A VARIATIONAL METHOD FOR NONLINEAR EIGENVALUE PROBLEMS

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Abstract. We present a new variational technique to characterize eigenvalues of several nonlinear problems. We apply it in detail to two different problems described by nonlinear ODE's. First we consider the bifurcation problem $u'' + \lambda u = N(u)$ with two point boundary conditions, where $N(u)$ is a general nonlinear term, and give a variational characterization for the eigenvalue λ. The second problem is the determination of the asymptotic speed, c, of propagation of fronts for the one dimensional reaction-diffusion equation $u_t = u_{xx} + f(u)$. We will obtain a variational characterization of c for different types of reaction terms $f(u)$. An extension to nonlinear eigenvalue problems described by PDE's will be briefly described.

1. Introduction

In this article we focus on three different types of problems which can be viewed generically as nonlinear eigenvalue problems. Our purpose in all three problems is to obtain good estimates on the nonlinear eigenvalue. In fact we will obtain, in each case, a variational characterization for the respective eigenvalues. Even though the problems are rather different in their nature, the techniques we use to find variational principles for the eigenvalues are very similar.

The first type of problem comes from classical mechanics and is epitomized by the simple pendulum (also by Euler's *Elastica*). Consider a particle of mass m attached to a fixed point through an ideal string of length ℓ, in a uniform gravitational field g. Let us denote by θ the angle that measures the deviation of the pendulum from the vertical. If we start the motion of the pendulum from the rest position at an angle θ_m, its motion is governed by

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0 \tag{1.1}$$

with $\theta(0) = \theta_m$ and $d\theta/dt(0) = 0$. The motion of the pendulum is periodic with period T, say. The problem here is to obtain the period T as a function of the amplitude θ_m. Certainly in this case $T(\theta_m)$ can be obtained in closed form in terms of complete elliptic integrals. However, in order to introduce the general type of problem we would like to treat here, let us transform the problem of the

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pendulum into a nonlinear eigenvalue problem. Because of the symmetry of the motion of the pendulum it is clear that $\theta(T/4) = 0$. Let $x = 4t/T$, so that if $t \in [0, T/4]$, $x \in [0, 1]$. After these considerations, the problem of finding the period of the pendulum has been reduced to finding a pair $(\theta, \lambda(\theta_m))$ satisfying the two point boundary value problem

$$\frac{d^2\theta}{dx^2} + \lambda \sin\theta = 0 \qquad \text{for } x \in (0,1), \tag{1.2}$$

with

$$\theta'(0) = 0, \qquad \theta(1) = 0, \qquad \text{and} \qquad \theta(0) \equiv \theta_m. \tag{1.3}$$

Henceforth $'$ denotes differentiation of a function with respect to its argument. Here, λ is given in terms of the period by $\lambda = T^2 g/16\ell$. We are only interested in the principal eigenvalue, corresponding to $\theta(x) \geq 0$. It is well known that in the θ_m-λ plane, the solution to the two point boundary value problem describes a curve $\lambda(\theta_m)$ that *bifurcates* from the trivial solution at the value $\lambda = \pi^2/4$, which is the principal eigenvalue obtained from (1.2) by approximating the nonlinear term $\sin\theta$ by θ. The value $\lambda = \pi^2/4$ gives $T = 2\pi\sqrt{\ell/g}$, which is the classical expression obtained by Galileo for the period of the pendulum. Concerning this first type of problem, we will find a variational characterization for $\lambda(\theta_m)$. More precisely we have the following,

THEOREM 1.1 ([**7**], [**9**]). *Let the pair (λ, u) be the principal solution (i.e., with $u(x) \geq 0$) of the two point boundary value problem*

$$\frac{d^2u}{dx^2} + \lambda u = N(u) \tag{1.4}$$

subject to $u'(0) = u(1) = 0$. Let $u_m = u(0)$, the sup-norm of the solution. Here, $N(u)$ is a general nonlinear term, which is continuous in $(0, u_m)$. Then

$$\lambda[u_m] = \max_{g \in \mathcal{D}} \left(\int_0^{u_m} N(u)g(u)\,du + \frac{1}{2}\left(\int_0^{u_m} g'(u)^{1/3}\,du\right)^3 \right) \Big/ \int_0^{u_m} ug(u)\,du, \tag{1.5}$$

where $\mathcal{D} = \{g \mid g \in C^1(0, u_m), g' > 0, g(0) = 0\}$. Moreover, the maximum is attained at some $\hat{g} \in \mathcal{D}$, which is unique up to a multiplicative constant.

REMARK 1.2. The two point boundary problem itself comes from minimizing an energy functional. Typically the eigenvalue λ is the derivative of the energy with respect to the L^2-norm of the solution u. Thus, the energy functional can not be used directly to yield estimates on the nonlinear eigenvalue λ. However, the variational principle given by Theorem 1.1 can be used for that purpose.

REMARK 1.3. To make (1.4) resemble the problem of the pendulum more closely we should have also multiplied the nonlinear term by λ. However, as should be clear from the proof of Theorem 1.1, this only produces straightforward changes in the variational characterization of the eigenvalue (see, nevertheless, Theorem 2.7 below).

REMARK 1.4. If the nonlinear term $N(u)$ is identically zero, (1.5) yields the inequality

$$\frac{\left(\int_0^{u_m} (g'(u))^{1/3}\,du\right)^3}{\int_0^{u_m} g(u)u\,du} \leq \frac{\pi^2}{2}.$$

which is a particular case of a family of inequalities for monotone functions recently derived by us [**10**].

The second type of problem originally came either from population dynamics or from combustion theory and flame propagation. In their simplest form, some of these problems can be described by the one-dimensional nonlinear diffusion equation

$$u_t = u_{xx} + f(u) \qquad \text{with } (x,t) \in \mathrm{R} \times \mathrm{R}_+. \tag{1.6}$$

Here, the reaction term $f(u)$ satisfies $f(u) \in C^1[0,1]$, $f(0) = f(1) = 0$. A typical example for the reaction term, in problems arising in population dynamics, is given by $f(u) = u(1-u)$. Eqn. (1.6) with this particular profile is known as Fisher's equation [**17**]). On the other hand for flame propagation a typical profile is given by $f(u) = (1-u)(\exp^{-\beta(1-u)} - \exp^{-\beta})$ [**34**]. Aronson and Weinberger [**1**] showed that any positive sufficiently localized (this means decaying faster than exponentially for $|x| \to \infty$) initial condition $u(x,0)$, with $u(x,0) \in [0,1]$, evolves into a travelling front propagating at a speed c^* (i.e., for large t, $u(x,t)$ behaves as $q(x - c^*t)$). The shape of the travelling front is determined by the following two point boundary value problem:

$$q'' + cq' + f(q) = 0, \tag{1.7}$$

with

$$\lim_{x\to-\infty} q(x) = 1 \qquad \text{and} \qquad \lim_{x\to+\infty} q(x) = 0. \tag{1.8}$$

Aronson and Weinberger characterized the asymptotic speed c^* as the minimum value of the parameter c in (1.7) for which the solution $q(x)$ is monotonic. For Fisher's equation, Kolmogorov, Petrovskii and Piskunov [**22**] (see also the translation of this article in [**23**]) showed that $c^* = c_{KPP} = 2$. In fact for any reaction term $f(u)$ such that $0 \le f(u) \le f'(0)u$, Kolmogorov, Petrovskii, and Piskunov showed that $c^* = 2\sqrt{f'(0)}$ (see [**22**], [**1**], [**15**], and Remark 3.1 below). The problem here is to determine the value of c^* for more general reaction terms. As in the first type of problem what we find is a variational characterization for $c^*(f)$.

In the sequel we will consider three types of reaction terms.

Case A: $f'(0) > 0$, $f(u) > 0$, $u \in (0,1)$. In population dynamics this case is known as *heterozygote intermediate*. This is the case considered in the classical work of Fisher [**17**] and Kolmogorov, Petrovskii and Piskunov [**22**].

Case B: there exists $a \in (0,1)$ such that $f(u) \le 0$ for all $u \in (0,a)$, $f(u) > 0$ for all $u \in (a,1)$ and $\int_0^1 f(u)\,du > 0$. This case is usually known as the *combustion case.*

Case C: There exists $a \in (0,1)$ such that $f(u) < 0$ in $(0,a)$, $f(u) > 0$ in $(a,1)$ and $\int_0^1 f(u)\,du > 0$ and $f'(0) < 0$. In population dynamics this case is known as *heterozygote inferior.* This is also known as the *bistable* case.

Our main results concerning the propagation of fronts are the following two theorems. The first one is valid only in Case A, while the second is valid for all three cases.

THEOREM 1.5 ([**6**], [**4**]). *Let $f \in C^1(0,1)$ with $f(0) = f(1) = 0$, $f'(0) > 0$ and $f(u) > 0$ for $u \in (0,1)$. Then*

$$c^* = J \equiv \sup\{I(g) \mid g \in \mathcal{E}\}, \tag{1.9}$$

where

$$I(g) = 2\,\frac{\int_0^1 \sqrt{fgh}\,du}{\int_0^1 g\,du}, \tag{1.10}$$

and $\mathcal{E}$ is the space of functions in $C^1(0,1)$ such that $g \geq 0$, $h \equiv -g' > 0$ in $(0,1)$, $g(1) = 0$, and $\int_0^1 g(u)\,du < \infty$. Moreover, if $c^* \neq 2\sqrt{f'(0)}$, J is attained at some $\hat{g} \in \mathcal{E}$, and $\hat{g}$ is unique up to a multiplicative constant.

Theorem 1.6 ([**8**]). *Let $f \in C^1(0,1)$ with $f(0) = f(1) = 0$, and $\int_0^1 f(u)\,du > 0$. Then,*

$$(c^*)^2 = L \equiv \sup\{M(g) \bigm| g \in \mathcal{F}\}, \tag{1.11}$$

where

$$M(g) = 2\frac{\int_0^1 fg\,du}{\int_0^1 (g^2/h)\,du}, \tag{1.12}$$

and $\mathcal{F}$ is the space of functions in $C^1(0,1)$ such that $g \geq 0$, $h \equiv -g' > 0$ in $(0,1)$, $g(1) = 0$, and the integrals in (1.12) *exist. Moreover, if f belongs to cases B or C, or to case A with $c^* \neq 2\sqrt{f'(0)}$, L is attained at some $\hat{g} \in \mathcal{F}$, and $\hat{g}$ is unique up to a multiplicative constant.*

It is important to remark here that the speed of propagation of fronts, c^*, is also characterized by a min-max principle (see [**18**], Theorem 8, pp. 257, and also [**33**]).

The first two problems presented in this introduction are described in terms of autonomous ordinary differential equations. Also, the method of proof of Theorems 1.1, 1.5, and 1.6 relies on the monotonicity of the solutions and on the reduction of the ordinary differential equations to the phase-plane. Thus, in principle, it would appear that the techniques developed here are not extendible to nonlinear eigenvalue problems described by partial differential equations. Fortunately this is not exactly so. In fact, the general scheme can be extended to at least some nonlinear eigenvalue problems described by nonlinear partial differential equations. We will briefly describe one such example in Section 4.

The rest of this article is organized as follows. In Section 2 we give in detail the proof of Theorem 1.1, and present several corollaries and extensions. In Section 3 we give the proofs of Theorems 1.5 and 1.6. Finally in Section 4, we will present, without a proof, the variational characterization for the eigenvalue of a nonlinear problem described by a partial differential equation.

2. Variational Characterization for the Eigenvalue of Nonlinear Hamiltonian Systems

We consider here the two point boundary value problem

$$u'' + \lambda u = N(u) \qquad \text{subject to} \qquad u'(0) = 0, \qquad u(1) = 0. \tag{2.1}$$

We are interested only in the principal solution, i.e., in the pair (λ, u) with $u \geq 0$. We denote by $u_m = u(0)$ the sup-norm of the solution. Here $N(u)$ is a general nonlinear term, which is assumed to be continuous in $(0, u_m)$. This equation, with different nonlinear terms $N(u)$, arises in the study of many elementary mechanics problems [**20, 24, 19, 21, 29**].

For the existence and regularity of nontrivial positive solutions to (2.1) we refer the reader to [**24**] (see also [**28**] for nonlinear eigenvalue problems related to general Sturm-Liouville operators). The positive solutions to (2.1) are decreasing and the problem can be reduced to a quadrature (see e.g., [**24**]). In fact,

$$\frac{1}{2}u'^2 + V(u) = E \tag{2.2}$$

where

$$V(u) = \frac{1}{2}\lambda u^2 - \int_0^u N(s)\,ds \tag{2.3}$$

and $E = V(u_m)$. From a physical point of view we can think of (2.1) as Newton's equation for a particle moving under a force $N(u) - \lambda u$. This is a conservative force which derives from the potential $V(u)$, and the energy of the particle is a constant, equal to E.

Let $p = -du/dx > 0$. Eqn. (2.1) is equivalent to the system of first order equations

$$\begin{aligned} u' &= -p \\ p' &= \lambda u - N(u). \end{aligned} \tag{2.4}$$

The functions $u(x)$, $p(x)$ corresponding to a solution of (2.4) trace out a trajectory in the (u,p)-plane (usually called the phase plane). Such a trajectory has slope

$$\frac{dp}{du} = -\frac{\lambda u}{p} + \frac{N(u)}{p}, \tag{2.5}$$

at any point where $p \neq 0$. Now, let $g \in C^1(0, u_m)$ with $g' \geq 0$ and $g(0) = 0$ but otherwise arbitrary. Multiplying (2.5) by $pg(u)$ and integrating in u from 0 to u_m we get,

(2.6)
$$\lambda \int_0^{u_m} ug(u)\,du - \int_0^{u_m} N(u)g(u)\,du = -\frac{1}{2}\int_0^{u_m} \frac{dp^2}{du}g(u)\,du = \frac{1}{2}\int_0^{u_m} p^2 g'(u)\,du$$

where the last equality follows by integrating by parts, using $p(u_m) = 0$ and $g(0) = 0$.

Now, notice that $1 = \int_0^1 dx = \int_0^{u_m} du/p(u)$. Thus, for any positive constant K we can write the right side of (2.6) as

$$R \equiv \frac{1}{2}\int_0^{u_m} p^2 g'(u)\,du = \int_0^{u_m} \left(\frac{1}{2}p^2 g'(u) + \frac{K}{p}\right)\,du - K. \tag{2.7}$$

Since $K > 0$ and $g' > 0$,

$$\frac{1}{2}p^2 g'(u) + \frac{K}{p} \geq \frac{3}{2}K^{2/3}(g')^{1/3} \tag{2.8}$$

The inequality is obtained by minimizing the left side with respect to p in $(0, \infty)$. Equality is attained in (2.8) if and only if

$$p^3 \frac{dg}{du} = K. \tag{2.9}$$

From (2.7) and (2.8) we get

$$R \geq \frac{3}{2} K^{2/3} \int_0^{u_m} (g')^{1/3}\, du - K. \tag{2.10}$$

Choosing $K = (\int_0^{u_m} g'(u)^{1/3}\, du)^3$, we obtain

$$R \geq \frac{1}{2} \left(\int_0^{u_m} g'(u)^{1/3}\, du \right)^3. \tag{2.11}$$

Hence, from (2.6), (2.7), and (2.11) we get

$$\lambda(u_m) \geq \left(\int_0^{u_m} N(u) g(u)\, du + \frac{1}{2} \left(\int_0^{u_m} g'(u)^{1/3}\, du \right)^3 \right) / \int_0^{u_m} u g(u)\, du, \tag{2.12}$$

for any $g \in \mathcal{D}$. To conclude with the proof of Theorem 1.1 we need to discuss the case of equality. Equality in (2.12) is attained if and only if the function g satisfies (2.9) for any positive constant K. Since $p = -u'$ satisfies (2.2), $p^3 = [2(E - V(u))]^{3/2}$. Thus, equality is attained in (2.12) if and only if $g(u)$ satisfies

$$\frac{dg}{du} = \frac{K}{[2(E - V(u))]^{3/2}} \tag{2.13}$$

in $(0, u_m)$ with $g(0) = 0$. Here K is any positive constant and $E = V(u_m)$. It is simple to prove that for $u < u_m$ the right side of (2.13) is Lipschitz and thus there exists a unique solution $g(u)$ of (2.13). Moreover, the solution is in $\mathcal{D}$. See [**11**] for details.

REMARK 2.1. Only for a few potentials $V(u)$ is it possible to obtain a solution to (2.13) in closed form. For the corresponding $N(u)$'s one can obtain $\lambda(u_m)$ exactly. In most cases (2.12) will provide a lower bound to $\lambda(u_m)$. This bound will be a good approximation to $\lambda(u_m)$ depending on whether the variational g is *close* to the solution of (2.13).

REMARK 2.2. The simplest case for which there is a closed form solution of (2.13) is $N(u) \equiv 0$. In that case $V(u)$ is quadratic in u and the corresponding g is given by

$$g_L(u) = \frac{u}{(u_m^2 - u^2)^{1/2}}$$

(we denote this function g_L because it is the exact g for the linear equation $u'' + \lambda u = 0$). Using the function g_L in the bound (2.12) gives the following lemma.

LEMMA 2.3. *For the principal solution* (λ, u) *of the two point boundary value problem* (1.4) *we have*

$$\lambda(u_m) \geq \left(\frac{\pi}{2} \right)^2 + \frac{4}{\pi u_m^2} \int_0^{u_m} \frac{s N(s)}{\sqrt{u_m^2 - s^2}}\, ds. \tag{2.14}$$

REMARK 2.4. The right side of (2.14) is a good approximation to the eigenvalue $\lambda(u_m)$ for small values of u_m. Consider for example the two point boundary value problem (1.4) with a nonlinear term given by $N(u) = -2u^3 + u^5$. Then from (2.14) we get $\lambda(u_m) \geq \pi^2/4 - 3u_m^2/2 + 5u_m^4/8$. This lower bound reproduces quite well all the features of the branch that bifurcates from the linear eigenvalue $\pi^2/4$. In Fig. 1 below, where u_m is plotted against λ for this particular example, the solid line gives $\lambda(u_m)$ obtained from numerical calculations while the dotted line represents the curve $\lambda(u_m) = \pi^2/4 - 3u_m^2/2 + 5u_m^4/8$ as calculated from (2.14). In particular our lower bound reproduces the turning point of this branch very accurately. We refer the reader to [**7**] where a different numerical example is presented.

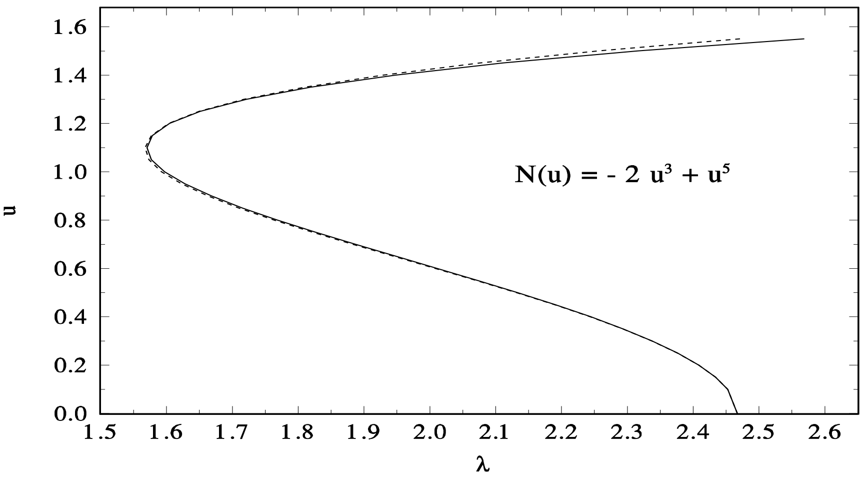

Fig. 1

REMARK 2.5. For nonlinearities behaving like $N(u) \approx au^n$ for large values of u, with $a > 0$ and $n > 1$, use the trial function $g_\alpha = u^\alpha$, compute the right side of (2.12) and let $\alpha \to \infty$. Then the right side becomes approximately au_m^{n-1}, which is the correct asymptotic behavior of $\lambda(u_m)$ for large values of u_m. Finally, for intermediate values of u_m (i.e., neither in the small amplitude region nor in the asymptotic region) one may introduce trial functions g depending on one or more parameters into the bound (2.12). By maximizing the right side of (2.12) with respect to the variational parameters, one can get accurate approximations to the eigenvalue $\lambda(u_m)$.

REMARK 2.6. There is a direct way of proving Lemma 2.3, without using our Theorem 1.1. Since $p = (2(E - V(U))^{1/2}$ and $1 = \int_0^1 dx = \int_0^{u_m} du/p(u)$ we have

$$\int_0^{u_m} \frac{1}{[2(E - V(U)]^{1/2}}\, du = 1, \tag{2.15}$$

with $E = V(u_m)$. Define the measure $d\mu = (2/\pi)du/\sqrt{u_m^2 - u^2}$. This is a probability measure in $(0, u_m)$. Separating appropriately the quadratic terms in $\sqrt{E - V(u)}$, using the fact that the mapping $x \to 1/\sqrt{x}$ is convex (for positive x), equation (2.15), and Jensen's inequality gives the lemma (see [**11**] for details).

Many authors have considered nonlinear problems for which the principal eigenvalue branch is not bifurcating from a linear eigenvalue. The methods we have described here to obtain bounds on the eigenvalues are also useful in this situation.

Consider the two point boundary value problem

$$-y'' = \lambda f(y) \qquad \text{in } (0, 1) \tag{2.16}$$

with

$$y'(0) = 0, \qquad y(1) = 0, \qquad \text{and} \qquad y(0) \equiv y_m. \tag{2.17}$$

Here $f(y)$ is positive and continuous but not necessarily 0 when $y = 0$. For example, this type of nonlinear two point boundary value problem arises in the study of heat generation and stability of temperature distribution of conducting plates [**20**]. Characteristic nonlinearities that appear in connection with that problem are $f(y) = \exp y$ and $f(y) = 1 + y + \delta y^2$. Under the assumption that f is positive and continuous, Laetsch [**24**] proved the existence and regularity of nontrivial solutions to (2.16), (2.17). As occurred with the problem (2.1), the positive solutions to (2.17) are decreasing and the problem can be reduced to a quadrature

$$\frac{1}{2}{y'}^2 + V(y) = E, \tag{2.18}$$

where $V(y) = \lambda \int_0^y f(s)\,ds$ and $E = V(y_m)$. Proceeding exactly as in the proof of Theorem 1.1, we obtain the following result concerning (2.16), (2.17).

Theorem 2.7. *Let the pair* (λ, y) *be the principal solution* (*i.e., with* $y(x) \geq 0$) *of the two point boundary value problem*

$$-\frac{d^2y}{dx^2} = \lambda f(y) \tag{2.19}$$

subject to $y'(0) = y(1) = 0$. *Let* $y_m = u(0)$ *be the sup-norm of the solution. Here,* $f(y)$ *is a general nonlinear term which is positive and continuous in* $(0, y_m)$. *Then*

$$\lambda[y_m] = \max_{g \in \mathcal{D}} \frac{1}{2} \frac{\left(\int_0^{y_m} g'(y)^{1/3}\,dy\right)^3}{\int_0^{y_m} f(y)g(y)\,dy}, \tag{2.20}$$

where $\mathcal{D} = \{g \mid g \in C^1(0, y_m), g' > 0, g(0) = 0\}$. *Moreover, the maximum is attained at a unique* (*up to a multiplicative constant*) $g \in \mathcal{D}$. *The maximizing* g *satisfies*

$$\frac{dg}{dy} = \frac{K}{(E - V(y))^{3/2}} \qquad \textit{and} \qquad g(0) = 0. \tag{2.21}$$

Since for most nonlinear terms $f(y)$ it is not possible to obtain $\lambda(y_m)$ in closed form, Eqn. (2.20) can be used to obtain lower bounds on $\lambda(y_m)$. There is a special function g that can be used to obtain good lower bounds on λ for small values of y_m. If $f(y) = c$, a constant, the potential $V(y)$ will be linear in y, and the solution to (2.21) in this case will be given by

$$g_c(y) = \frac{1}{\sqrt{y_m - y}} - \frac{1}{\sqrt{y_m}}. \tag{2.22}$$

Using this function g_c in (2.20) gives the following lemma.

Lemma 2.8. *For the principal solution* (λ, y) *of the two point boundary value problem* (2.16), (2.17) *we have*

$$\lambda(y_m) \geq \frac{2y_m^{3/2}}{\int_0^{y_m} f(y)[(y_m - y)^{-1/2} - {y_m}^{-1/2}]\,dy}. \tag{2.23}$$

REMARK 2.9. The right-side of (2.23) is a good approximation to $\lambda(y_m)$ for small values of y_m in the case of nonlinear functions $f(y)$ which go to a positive constant as $y \to 0$. For example, using the bound (2.23) for the case of $f(y) = 1 + y + \delta y^2$ studied by D. D. Joseph in [**20**], yields

$$\lambda(y_m) \geq 2y_m/(1 + \frac{5}{6}y_m + \frac{11\delta}{15}y_m^2),$$

which is precisely the lower bound obtained by him using different methods (see [**20**], Eqn. (11), pp. 284).

REMARK 2.10. The two point boundary value problem (2.16), (2.17) with many different classes of f's has been studied by Biollay [**13**], who has an alternative method of providing bounds on the nonlinear eigenvalues.

3. The Speed of Propagation of Travelling Fronts

In this section we present the proofs of Theorems 1.5 and 1.6, and we give a few applications. As shown by Aronson and Weinberger [**1**], the asymptotic speed of propagation of fronts of the nonlinear reaction-diffusion equation (1.6), c^*, is the minimum value c for which there is a solution of the two point boundary value problem described by (1.7) and (1.8). Before we go into the details of the proof of Theorem 1.1, a heuristic argument concerning the solutions of (1.7) is in order. In all three cases (1.7) can be viewed as Newton's equation for a particle moving in one-dimension under the action of the force $-f(q) - cq'$ (the variable x now plays the role of time). The force $-f(q)$ is conservative and derives from the potential $V(q) = \int_0^q f(s)\, ds$. If $f(q)$ belongs to Case A, the potential has a minimum at the point $q = 0$ and a maximum at $q = 1$. The second term, i.e., $-cq'$ represents a viscous force, and, with this picture in mind the parameter c represents the viscosity or viscous coefficient. In summary, (1.7) is Newton's equation for a particle coming down from the top of the potential at $q = 1$ to the bottom of it at $q = 0$ in the presence of a viscous force. If the viscosity (the parameter c, which is exactly our object of interest here) is small the particle will oscillate near the bottom of the well before it settles down at the minimum (i.e., at $q = 0$). If we increase the viscosity, we will reach a value for which there will not be oscillations any more. That is, the particle will come down monotonically from $q = 1$ to $q = 0$ (in mechanics this is usually known as critical damping). It is intuitively clear that if you increase the viscosity even further, the particle will also go down monotonically, and that at every point of its trajectory will have less and less speed. So, there is a critical value of c, which we denote by c^*, such that for $c \geq c^*$, there will be monotonic decreasing solutions of (1.7), (1.8). This fact was proven rigorously by Aronson and Weinberger, and, more importantly, they showed that this critical value c^* will be the speed of propagation of fronts of the reaction-diffusion equation (1.6). If $f(u)$ belongs to Case C, the potential has two maxima, at $q = 0$ and $q = 1$, and a minimum at $q = a$. The particle starts at $q = 1$ at $x = -\infty$ and it should get to $q = 0$ at $x = \infty$ with zero speed. Because of energy conservation, the maximum at $q = 0$ must be smaller than the maximum at $q = 1$ (otherwise the particle would never reach $q = 0$). This immediately requires the condition $\int_0^1 f(q)\, dq > 0$, on $f(u)$. Now, it should be intuitively clear that there is only one value of the viscosity c for which the particle will go down from the maximum at $q = 1$ to the valley at $q = a$ and then up to the maximum at $q = 0$. This is precisely the value

c^*. If c is smaller than c^* the particle will be trapped forever at the valley. On the other hand, if $c > c^*$ the particle will overshoot at $q = 1$. All these facts were proved in [**1**].

Hence, we are interested in the calculation of the minimal speed for which equation (1.6) has a monotonic travelling front $u(x,t) = q(z)$ with $z = x - ct$ joining $u = 1$ to $u = 0$. Since the selected speed corresponds to that of a decreasing monotonic front, it is convenient to work in phase space. Calling $p(q) = -dq/dz$, where the minus sign is included so that p is positive, we find that the monotonic fronts are solutions of

$$p\,\frac{dp}{dq} - c\,p(q) + f(q) = 0, \tag{3.1}$$

with

$$p(0) = 0, \qquad p(1) = 0, \qquad p > 0 \quad \text{in} \quad (0,1). \tag{3.2}$$

Aronson and Weinberger have proved that there is a (unique) p satisfying (3.1) and (3.2) for $c = c^*$ (see [**1**], Section 4). Moreover, the solution p is such that $p(q) \sim |m| q$ near $q = 0$, where $|m|$ is the largest root of the equation

$$x^2 - c^* x + f'(0) = 0, \tag{3.3}$$

i.e.,

$$|m| = \frac{1}{2}(c^* + \sqrt{c^{*2} - 4f'(0)}).$$

We find it convenient to introduce the parameter σ defined as $\sigma = c^*/|m|$. In terms of σ one can write

$$c^* = \sigma\sqrt{f'(0)/(\sigma - 1)} \qquad \text{and} \qquad |m| = \sqrt{f'(0)/(\sigma - 1)}. \tag{3.4}$$

It is straightforward to verify that whenever $1 < \sigma < 2$ the value of $|m|$ given by (3.4) corresponds to the largest root of (3.5) and therefore to the asymptotic slope at the origin of the selected front [**2**]. At $\sigma = 2$ the speed c^* attains the linear value c_L.

Proof of Theorem 1.5. The proof of the theorem is done in two steps. First we show that

$$c^* \geq J, \tag{3.5}$$

and then we show that equality actually holds in (3.5).To prove (3.5) we need only to show that

$$c^* \geq I(g)$$

for all $g \in \mathcal{E}$. This latter fact has been proven by us in [**2**]. We repeat here the argument for completeness. Let g be any function in $\mathcal{E}$. Multiplying equation (3.1) by g/p and integrating with respect to q we find, after integrating by parts, that

$$c^* = \frac{\int_0^1 \left(h\,p + \frac{f(q)}{p}\,g\right) dq}{\int_0^1 g(q)dq}. \tag{3.6}$$

However, since p, h, f and g are positive, for every fixed q,

$$h\,p + \frac{f(q)\,g}{p} \geq 2\sqrt{f\,g\,h}. \tag{3.7}$$

Hence

$$c^* \geq 2\,\frac{\int_0^1 \sqrt{f\,g\,h}\,dq}{\int_0^1 g\,dq}, \tag{3.8}$$

which proves (3.5)

To finish the proof of the theorem we have to show that the equality holds in (3.5). From the results of Aronson and Weinberger [**1**] it follows that $c^* \geq 2\sqrt{f'(0)}$. We separate the proof that equality holds in (3.5) into two cases: Case i) $c^* = 2\sqrt{f'(0)}$ and Case ii) $c^* > 2\sqrt{f'(0)}$.

Case i) $c^* = 2\sqrt{f'(0)}$. Take $g_\alpha(u) = \alpha u^{\alpha-1}$ for $0 < \alpha < 1$. Then $g_\alpha \in \mathcal{E}$ and it is straightforward to verify that $I(g_\alpha) \to 2\sqrt{f'(0)}$ as $\alpha \to 0$ so that $J \geq 2\sqrt{f'(0)}$. Hence, from (3.5) we have

$$c^* \geq J \geq 2\sqrt{f'(0)},$$

which implies $J = c^*$ in this case.

Case ii) $c^* > 2\sqrt{f'(0)}$. In this case we will not only prove that equality in (3.5) holds but also that there exists $\hat{g} \in \mathcal{E}$ such that $c^* = I(\hat{g})$.

Let $p(q)$ be the positive solution of (3.1) satisfying (3.2). The existence of such a solution has been established in [**1**]. Moreover, $p(q) \approx |m|q$ near $q = 0$.

A function $\hat{g}$ will saturate the bound (3.8) if and only if (3.7) is satisfied as an equality, i.e., if $\hat{g}$ is such that

$$h\,p = \frac{f\,\hat{g}}{p}, \tag{3.9}$$

where $h = -\hat{g}'$ and $p(q)$ is the solution of (3.1) mentioned above. From (3.1) and (3.9) we get

$$-\frac{\hat{g}'}{\hat{g}} = \frac{c^*}{p} - \frac{p'}{p},$$

which can be integrated to yield

$$\hat{g}(q) = \frac{p(q)}{c^*} \exp\left\{ \int_q^{q_0} \frac{c^*}{p}\,dq \right\} \tag{3.10}$$

for some fixed $0 < q_0 < 1$. To complete the argument we need to show that $\hat{g} \in \mathcal{E}$. It follows from (3.9) and (3.10) that

$$h(q) = \frac{f(q)}{c^*\,p(q)} \exp\left\{ \int_q^{q_0} \frac{c^*}{p}\,dq \right\}$$

in $(0,1)$. Moreover, since $p(q) > 0$ in $(0,1)$ and $p \in C^1(0,1)$ we have that $\hat{g} \in C^1(0,1)$. Thus $\hat{g}$ is a continuous, positive and decreasing function in $(0,1)$. Hence $\hat{g}$ is bounded away from the origin. In order to show that $\int_0^1 \hat{g}(q)\,dq$ is finite we have to determine the behavior of $\hat{g}$ near $q = 0$. Since $p \approx |m|q$ near 0, we have from (3.10) that

$$\hat{g}(q) \approx \frac{1}{\sigma}\frac{1}{q^{\sigma-1}}$$

near 0. Therefore, if $\sigma < 2$ (i.e., if $c^* > 2\sqrt{f'(0)}$, we have $\int_0^1 \hat{g}(q)\,dq < \infty$ and $\hat{q} \in \mathcal{E}$. $\square$

REMARK 3.1. In [**1**], Aronson and Weinberger showed that for reaction terms of type A

$$2\sqrt{f'(0)} \le c^* \le \sup_u 2\sqrt{\frac{f(u)}{u}}, \tag{3.11}$$

and, as a consequence, if f is such that $f(u) \le f'(0)\,u$, $c^* \equiv 2\sqrt{f'(0)}$, the KPP value. It is interesting to note that the upper bound obtained by Aronson and Weinberger follows from Theorem 1.5. In fact

$$\begin{aligned} c^* = \max_{g\in\mathcal{E}} 2\,\frac{\int_0^1 \sqrt{fgh}\,du}{\int_0^1 g\,du} &= \max_{g\in\mathcal{E}} 2\,\frac{\int_0^1 \sqrt{(fh/g)}\,g\,du}{\int_0^1 g\,du} \\ &\le 2\max_{g\in\mathcal{E}} \sqrt{\frac{\int_0^1 (fh/g)\,g\,du}{\int_0^1 g\,du}} = 2\max_{g\in\mathcal{E}} \sqrt{\frac{\int_0^1 fh\,du}{\int_0^1 g\,du}}, \end{aligned} \tag{3.12}$$

where the inequality follows from Jensen's inequality and the concavity of $x \to \sqrt{x}$ for positive x. Now, since h is nonnegative,

$$\frac{\int_0^1 f\,h\,du}{\int_0^1 g\,du} = \frac{\int_0^1 (f/u)\,h\,u\,du}{\int_0^1 g\,du} \le \sup_u \frac{f(u)}{u}\,\frac{\int_0^1 h\,u\,du}{\int_0^1 g\,du} = \sup_u \frac{f(u)}{u} \tag{3.13}$$

for all $g \in \mathcal{E}$. Here we have used that $\int_0^1 h\,u\,du = \int_0^1 g\,du$, which follows by integration by parts. From (3.12) and (3.13) the bound of Aronson and Weinberger (3.11) follows.

REMARK 3.2. An important case in combustion theory is characterized by nonlinear terms $f(u)$ which are highly peaked around $u = 1$. A well studied example is given by $f(u) = (1-u)(\exp^{-\beta(1-u)} - \exp^{-\beta})$ for large values of the parameter β (see, e.g., [**34**], and also [**5**] and references therein). In 1938, Zeldovich and Frank-Kamenetskii [**34**], using boundary-layer techniques, showed that for functions highly peaked about $u = 1$ the speed of propagation of fronts is approximately given by

$$c_{ZFK} = \sqrt{2\int_0^1 f(u)\,du}.$$

In 1992 Berestycki and Nirenberg proved directly from the ODE (1.7) that for any $f(u)$ of type A,

$$c^* \ge c_{ZFK} = \sqrt{2\int_0^1 f(u)\,du} \tag{3.14}$$

([**14**], Remark 10.2, pp. 567). It is interesting to note that this result follows directly from (3.8). In fact, take the trial function

$$g = \sqrt{2\int_u^1 f(s)\,ds},$$

which is in $\mathcal{E}$ and satisfies $g\,h = f$. Inserting this variational function in (3.8) we obtain

$$c^* \ge 2\,\frac{\int_0^1 \sqrt{f\,g\,h}\,dq}{\int_0^1 g\,dq} = 2\,\frac{\int_0^1 f(q)\,dq}{\int_0^1 g\,dq} \ge 2\,\frac{\int_0^1 f(q)\,dq}{g(0)} = \sqrt{2\int_0^1 f(q)\,dq} = c_{ZFK},$$

since $g(u)$ is decreasing and thus $g(u) \geq g(0)$ for all $u \in (0,1)$.

REMARK 3.3. The uniqueness of the maximizer follows from the fact that we need to choose $\hat{g}$ in such a way that (3.7) is satisfied as an equality. Alternatively, we can prove uniqueness directly from the concavity of $I(g)$. From the homogeneity of $I(g)$ in g we can restrict, without loss of generality, to g's such that $\int_0^1 g(q)\,dq = 1$. Assume $c^* > c_L$ and that there are two g's in $\mathcal{E}$, g_1 and g_2 say, such that $I(g_1) = I(g_2) = c^*$ (and $\int g_1 = \int g_2 = 1$). Then let $g = \alpha g_1 + (1-\alpha) g_2$, with $0 < \alpha < 1$. Clearly, $\int g = 1$, and $h \equiv -g' = \alpha h_1 + (1-\alpha) h_2 > 0$. Therefore,

$$I(g) = \int_0^1 \sqrt{f(\alpha g_1 + (1-\alpha) g_2)(\alpha h_1 + (1-\alpha) h_2)}\,dq.$$

However, by the Cauchy-Schwarz inequality,

$$\sqrt{(\alpha g_1 + (1-\alpha) g_2)}\sqrt{(\alpha h_1 + (1-\alpha) h_2} \geq \alpha\sqrt{h_1 g_1} + (1-\alpha)\sqrt{h_2 g_2}$$

with equality if and only if $g_1 = \beta h_1$ and $g_2 = \beta h_2$. Thus,

$$c^* \geq I(g) \geq \alpha I(g_1) + (1-\alpha) I(g_2) = c^*,$$

which implies $h_1/g_1 = h_2/g_2$, and therefore $g_1 \equiv g_2$.

It is clear that Theorem 1.5 is not valid if f is either of type B or of type C because (3.7) does not hold if f is negative. In spite of this fact we can still derive a variational characterization of c^* valid in this situation.

PROOF OF THEOREM 1.6. The proof follows in a simple way from (3.1). Let $g \in \mathcal{F}$ and multiply (3.1) by $g(q)$ and integrate in q between 0 and 1. After integration by parts we obtain

$$\int_0^1 f\,g\,dq = c\int_0^1 p\,g\,dq - \int_0^1 \frac{1}{2}\,h\,p^2\,dq, \tag{3.15}$$

with $h \equiv -g' > 0$. Since c, p, g, and h are nonnegative we have

$$c\,p\,g - \frac{1}{2}\,h\,p^2 \leq c^2\,\frac{g^2}{2\,h}. \tag{3.16}$$

Hence, from (3.15) and (3.16) we get

$$c^2 \geq 2\,\frac{\int_0^1 f\,g\,dq}{\int_0^1 (g^2/h)\,dq}, \tag{3.17}$$

and therefore

$$(c^*)^2 \geq L. \tag{3.18}$$

To conclude the proof we must show that equality actually holds in (3.18). From (3.16) we notice that equality will be attained when $\hat{g}$ is the solution to the ordinary differential equation

$$c\,\frac{\hat{g}}{\hat{h}} \equiv -c\,\frac{\hat{g}}{\hat{g}'} = p. \tag{3.19}$$

This equation can be integrated to yield,

$$\hat{g} = \exp\left(-\int_{q_0}^{q} \frac{c}{p}\,dq\right), \tag{3.20}$$

with $0 < q_0 < 1$. We have to verify that this solution actually belongs to $\mathcal{F}$. Evidently $\hat{g}$ is positive, monotonic decreasing, and moreover $\hat{g}(1) = 0$. Near $q = 0$,

$\hat{g}$ diverges. We must ensure that the integrals in the right side of (3.17) exist. To verify this we recall that in the three cases, A, B, and C the front approaches $q = 0$ exponentially [**1**]. Therefore, near $q = 0$,

$$p \approx \frac{1}{2}\left[c + \sqrt{c^2 - 4f'(0)}\right] q \equiv mq.$$

Thus, from (3.20) we obtain $\hat{g}(q) \approx q^{-c/m}$, near $q = 0$: Hence, $f\,\hat{g}$ and $\hat{g}^2/\hat{h}$ diverges at most as $q^{1-c/m}$ near $q = 0$. Therefore, the integrals in the right side of (3.17) exist if $m/c > 1/2$. This condition is always satisfied when $f'(0) \le 0$, that is, in cases B and C. In case A this condition is satisfied provided $c > 2\sqrt{f'(0)}$. This concludes the proof of the theorem in Cases B, C and also in case A for $c^* \neq c_{KPP}$. Finally, in case A, if $(c^*)2 = c_{KPP}^2 = 4\,f'(0)$, one can take the maximizing sequence $g_\alpha(q) = \alpha(2-\alpha)q^{\alpha-2}$, which is in $\mathcal{F}$ for $0 < \alpha < 1$, and let $\alpha \to 0$. One can verify that $M(g_\alpha) \to 4f'(0)$ as $\alpha \to 0$ and we are done. □

4. A Nonlinear Eigenvalue Problem Described by a Partial Differential Equation

It should be clear from the proofs of Theorems 1.1, 1.5, and 1.6 that the method we have used in Sections 2 and 3, as it stands, only applies to nonlinear eigenvalue problems described by autonomous ordinary differential equations. Is it possible to extend the method in such a way that it can be applied to more general nonlinear eigenvalue problems? Although we do not know what is the more general setting for which the ideas developed above can be applied, we do have several results for nonlinear eigenvalue problems described by partial differential equations. In this section we present results analogous to the ones obtained in the previous sections for a well-known nonlinear eigenvalue problem described by a PDE. Here the nonlinear eigenvalue is *bifurcating from infinity*, and instead of using the sup-norm of the solution we use the L^1 norm. Nevertheless, the variational characterization we obtain for the eigenvalue is closely related with the variational characterizations obtained in the previous sections.

The nonlinear problem we consider here is the Thomas-Fermi model, which is a nonlinear eigenvalue problem. The eigenvalue is the *chemical potential*, which we would like to consider as a function of the number of electrons which turns out to be the L^1 norm of the solution to the Thomas-Fermi equation. For this example there are no linear eigenvalues, and the nonlinear eigenvalue can be viewed as *bifurcating from infinity*. Before we give the main results concerning this problem, we give a brief introduction to the Thomas-Fermi model of atoms and molecules.

The Thomas-Fermi model [**32, 16**] is defined by the energy functional

$$\mathcal{E}(\rho) = \frac{3}{5}\int \rho^{5/3}(x)\,d^3x - \int V(x)\rho(x)\,d^3x + D(\rho,\rho), \tag{4.1}$$

where

$$D(\rho,\rho) = \frac{1}{2}\int \rho(x)|x-y|^{-1}\rho(y)\,d^3x\,d^3y, \tag{4.2}$$

and

$$V(x) = \sum_{j=1}^{k} z_j|x-R_j|^{-1}. \tag{4.3}$$

We are using units for which $h^2(8m)^{-1}(3/\pi)^{2/3} = 1$ and $|e| = 1$, where e and m are the electron charge and mass, respectively, and h is Planck's constant. Here $z_1, \ldots, z_k \geq 0$ are the charges of k fixed nuclei located at $R_1, \ldots, R_k$. $\mathcal{E}(\rho)$ is defined for electronic densities $\rho(x) \geq 0$ such that $\int \rho \, d^3x$ and $\int \rho^{5/3} \, d^3x$ are finite. The Thomas-Fermi electronic energy for N (not necessarily an integer) electrons is defined by

$$E(N) = \min\{\mathcal{E}(\rho) \mid \int \rho \, d^3x = N\}. \tag{4.4}$$

It is known [**25, 26**] that for $N \leq Z \equiv \sum_{j=1}^{k} z_j$ there is a unique minimizing ρ for (4.4). It is the unique solution to the Thomas-Fermi equation

$$\rho(x)^{2/3} = \max(\varphi + \epsilon_F, 0) \tag{4.5}$$

for some $\epsilon_F \leq 0$ and with

$$\varphi(x) = V(x) - \int \frac{\rho(y)}{|x-y|} \, d^3y. \tag{4.6}$$

$\epsilon_F(N; z_1, \ldots, z_k)$ is the chemical potential, i.e.,

$$\epsilon_F = \frac{dE}{dN}. \tag{4.7}$$

There is an extensive literature on the Thomas-Fermi model, and we refer the interested reader to the review articles by Lieb [**27**] and Spruch [**30**] for details.

Our interest here is to give a variational characterization of the chemical potential ϵ_F. A minimax and a maximin principle for ϵ_F were proven by Lieb and Simon (see [**26**], Theorems II.28 and II.29). In short, their maximin principle reads as follows: let ρ be an admissible density, define $\phi_\rho(x) = \int \rho(y)|x-y|^{-1} \, d^3y$. Let $T(\rho) = \inf_x(\rho^{2/3}(x) - \phi_\rho)$. Then $\epsilon_F(N, Z) = \sup\{T(\rho) \mid \int \rho \, d^3x = N\}$. Their minimax principle is analogous, with the operations on x and ρ reversed. In spirit, these minimax principles are of the same type as Barta's (or Duffin's) principle for the Schrödinger equation (see e.g., [**31**], pp. 154–155). On the other hand, the variational characterization we give here for ϵ_F is closer in spirit to the Rayleigh-Ritz principle for the ground state energy of the Schrödinger equation.

The main results concerning the Thomas-Fermi model are the following two theorems, which we state without proof.

THEOREM 4.1 (Atomic Case, k=1, [**12**]). *Let $\mathcal{H}$ be the set of nonnegative, radially symmetric, bounded, subharmonic functions g on $\Re^3$ (i.e., with $\Delta g \geq 0$), with piecewise continuous second derivatives and such that both Δg and $(\Delta g)^3/g^2$ are integrable. For any $g \in \mathcal{H}$ let*

$$\mathcal{I}(g) = (\int \Delta g \, d^3x)^{-1} \left(4\pi g_\infty (Z-N) - 4\pi g_0 Z - \frac{1}{108\pi^2} \int \frac{(\Delta g)^3}{g^2} \, d^3x\right), \tag{4.8}$$

where g_0 is the value of g at the nucleus and g_∞ is the limit of $g(x)$ as $|x| \to \infty$ (such a limit always exists for $g \in \mathcal{D}$).

Then

$$-\epsilon_F(N, Z) = \max\{\mathcal{I}(g) \mid g \in \mathcal{D}\}, \tag{4.9}$$

and the maximum is attained at some $\hat{g} \in \mathcal{H}$, which is unique up to a multiplicative constant.

THEOREM 4.2 (Molecular Case, $k > 1$, [**12**]). *Let $\tilde{\mathcal{H}}$ be the set of nonnegative, bounded, subharmonic functions g on $\Re^3$ (i.e., with $\Delta g \geq 0$), with piecewise continuous second derivatives and such that both Δg and $(\Delta g)^3/g^2$ are integrable. For any $g \in \tilde{\mathcal{H}}$ let*

(4.10)

$$\mathcal{I}(g) = (\int \Delta g \, d^3x)^{-1} \left(4\pi g_\infty (Z - N) - 4\pi \sum_{j=1}^{k} g_j z_j - \frac{1}{108\pi^2} \int \frac{(\Delta g)^3}{g^2} \, d^3x \right),$$

where g_j is the value of g at the nucleus located at R_j and g_∞ is the limit of the spherical average of $g(x)$, over a ball of radius R enclosing all the nuclei, as $R \to \infty$ (such a limit always exists for $g \in \tilde{\mathcal{H}}$).

Then

$$-\epsilon_F(N; z_1, \ldots, z_k) = \max\{\mathcal{I}(g) \mid g \in \tilde{\mathcal{H}}\}, \tag{4.11}$$

and the maximum is attained at some $\hat{g} \in \tilde{\mathcal{H}}$, which is unique up to a multiplicative constant.

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