ldots, K_6$). The value of C may change from line to line.

We have already checked that (H4–H5) hold for $j = 1$, so we can restrict attention to $j \geq 2$, which implies $m(k) \geq j > 1$ and hence (recall (2.8)) $1 - \hat{D}(k) \leq \frac{\gamma}{2} \leq \frac{1}{2}$. Since $D_0 = 1$, we have

$$|D_i - 1| \leq CK_2\beta, \qquad 1 \leq i \leq j, \tag{2.16}$$

by (H2), so all factors in the product in (H3) are strictly positive when β is sufficiently small. Using $1 + x \leq e^x$, we therefore have

$$\begin{aligned} 0 \leq A_j(k) &= \prod_{i=1}^{j} \Big[1 - D_i[1 - \hat{D}(k)] + E_i(k)\Big][1 + F_i] \\ &\leq \exp\Big[\sum_{i=1}^{j} \Big\{ - D_i[1 - \hat{D}(k)] + |E_i(k)| + |F_i|\Big\}\Big]. \end{aligned} \tag{2.17}$$

The bounds of (H3) now give

$$\begin{aligned} |A_j(k)| &\leq \exp\Big[- j(1 - CK_2\beta)[1 - \hat{D}(k)] + CK_3\beta j^{1-\delta'} k^2 + CK_3\beta\Big] \\ &\leq 2\exp\Big[- j(1 - C(K_2 + K_3)\beta)[1 - \hat{D}(k)]\Big], \end{aligned} \tag{2.18}$$

where we use (2.16), $0 < \delta' < 1$, and the inequality $1 - \hat{D}(k) \geq Ck^2$.

For $rm(k) \leq j \leq m(k)$, the right-hand side of (2.18) is maximal at $j = rm(k)$, while the bound of (H4) is minimal at $j = m(k)$. To obtain (H4), it therefore suffices to show that K_4 can be chosen large enough to guarantee that

$$2\exp\Big[- rm(k)(1 - C(K_2 + K_3)\beta)[1 - \hat{D}(k)]\Big] \leq K_4 m(k)^{-2-\epsilon} k^{-4-\delta}. \tag{2.19}$$

The left-hand side equals $2[1 - \hat{D}(k)]^{r\gamma(1-C(K_2+K_3)\beta)}$. For $k \to 0$, this term behaves like a multiple of $k^{2r\gamma(1-C(K_2+K_3)\beta)}$, while the right-hand side behaves like a multiple of $k^{2\epsilon-\delta}(\log \frac{1}{k^2})^{2+\epsilon}$. Thus (2.19) holds for all k, provided $K_4 \gg 1$ and

$$2r\gamma(1 - C(K_2 + K_3)\beta) > 2\epsilon - \delta. \tag{2.20}$$

For β small and for r sufficiently close to 1, (2.20) is satisfied for any γ obeying the bound in (2.6). This completes the derivation of (H4) from (H3) for $rm(k) \leq j \leq m(k)$.

To obtain (H5), we start from the expression

$$A_j(k) - A_{j-1}(k) = A_{j-1}(k)\Big\{\Big[1 - D_j[1 - \hat{D}(k)] + E_j(k)\Big][1 + F_j] - 1\Big\}. \tag{2.21}$$

For $rm(k) \leq j \leq m(k)$, the absolute value of the factor multiplying $A_{j-1}(k)$ on the right-hand side can be estimated, using (H3), by

$$[1 + C(K_2 + K_3)\beta][1 - \hat{D}(k)] + CK_3\beta j^{-1-\epsilon} \leq 2[1 - \hat{D}(k)] + CK_3\beta m(k)^{-1-\epsilon}. \tag{2.22}$$

The right-hand side is bounded above by a multiple of k^2, since $m(k) \leq Ck^{-2}[1 \vee \log k^{-2}]$ (recall (2.8)). Since we have already shown that (H4) follows from (H3) for $rm(k) \leq j \leq m(k)$, and since the difference between (H4) and (H5) is a factor $K_5 k^2/K_4$, it follows that (H5) holds provided $K_5 \gg K_4$.

2.4 Preparations: bounds on $\hat{\pi}_m(k)$, $2 \le m \le n+1$

In this section, we use the induction hypotheses (H1–H6) to prove bounds on $\hat{\pi}_m(k)$ for $2 \le m \le n+1$. This will be the driving force behind the advancement of the induction in Section 3.

Lemma 2.1 *Assume (H2–H4) and (H6).*
(i) $\max_{1\le j\le n} A_j(0) \le e^{CK_3\beta}$.
(ii) $||A_j||_1 \le Kj^{-\frac{d}{2}}$ *for* $1 \le j \le n$ *with* $K = C(1+K_4)$.
(iii) $|\nabla^2 A_j(0)| \le K^* j$ *for* $1 \le j \le n$ *with* $K^* = e^{C(K_2+K_3)\beta} + K_6\beta$.

Proof. (i) This is an immediate consequence of (H3) with $k=0$.
(ii) Fix $1 \le j \le n$, and define

$$\begin{aligned} R_j &= \{k \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \times (-\pi,\pi]^{d-1} : m(k) \ge j\} \\ R_j^c &= \{k \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \times (-\pi,\pi]^{d-1} : m(k) < j\}. \end{aligned} \tag{2.23}$$

Since $A_j(k+\vec{\pi}) = (-1)^j A_j(k)$, we have

$$||A_j||_1 = 2\int_{R_j} |A_j(k)| dk + 2\int_{R_j^c} |A_j(k)| dk. \tag{2.24}$$

Using (2.18) on R_j (which follows from (H2–H3)) and the inequality $1-\hat{D}(k) \ge Ck^2$, and using (H4) on R_j^c, we get

$$||A_j||_1 \le 4\int_{R_j} e^{-Cjk^2} dk + 2\int_{R_j^c} K_4 j^{-2-\epsilon} k^{-4-\delta} dk. \tag{2.25}$$

The first term on the right-hand side is bounded above by $Cj^{-d/2}$.

Since $2+\epsilon = \frac{d}{2}+p$, to complete the proof it suffices to show that

$$j^{-p}\int_{R_j^c} k^{-4-\delta} dk \le C. \tag{2.26}$$

The integral is bounded uniformly in j when $4+\delta < d$, so we need only consider the case $4+\delta \ge d$. By (2.8), $R_j^c \subset \{k: 1-\hat{D}(k) \ge C\frac{\log j}{j}\}$. But $1-\hat{D}(k) \le \frac{k^2}{2d}$, and hence $R_j^c \subset \{k: k^2 \ge C\frac{\log j}{j}\}$. Therefore, when $4+\delta > d$, the left-hand side of (2.26) is bounded above by

$$j^{-p}\int_{k^2 \ge C\frac{\log j}{j}} k^{-4-\delta} dk \le Cj^{-p}\left(\frac{\log j}{j}\right)^{\frac{d-4-\delta}{2}} = Cj^{-\epsilon+\frac{\delta}{2}} (\log j)^{\frac{d-4-\delta}{2}} \le C, \tag{2.27}$$

where we use that $\delta < 2\epsilon$ (recall (2.6)). A similar calculation applies in the borderline case $4+\delta = d$, yielding the bound $Cj^{-p}\sqrt{\log j} \le C$.
(iii) This is an immediate consequence of (H2–H3) and (H6). □

Lemma 2.2 *Assume (H1). Then $I_1 \supset I_2 \supset \cdots \supset I_n$.*

Proof. Recall (2.9). Suppose $z \in I_j$ for some $2 \le j \le n$. Then by (H1),

$$|z - z_{j-1}| \le |z - z_j| + |z_j - z_{j-1}| \le \frac{K_1\beta}{j^{1+\epsilon}} + \frac{K_1\beta}{j^{2+\epsilon}} \le \frac{K_1\beta}{(j-1)^{1+\epsilon}}, \tag{2.28}$$

and hence $z \in I_{j-1}$. □

The next lemma is the key to our induction step, as it provides bounds, in particular, on $\hat{\pi}_{n+1}(k)$.

Lemma 2.3 *Assume (H1–H4) and (H6). For $2 \le m \le n+1$, $z \in I_n$, and $k \in (-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\pi, \pi]^{d-1}$,*
(i)

$$|\hat{\pi}_m(k)|\, z^m \le CK\beta m^{-2-\epsilon}, \tag{2.29}$$

(ii)

$$|\nabla^2 \hat{\pi}_m(k)| z^m \le CK\beta m^{-1-\epsilon}, \tag{2.30}$$

(iii)

$$\Big|\hat{\pi}_m(k) - \hat{\pi}_m(0) - [1 - \hat{D}(k)]\nabla^2 \hat{\pi}_m(0)\Big| z^m \le CK\beta k^{2+2\tilde{\epsilon}} m^{-1-\epsilon+2\tilde{\epsilon}}, \tag{2.31}$$

where K is the constant in Lemma 2.1(ii) and $0 \le \tilde{\epsilon} \le 1$ is arbitrary.

Proof. (i) By Lemma A.2(i),

$$|\hat{\pi}_m(k)| \le 2d\lambda(m)\|c_{m-1}\|_\infty + \sum_{N=2}^{\infty} (2N-1)2^{N-1} \sum \lambda(m^*)\|c_{m^*}\|_\infty \prod_{l=1}^{N-1} \lambda(m'_l)\|c_{m'_l}\|_\infty c_{m_l}, \tag{2.32}$$

where $\lambda(m) = 1 - e^{-\beta/m^p} \le \beta m^{-p}$, $c_j = \sum_x c_j(x)$, $\|c_j\|_\infty = \sup_x c_j(x)$, and the unlabelled sum is over the set of $m^*, m_1, m'_1, \ldots, m_{N-1}, m'_{N-1}$ whose sum is m and for which m^* is maximal and $m'_l \ge m_l$ for each l. The conditions on the unlabelled sum imply that all c_j on the right-hand side of (2.32) involve $1 \le j \le m-1 \le n$ only.

We multiply both sides of (2.32) by z^m and associate a factor z^j to each c_j on the right-hand side. By Lemma 2.2, $z \in I_j$ for each $1 \le j \le n$. Using the relations $c_j = z^{-j} A_j(0)$, $\|c_j\|_\infty \le (2\pi)^{-d} z^{-j} \|A_j\|_1$, we then obtain

$$\begin{aligned} |\hat{\pi}_m(k)| z^m \ \le\ & C\Big[\beta m^{-p}\|A_{m-1}\|_1 + \sum_{N=2}^{\infty}(2N-1)2^{N-1}\sum \beta(m^*)^{-p}\|A_{m^*}\|_1 \\ & \times \prod_{l=1}^{N-1} \beta(m'_l)^{-p}\|A_{m'_l}\|_1 A_{m_l}(0)\Big]. \end{aligned} \tag{2.33}$$

By Lemma 2.1(i,ii), and the fact that $m^* \ge (2N-1)^{-1}m$, the unlabelled sum is bounded above by

$$\begin{aligned} & K^N \beta^N (2N-1)^{\frac{d}{2}+p} m^{-\frac{d}{2}-p} \left(\sum_{m'=1}^{\infty} \sum_{m=0}^{m'} (m')^{-\frac{d}{2}-p} \right)^{N-1} e^{CK_3\beta(N-1)} \\ & \le K^N \beta^N (2N-1)^{2+\epsilon} m^{-2-\epsilon} C^{N-1}, \end{aligned} \tag{2.34}$$

where we also insert $\frac{d}{2} + p = 2 + \epsilon$. Hence, for β sufficiently small, the sum over N in (2.33) converges and is bounded above by $CK^2\beta^2 m^{-2-\epsilon}$. The first term in (2.33) is bounded above by

$$C\beta m^{-p} K(m-1)^{-\frac{d}{2}} \le CK\beta m^{-2-\epsilon}, \tag{2.35}$$

which dominates the second term for β sufficiently small because it has one factor β less. This proves the claim.

(ii) The proof is similar, and uses Lemma A.2(ii) and Lemma 2.1(iii).

(iii) This follows immediately from (ii) and Lemma A.2(iii), where also the restriction on $\tilde{\epsilon}$ appears. □

3 The induction advanced

In this section we advance the induction hypotheses (H1–H6) one by one. The computations are technical but not difficult. Throughout this section, in accordance with the uniformity condition on (H2–H6), we have $z \in I_{n+1}$.

3.1 Advancement of (H1)

By (2.1) and the mean-value theorem,

$$\begin{aligned} z_{n+1} - z_n &= -\frac{1}{2d}\left[\sum_{m=2}^{n} \hat{\pi}_m(0)(z_n^m - z_{n-1}^m) + \hat{\pi}_{n+1}(0) z_n^{n+1}\right] \\ &= -\frac{1}{2d}\left[(z_n - z_{n-1})\sum_{m=2}^{n} m\hat{\pi}_m(0) y_n^{m-1} + \hat{\pi}_{n+1}(0) z_n^{n+1}\right], \end{aligned} \tag{3.1}$$

where y_n is between z_n and z_{n-1}. By (H1) and (2.9), $y_n \in I_n$. It further follows from (H1) that $y_n^{-1} \le C$. Hence, by Lemma 2.3(i) and (H1), we have that

$$\begin{aligned} |z_{n+1} - z_n| &\le \frac{1}{2d}\Big[K_1\beta n^{-2-\epsilon}\sum_{m=2}^{n} CK\beta m^{-1-\epsilon} + K\beta(n+1)^{-2-\epsilon}\Big] \\ &\le \frac{1}{2d}(CK_1K\beta^2 + K\beta)(n+1)^{-2-\epsilon}. \end{aligned} \tag{3.2}$$

Thus (H1) holds for $n+1$, provided β is small enough and $K_1 > \frac{K}{2d}$. Since $K = C(1+K_4)$, it therefore suffices that $K_1 \gg K_4$.

Now that (H1) holds for $n+1$, it follows that $I_{n+1} \subset I_n$, as in the proof of Lemma 2.2. For $n \ge 0$, define

$$\zeta_{n+1} = -1 + 2dz + \sum_{m=2}^{n+1} \hat{\pi}_m(0) z^m. \tag{3.3}$$

As usual, we do not make the z-dependence explicit in the notation, and we recall that $z \in I_{n+1}$. The following lemma, whose proof makes use of (H1) for $n+1$, will be needed in Sections 3.3–3.5.

Lemma 3.1 *Uniformly for $z \in I_{n+1}$,*

$$|\zeta_{n+1}| \leq CK_1\beta(n+1)^{-1-\epsilon}. \tag{3.4}$$

Proof. By (2.1) and the mean-value theorem,

$$\begin{aligned} |\zeta_{n+1}| &= \Big|2d(z-z_{n+1}) + \sum_{m=2}^{n+1} \hat{\pi}_m(0)(z^m - z_n^m)\Big| \\ &= \Big|2d(z-z_{n+1}) + (z-z_n)\sum_{m=2}^{n+1} m\hat{\pi}_m(0)y_n^{m-1}\Big|, \end{aligned} \tag{3.5}$$

where y_n is between z and z_n. Also, $z \in I_{n+1} \subset I_n$ and $z_n \in I_n$, and hence $y_n \in I_n$. Therefore, by Lemma 2.3(i),

$$\begin{aligned} |\zeta_{n+1}| &\leq 2dK_1\beta(n+1)^{-1-\epsilon} + K_1\beta n^{-1-\epsilon} C\sum_{m=2}^{n+1} CK\beta m^{-1-\epsilon} \\ &\leq CK_1\beta(1+K\beta)(n+1)^{-1-\epsilon} \end{aligned} \tag{3.6}$$

(recall that $y_n^{-1} \leq C$). □

3.2 Advancement of (H2)

The definition of D_n in (2.5) implies that

$$D_{n+1} - D_n = \frac{1}{1+C_{n+1}}(B_{n+1}-B_n) - \frac{B_n}{(1+C_n)(1+C_{n+1})}(C_{n+1}-C_n), \tag{3.7}$$

where, by (2.3–2.4),

$$B_{n+1} - B_n = -\nabla^2\hat{\pi}_{n+1}(0)z^{n+1}, \quad C_{n+1} - C_n = n\hat{\pi}_{n+1}(0)z^{n+1}. \tag{3.8}$$

By Lemma 2.3(i,ii) and the fact that $z \in I_{n+1} \subset I_n$, both differences are bounded above by $CK\beta n^{-1-\epsilon}$. In addition, $|B_n - 1| \leq C(K_1+K)\beta$ and $|C_n|, |C_{n+1}| \leq CK\beta$. This leads to

$$|D_{n+1} - D_n| \leq 2(1-CK\beta)^{-2}K\beta(n+1)^{-1-\epsilon}, \tag{3.9}$$

and hence (H2) holds for $n+1$ provided β is small enough and $K_2 > K$. Since $K = C(1+K_4)$, it therefore suffices that $K_2 \gg K_4$.

3.3 Advancement of (H3)

This section, which involves our principal induction hypothesis, is the most technical. Throughout this section, we fix k and $n+1 \leq m(k)$. Because (H3) has already been verified for $n=1$, we need only consider $m(k) \geq 2$, which implies that

$$1 - \hat{D}(k) \leq \frac{\gamma}{2}. \tag{3.10}$$

1. The induction step will be achieved as soon as we are able to write $A_{n+1}(k)/A_n(k)$ as $\Big[1 - D_{n+1}[1-\hat{D}(k)] + E_{n+1}(k)\Big][1+F_{n+1}]$, and show that $E_{n+1}(k)$ and F_{n+1} satisfy the required bounds. For this, we will write

$$\frac{A_{n+1}(k)}{A_n(k)} = 1 - D_{n+1}[1-\hat{D}(k)] + E'_{n+1}(k) + F_{n+1} \tag{3.11}$$

and then set

$$E_{n+1}(k) = [1+F_{n+1}]^{-1}\left[E'_{n+1}(k) + D_{n+1}[1-\hat{D}(k)]F_{n+1}\right]. \tag{3.12}$$

To begin, we divide the recursion relation (2.10) by $A_n(k)$, and use (3.3), to obtain

$$\frac{A_{n+1}(k)}{A_n(k)} = 1 - 2dz[1-\hat{D}(k)] + \sum_{m=2}^{n+1} \hat{\pi}_m(k) z^m \frac{A_{n+1-m}(k)}{A_n(k)} - \sum_{m=2}^{n+1}\hat{\pi}_m(0)z^m + \zeta_{n+1}. \tag{3.13}$$

Using (2.3–2.4), we can rewrite (3.13) as

$$\frac{A_{n+1}(k)}{A_n(k)} = 1 \ - (B_{n+1} - D_{n+1}C_{n+1})[1-\hat{D}(k)] + E'_{n+1}(k) + F_{n+1}, \tag{3.14}$$

where

$$E'_{n+1}(k) = I + II + III + IV \quad \text{and} \quad F_{n+1} = V + \zeta_{n+1} \tag{3.15}$$

with

$$I = \sum_{m=2}^{n+1}\Big[\hat{\pi}_m(k) - \hat{\pi}_m(0) - [1-\hat{D}(k)]\nabla^2\hat{\pi}_m(0)\Big]z^m, \tag{3.16}$$

$$II = \sum_{m=2}^{n+1}\hat{\pi}_m(0)z^m\left[\frac{A_{n+1-m}(k)}{A_n(k)} - \frac{A_{n+1-m}(0)}{A_n(0)} - (m-1)D_{n+1}[1-\hat{D}(k)]\right], \tag{3.17}$$

$$III = \sum_{m=2}^{n+1}[\hat{\pi}_m(k) - \hat{\pi}_m(0)]\, z^m \left[\frac{A_{n+1-m}(k)}{A_n(k)} - \frac{A_{n+1-m}(0)}{A_n(0)}\right], \tag{3.18}$$

$$IV = \sum_{m=2}^{n+1}[\hat{\pi}_m(k) - \hat{\pi}_m(0)]\, z^m \left[\frac{A_{n+1-m}(0)}{A_n(0)} - 1\right], \tag{3.19}$$

$$V = \sum_{m=2}^{n+1}\hat{\pi}_m(0)z^m\left[\frac{A_{n+1-m}(0)}{A_n(0)} - 1\right]. \tag{3.20}$$

Since $B_{n+1} - D_{n+1}C_{n+1} = D_{n+1}$ by (2.5), indeed (3.14) yields (3.11).

2. Beginning with F_{n+1}, we first note from Lemma 3.1 that $|\zeta_{n+1}| \leq CK_1\beta(n+1)^{-1-\epsilon}$. To estimate V, and for later purposes, we make use of the following elementary bounds. For a vector $x = (x_l)$ with $\sup_l |x_l| < 1$, define $\chi(x) = \sum_l \frac{|x_l|}{1-|x_l|}$. The bound $(1-t)^{-1} \leq \exp[t(1-t)^{-1}]$, together with Taylor's Theorem applied to $f(t) = \prod_l \frac{1}{1-tx_l}$, gives

$$\left|\prod_l \frac{1}{1-x_l}\right| \leq e^{\chi(x)}, \quad \left|\prod_l \frac{1}{1-x_l} - 1\right| \leq \chi(x)e^{\chi(x)}, \tag{3.21}$$

$$\left| \prod_l \frac{1}{1-x_l} - 1 - \sum_l x_l \right| \leq \frac{3}{2}\chi(x)^2 e^{\chi(x)}. \tag{3.22}$$

Applying (H3), the second estimate of (3.21), and Lemma 2.3(i) to estimate V, we obtain

$$\begin{aligned} |V| &= \left| \sum_{m=2}^{n+1} \hat{\pi}_m(0) z^m \left[\prod_{j=n+2-m}^{n} \frac{1}{1+F_j} - 1 \right] \right| \\ &\leq \sum_{m=2}^{n+1} \frac{CK\beta}{m^{2+\epsilon}} \sum_{j=n+2-m}^{n} \frac{CK_3\beta}{j^{1+\epsilon}} e^{CK_3\beta} \leq \frac{CKK_3\beta^2}{(n+1)^{1+\epsilon}}. \end{aligned} \tag{3.23}$$

Therefore, if we take $K_3 \gg K_1$ and β sufficiently small, then

$$|F_{n+1}| \leq |\zeta_{n+1}| + |V| \leq \frac{CK_1\beta + CKK_3\beta^2}{(n+1)^{1+\epsilon}} \leq \frac{K_3\beta}{(n+1)^{1+\epsilon}}. \tag{3.24}$$

This advances the bound on F_{n+1} of (H3).

3. We next consider the contributions $I, \ldots, IV$ to $E'_{n+1}(k)$, beginning with the simplest terms I and IV.

For I, by Lemma 2.3(iii) we have

$$\begin{aligned} |I| &\leq \sum_{m=2}^{n+1} \left| \hat{\pi}_m(k) - \hat{\pi}_m(0) - [1-\hat{D}(k)]\nabla^2\hat{\pi}_m(0) \right| z^m \\ &\leq CK\beta k^{2+2\tilde{\epsilon}} \sum_{m=2}^{n+1} \frac{1}{m^{1+\epsilon-2\tilde{\epsilon}}} \leq CK\beta k^{2+2\tilde{\epsilon}}, \end{aligned} \tag{3.25}$$

for any $\tilde{\epsilon} < \frac{1}{2}\epsilon$. By (2.8), since $n+1 \leq m(k)$, we have

$$k^2 \leq C\frac{\log m(k)}{m(k)} \leq C\frac{\log(n+1)}{n+1}. \tag{3.26}$$

So

$$|I| \leq \frac{CK\beta k^2}{(n+1)^{\delta'}} \tag{3.27}$$

for any $\delta' < 2\tilde{\epsilon}$ (compatible with (2.6)).

For IV, we first combine Lemma 2.3(ii,iii) with $\tilde{\epsilon} = 0$ to obtain $|\hat{\pi}_m(k) - \hat{\pi}_m(0)| \leq CK\beta k^2 m^{-1-\epsilon}$. Then we argue as for V, to obtain

$$|IV| \leq \sum_{m=2}^{n+1} \frac{CK\beta k^2}{m^{1+\epsilon}} \sum_{j=n+2-m}^{n} \frac{CK_3\beta}{j^{1+\epsilon}} e^{CK_3\beta} \leq \frac{CKK_3\beta^2 k^2}{(n+1)^{\epsilon}}. \tag{3.28}$$

4. For II, we first simplify the notation by defining

$$\Delta_{m,n}(k) = \prod_{j=n+2-m}^{n} \left[1 - D_j[1-\hat{D}(k)] + E_j(k) \right]^{-1} - 1 - \sum_{j=n+2-m}^{n} \left[D_j[1-\hat{D}(k)] - E_j(k) \right]. \tag{3.29}$$

Since by (H3),

$$\frac{A_{n+1-m}(k)}{A_n(k)} = \frac{A_{n+1-m}(0)}{A_n(0)} \prod_{j=n+2-m}^{n} \Big[1 - D_j[1-\hat{D}(k)] + E_j(k)\Big]^{-1}, \tag{3.30}$$

we can decompose II as

$$II = II_1 + II_2 + II_3 \tag{3.31}$$

with

$$II_1 = \sum_{m=2}^{n+1} \hat{\pi}_m(0) z^m \frac{A_{n+1-m}(0)}{A_n(0)} \Delta_{m,n}(k), \tag{3.32}$$

$$II_2 = \sum_{m=2}^{n+1} \hat{\pi}_m(0) z^m \frac{A_{n+1-m}(0)}{A_n(0)} \sum_{j=n+2-m}^{n} \Big[(D_j - D_{n+1})[1-\hat{D}(k)] - E_j(k)\Big], \tag{3.33}$$

$$II_3 = \sum_{m=2}^{n+1} \hat{\pi}_m(0) z^m \left[\frac{A_{n+1-m}(0)}{A_n(0)} - 1\right] (m-1) D_{n+1}[1-\hat{D}(k)]. \tag{3.34}$$

The terms II_2 and II_3 can be estimated with the help of (H3), Lemma 2.3(i) and (3.21). Namely, as in (3.23),

$$|II_3| \le \sum_{m=2}^{n+1} \frac{CK\beta}{m^{1+\epsilon}} \sum_{j=n+2-m}^{n} \frac{CK_3\beta}{j^{1+\epsilon}} e^{CK_3\beta} Ck^2 \le \frac{CKK_3\beta^2 k^2}{(n+1)^{\epsilon}}, \tag{3.35}$$

where we use the inequality $1-\hat{D}(k) \le \frac{k^2}{2d}$, and the bounds $D_{n+1} \le C$ by (H2) (which was advanced in Section 3.2). Also, using $A_{n+1-m}(0)/A_n(0) \le e^{CK_3\beta}$ by (H3), and the fact that $\delta' < \epsilon$ by (2.6), we have

$$|II_2| \le \sum_{m=2}^{n+1} \frac{CK\beta}{m^{2+\epsilon}} e^{CK_3\beta} \sum_{j=n+2-m}^{n} \left[\frac{CK_2\beta k^2}{j^{\epsilon}} + \frac{K_3\beta k^2}{j^{\delta'}}\right] \le \frac{CK(K_2+K_3)\beta^2 k^2}{(n+1)^{\delta'}}. \tag{3.36}$$

5. To deal with II_1, we use (3.22) to estimate $\Delta_{m,n}(k)$. This gives

$$|II_1| \le \sum_{m=2}^{n+1} \frac{CK\beta}{m^{2+\epsilon}} e^{CK_3\beta} \frac{3}{2} \left[\chi_{m,n}(k)\right]^2 e^{\chi_{m,n}(k)} \tag{3.37}$$

with

$$\chi_{m,n}(k) = \sum_{j=n+2-m}^{n} \frac{[1-\hat{D}(k)]D_j + |E_j(k)|}{1 - [1-\hat{D}(k)]D_j - |E_j(k)|}. \tag{3.38}$$

But, by (H2–H3), (3.10) and (2.16), for sufficiently small β we have

$$|\chi_{m,n}(k)| \le (m-1)[1-\hat{D}(k)]q \quad \text{with } q = q(k) = (1 + C(K_2+K_3)\beta)(1+Ck^2). \tag{3.39}$$

In particular, since $m \le n+1 \le m(k)$, it follows via (2.8) that

$$e^{\chi_{m,n}(k)} \le e^{m(k)[1-\hat{D}(k)]q} = [1-\hat{D}(k)]^{-\gamma q}. \tag{3.40}$$

Therefore, again using $n+1 \le m(k)$ and (2.8), we have

$$|II_1| \le CK\beta q^2[1-\hat{D}(k)]^{2-\gamma q} \sum_{m=2}^{n+1} \frac{1}{m^\epsilon} \le CK\beta k^{4-2\gamma q} m(k)^{0\vee(1-\epsilon)}. \tag{3.41}$$

Inserting the definition of q, we find

$$|II_1| \le CK\beta k^{2+2\delta'} \left(k^{-C\gamma(K_2+K_3)\beta} k^{-C\gamma k^2} k^{2-2\gamma-2\delta'} m(k)^{0\vee(1-\epsilon)} \right). \tag{3.42}$$

(A harmless factor $\log m(k)$ should also appear in the right sides of (3.41) and (3.42) when $\epsilon = 1$.) Now, $k^{-C\gamma k^2}$ is bounded and $m(k) \le Ck^{-2}[1 \vee \log k^{-2}]$ (recall (2.8)). Therefore, and in view of (2.6), the quantity in parentheses is bounded by a strictly positive power of k (provided β is sufficiently small). Since $n+1 \le m(k)$, it follows from (3.26) that $k^2 \le C(n+1)^{-1} \log m(k)$, and (3.42) then gives

$$|II_1| \le \frac{CK\beta k^2}{(n+1)^{\delta'}}. \tag{3.43}$$

6. To estimate III, we use the bound $|\hat{\pi}_m(k) - \hat{\pi}_m(0)| \le CK\beta k^2 m^{-1-\epsilon}$, (3.30), (H3), and (3.21), to obtain

$$|III| \le \sum_{m=2}^{n+1} \frac{CK\beta k^2}{m^{1+\epsilon}} e^{CK_3\beta} \chi_{m,n}(k) e^{\chi_{m,n}(k)}. \tag{3.44}$$

By (3.39) and (3.40) therefore,

$$|III| \le CK\beta k^2 [1-\hat{D}(k)]^{1-\gamma q} \sum_{m=2}^{n+1} \frac{1}{m^\epsilon}. \tag{3.45}$$

This is equivalent to the first bound in (3.41), and therefore III also obeys (3.43).

7. Combining the above bounds on $I, \ldots, IV$, and recalling that $\delta' < \epsilon$, we have

$$|E'_{n+1}(k)| \le \frac{CK\beta k^2}{(n+1)^{\delta'}}. \tag{3.46}$$

(Note that the bounds on II_2, II_3, IV have an extra factor β and therefore do not show up in the constant for β sufficiently small.) In view of (3.12), this advances the bound on $E_{n+1}(k)$ of (H3), provided $K_3 \gg K$, which means $K_3 \gg K_4$ because $K = C(1+K_4)$.

3.4 Advancement of (H4–H5)

Recall that, in Section 2.3, (H4–H5) were shown to follow from (H2–H3) for $rm(k) \le j \le m(k)$ when r is sufficiently close to 1. Therefore (H4–H5) may in fact be assumed to hold for $rm(k) \le j \le n$. In this section, we fix $n+1 \ge m(k)$, and obtain (H4–H5) for $n+1$, using (H4–H5) for $rm(k) \le j \le n$. We will also use (H3).

1. We begin by rewriting (3.13) as (recall (2.3))

$$A_{n+1}(k) = A_n(k)\Big\{1 \ - B_{n+1}[1-\hat{D}(k)] + I + \zeta_{n+1}\Big\} + II, \tag{3.47}$$

with

$$\begin{aligned} I &= \sum_{m=2}^{n+1} [\hat{\pi}_m(k) - \hat{\pi}_m(0) - [1-\hat{D}(k)]\nabla^2 \pi_m(0)] z^m, &(3.48)\\ II &= \sum_{m=2}^{n+1} \hat{\pi}_m(k) z^m \left[A_{n+1-m}(k) - A_n(k)\right]. &(3.49)\end{aligned}$$

By Lemma 3.1, $|\zeta_{n+1}| \leq CK_1\beta(n+1)^{-1-\epsilon}$. By (3.25), $|I| \leq CK\beta k^{2+2\tilde{\epsilon}}$, for any $\tilde{\epsilon} < \frac{1}{2}\epsilon$. It therefore remains only to estimate II.

2. To that end, we rewrite II as

$$II = \sum_{j=0}^{n-1} \hat{\pi}_{n+1-j}(k) z^{n+1-j} \sum_{l=j+1}^{n} [A_{l-1}(k) - A_l(k)]. \tag{3.50}$$

We divide the sum over j into two parts, II_1 and II_2, corresponding respectively to $0 \leq j \leq rm(k)$ and $rm(k) < j \leq n-1$. Applying Lemma 2.3(i), we may then estimate

$$\begin{aligned} |II_1| &\leq \sum_{j=0}^{rm(k)} \frac{CK\beta}{(n+1-j)^{2+\epsilon}} \sum_{l=j+1}^{n} [A_{l-1}(k) - A_l(k)] &(3.51)\\ |II_2| &\leq \sum_{j=rm(k)+1}^{n-1} \frac{CK\beta}{(n+1-j)^{2+\epsilon}} \sum_{l=j+1}^{n} [A_{l-1}(k) - A_l(k)]. &(3.52)\end{aligned}$$

The term II_2 is easy. Namely, by (H5),

$$\begin{aligned} |II_2| &\leq \sum_{j=rm(k)+1}^{n-1} \frac{K\beta}{(n+1-j)^{2+\epsilon}} \sum_{l=j+1}^{n} \frac{CK_5}{k^{2+\delta} l^{2+\epsilon}} \\ &\leq \frac{CKK_5\beta}{k^{2+\delta}(n+1)^{2+\epsilon}}. &(3.53)\end{aligned}$$

3. For II_1, we divide the sum over l into two parts, II_1' and II_1'', corresponding respectively to $j+1 \leq l \leq rm(k)$ and $rm(k) < l \leq n$. These can be estimated with the help of (H3) respectively (H5). Beginning with II_1'', we have

$$|II_1''| \leq \sum_{j=0}^{rm(k)} \frac{CK\beta}{(n+1-j)^{2+\epsilon}} \sum_{l=rm(k)+1}^{n} \frac{K_5}{k^{2+\delta} l^{2+\epsilon}} \leq \frac{CKK_5\beta}{k^{2+\delta} n^{2+\epsilon}} \sum_{j=0}^{rm(k)} \sum_{l=rm(k)+1}^{n} \frac{1}{l^{2+\epsilon}}. \tag{3.54}$$

The double sum is bounded uniformly in n and k, and hence (recall that $m(k) \leq n+1$)

$$|II_1''| \leq \frac{CKK_5\beta}{k^{2+\delta}(n+1)^{2+\epsilon}}. \tag{3.55}$$

For II_1', we require an estimate for $|A_{l-1}(k) - A_l(k)|$ valid for $1 \leq l \leq rm(k)$. For this range of l, it follows from (H3) that

$$|A_{l-1}(k) - A_l(k)| \leq Ce^{-Ck^2 l}\left(k^2 + \frac{K_3\beta}{l^{1+\epsilon}}\right). \tag{3.56}$$

Thus we have

$$|II_1'| \leq \frac{CK\beta}{n^{2+\epsilon}} \sum_{j=0}^{rm(k)} \sum_{l=j+1}^{rm(k)} e^{-Ck^2 l}\left(k^2 + \frac{K_3\beta}{l^{1+\epsilon}}\right) \leq \frac{CK\beta}{(n+1)^{2+\epsilon}} \frac{C}{k^2}(1 + K_3\beta). \tag{3.57}$$

Summarizing the above bounds, we have

$$|II| \leq |II_1'| + |II_1''| + |II_2| \leq \frac{CK\beta}{(n+1)^{2+\epsilon}} \frac{1}{k^{2+\delta}}(K_5 + 1 + K_3\beta). \tag{3.58}$$

4. We are now in a position to advance (H5). For this, we use (3.47), (H4), the inequality $1 - \hat{D}(k) \leq \frac{k^2}{2d}$, and the bounds found above, to obtain

$$\begin{aligned} \Big|A_{n+1}(k) - A_n(k)\Big| &\leq |A_n(k)|\, | - B_{n+1}[1 - \hat{D}(k)] + I + \zeta_{n+1}| + |II| \\ &\leq \frac{K_4}{(n+1)^{2+\epsilon}k^{4+\delta}}\left(\frac{k^2}{2d}B_{n+1} + CK\beta k^{2+2\tilde{\epsilon}} + \frac{CK_1\beta}{(n+1)^{1+\epsilon}}\right) \\ &\quad + \frac{CK\beta(1+K_5)}{(n+1)^{2+\epsilon}k^{2+\delta}}. \end{aligned} \tag{3.59}$$

Since $|B_{n+1} - 1| \leq C(K_1 + K)\beta$, and $(n+1)^{-1-\epsilon} \leq m(k)^{-1-\epsilon} \leq Ck^2$, (H5) follows for $n+1$ provided $K_5 > K_4/2d$ and β is sufficiently small.

5. To advance (H4), we begin by using (H4) and arguing as above, to obtain

$$\begin{aligned} |A_{n+1}(k)| &\leq |A_n(k)| \Big|1 - B_{n+1}[1 - \hat{D}(k)] + I + \zeta_{n+1}\Big| + |II| \\ &\leq \frac{K_4}{n^{2+\epsilon}k^{4+\delta}}\Big\{|1 - B_{n+1}[1 - \hat{D}(k)]| + CK\beta k^{2+2\tilde{\epsilon}} + \frac{CK_1\beta}{(n+1)^{1+\epsilon}}\Big\} \\ &\quad + \frac{CK\beta(1+K_5)}{(n+1)^{2+\epsilon}k^{2+\delta}}. \end{aligned} \tag{3.60}$$

We need to argue that the right-hand side is no larger than $K_4(n+1)^{-2-\epsilon}k^{-4-\delta}$. To achieve this, we will use separate arguments for $1 - \hat{D}(k) \leq \frac{1}{2}$ and $1 - \hat{D}(k) > \frac{1}{2}$. These arguments will be valid only when n is large enough. However, this is sufficient, since (H4) clearly holds for any finite set of values of n, if K_4 is taken to be large enough.

Suppose that $1 - \hat{D}(k) \leq \frac{1}{2}$. For β sufficiently small,

$$1 - B_{n+1}[1 - \hat{D}(k)] \geq 0. \tag{3.61}$$

Hence, the absolute value signs on the right side of (3.60) may be removed. To obtain (H4) for $n+1$, it now suffices to show that

$$1 - c[1-\hat{D}(k)] + \frac{CK_1\beta}{(n+1)^{1+\epsilon}} \leq \frac{n^{2+\epsilon}}{(n+1)^{2+\epsilon}}, \tag{3.62}$$

for c within order β of 1. The term $c[1-\hat{D}(k)]$ has been introduced to absorb $B_{n+1}[1-\hat{D}(k)]$ and the order β terms in (3.60) involving $k^{2+2\tilde{\epsilon}}$ and $(1+K_5)$. By (2.8), $1-\hat{D}(k) \geq Cm(k)^{-1}\log m(k) \geq C(n+1)^{-1}\log(n+1)$. Thus (3.62) holds for n sufficiently large and β sufficiently small.

Suppose, on the other hand, that $1-\hat{D}(k) > \frac{1}{2}$. Since $k \in (-\frac{\pi}{2}, \frac{\pi}{2}] \times (-\pi,\pi]^{d-1}$, we have $\frac{1}{2} < 1-\hat{D}(k) \leq 2-\frac{1}{d}$. Therefore

$$|1 - B_{n+1}[1-\hat{D}(k)]| \leq |\hat{D}(k)| + |B_{n+1}-1|\,|1-\hat{D}(k)| \leq (1-\tfrac{1}{d}) \vee \tfrac{1}{2} + |B_{n+1}-1|(2-\tfrac{1}{d}). \tag{3.63}$$

Hence

$$|1 - B_{n+1}[1-\hat{D}(k)]| + CK\beta k^{2+2\tilde{\epsilon}} + \frac{CK_1\beta}{(n+1)^{1+\epsilon}} \leq (1-\tfrac{1}{d}) \vee \tfrac{1}{2} + C\beta, \tag{3.64}$$

and the right side of (3.60) is no greater than

$$\frac{K_4}{n^{2+\epsilon}k^{4+\delta}}\left[(1-\tfrac{1}{d}) \vee \tfrac{1}{2} + C\beta\right] + \frac{CK\beta(1+K_5)}{(n+1)^{2+\epsilon}k^{2+\delta}} \leq \frac{K_4}{n^{2+\epsilon}k^{4+\delta}}\left[(1-\tfrac{1}{d}) \vee \tfrac{1}{2} + C'\beta\right]. \tag{3.65}$$

This is less than the required bound $K_4(n+1)^{-2-\epsilon}k^{-4-\delta}$ if β is sufficiently small and n is large enough.

3.5 Advancement of (H6)

We begin by adding $-\nabla^2 A_n(0) + D_{n+1}A_n(0)$ to both sides of (2.13), and then using $D_{n+1} = B_{n+1} - D_{n+1}C_{n+1}$ on the right-hand side, together with (2.3–2.4) and (3.3). This leads us to

$$\nabla^2 A_{n+1}(0) - \nabla^2 A_n(0) + D_{n+1}A_n(0) = I + II + \zeta_{n+1}\nabla^2 A_n(0) \tag{3.66}$$

with

$$I = \sum_{m=2}^{n+1} \hat{\pi}_m(0) z^m \left[\nabla^2 A_{n+1-m}(0) - \nabla^2 A_n(0) - (m-1)D_{n+1}A_n(0)\right], \tag{3.67}$$

$$II = \sum_{m=2}^{n+1} \nabla^2 \hat{\pi}_m(0) z^m \left[A_{n+1-m}(0) - A_n(0)\right]. \tag{3.68}$$

To estimate I, we use (H6), Lemma 2.3(i), (H2) (which was advanced in Section 3.2), and (H3) for $k = 0$, to obtain

$$\begin{aligned} |I| &\leq \sum_{m=2}^{n+1} \frac{CK\beta}{m^{2+\epsilon}} \sum_{j=n+2-m}^{n} \left\{ |D_j A_{j-1}(0) - D_{n+1} A_n(0)| + K_6 \beta j^{-\epsilon} \right\} \\ &\leq \sum_{m=2}^{n+1} \frac{CK\beta}{m^{2+\epsilon}} \sum_{j=n+2-m}^{n} \left\{ D_j |A_{j-1}(0) - A_n(0)| + |D_j - D_{n+1}| A_n(0) + K_6 \beta j^{-\epsilon} \right\} \\ &\leq CK(K_2 + K_3 + K_6)\beta^2 (n+1)^{-\epsilon}. \end{aligned} \tag{3.69}$$

To estimate II, we use Lemma 2.3(ii), and (H3) for $k = 0$, to obtain

$$|II| \leq CKK_3 \beta^2 (n+1)^{-\epsilon}. \tag{3.70}$$

Hence $I + II$ is bounded by a multiple of $\beta^2(n+1)^{-\epsilon}$, which is a factor β smaller than the bound in (H6). Thus, the main term in (3.66) is $\zeta_{n+1} \nabla^2 A_n(0)$, which by Lemma 2.1(iii) and Lemma 3.1 is bounded above by $CK^* K_1 \beta (n+1)^{-\epsilon}$. Since $K^* = C + K_6 \beta$, (H6) holds for $n+1$ provided β is small enough and $K_6 \gg K_1$.

4 Proof of the main theorems

Theorem 1.1 is proved in Section 4.1 and Theorem 1.2 is proved in Section 4.2. As a consequence of the completed induction, we now know that there exist constants μ, A and D such that the following estimates hold for $n \to \infty$ uniformly in $z \in I_n$:

$$z - \mu^{-1} = \mathcal{O}(n^{-1-\epsilon}) \tag{4.1}$$

$$A_n(0) - A = \mathcal{O}(n^{-\epsilon}) \tag{4.2}$$

$$D_n - D = \mathcal{O}(n^{-\epsilon}). \tag{4.3}$$

To see the first of these statements, note that Lemma 2.2 gives $I_1 \supset I_2 \supset I_3 \supset \cdots$, so $\cap_{n=1}^{\infty} I_n$ consists of a single point, which we call μ^{-1}. Clearly, (4.1) follows from (H1) and (2.9). The second statement follows from (H3), and the third from (H2). The constants μ, A and D are identified in Section 4.3 in terms of $\hat{\pi}_m(0)$ and $\nabla^2 \hat{\pi}_m(0)$.

4.1 Proof of Theorem 1.1

Proof of Theorem 1.1(a): By (1.5), (2.2) and (4.1–4.2), we have

$$c_n = z^{-n} A_n(0) = \mu^n A[1 + \mathcal{O}(n^{-\epsilon})]. \tag{4.4}$$

□

Proof of Theorem 1.1(b): Using $\nabla^2 A_0(0) = 0$, (H6) and (4.2–4.3), we have

$$\begin{aligned}
\frac{1}{c_n}\sum_x x^2 c_n(x) &= -\frac{1}{A_n(0)}\nabla^2 A_n(0) \\
&= \frac{1}{A_n(0)}\sum_{j=1}^{n}[D_j A_{j-1}(0) + \mathcal{O}(j^{-\epsilon})] \\
&= \begin{cases} Dn[1+\mathcal{O}(n^{-1\wedge\epsilon})] & \epsilon \neq 1 \\ Dn[1+\mathcal{O}(n^{-1}\log n)] & \epsilon = 1 \end{cases}.
\end{aligned} \tag{4.5}$$

□

Proof of Theorem 1.1(c): Suppose $k \in R_n$. Then, by (2.23), $n \leq m(k)$ and hence (H3) applies. Therefore, using (4.3), $\delta' < 1 \wedge \epsilon \leq 1$ (recall (2.6)), and the fact that $1 - \hat{D}(k) = \frac{k^2}{2d} + \mathcal{O}(k^4)$, we obtain

$$\begin{aligned}
\frac{\hat{c}_n(k)}{c_n} &= \frac{A_n(k)}{A_n(0)} \\
&= \prod_{i=1}^{n}\Big[1 - D_i\Big[1-\hat{D}(k)\Big] + \mathcal{O}(\beta k^2 i^{-\delta'}))\Big] \\
&= e^{-\frac{k^2}{2d}Dn} e^{\mathcal{O}(k^4 n + k^2 n^{1-\delta'})}.
\end{aligned} \tag{4.6}$$

But by (2.8), $k \in R_n$ if n is sufficiently large and k^2 is less than a sufficiently small multiple of $n^{-1}\log n$. Hence for $k/\sqrt{Dn}$ less than a sufficiently small multiple of $\log n$, we have

$$\frac{1}{c_n}\hat{c}_n\Big(\frac{k}{\sqrt{Dn}}\Big) = e^{-\frac{k^2}{2d}[1+\mathcal{O}(n^{-\delta'})]}, \tag{4.7}$$

as required. □

4.2 Proof of Theorem 1.2

Our starting point for Theorem 1.2 is the relation

$$\frac{c_n(x)}{c_n} = \frac{1}{(2\pi)^d}\int_{(-\pi,\pi]^d}\frac{\hat{c}_n(k)}{c_n}e^{-ik\cdot x}dk, \tag{4.8}$$

which we rewrite, using symmetry, as

$$\frac{c_n(x)}{c_n} = \Big(1+(-1)^{n+||x||_1}\Big)\frac{1}{(2\pi)^d}\int_{(-\frac{1}{2}\pi,\frac{1}{2}\pi]\times(-\pi,\pi]^{d-1}}\frac{\hat{c}_n(k)}{c_n}e^{-ik\cdot x}dk. \tag{4.9}$$

We split the integral in (4.9) into the regions R_n and R_n^c, with these sets defined in (2.23).

Equation (4.6) can be used to write the integral over R_n in (4.9) as

$$\frac{1}{(2\pi)^d}\int_{R_n} e^{-\frac{k^2}{2d}Dn[1+\mathcal{O}(n^{-(1\wedge\delta')})]}e^{-ik\cdot x}dk \sim \left(\frac{d}{2\pi Dn}\right)^{\frac{d}{2}} e^{-\frac{dx^2}{2Dn}} \quad \text{as } n\to\infty. \tag{4.10}$$

The asymptotic formula in (4.10) is uniform in $x \in \mathbb{Z}^d$ provided $\|x\|_1(n\log n)^{-1/2}$ is sufficiently small. For the integral over R_n^c, we note that $\hat{c}_n(k)/c_n = A_n(k)/A_n(0)$, and use (H4) and (4.2), to obtain as upper bound

$$\frac{CK_4}{n^{2+\epsilon}}\int_{R_n^c}\frac{1}{k^{4+\delta}}dk = \frac{CK_4}{n^{\frac{d}{2}+p}}\int_{R_n^c}\frac{1}{k^{4+\delta}}dk. \tag{4.11}$$

The integral in the right-hand side of (4.11) was estimated in (2.26–2.27). There it was shown that either the integral times n^{-p} decays as a power of n, or the integral converges. In the former case, (4.11) represents an error term compared to (4.10). In the latter case, (4.11) represents an error term compared to (4.10) if $p>0$, but not if $p=0$. This proves (1.12).

For $d>4$, $p=0$, the integral in (4.11) converges, and hence (4.11) is bounded by a multiple of $n^{-d/2}$. This proves (1.13). □

4.3 Identification of μ, A, D

In this section we abbreviate $A_n = A_n(0)$. The formulas appearing in the following theorem were first derived in Brydges and Spencer [1] (see also Madras and Slade [11]) for the case $d>4$, $p=0$.

Theorem 4.1 *The limits μ^{-1}, A, D of z_n, A_n, D_n satisfy*

$$1 = 2d\mu^{-1} + \sum_{m=2}^{\infty}\hat{\pi}_m(0)\mu^{-m} \tag{4.12}$$

$$A = \left[2d\mu^{-1} + \sum_{m=2}^{\infty} m\hat{\pi}_m(0)\mu^{-m}\right]^{-1} \tag{4.13}$$

$$D = A\left[2d\mu^{-1} - \sum_{m=2}^{\infty}\nabla^2\hat{\pi}_m(0)\mu^{-m}\right]. \tag{4.14}$$

Proof. We have already observed that $\cap_{n=1}^{\infty} I_n = \{\mu^{-1}\}$. Therefore (4.12) follows after we let $n\to\infty$ in (3.3), with $z=\mu^{-1}$, and use Lemma 3.1.

To determine A, we need a summation argument because the recurrence relation (2.10) for A_n is linear. Fix $z=\mu^{-1}$. For $n\geq 1$, (2.10) gives

$$A_n = 2d\mu^{-1}A_{n-1} + \sum_{m=2}^{n}\hat{\pi}_m(0)\mu^{-m}A_{n-m}. \tag{4.15}$$

Defining $S_n = \sum_{k=0}^{n} A_k$, and combining (4.15) with $A_0 = 1$, we find

$$\begin{aligned} S_n &= 1 + \sum_{k=1}^{n} A_k = 1 + \sum_{k=1}^{n} \Big(2d\mu^{-1} A_{k-1} + \sum_{m=2}^{k} \hat{\pi}_m(0)\mu^{-m} A_{k-m} \Big) \\ &= 1 + 2d\mu^{-1} S_{n-1} + \sum_{m=2}^{n} \hat{\pi}_m(0)\mu^{-m} S_{n-m}. \end{aligned} \tag{4.16}$$

By (3.3) with $z = \mu^{-1}$, this gives

$$S_n - S_{n-1} = 1 - \sum_{m=2}^{n} \hat{\pi}_m(0)\mu^{-m}(S_{n-1} - S_{n-m}) + S_{n-1}\zeta_n, \tag{4.17}$$

which is the same as

$$A_n = 1 - \sum_{m=2}^{n} \hat{\pi}_m(0)\mu^{-m} \left(\sum_{k=n-m+1}^{n-1} A_k \right) + \left(\sum_{k=0}^{n-1} A_k \right) \zeta_n. \tag{4.18}$$

Finally, we use $\lim_{n\to\infty} A_n = A$, and note from Lemma 3.1 that the last term in (4.18) vanishes, to obtain

$$A = 1 - \sum_{m=2}^{\infty} \hat{\pi}_m(0)\mu^{-m}(m-1)A. \tag{4.19}$$

This gives $A = [1 + \sum_{m=2}^{\infty}(m-1)\hat{\pi}_m(0)z^m]^{-1}$, which by (4.12) gives (4.13).

The proof of (4.14) is straightforward. Use (4.3), (2.3–2.5) with $z = \mu^{-1}$, and Lemma 2.3. □

4.4 Discussion

Our method has used induction on the number of steps in the walk to provide a direct proof of Gaussian behaviour. The use of generating functions has been avoided. We have used the Fourier transform, but this seems harmless enough. There remains the possibility that induction hypotheses could be formulated directly in x–space rather than in k–space. However, this would likely make the argument more technical.

The induction hypotheses (H1–H6) have a universal character: they explicitly involve few parameters (essentially only ϵ and β) and do not involve any detailed information about the nature of the self-avoidance interaction. The hard part of the analysis sits in guessing the precise form of (H1–H6). Once these are adequately chosen, the proof of the induction step is mechanical. In guessing (H1–H6), we were partially guided by earlier work on the problem, predominantly when setting up the definitions in Section 2.1. Interestingly, (H1–H6) provide us with quite detailed information about the approach to Gaussian behaviour for finite n.

We have left open the important problem of proving the local central limit theorem for $d > 4$, $p = 0$, where our method is inadequate. We have also not treated the case

$d > 4$, $-\frac{d-4}{2} < p < 0$, in which loops receive a penalty which increases, rather than decreases, with their length.

It is possible that our method could be applied to obtain an alternate proof of Gaussian behaviour of the strictly self-avoiding walk ($\beta = \infty$) in all dimensions $d \geq 5$ (Hara and Slade [6]), but this would involve serious effort. The role of small β would have to be taken over by $\frac{1}{2d}$, and this would involve, among other things, a more delicate choice of the constants $K_1, \ldots, K_6$.

It would be of interest to extend our method to lattice trees and percolation. In both these models, the inversion of generating functions poses serious technical problems (Derbez and Slade [3, 4]; Hara and Slade [8]), and their removal would lead to improved results. Our method would require the formulation of induction hypotheses suitable for convergence to integrated super-Brownian excursion (ISE), rather than to Brownian motion, as this is what arises as the scaling limit in these two models. Perhaps the previous work on application of the lace expansion to lattice trees and percolation can be helpful in the formulation of appropriate replacements for (H1-H6).

Finally, it may also be possible to extend the methods and results of Nguyen and Yang [12, 13] for high-dimensional oriented percolation, by a reformulation in a similar inductive scheme.

A The lace expansion

This section contains standard material on the lace expansion, and consists of the minimum necessary to make our paper self-contained. The lace expansion was introduced in Brydges and Spencer [1] and is discussed at length in Madras and Slade [11]. A brief discussion with a more combinatorial flavour is given in Zeilberger [15].

A.1 Definition of $\hat{\pi}_m(k)$

In this section, we define $\hat{\pi}_m(k)$ and prove (1.7). This requires the introduction of the following standard terminology.

Given an interval $I = [a, b]$ of integers with $0 \leq a \leq b$, we refer to a pair $\{s, t\}$ ($s < t$) of elements of I as an *edge*. To abbreviate the notation, we write st for $\{s, t\}$. A set of edges is called a *graph*. A graph Γ on $[a, b]$ is said to be *connected* if both a and b are endpoints of edges in Γ and if, in addition, for any $c \in (a, b)$ there is an edge $st \in \Gamma$ such that $s < c < t$. The set of all graphs on $[a, b]$ is denoted $\mathcal{B}[a, b]$, and the subset consisting of all connected graphs is denoted $\mathcal{G}[a, b]$. A *lace* is a minimally connected graph, i.e., a connected graph for which the removal of any edge would result in a disconnected graph. The set of laces on $[a, b]$ is denoted $\mathcal{L}[a, b]$, and the set of laces on $[a, b]$ consisting of exactly N edges is denoted $\mathcal{L}^{(N)}[a, b]$.

Given a connected graph Γ, the following prescription associates to Γ a unique lace L_Γ: The lace L_Γ consists of edges $s_1t_1, s_2t_2, \ldots$, with $t_1, s_1, t_2, s_2, \ldots$ determined, in that order, by

$$t_1 = \max\{t : at \in \Gamma\}, \qquad s_1 = a,$$

$$t_{i+1} = \max\{t : \exists s < t_i \text{ such that } st \in \Gamma\}, \qquad s_{i+1} = \min\{s : st_{i+1} \in \Gamma\}.$$

Given a lace L, the set of all edges $st \notin L$ such that $\mathsf{L}_{L\cup\{st\}} = L$ is denoted $\mathcal{C}(L)$. Edges in $\mathcal{C}(L)$ are said to be *compatible* with L.

For integers $0 \leq s < t$, define (recall (1.2–1.3))

$$V_{st}(\omega) = \lambda_{st} U_{st}(\omega), \tag{A.1}$$

and, for integers $0 \leq a < b$,

$$K[a,b](\omega) = \prod_{a \leq s < t \leq b} (1 - V_{st}(\omega)). \tag{A.2}$$

Then

$$c_n(x) = \sum_{\substack{\omega\,:\,0 \to x \\ |\omega| = n}} K[0,n](\omega), \tag{A.3}$$

where the sum is over all n-step simple random walk paths from 0 to x.

Expanding the product in the definition of $K[a,b](\omega)$, we get

$$K[a,b](\omega) = \sum_{\Gamma \in \mathcal{B}[a,b]} \prod_{st \in \Gamma} (-V_{st}(\omega)). \tag{A.4}$$

For $0 \leq a < b$ we define an analogous quantity, in which the sum over graphs is restricted to connected graphs, namely,

$$J[a,b](\omega) = \sum_{\Gamma \in \mathcal{G}[a,b]} \prod_{st \in \Gamma} (-V_{st}(\omega)). \tag{A.5}$$

This allows us to define the key quantity in the lace expansion:

$$\pi_m(x) = \sum_{\substack{\omega\,:\,0 \to x \\ |\omega| = m}} J[0,m](\omega), \qquad m \geq 2. \tag{A.6}$$

The identity in (1.7) now follows by taking the Fourier transform of the identity given in the following lemma.

Lemma A.1 *For $n \geq 0$,*

$$c_{n+1}(x) = \sum_{y:\|y\|_1=1} c_n(x-y) + \sum_{m=2}^{n+1} \sum_{v \in \mathbb{Z}^d} \pi_m(v) c_{n+1-m}(x-v). \tag{A.7}$$

Proof. Suppress ω in the notation. It suffices to show that

$$K[0,n+1] = K[1,n+1] + \sum_{m=2}^{n+1} J[0,m]K[m,n+1], \tag{A.8}$$

since (A.7) is obtained after insertion of (A.8) into (A.3) followed by factorisation of the sum over ω.

To prove (A.8), we note from (A.4) that the contribution to $K[0,n+1]$ from all graphs Γ for which 0 is not in an edge is exactly $K[1,n+1]$. To resum the contribution from the remaining graphs, we proceed as follows. When Γ does contain an edge ending at 0, we let $m(\Gamma)$ denote the largest value of m such that the set of edges in Γ with at least one end in the interval $[0,m]$ forms a connected graph on $[0,m]$. We lose nothing by taking $m \geq 2$, since $V_{a,a+1} = 0$ for all a. Then resummation over graphs on $[m,n+1]$ gives

$$K[0,n+1] = K[1,n+1] + \sum_{m=2}^{n+1} \sum_{\Gamma \in \mathcal{G}[0,m]} \prod_{st \in \Gamma} (-V_{st})\, K[m,n+1], \tag{A.9}$$

which with (A.5) proves (A.8). $\square$

We next rewrite (A.6) in a form that can be used to obtain good bounds on $\pi_m(x)$. For this, we begin by partially resumming the right-hand side of (A.5), to obtain

$$\begin{aligned} J[a,b] &= \sum_{L \in \mathcal{L}[a,b]} \sum_{\Gamma : \mathsf{L}_\Gamma = L} \prod_{st \in L} (-V_{st}) \prod_{s't' \in \Gamma \setminus L} (-V_{s't'}) \\ &= \sum_{L \in \mathcal{L}[a,b]} \prod_{st \in L} (-V_{st}) \prod_{s't' \in \mathcal{C}(L)} (1 - V_{s't'}). \end{aligned} \tag{A.10}$$

For $0 \leq a < b$, we define $J^{(N)}[a,b]$ to be the contribution to (A.10) coming from laces consisting of exactly N bonds:

$$J^{(N)}[a,b] = \sum_{L \in \mathcal{L}^{(N)}[a,b]} \prod_{st \in L} (-V_{st}) \prod_{s't' \in \mathcal{C}(L)} (1 - V_{s't'}), \qquad N \geq 1. \tag{A.11}$$

Then

$$J[a,b] = \sum_{N=1}^{\infty} J^{(N)}[a,b] \tag{A.12}$$

and by (A.6),

$$\pi_m(x) = \sum_{N=1}^{\infty} (-1)^N \pi_m^{(N)}(x), \tag{A.13}$$

where we define

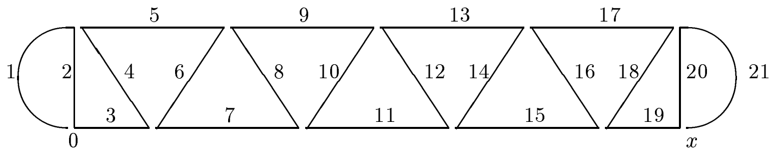
$$\begin{aligned} \pi_m^{(N)}(x) &= (-1)^N \sum_{\substack{\omega : 0 \to x \\ |\omega| = m}} J^{(N)}[0,m](\omega) \\ &= (-1)^N \sum_{\substack{\omega : 0 \to x \\ |\omega| = m}} \sum_{L \in \mathcal{L}^{(N)}[0,m]} \prod_{st \in L} V_{st}(\omega) \prod_{s't' \in \mathcal{C}(L)} (1 - V_{s't'}(\omega)). \end{aligned} \tag{A.14}$$

A.2 Bounds on $\hat{\pi}_m(k)$

In this section, we obtain bounds on $\hat{\pi}_m(k)$ in terms of $c_j = \sum_x c_j(x)$ and $\|c_j\|_\infty = \sup_x c_j(x)$ with $0 \leq j < m$. This serves as a key step in the proof of Lemma 2.3.

A lace L is a collection of edges $s_1t_1, \ldots, s_Nt_N$. Let $\sigma_0, \sigma_1, \ldots, \sigma_{2N-1}$ represent an ordered relabelling of the s_i and t_j. For a lace L on $[0, m]$, by definition $\sigma_0 = 0$ and $\sigma_{2N-1} = m$. Define the intervals $\tilde{I}_j = [\sigma_{j-1}, \sigma_j]$ $(j = 1, \ldots, 2N-1)$, and write $|\tilde{I}_j| = \sigma_j - \sigma_{j-1}$. Note that $|\tilde{I}_j| = 0$ is possible if and only if $N \geq 3$ and $j = 2l+1$ for some $l \in \{1, \ldots, N-2\}$. Define j^* to be the smallest j for which $|\tilde{I}_{j^*}| = \max_{j \in \{1,\ldots,2N-1\}} |\tilde{I}_j|$.

A walk giving a nonzero contribution to (A.14) must intersect itself N times, to ensure that $U_{st} \neq 0$ for each $st \in L$. For example, when $N = 11$ the walk must undergo a trajectory of the form



where the labels on subwalks correspond to the labels of the intervals $\tilde{I}_j$. Here, any of the subwalks labelled 3, 5, 7, 9, 11, 13, 15, 17, 19 can have length zero.

Given a lace L with its corresponding j^*, denote by $\hat{I}_1, \ldots, \hat{I}_{2N-2}$ the ordered set of intervals $\tilde{I}_j$ with $\tilde{I}_{j^*}$ removed. For each $i = 1, \ldots, N-1$, define $I'_i = \hat{I}_{2i-1}$ and $I_i = \hat{I}_{2i}$ if $|\hat{I}_{2i-1}| \geq |\hat{I}_{2i}|$, otherwise define $I'_i = \hat{I}_{2i}$ and $I_i = \hat{I}_{2i-1}$. We have thus partitioned our original $2N-1$ intervals $\tilde{I}_j$ into a maximal interval $\tilde{I}_{j^*}$ and $N-1$ pairs of intervals I_i, I'_i in which the maximal interval in each pair has been associated with a prime. Note that, by construction,

$$|\tilde{I}_{j^*}| \geq \frac{m}{2N-1}. \tag{A.15}$$

Since λ_{st} is a function of $|s-t|$, we write $\lambda_{st} = \lambda(|s-t|)$ in the following lemma.

Lemma A.2 *For any $k \in (-\pi, \pi]^d$ and $m \geq 2$, the following hold:*
(i) For $N = 1$,

$$0 \leq \hat{\pi}_m^{(1)}(k) = \pi_m^{(1)}(0) \leq 2d\lambda(m)\|c_{m-1}\|_\infty. \tag{A.16}$$

For $N \geq 2$,

$$|\hat{\pi}_m^{(N)}(k)| \leq (2N-1)2^{N-1} \sum \lambda(m^*)\|c_{m^*}\|_\infty \prod_{l=1}^{N-1} \lambda(m'_l)\|c_{m'_l}\|_\infty c_{m_l}, \tag{A.17}$$

where the unlabelled sum is over the set of $m^*, m_1, m'_1, \ldots, m_{N-1}, m'_{N-1}$ whose sum is m and for which m^* is maximal and $m'_l \geq m_l$ for each l. Possibly $m_l = 0$, but $m'_l > 0$ for all l, and $m^* > 0$.

(ii) For $N \geq 1$,

$$|\nabla^2 \hat{\pi}_m^{(N)}(k)| \;\leq\; (2N-1)2^{N-1}(N-1)\sum \lambda(m^*)\|c_{m^*}\|_\infty \times \sum_{r=1}^{N-1} \lambda(m'_r)\|c_{m'_r}\|_\infty \nabla^2 \hat{c}_{m_r}(0) \prod_{\substack{1 \leq l \leq N-1 \\ l \neq r}} \lambda(m'_l)\|c_{m'_l}\|_\infty c_{m_l}. \tag{A.18}$$

(iii) For $N \geq 1$, and for any $0 \leq \tilde{\epsilon} \leq 1$,

$$\left|\hat{\pi}_m^{(N)}(k) - \hat{\pi}_m^{(N)}(0) - \frac{k^2}{2d}\nabla^2 \hat{\pi}_m^{(N)}(0)\right| \leq Ck^{2+2\tilde{\epsilon}} m^{2\tilde{\epsilon}} \times [\text{R.H.S. (A.18)}]. \tag{A.19}$$

The same bound holds with $k^2/2d$ replaced by $1 - \hat{D}(k)$ on the left-hand side.

Proof. (i) The equality in (A.16) is a consequence of the fact that $\pi_m^{(1)}(x)$ is nonzero only for $x = 0$. The inequality in (A.16) follows from

$$\pi_m^{(1)}(0) = \lambda(m) \sum_{\substack{\omega : 0 \to 0 \\ |\omega| = m}} \prod_{\substack{0 \leq s' < t' \leq m \\ (s',t') \neq (0,m)}} (1-\lambda_{s't'}U_{s't'}(\omega)) \leq \lambda(m) \sum_{y:|y|=1} c_{m-1}(y). \tag{A.20}$$

For (A.17), we begin with the bound

$$|\hat{\pi}_m^{(N)}(k)| \leq \sum_x |\pi_m^{(N)}(x)|. \tag{A.21}$$

The indices $m^*, m_1, m'_1, \ldots, m_{N-1}, m'_{N-1}$ in (A.17) represent the lengths of the corresponding lace subintervals $\tilde{I}_{j^*}, I_1, I'_1, \ldots, I_{N-1}, I'_{N-1}$. The factor $2N-1$ arises from the number of ways of choosing which of the $2N-1$ subintervals has maximal length. For each of the remaining $N-1$ pairs of subintervals, there is a factor 2 associated with the choice of the longer subinterval, which explains the overall factor 2^{N-1}. Suppose now that the lengths of all the subintervals are fixed. So, in particular, it is known which are the maximal intervals.

Using $1 - V_{s't'} \leq 1$ in (A.14) whenever s' and t' belong to different subwalks, we get an upper bound in which distinct subwalks no longer interact. However, each subwalk remains self-interacting. The norms appearing in (A.17) arise when bounding the sum over $N-1$ diagram vertices (an additional vertex is fixed at 0). Rather than writing down a formal proof, we illustrate the bound with an example. Consider the case $N = 7$ and suppose that $\tilde{I}_{j^*} = \tilde{I}_6$, $I_1 = \tilde{I}_1$, $I'_1 = \tilde{I}_2$, $I_2 = \tilde{I}_4$, $I'_2 = \tilde{I}_3$, $I_3 = \tilde{I}_7$, $I'_3 = \tilde{I}_5$, $I_4 = \tilde{I}_9$, $I'_4 = \tilde{I}_8$, $I_5 = \tilde{I}_{10}$, $I'_5 = \tilde{I}_{11}$, $I_6 = \tilde{I}_{12}$, $I'_6 = \tilde{I}_{13}$. The relevant diagram is bounded by

$$\begin{aligned}\sum_{x_1,x_2,x_3,x_4,x_5,x_6} & c_{m_1}(x_1)c_{m'_1}(x_1)c_{m_2}(x_2-x_1)c_{m'_2}(x_2)c_{m_3}(x_4-x_2)c_{m'_3}(x_3-x_1) \\ & \times c_{m^*}(x_3-x_2)c_{m_4}(x_5-x_3)c_{m'_4}(x_4-x_3) \\ & \times c_{m_5}(x_5-x_4)c_{m'_5}(x_6-x_4)c_{m_6}(x_6-x_5)c_{m'_6}(x_6-x_5).\end{aligned} \tag{A.22}$$

We bound each of the factors corresponding to the maximal subintervals by using the supremum norm. This leaves

$$\sum_{x_1,x_2,x_3,x_4,x_5,x_6} c_{m_1}(x_1)c_{m_2}(x_2-x_1)c_{m_3}(x_4-x_2)c_{m_4}(x_5-x_3)c_{m_5}(x_5-x_4)c_{m_6}(x_6-x_5). \quad (A.23)$$

We then do the sum in the order $x_6, x_3, x_5, x_4, x_2, x_1$.

There is a factor of λ associated with each diagram loop, evaluated at the length of the loop. Since $\lambda(m)$ is monotone decreasing in m, the factor λ associated to a loop can be bounded by λ evaluated at an appropriate subwalk of the loop. It is not difficult to see that it is always possible to choose the subwalks corresponding to $\tilde{I}_{j^*}, I'_1, \ldots, I'_{N-1}$. (In the above example, we get a factor $\lambda(m'_1)$ from the first loop and $\lambda(m'_2)$ from the second. Since $I'_3 = \tilde{I}_5$, we take a factor $\lambda(m'_3)$ from the third, rather than taking $\lambda(m^*)$, which we take instead from the fourth loop. The remaining loops are straightforward.)

(ii) We begin with the bound

$$|\nabla^2 \hat{\pi}_m^{(N)}(k)| \leq \sum_x x^2 |\pi_m^{(N)}(x)|. \quad (A.24)$$

The displacement x can be written as a sum of subwalk displacements y_j, and we can always choose these displacements from among the nonmaximal subwalks, i.e., from $I_1, I_2, \ldots, I_{N-1}$. (In the above example, we would write $x = x_1 + (x_2 - x_1) + (x_4 - x_2) + (x_5 - x_4) + (x_6 - x_5)$.) We use the Cauchy–Schwarz inequality in the form $(\sum_{j=1}^{N-1} y_j)^2 \leq (N-1)\sum_j y_j^2$. Then the argument proceeds as for (A.17).

(iii) We start with the observation that, by symmetry,

$$\hat{\pi}_m^{(N)}(k) - \hat{\pi}_m^{(N)}(0) - \frac{k^2}{2d}\nabla^2 \hat{\pi}_m^{(N)}(0) = \sum_x \Big[\cos(k\cdot x) - 1 + \frac{1}{2}(k\cdot x)^2\Big]\pi_m(x). \quad (A.25)$$

Using $|\cos t - 1 + \frac{1}{2}t^2| \leq Ct^{2+2\tilde{\epsilon}}$ (valid for any $0 \leq \tilde{\epsilon} \leq 1$), and $(k\cdot x)^{2+2\tilde{\epsilon}} \leq k^{2+2\tilde{\epsilon}} x^{2+2\tilde{\epsilon}} \leq k^{2+2\tilde{\epsilon}} m^{2\tilde{\epsilon}} x^2$, we obtain

$$\Big|\hat{\pi}_m^{(N)}(k) - \hat{\pi}_m^{(N)}(0) - \frac{k^2}{2d}\nabla^2 \hat{\pi}_m^{(N)}(0)\Big| \leq Ck^{2+2\tilde{\epsilon}} m^{2\tilde{\epsilon}} \sum_x x^2 |\pi_m^{(N)}(x)|. \quad (A.26)$$

But the last sum is what we bounded in (ii). Replacement of $k^2/(2d)$ by $1 - \hat{D}(k)$ is possible because $1 - \hat{D}(k) = k^2/2d + O(k^4)$. $\square$

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