Anderson Model with decaying randomness : Mobility Edge

W. Kirsch*
Institut für Mathematik
and Sonderforschungsbereich 237
Ruhr-Universität Bochum, 44780 Bochum, Germany

M. Krishna†
Institute of Mathematical Sciences
Taramani, Chennai 600 113, India

J. Obermeit‡
Institut für Mathematik
and Sonderforschungsbereich 237
Ruhr-Universität Bochum, 44780 Bochum, Germany

Abstract

In this paper we consider the Anderson model with decaying randomness $a_n q_\omega(n)$, $a_n > 0, n \in \mathbb{Z}^\nu$ and $q_\omega(n)$, i.i.d random variables with an absolutely continuous distribution μ. For a class of μ we show the following results on a set ω of full measure.

(i) If $|a_n| \to 0$ as $|n| \to \infty$, then $\sigma_c(H_\omega) \subseteq [-2\nu, 2\nu]$

(ii) $\sigma(H_\omega) = \mathbb{R}$.

(iii) If $|a_n| \leq (|n|^{-1-\epsilon})$ for large $|n|$ and $\nu \geq 3$, the mobility edges are the two points $\{-2\nu, 2\nu\}$.

1 Introduction

The Anderson model (see below for a definition) is expected to have interesting spectral properties and Anderson [3] gave the initial argument for the

*E-mail: Werner.Kirsch@mathphys.ruhr-uni-bochum.de

†E-mail:krishna@imsc.ernet.in

‡E-mail: joerg@mathphys.ruhr-uni-bochum.de

absence of diffusion in the model for large disorder. In particular in dimension higher than two, the model is expected to have both absolutely continuous and pure point spectra for small coupling constant. The case of dimension two is still unclear. The existence of pure point spectrum for large λ or large energies has already been shown by Fröhlich-Martinelli-Scoppola-Spencer [11], Delyon-Levy-Souillard [7] and Simon-Wolff [25] based on the breakthrough obtained by Fröhlich-Spencer [12] who obtained the necessary decay of the Green function of the Anderson operator.

The result is further improved and simplified by Carmona-Klein-Martinelli [4], Dreifus-Klein [9] and a very simple proof of the decay estimates on the Green function is recently obtained by Aizenman-Molchanov [2], which was applied to the case of the Bethe lattice also by Aizenman [1].

In this context it is also expected that in higher dimension for small λ, there is absolutely continuous spectrum in an interval containing zero, the length of the interval depending upon λ, outside which it is expected that there is only pure point spectrum. The boundary points of this interval are called the mobility edges. In [1], Aizenman studies the regime where point spectrum exists in the limit $\lambda \to 0$, thus getting upper bounds on the mobility edge as a function of λ.

In one dimension Jacobi matrices with decaying random potentials were first studied by Simon [22], Delyon-Simon-Souillard [8] for Jacobi matrices and Kotani-Ushiroya [18] for Schrödinger operators. Recently there are several further interesting results in the one dimensional decaying randomness case exhibiting various types of spectra. we refere to Kislaev [15], Last-Simon [20] and Kiselev-Last-Simon [16], for example, and the literature therein for further references for Schrödinger/Jacobi operators.

However, the existence of continuous spectrum (let alone the absolutely continuous spectrum) is still elusive in the higher dimensional models, though in a very general context Simon [23] showed that the operators having purely singular continuous spectrum form a dense G_δ in an appropriate metric space of operators.

For the case of decaying randomness with sufficiently fast decay, one of us [19] showed that there is absolutely continuous spectrum in the interval [-2ν, 2ν] for $\nu \geq 3$ and recently Klein [17] showed the existence of absolutely continuous spectrum for small λ in the Bethe lattice.

In this work we are mainly concerned with dimension $\nu > 1$ and decaying randomness and show that outside the interval [-2ν,2ν] there is only pure point spectrum, if any. We use the Aizenman-Molchanov technique [2] (see also [1], [13] and [14]) together with the Simon-Wolff criterion for showing the absence of continuous spectrum. In particular we prove that on certain assumptions on the decaying random parameters $\pm 2\nu$ are the mobility edges of the model, i.e. the transition points from pure point to absolutely continuous spectrum.

We refer to the books of Cycon- Froese-Kirsch-Simon [6], Carmona-Lacroix

[5], Figotin-Pastur [10], Reed-Simon [21] and Weidmann [26] for the general introduction of the subject and for facts used without proof in this paper.

Acknowledgment: MK thanks the Ruhr-University Bochum for an invitation and Sonderforschungsbereich 237 for financial support, which made this work possible.

2 Model and Results

In the following we consider the Anderson model $H_\omega = H_0 + V_\omega$ on $l^2(\mathbb{Z}^\nu)$ with

$$(H_0 u)(n) = \sum_{|i|=1} u(n+i) \quad \text{and} \quad V_\omega(n) = a_n q_n(\omega), \tag{1}$$

where $\{q_n\}$ are independent, identically distributed (iid) random variables with an absolutely continuous probability distribution μ. If not stated otherwise we will suppose $a_n > 0, n \in \mathbb{Z}^\nu$, and a_n bounded.

To state our results we have to introduce some additional properties on the probability measure μ.

Definition 2.1 *Let $\{a_n\}$ be a bounded, positive sequence on $\mathbb{R}$. Then we say that $x \in \{a_n\}-supp\,\mu \equiv M$ if $\sum_n \mu(\frac{1}{a_n}(x-\epsilon, x+\epsilon)) = \infty$ for all $\epsilon > 0$. We call a probability measure μ* **asymptotically large** *wrt. a_n if $\{a_{kn}\} - supp\,\mu = \mathbb{R}$ for all $k \in \mathbb{Z}^+$. μ is said to be asymptotically large at $+\infty$ (resp. $-\infty$) wrt. a_n if $\{a_{kn}\} - supp\,\mu \supset [0,\infty)$ (resp. $(-\infty, 0]$) for all $k \in \mathbb{Z}^+$.*

For the application of the Aizenman-Molchanov method, we need the following definition (see [2]).

Definition 2.2 *A probability measure μ, on the real line, is said to be τ-regular (with $0 < \tau \leq 1$) if, with some $v > 0$ and $C < \infty$,*

$$\mu([z-\delta, z+\delta]) \leq C|\delta|^\tau \mu([z-v, z+v]) \tag{2}$$

for all $0 < \delta < 1$ and $z \in \mathbb{R}$.

Remark: If μ is τ-regular then μ is obviously τ'-regular with $0 < \tau' \leq \tau$.

In this paper we prove the following theorems.

Theorem 2.3 *Let $\nu \geq 1$ and consider the operator H_ω of (1). Let μ be 1-regular and $\int |x|\mu(dx) < \infty$. Furthermore assume that $a_n \to 0$ as $|n| \to \infty$. Then*

$$\sigma_c(H_\omega) \subseteq [-2\nu, 2\nu] \ \mathbb{P} - a.s.$$

Remark:

(i) The assumptions on the measure are made to ensure that it is (1,1)-regular in the sense of Aizenman-Molchanov [2]. The 1-regularity also ensures that $\sup_x \mu(x-a, x+a) \leq C|a|$ required for the apriori bound on the average Green function.

(ii) The above theorem does not assert the existence of continuous spectrum, though it asserts that any spectrum outside $[-2\nu, 2\nu]$ is pure point almost surely. In the case when a_n and μ satisfy the conditions: $\sum_n |a_n|^p < \infty$ with

$$p \equiv \sup \left\{ k \in \mathbb{R}^+, \int |x|^k \mu(dx) < \infty \right\}$$

there is no essential spectrum outside $[-2\nu, 2\nu]$, (see Delyon-Simon-Souillard [8]).

The interesting case of the above theorem is contained in the following theorem.

Theorem 2.4 *Let $\nu \geq 1$ and let H_ω be the operator as in (1) and μ be a probability measure with $M = \cap_{k \in \mathbb{Z}^+}(a_{kn} - supp\mu)$. Then we have*

$$\bigcup_{\lambda \in M} \sigma(H_0 + \lambda \delta_0) \equiv S \subset \sigma_{ess}(H_\omega)\ \mathbb{P} - a.s..$$

As a corollary of Theorem 2.4 we get

Corollary 2.5 *Let $\{a_n\}, \mu$ and M be as in the above theorem. If in addition $M = \mathbb{R}^+$ (resp. $M = \mathbb{R}^-$) then we have*

$$\sigma_{ess}(H_\omega) = [-2\nu, 2\nu] + \mathbb{R}^+ \quad (resp\ \sigma_{ess}(H_\omega) = [-2\nu, 2\nu] + \mathbb{R}^-). \tag{3}$$

Our final theorem is a corollary of the above two theorems, together with a theorem of Krishna [19] which we quote here for completeness.

Theorem 2.6 *Consider the operator $H_\omega = H_0 + V_\omega$ on $l^2(\mathbb{Z}^d), \nu \geq 3$ and $|a_n| \leq |n|^{-\alpha}, \alpha > 1$, as $|n| \to \infty$. Suppose that the corresponding probability measure satisfies the following conditions*

$$\int x d\mu(x) = 0 \qquad and \qquad \int x^2 d\mu(x) = \sigma^2 < \infty, \tag{4}$$

then we have

$$[-2\nu, 2\nu] \subseteq \sigma_{ac}(H_\omega). \tag{5}$$

Theorem 2.7 *Let $\nu \geq 3$ and H_ω be as in (1) with $|a_n| \leq C|n|^{-\alpha}$ as $|n| \to \infty$ for some $\alpha > 1$. Suppose that μ is a 1-regular asyptotically large (with respect to $\{a_n\}$) probability measure with $\int |x|^2 \mu(dx) < \infty$. Then we have*

(i) $\sigma(H_\omega) = \mathbb{R}, \quad \sigma_c(H_\omega) \subseteq [-2\nu, 2\nu], \quad \sigma_{ac}(H_\omega) = [-2\nu, 2\nu]$ $\mathbb{P}$-*a.s.*

(ii) *The mobility edges are precisely the points $\{-2\nu, 2\nu\}$.*

Remark: We note that we cannot yet rule out the presence of singular continuous spectrum or the presence of point spectrum in $[-2\nu, 2\nu]$.
Example: Let $\mu = C\chi_{[d,\infty)}(x)x^\beta dx$, $d > 0$ be a probability measure. The asymptotically large condition at $+\infty$ is satisfied for the pair a_n,μ if $a_n \approx |n|^{-\alpha}, |n| \to \infty$ if $\beta < -3$ and $\alpha(\beta + 1) \leq \nu$. Using this prototypical example one can construct a wide class of them.

Before we actually get down to the proofs, let us explain the idea behind them. The Simon-Wolff criterion asserts that if (a, b) is an interval and H_ω are operators as above, then the condition for absence of continuous spectrum of H_ω in (a, b) is that the resolvent kernel $(H_\omega - E + i0)^{-1}(n, m)$ be square summable in m, for each $n \in \mathbb{Z}^\nu$ for almost every (ω, E). (Actually if this happens for a fixed n, it implies that the spectral measure of H_ω associated with the vector δ_n has no continuous component in (a, b) for almost all ω). The criterion does not guarantee the existence of spectrum, as for example the criterion is trivially valid if (a, b) is in the resolvent set of H_ω for a set of ω of full measure.

We use the Aizenman-Molchanov technique to verify the Simon-Wolff criterion outside $[-2\nu, 2\nu]$ obtaining theorem (2.3) to rule out the continuous component of the spectrum there.

We independently show that there is spectrum outside the above interval by a probabilistic analysis using the conditions on the distribution, its support properties and the moment condition on the distribution for proving theorem 2.4.

While both of the above are valid in more general situations, the limitations in theorem (2.7) come from [19], where the existence of the absolutely continuous spectrum is shown for a restricted class of a_n.

3 Proof of Theorem 2.3

The proof of Theorem 2.3 will mainly follow the proof of Aizenman-Molchanov given in [2]. Where necessary we shall introduce the changes that have to be made in order that Aizenman-Molchanov method works.

Even though we consider the unbounded operators H_ω, we do not need to worry about domain questions, since in most of the computations in the proofs, only the random variable $V_\omega(m)$ for some m occurs and this is defined almost everywhere.

For the proof of the theorem, we verify the Simon-Wolff criterion, namely: if for all $n \in \mathbb{Z}^\nu$, and Lebesgue-a.e. $E \in (a,b)$:

$$\lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^\nu} |G_{E+i\epsilon}(n,m)|^2 < \infty. \tag{6}$$

for almost every realisation of $\{V_n\}_{n \in \mathbb{Z}^\nu}$, then almost surely the spectral measure of the operator H_ω associated with the vector δ_n has no continuous component in the interval (a,b). Since the method works for all the vectors $\delta_n, n \in \mathbb{Z}^\nu$, it follows that there is no continuous component of the spectrum in the stated interval. (We note that unlike the case of $a_n \equiv\ const.$, where ergodicity is used, here we have to argue for each n.)

The need for looking at low moments of the integral kernel $|G_z(n,m)|$ is already sufficiently explained in Aizenman-Molchanov [2] and Simon [24], so we will not stress it further here.

So, if for $0 < s < 1$, $\sum_{m \in \mathbb{Z}^\nu} \mathbb{E}(|G_{E+i\epsilon}(n,m)|^s) < \infty$ for all $E \in (a,b)$ Fubini's theorem implies that for a.e. (ω, E) we have $\sum_{m \in \mathbb{Z}^\nu} |G_{E+i\epsilon}(n,m)|^s < C(\omega,E) < \infty$. Then, for a.e. (ω,E) we get

$$\begin{aligned} \sum_{m \in \mathbb{Z}^\nu} |G_{E+i\epsilon}(n,m)|^2 &= \sum_{m \in \mathbb{Z}^\nu} \left(|G_{E+i\epsilon}(n,m)|^s\right)^{\frac{2}{s}} \\ &\leq \left(\sum_{m \in \mathbb{Z}^\nu} |G_{E+i\epsilon}(n,m)|^s \right)^{\frac{2}{s}} \\ &\leq C(\omega,E)^{\frac{2}{s}} < \infty \end{aligned}$$

and therefore Simon-Wolff criterion holds for the interval (a,b).

In the following, for the application of Aizenman-Molchanov method, we need only a weaker condition on the measure, recall the definition 2.2 The following lemmas will provide the key tools for proving Theorem (2.3). We start with a decoupling principle.

Lemma 3.1 *(Aizenman-Molchanov) Let μ be a 1-regular, absolutely continuous (probability) measure on $\mathbb{R}$. If*

$$\int |x|^p \, d\mu(x) < \infty \tag{7}$$

then for $s < p$ and $s < 1$

$$\int |x-\eta|^s \frac{\alpha}{|x-\beta|^s} \, d\mu(x) \geq \left[\Theta_s(|\eta|)\right]^s \int \frac{\alpha}{|x-\beta|^s} \, d\mu(x), \tag{8}$$

with $\Theta_s(\cdot)$ an increasing positive function on $\mathbb{R}^+$, satisfying

$$\lim_{\eta \to \infty} \frac{\Theta_s(\eta)}{\eta} = 1. \tag{9}$$

Proof: We note that the hypothesis on the density of the measure ensures that it is s-regular for each $0 < s \leq 1$, so the proof follows from Lemma 3.1 in [2].

The properties of the function Θ in the above lemma play an essential role in the proof of Theorem 2.3 via the following lemma.

Lemma 3.2 *Let $a_n, n \in \mathbb{Z}^\nu$ be a sequence satisfying the conditions of theorem (2.3) and let $\Theta_s(\cdot)$ be as in the above lemma. Then,*

(i) *for $E \neq 0$, given any number $0 < u < 1$, there exists an $R = R(E, s, u) \in \mathbb{Z}^+$ with*

$$a_m \left[\Theta_s\left(\frac{|E|}{|a_m|}\right)\right] > u|E|, \quad \forall m \in \mathbb{Z}^+ \quad \text{with} \quad |m| > R. \tag{10}$$

(ii) *For $|n - m| > 2R$*

$$\mathbb{E}\left(|G_{E+i0}\,(n,m)\,|^s\right) \leq D_{R(n,m)} e^{-c\left(\frac{|m|-R}{2}\right)}, \tag{11}$$

where $D_{R(n,m)}$ is a constant independent of E, ϵ, but is polynomially bounded in $|n|$ and $|m|$.

Proof: The estimate in equation (10) follows from the properties of Θ_s of the previous lemma and the assumptions on a_m and μ, since E is non-zero and $s < 1$.

To prove the rest, we set $z = E + i\epsilon$ and consider the case $n = 0$. The proof is similar for any n. We start with

$$\begin{aligned}\delta_{0,m} &= \langle \delta_0, (H_\omega - z)^{-1}\,(H_\omega - z)\,\delta_m\rangle \\ &= (V_\omega\,(m) - z)\,\langle \delta_0, (H_\omega - z)^{-1}\,\delta_m\rangle + \sum_{|i|=1} \langle \delta_0, (H_\omega - z)^{-1}\,\delta_{m+i}\rangle,\end{aligned}$$

i.e.

$$\begin{aligned}\mathbb{E}\left(|V_\omega\,(m) - z|^s\,|G_z\,(0,m)\,|^s\right) &\leq \mathbb{E}\left(|\delta_{0,m}|^s\right) + \mathbb{E}\left(\sum_{|i|=1} |G_z\,(0,m+i)\,|^s\right) \\ &= \delta_{0,m} + \sum_{|i|=1} \mathbb{E}\left(|G_z\,(0,m+i)\,|^s\right).\end{aligned}$$

By Lemma 3.1 we know that

$$\mathbb{E}\left(|V_\omega\,(m) - z|^s |G_z\,(0,m)\,|^s\right) \geq |a_m|^s \left[\Theta_s\left(\frac{|E|}{|a_m|}\right)\right]^s \mathbb{E}\left(|G_z\,(0,m)\,|^s\right) \tag{12}$$

which implies

$$
\begin{aligned}
\mathbb{E}\left(\left|G_z\left(0,m\right)\right|^s\right) &\leq \frac{2\nu}{|a_m|^s\left[\Theta_s\left(\frac{|E|}{|a_m|}\right)\right]^s}\left(\frac{1}{2\nu}\sum_{|i|=1}\mathbb{E}\left(\left|G_z\left(0,m+i\right)\right|^s\right)\right) \\
&= \underbrace{\left[\frac{(2\nu)^{1/s}}{|E|\left[\frac{\Theta_s\left(\frac{|E|}{|a_m|}\right)}{\frac{|E|}{|a_m|}}\right]}\right]^s}_{\equiv\kappa_m}\left(\frac{1}{2\nu}\sum_{|i|=1}\mathbb{E}\left(\left|G_z\left(0,m+i\right)\right|^s\right)\right) \qquad (13)
\end{aligned}
$$

where $\Theta_s(\cdot)$ is as in Lemma 3.1.

We fix $0 < s < 1$, $E \in \mathbb{R}\backslash[-(2\nu)^{1/s},(2\nu)^{1/s}]$, define $\gamma_{E,\nu} \equiv \frac{(2\nu)^{1/s}}{|E|}$. Clearly $\gamma_{E,\nu} < 1$.

Now we take, $u = \gamma_{E,\nu}(1+\gamma(E,\nu))/2$ in equation (10), then from part (i) of the lemma we get the existence of an $R(E,\nu,\gamma(E,\nu))$ such that

$$
\frac{\Theta_s\left(\frac{|E|}{|a_m|}\right)}{\frac{|E|}{|a_m|}} \geq \gamma_{E,\nu} + \frac{(1-\gamma_{E,\nu})}{2}, \quad \forall\ m \in \mathbb{Z}^\nu \text{ with } |m| > R.
$$

We therefore get an upper bound for κ_m, $m \notin B_{2R}(0)$

$$
\kappa_m = \frac{\gamma_{E,\nu}}{\frac{\Theta_s\left(\frac{|E|}{|a_m|}\right)}{\frac{|E|}{|a_m|}}} \leq \frac{\gamma_{E,\nu}}{\gamma(E,\nu)+\frac{(1-\gamma_{E,\nu})}{2}} \equiv \gamma_0^{1/s} < 1. \qquad (14)
$$

By repeating the above estimate $\frac{|m|-R}{2}$ times, for $m \notin B_{2R}(0)$, we get

$$
\mathbb{E}\left(\left|G_z\left(0,m\right)\right|^s\right) \leq \gamma_0^{\frac{|m|-R}{2}}\left[\sup_{k\in B_{\frac{|m|-R}{2}}(m)}\mathbb{E}\left(\left|G_z\left(0,k\right)\right|^s\right)\right]. \qquad (15)
$$

Suppose $\sup_k \mathbb{E}\left(\left|G_z\left(0,k\right)\right|^s\right)$ is polynomially bounded in m and R independent of z.

Then setting $\gamma_0 = e^{-\ell}$, we finally have,

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}^\nu}\mathbb{E}\left(\left|G_z\left(0,m\right)\right|^s\right) &\leq \sum_{|m|\leq 2R}\mathbb{E}\left(\left|G_z\left(0,m\right)\right|^s\right) + \sum_{|m|>2R}\mathbb{E}\left(\left|G_z\left(0,m\right)\right|^s\right) \\
&\leq \sum_{|m|\leq 2R} D_{0,m} + \sum_{|m|>2R} e^{-\ell\left(\frac{|m|-R}{2}\right)} \\
&\leq (4R)^\nu D_{0,2R} + \sum_{|m|>2R} e^{-\ell\left(\frac{|m|-R}{2}\right)} < \infty
\end{aligned}
\qquad (16)
$$

where $D_{0,\cdot}$ denotes the polynomial bound.

To prove the assumed polynomial boundedness of $\mathbb{E}\left(|G_z(0,m)|^s\right)$ we need the following where we set $G_z(n,m) \equiv (H_\omega - z)^{-1}(n,m)$. Henceforth $\mathbb{P}(A)$ will stand for the probability of the event A.

Lemma 3.3 *Suppose μ is a an absolutely continuous probability measure with bounded density and let H_ω be as in (1). Then for any pair of sites $n, m \in \mathbb{Z}^\nu$*

$$\mathbb{P}\left(|G_z(n,m)| \geq t\right) \leq \left[\frac{C}{a_n} + \frac{C}{a_m}\right]\frac{1}{|t|}.$$

Proof (of Lemma 3.3): We present here only the part of the proof of this lemma that requires an argument, the proof essentially follows Theorem II.1 in [2] with $\lambda = 1$. We note here that since the measure μ has bounded density, we have the estimate $\sup_u \mu([u, u+t]) \leq C|t|$ required by Theorem II.1 of [2]. We present only the case of $n \neq m$, since the case $n = m$ is easy. We essentially need to calculate,

$$\begin{aligned}
&\mathbb{P}\left(d_1\left((V_n, V_m), S\right) \leq \frac{1}{t}\right) \\
&\leq \mathbb{P}\left(|V_m - S_1(n)| \leq \frac{\sqrt{2}}{t}\right) + \mathbb{P}\left(|V_n - S_2(m)| \leq \frac{\sqrt{2}}{t}\right) \\
&= \mathbb{P}\left(|a_m q_m - S_1(n)| \leq \frac{\sqrt{2}}{t}\right) + \mathbb{P}\left(|a_n q_n - S_2(m)| \leq \frac{\sqrt{2}}{t}\right) \\
&= \int_{\frac{S_1}{a_m} - \frac{\sqrt{2}}{a_m t}}^{\frac{S_1}{a_m} + \frac{\sqrt{2}}{a_m t}} d\mu\,(q_n) + \int_{\frac{S_2}{a_n} - \frac{\sqrt{2}}{a_n t}}^{\frac{S_2}{a_n} + \frac{\sqrt{2}}{a_n t}} d\mu\,(q_m) \\
&= \mu\left(\left[\frac{S_1}{a_m} - \frac{\sqrt{2}}{a_m t}, \frac{S_1}{a_m} + \frac{\sqrt{2}}{a_m t}\right]\right) + \mu\left(\left[\frac{S_2}{a_n} - \frac{\sqrt{2}}{a_n t}, \frac{S_2}{a_n} + \frac{\sqrt{2}}{a_n t}\right]\right) \\
&\leq \left(\frac{2\sqrt{2}}{a_n t} + \frac{2\sqrt{2}}{a_m t}\right)
\end{aligned}$$

using notations (for d_1, S, S_1 and S_2) as given in Theorem II.1 in [2].

With the help of Lemma 3.3 we finally prove the polynomial bound for $\mathbb{E}(|G_z(0,n)|^s)$. We get

$$\begin{aligned}
\mathbb{E}\left(|G_z(n,m)|^s\right) &= \int_0^\infty \mathbb{P}\left(|G_z(n,m)|^s \geq t\right) dt \\
&= \int_0^\infty \mathbb{P}\left(|G_z(n,m)| \geq t^{1/s}\right) dt \\
&\leq \int_0^\infty \min\left(1, \frac{2\sqrt{2}}{a_n t^{1/s}}\right) dt + \int_0^\infty \min\left(1, \frac{2\sqrt{2}}{a_m t^{1/s}}\right) dt \\
&\leq C\left(a_n^{-1} + a_m^{-1}\right) \int_0^\infty \min\left(1, \frac{2\sqrt{2}}{t^{1/s}}\right) dt \\
&\equiv D_{n,m} < \infty.
\end{aligned}$$

Thus, $D_{n,m}$ is polynomial bounded and independent of z, but not of n and m. From this the stated polynomial boundedness assumed after equation (15) is clear.

We showed that $\Omega_s = \left\{\omega : \sigma_c(H_\omega) \subseteq [-(2\nu)^{1/s}, (2\nu)^{1/s}]\right\}$ has probability 1 for each $0 < s < 1$. Now taking a subsequence $s_n \uparrow 1$ and taking $\omega \in \cap \Omega_{s_n}$, we can conclude the theorem.

4 Proof of Theorem 2.4 and Corollary 2.5

In this section we prove the existence of essential spectrum of the operators H_ω outside the interval $[-2\nu, 2\nu]$. We start with the proof of Theorem 2.4

Proof of Theorem 2.4: Let $\lambda \in \cap_{k\in\mathbb{Z}^+}(\{a_{kn}\} - \operatorname{supp}\mu)$ and consider the "model-operator" $H(\lambda) = H_0 + \lambda\delta_0$. We prove that if $E \in \sigma(H(\lambda))$ then $E \in \sigma_{ess}(H_\omega)$.

As a direct conclusion from our definition of the $\{a_n\} - \operatorname{supp}\mu$ we know for all $\epsilon > 0$ that

$$\sum_{n\in k\mathbb{Z}^\nu} \mu\left(a_n^{-1}(\lambda-\epsilon, \lambda+\epsilon)\right) = \infty. \tag{17}$$

We define,

$$\begin{aligned}\Omega_{\epsilon,k} = \{\omega \mid \text{ For } \infty\text{-many } n \text{ we have: } q_n \in a_n^{-1}(\lambda-\epsilon,\lambda+\epsilon) \\ \text{and } q_j \in a_j^{-1}(-\epsilon,\epsilon) \text{ for } 0 < |j-n| < k\}.\end{aligned} \tag{18}$$

We now cover the lattice $\mathbb{Z}^\nu$ with cubes of side 2k centered at the points of $k\mathbb{Z}^\nu$ for an arbitrary but fixed $k \in \mathbb{Z}^+$. (This covering is done to obtain a collection of independent events indexed by points in $k\mathbb{Z}^\nu$ to apply the Borel-Cantelli lemma). An application of the Borel-Cantelli lemma implies that

$$\mathbb{P}\left(\Omega_{\epsilon,k}\right) = 1. \tag{19}$$

Now let

$$\overline{\Omega} = \bigcap_n \Omega_{\frac{1}{n},n}. \tag{20}$$

Then $\mathbb{P}(\overline{\Omega}) = 1$, since $\overline{\Omega}$ is a countable intersection of sets of measure 1.

Now, suppose $E \in \sigma(H(\lambda))$ and $E \neq 0$. Then there exists a Weyl sequence ψ_n in $l^2(\mathbb{Z}^\nu)$ for $H(\lambda)$ such that: $||\psi_n|| = 1$ and $||H(\lambda)\psi_n - E\psi_n|| < \frac{1}{n}$ and $\operatorname{supp}\psi_n$ is compact.

Then find cubes $\Lambda_n(\varrho_n)$ with

$$|V_\omega(\varrho_n) - E| < \frac{1}{n} \quad \textit{and} \quad |V_\omega(k)| < \frac{1}{n} \qquad , \ k \in \Lambda_n(\varrho_n) \setminus \{\varrho_n\}$$

where Λ_n is large enough so that $\operatorname{supp}\psi_n \subset \Lambda_n(0)$. By $\Lambda_n(\varrho_n)$ we denote the cube of side n with center ϱ_n.

Now we define a new sequence $\varphi_n(j) = \psi_n(j - \varrho_n)$.

Thus,

$$\begin{aligned} \|\,(H_0 + V_\omega)\,\varphi_n - E\varphi_n\| &= \|\,(H_0 + V_\omega\,(\cdot + \varrho_n))\,\psi_n - E\psi_n\| \\ &\leq \|\,(H_0 + \lambda\delta_0)\,\psi_n - E\psi_n\| \\ &\quad + \|\,(V_\omega\,(\cdot + \varrho_n) - \lambda\delta_0\,(\cdot))\,\psi_n\| \\ &\leq \frac{1}{n} + \frac{1}{n} \end{aligned}$$

Because of (20) we are able to choose ϱ_n and $\Lambda_n(\varrho_n)$ such that $\operatorname{supp}\varphi_n \cap \operatorname{supp}\varphi_k = \emptyset$, $k = 1, \ldots, n-1$. E.g. choose ϱ_n so large so that

$$\Lambda_{R_n}(\varrho_n) \cap K_{n-1} = \emptyset \tag{21}$$

where $K_{n-1} = \bigcup_{j=1}^{n-1} \operatorname{supp}\varphi_j$.

Therefore $\varphi_n \overset{w}{\to} 0$. Thus φ_n is a Weyl sequence for E for H_ω, so $E \in \sigma(H_\omega)$.

Proof of Corollary 2.5: We just have to prove that if $\{a_n\} - \operatorname{supp}\mu = \mathbb{R}^+$ (resp. $\mathbb{R}^-$) then $\sigma_{ess}(H_\omega) \supset \mathbb{R}^+$ (resp. $\mathbb{R}^-$).

Let $2\nu < E \in \mathbb{R}^+$. Then there exists a $\lambda \in \mathbb{R}^+$ such that $E \in \sigma(H_0 + \lambda\delta_0)$. Furthermore, it is possible to find some $\lambda \in \mathbb{R}^+$ with $E = \sup\sigma(H_0 + \lambda\delta_0)$. Now define $s_\lambda \equiv \sup\sigma(H_0 + \lambda\delta_0) = \sup_{\|\varphi\|=1}\langle\varphi, (H_0 + \lambda\delta_0)\varphi\rangle$. By the min-max-principle it is easy to see, that s_λ is continuous in λ. Therefore it is sufficient to prove that $s_\lambda \to \infty$ if $\lambda \to \infty$.

We have

$$\begin{aligned} s_\lambda &= \sup_{\|\varphi\|=1} \left(\langle\varphi, H_0\varphi\rangle + \lambda\langle\varphi, \delta_0\varphi\rangle\right) \\ &\geq \lambda \sup_{\|\varphi\|=1} \langle\varphi, \delta_0\varphi\rangle - 2\nu \\ &= \lambda - 2\nu \to \infty \text{ as } \lambda \to \infty. \end{aligned} \tag{22}$$

To prove the other statement we just have to take $-2\nu > E \in \mathbb{R}^-$ and to change sup to inf in (22).

References

[1] M. Aizenman. Localization at weak disorder: Some elementary bounds. *Rev. Math. Phys.*, 6:1163–1182, 1994.

[2] M. Aizenman and S. Molchanov. Localization at large disorder and at extreme energies: an elementary derivation. *Commun. Math. Phys.*, 157:245–278, 1993.

[3] P. Anderson. Absence of diffusion in certain random lattices. *Phys. Rev.*, 109:1492–1505, 1958.

[4] R. Carmona, A. Klein, and F. Martinelli. Anderson localization for Bernoulli and other singular potentials. *Commun. Math. Phys.*, 108:41–66, 1987.

[5] R. Carmona and J. Lacroix. *Spectral theory of random Schrödinger operators.* Birkhäuser Verlag, Boston, 1990.

[6] H. Cycon, R. Froese, W. Kirsch, and B. Simon. *Topics in the Theory of Schrödinger operators.* Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[7] F. Delyon, Y. Levy, and B. Souillard. Anderson localization for multi dimensional systems at large disorder or low energy. *Commun. Math. Phys.*, 100:463–470, 1985.

[8] F. Delyon, B. Simon, and B. Souillard. From power pure point to continuous spectrum in disordered systems. *Ann. Institute H. Poincaré*, 42:283–309, 1985.

[9] H. v. Dreifus and A. Klein. A new proof of localization in the Anderson tight binding model. *Commun. Math. Phys.*, 124:285–299, 1989.

[10] A. Figotin and L. Pastur. *Spectral properties of disordered systems in the one body approximation.* Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[11] J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer. Constructive proof of localization in the Anderson tight binding model. *Commun. Math. Phys.*, 101:21–46, 1985.

[12] J. Fröhlich and T. Spencer. Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Commun. Math. Phys.*, 88:151–184, 1983.

[13] G.M. Graf. Anderson localization and the space-time characteristic of continuum states. *J. Stat. Phys.*, 75:337–346, 1994.

[14] D. Hundertmark. On the time-dependent approach to Anderson localization. *Preprint*, 1997.

[15] A. Kiselev. Absolutely continuous spectrum of one dimensional Schrödinger operators and Jacobi matrices with slowly decreasing potentials. *Commun. Math. Phys.*, 179:377–400, 1996.

[16] A. Kiselev, Y. Last and B. Simon. Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators. *Caltech Preprint*, 1997.

[17] A. Klein. Extended states in the Anderson model on the Bethe lattice. Preprint, Department of Mathematics, University of California, Irvine, 1994.

[18] S. Kotani and N. Ushiroya. One-dimensional Schrödinger operators with random decaying potentials. *Commun. Math. Phys.*, 115:247–266, 1988.

[19] M. Krishna. Anderson Model with decaying randomness - extended states. *Proc. Indian. Acad. Sci. (MathSci.)*, 100(4):285–294, 1990.

[20] Y. Last and B. Simon. Eigenfunctions, transfer matrices and absolutely continuous spectrum of one dimensional Schrödinger operators. *Caltech Preprint*, 1996.

[21] M. Reed and B. Simon. *Methods of modern Mathematical Physics: Functional Analysis.* Academic Press, New York, 1975.

[22] B. Simon. Some Jacobi matrices with decaying potentials and dense point spectrum. *Comm. Math. Phys.*, 87:253–258, 1982.

[23] B. Simon. Operators with singular continuous spectrum: General operators. *Ann. Math.*, 141:131–145, 1995.

[24] B. Simon. Spectral analysis of rank one perturbations and applications. In J. Feldman, R. Froese, and L. Rosen, editors, *CRM Lecture Notes Vol. 8*, pages 109–149, Amer. Math. Soc., Providence, RI, 1995.

[25] B. Simon and T. Wolff. Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians. *Comm. Pure Appl. Math.*, 39:75–90, 1986.

[26] J. Weidman. *Linear Operators in Hilbert spaces, GTM-68.* Springer-Verlag, Berlin, 1987.