Some Properties of Riesz Means and Spectral Expansions

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Abstract

It is well known that short-time expansions of heat kernels correlate to formal high-frequency expansions of spectral densities. It is also well known that the latter expansions are generally not literally true beyond the first term. However, the terms in the heat-kernel expansion correspond rigorously to quantities called Riesz means of the spectral expansion, which damp out oscillations in the spectral density at high frequencies by dint of performing an average over the density at all lower frequencies. In general, a change of variables leads to new Riesz means that contain different information from the old ones. In particular, for the standard second-order elliptic operators, Riesz means with respect to the square root of the spectral parameter correspond to terms in the asymptotics of elliptic and hyperbolic Green functions associated with the operator, and these quantities contain "nonlocal" information not contained in the usual Riesz means and their correlates in the heat kernel. Here the relationship between these two sets of Riesz means is worked out in detail; this involves just classical one-dimensional analysis and calculation, with no substantive input from spectral theory or quantum field theory. This work provides a general framework for calculations that are often carried out piecemeal (and without precise understanding of their rigorous meaning) in the physics literature.

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I. PHYSICAL MOTIVATION

Let H be a positive, self-adjoint, elliptic, second-order partial differential operator. For temporary expository simplicity, assume that H has purely discrete spectrum, with eigenvalues λ_n and normalized eigenfunctions $\phi_n(x)$. Quantum field theorists, especially those working in curved space, are accustomed to calculating (1) the *heat kernel*,

$$K(t,x,y) = \sum_{n=1}^{\infty} \phi_n(x)\phi_n(y)^* e^{-t\lambda_n}; \tag{1}$$

(2) the *Wightman function*,

$$W(t,x,y) = \sum_{n=1}^{\infty} \phi_n(x)\phi_n(y)^* \frac{1}{2\sqrt{\lambda_n}} e^{-it\sqrt{\lambda_n}}, \tag{2}$$

which determines the vacuum energy density of a quantized field in a static space-time background. Of greatest interest are the behaviors of these functions with $y = x$ (either evaluated pointwise, or integrated over the whole spatial manifold) in the limit $t \to 0$.

Roughly speaking, the small-t asymptotics of both (1) and (2) are determined by the large-λ (ultraviolet) asymptotics of the density of eigenvalues and of the eigenfunctions. However, there is a major difference between the heat kernel and the Wightman function in this regard. *All* the coefficients $b_s(x)$ in the heat-kernel expansion

$$K(t,x,x) \sim \sum_{s=0}^{\infty} b_s(x) t^{-\frac{m}{2}+\frac{s}{2}} \tag{3}$$

(m = dimension) are *locally determined* by the coefficient functions in the differential operator H *at the point* x. The same is true of the leading, singular terms in an expansion of $W(t,x,y)$, which are removed by renormalization; the finite residue, however, is a *nonlocal* functional of the coefficients in H, containing information about, for example, the global structure of the manifold — this is what makes vacuum energy interesting and nontrivial. It follows that this nonlocal information is somehow contained in the ultraviolet asymptotics of the spectrum, although it is lost in passing (pointwise) to $K(t,x,x)$. The Wightman function is more typical of Green functions associated with H; the striking and somewhat mysterious thing is how *special* the heat kernel is.

The primary purpose of this paper is to point out that many of the facts of this subject have nothing to do specifically with partial differential operators (much less with quantum field theory). Rather, they result from some classical real analysis (in one dimension) concerning the summability of series and integrals, much of which was developed by M. Riesz and G. H. Hardy early in the twentieth century. One consequence is that many formal relationships between the asymptotic expansions of Green functions and those of the associated spectral measures, and between the asymptotic expansion of one Green function and that of another, can be worked out in the abstract without reference to the detailed spectral theory of any particular operator. The same is true, qualitatively, of the *limitations* of such relationships: The rigorous asymptotic expansion of a heat kernel suggests a formal asymptotic expansion of an associated spectral measure, but the latter is usually valid only in some averaged sense. It can be translated into rigorous statements about the *Riesz means* of the measure. The construction of Riesz means not only washes out oscillations in the measure at the ultraviolet end, but also incorporates some information about the measure at small or moderate values of λ. The information contained in a Riesz mean depends on the variable of integration; for example, the Riesz means with respect to $\sqrt{\lambda}$ contain more information than those with respect to λ. The difference between the nonlocal asymptotics of the Wightman function and the local asymptotics of the heat kernel is, at root, an example of this phenomenon.

These relationships can be made yet more precise by introducing some concepts from distribution theory[1,2].

II. SURVEY OF THE PHENOMENA

To provide a concrete context for the later sections of the paper, we display here some Green functions and asymptotic expansions associated with a constant-coefficient differential operator in one dimension. (The later sections are logically independent of this one and much more general).

Here and in Sections VI and VII we use as a surrogate for the Wightman function a technically simpler Green function that we call the *cylinder kernel*. This is the integral kernel $T(t,x,y)$ of the operator $e^{-t\sqrt{H}}$, and it is related to the elliptic equation

$$\frac{\partial^2\Psi}{\partial t^2} = H\Psi \tag{4}$$

in the same way that the heat kernel (the integral kernel of e^{-tH}) is related to the parabolic equation

$$-\frac{\partial\Psi}{\partial t} = H\Psi. \tag{5}$$

That is,

$$\Psi(t,x) = \int_{\mathcal{M}} T(t,x,y) f(y)\, dy \tag{6}$$

solves (4) in the infinite half-cylinder $(0,\infty)\times\mathcal{M}$ with the initial data

$$\lim_{t\downarrow 0}\Psi(t,x) = f(x) \tag{7}$$

on the manifold $\mathcal{M}$. The cylinder kernel displays the same kind of nonlocal short-time asymptotics as the Wightman function.

In this section we consider

$$H = -\frac{\partial^2}{\partial x^2} \tag{8}$$

on various one-dimensional manifolds $\mathcal{M}$.

Case $\mathcal{M} = \mathbb{R}$

The heat kernel is

$$K(t,x,y) = (4\pi t)^{-1/2} e^{-(x-y)^2/4t}. \tag{9}$$

As $t \downarrow 0$,

$$K(t,x,y) \sim \begin{cases} (4\pi t)^{-1/2} + O(t^\infty) & \text{if } y = x, \\ O(t^\infty) & \text{if } y \neq x; \end{cases} \tag{10}$$

that is, all terms in the expansion (3) beyond the first vanish.

The cylinder kernel is

$$T(t,x,y) = \frac{t}{\pi} \frac{1}{(x-y)^2 + t^2}\,. \tag{11}$$

As $t \downarrow 0$,

$$T(t,x,y) \sim \begin{cases} \dfrac{1}{\pi t} & \text{if } y = x, \\ \dfrac{t}{\pi (x-y)^2} \displaystyle\sum_{k=0}^{\infty} (-1)^k \left(\frac{t}{x-y}\right)^{2k} & \text{if } y \neq x. \end{cases} \tag{12}$$

(For the distributional, rather than pointwise, limit, see Ref. 1.)

Case $\mathcal{M} = \mathbb{R}^+$

We consider the operator (8) on the interval $(0,\infty)$, with either the Dirichlet or the Neumann boundary condition at 0. The Green functions are most easily obtained by the method of images from the previous case. The heat kernel is

$$K(t,x,y) = (4\pi t)^{-1/2} \left[e^{-(x-y)^2/4t} \mp e^{-(x+y)^2/4t} \right], \tag{13}$$

where *the upper and lower signs are for the Dirichlet and Neumann cases, respectively.* Because of the rapid decay of the image term, the asymptotic behavior is still described exactly by (10); the heat kernel in the interior does not sense the existence of the boundary. The cylinder kernel is

$$T(t,x,y) = \frac{t}{\pi} \left[\frac{1}{(x-y)^2 + t^2} \mp \frac{1}{(x+y)^2 + t^2} \right]. \tag{14}$$

As $t \downarrow 0$,

$$T(t,x,y) \sim \begin{cases} \dfrac{1}{\pi t} \mp \dfrac{t}{\pi (2x)^2} \displaystyle\sum_{k=0}^{\infty} (-1)^k \left(\frac{t}{2x}\right)^{2k} & \text{if } y = x, \\ \dfrac{t}{\pi (x-y)^2} \displaystyle\sum_{k=0}^{\infty} (-1)^k \left(\frac{t}{x-y}\right)^{2k} \mp \dfrac{t}{\pi (x+y)} \displaystyle\sum_{k=0}^{\infty} (-1)^k \left(\frac{t}{x+y}\right)^{2k} & \text{if } y \neq x. \end{cases} \tag{15}$$

Because of the slow decay of the basic kernel in its role as image term, the expansion differs from (12) beyond the leading $O(t^{-1})$ term; the cylinder kernel senses the presence of the boundary, the type of boundary condition, and the distance to the boundary (more precisely, the length, $x+y$, of a path between the two arguments that bounces off the endpoint[3]).

Case $\mathcal{M} = S^1$

Consider (8) on the interval $(-L, L)$ with periodic boundary conditions. The heat kernel is the well known theta function[4]

$$K(t,x,y) = (4\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} e^{-(x-y-2nL)^2/4t}. \tag{16}$$

The expansion (10) is still valid. The cylinder kernel could also be expressed as an infinite image sum, but its Fourier representation can be expressed in closed form (via the geometric series):

$$\begin{aligned} T(t,x,y) &= \frac{1}{2L} \sum_{k=-\infty}^{\infty} e^{i\pi k(x-y)/L} e^{-\pi|k|t/L} \\ &= \frac{1}{2L} \frac{\sinh(\pi t/L)}{\cosh(\pi t/L) - \cos(\pi(x-y)/L)}\,. \end{aligned} \tag{17}$$

The first few terms of the expansion as $t \downarrow 0$ are

$$T(t,x,y) \sim \begin{cases} \dfrac{1}{\pi t}\left[1 + \dfrac{1}{12}\left(\dfrac{\pi t}{L}\right)^2 - \dfrac{1}{720}\left(\dfrac{\pi t}{L}\right)^4 + O(t^6)\right] & \text{if } y = x, \\ \dfrac{\pi t}{2L^2} \dfrac{1}{1-\kappa}\left[1 + \left(\dfrac{\pi t}{L}\right)^2 \left(\dfrac{1}{6} - \dfrac{1}{2(1-\kappa)}\right) + O(t^4)\right] & \text{if } y \neq x, \end{cases} \tag{18}$$

where

$$\kappa = \cos \frac{\pi(x-y)}{L}\,. \tag{19}$$

Thus the cylinder kernel is locally sensitive to the size of the manifold. (In the limit of large L, (18) matches (12), as it should.)

In this case it is possible to "trace" the Green functions over the manifold:

$$K(t) \equiv \int_{-L}^{L} K(t,x,x)\,dx \sim (4\pi t)^{-1/2}(2L) + O(t^\infty) \tag{20}$$

(since (10) is uniform in $\mathcal{M}$), and

$$T(t) \equiv \int_{-L}^{L} T(t,x,x)\,dx = \frac{\sinh(\pi t/L)}{\cosh(\pi t/L) - 1} \sim \frac{2L}{\pi t}\left[1 + \frac{1}{12}\left(\frac{\pi t}{L}\right)^2 + \cdots\right]. \tag{21}$$

Both have leading terms proportional to the volume of the manifold, but (21) has higher-order correction terms analogous to those in (12).

Case $\mathcal{M} = I$

We consider (8) on the interval $I = (0, L)$. For brevity we consider only Dirichlet boundary conditions and the expansions on the diagonal (coincidence limit). The heat kernel is

$$K(t,x,y) = (4\pi t)^{-1/2} \sum_{n=-\infty}^{\infty} \left[e^{-(x-y-2nL)^2/4t} - e^{-(x+y-2nL)^2/4t} \right]. \tag{22}$$

Expansion (10) is valid in the interior (but not uniformly near the endpoints). The cylinder kernel is

$$T(t,x,y) = \frac{1}{2L} \left[\frac{\sinh(\pi t/L)}{\cosh(\pi t/L) - \cos(\pi(x-y)/L)} - \frac{\sinh(\pi t/L)}{\cosh(\pi t/L) - \cos(\pi(x+y)/L)} \right]. \tag{23}$$

Not surprisingly, its expansion combines the features of (15) and (18).

$$\begin{aligned} T(t,x,x) &\sim \frac{1}{\pi t} \left[1 + \frac{1}{12} \left(\frac{\pi t}{L} \right)^2 - \frac{1}{720} \left(\frac{\pi t}{L} \right)^4 + \cdots \right] \\ &\quad - \frac{\pi t}{2L^2} \frac{1}{1 - \cos(2\pi x/L)} \left[1 + \left(\frac{\pi t}{L} \right)^2 \left(\frac{1}{6} - \frac{1}{2(1 - \cos(2\pi x/L))} \right) + \cdots \right] \\ &= \frac{1}{\pi t} \left[1 + \left(\frac{\pi t}{L} \right)^2 \left(\frac{1}{12} - \frac{1}{2(1 - \cos(2\pi x/L))} \right) \right. \\ &\quad \left. + \left(\frac{\pi t}{L} \right)^4 \left(\frac{-1}{720} - \frac{1}{12(1 - \cos(2\pi x/L))} + \frac{1}{4(1 - \cos(2\pi x/L))^2} \right) + \cdots \right]. \end{aligned} \tag{24}$$

(Compare (15) in the form

$$T(t,x,x) \sim \frac{1}{\pi t} \left[1 - \frac{t^2}{4x^2} + \frac{t^4}{16x^4} + \cdots \right], \tag{25}$$

which (24) matches as $x \to 0$ or $L \to \infty$.)

For the traces in this case one has

$$K(t) \equiv \int_0^L K(t,x,x)\, dx \sim (4\pi t)^{-1/2} L - \tfrac{1}{2} + O(t^\infty), \tag{26}$$

a well known result[5], and

$$\begin{aligned} T(t) &\equiv \int_0^L T(t,x,x)\, dx \\ &= \frac{1}{2} \frac{\sinh(\pi t/L)}{\cosh(\pi t/L) - 1} - \frac{\sinh(\pi t/L)}{2L} \int_0^L \frac{dx}{\cosh(\pi t/L) - \cos(2\pi x/L)} \\ &= \frac{1}{2} \frac{\sinh(\pi t/L)}{\cosh(\pi t/L) - 1} - \frac{1}{2} \\ &\sim \frac{L}{\pi t} \left[1 - \frac{\pi t}{2L} - \frac{1}{12} \left(\frac{\pi t}{L} \right)^2 + O(t^4) \right]. \end{aligned} \tag{27}$$

In comparison with (20) and (21), the leading terms of (26) and (27) have adjusted to reflect the smaller size of the manifold, and the new second (t-independent) terms are the effect of the boundary. The minus sign on those terms distinguishes the Dirichlet boundary condition from the Neumann. (A less trivial differential operator would yield more complicated expansions, with higher-order terms exhibiting an interaction between the boundary condition and the coefficients (potential, curvature) (see Ref. 5).)

It is worth noting that the most important dependence of (21) or (27) on the size of the manifold comes from the integration, not from the form of the integrand. (Indeed, the L-dependence of (18) and (24) as written is downright misleading in this respect.) In ignorance of both the size of $\mathcal{M}$ and the nature and location of any boundaries, knowledge of the $O(t)$ term in $T(t,x,x)$ on a small interval of x would be a rather useless tool for the inverse problem.

III. NOTATION AND BASIC FORMULAS

The basic references for this section and much of the next are Hardy[6–8] and Hörmander[9,10]. (The formulations given here are somewhat new, however.)

Let $f(\lambda)$ be a function of locally bounded variation on $[0,\infty)$ such that

$$f(0) = 0. \tag{28}$$

Typically, f will be defined as a Stieltjes integral

$$f(\lambda) = \int_0^\lambda a(\sigma)\, d\mu(\sigma), \tag{29}$$

where $\mu(\lambda)$ is another function of the same kind, and $a(\lambda)$ is (say) a continuous function. By convention we take functions of locally bounded variation to be continuous from the left:

$$f(\lambda) \equiv \lim_{\varepsilon\downarrow 0} f(\lambda-\varepsilon); \tag{30}$$

$$\int_a^b a(\sigma)\, d\mu(\sigma) \equiv \lim_{\varepsilon\downarrow 0} \int_{a-\varepsilon}^{b-\varepsilon} a(\sigma)\, d\mu(\sigma). \tag{31}$$

By $\partial_\lambda^\alpha f$ we denote the αth derivative of f with respect to λ. Since iterated indefinite integrals can be regarded as derivatives of negative order, we define

$$\partial_\lambda^{-\alpha} f(\lambda) \equiv \int_0^\lambda d\sigma_1 \cdots \int_0^{\sigma_{\alpha-1}} d\sigma_\alpha\, f(\sigma_\alpha). \tag{32}$$

As is well known, the iterated integral is equal to the single integral

$$\partial_\lambda^{-\alpha} f(\lambda) = \frac{1}{(\alpha-1)!} \int_0^\lambda (\lambda-\sigma)^{\alpha-1} f(\sigma)\, d\sigma, \tag{33}$$

which in turn may be converted to

$$\partial_\lambda^{-\alpha} f(\lambda) = \frac{1}{\alpha!} \int_0^\lambda (\lambda-\sigma)^{\alpha}\, df(\sigma). \tag{34}$$

This last formula remains meaningful for $\alpha = 0$, yielding the natural definition

$$\partial_\lambda^0 f(\lambda) = \int_0^\lambda df(\sigma) = f(\lambda). \tag{35}$$

The αth *Riesz mean* of f is defined by

$$R_\lambda^\alpha f(\lambda) \equiv \alpha!\,\lambda^{-\alpha}\partial_\lambda^{-\alpha} f(\lambda) = \int_0^\lambda \left(1 - \frac{\sigma}{\lambda}\right)^\alpha df(\sigma). \tag{36}$$

We call $\partial_\lambda^{-\alpha} f(\lambda)$ the αth *Riesz integral* of f, and we call

$$\lim_{\lambda\to\infty} R_\lambda^\alpha f(\lambda), \tag{37}$$

if it exists, the αth *Riesz limit* of f. If such a limit exists for f defined by (29), we say that the integral (29) is *summable by Riesz means of order α*. Eq. (34) may be used to define all these things for any $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > -1$, but *here we shall consider only nonnegative integer α.*

From (32) and (33) with f replaced by $\partial_\lambda^{-\beta} f$, we have (for $\alpha > 0$, $\beta \geq 0$)

$$\partial_\lambda^{-(\alpha+\beta)} f(\lambda) = \frac{1}{(\alpha-1)!}\int_0^\lambda (\lambda-\sigma)^{\alpha-1}\partial_\sigma^{-\beta} f(\sigma)\,d\sigma, \tag{38}$$

hence

$$R_\lambda^{\alpha+\beta} f(\lambda) = \alpha\binom{\alpha+\beta}{\alpha}\lambda^{-(\alpha+\beta)}\int_0^\lambda (\lambda-\sigma)^{\alpha-1}\sigma^\beta R_\sigma^\beta f(\sigma)\,d\sigma. \tag{39}$$

From this is proved Hardy's *first theorem of consistency*:

> If an integral (29) is summable by Riesz means of order α_0, then it is summable to the same value of means of order α for any $\alpha \geq \alpha_0$. In particular, if the integral is convergent, then it is summable by means of any order $\alpha \geq 0$. (40)

The proof is similar to that of (45), below.

Although Riesz and Hardy were primarily interested in defining numerical values for nonconvergent series and integrals, our concern (following Hörmander) is to use Riesz means as a way of organizing the asymptotic information contained in a function, without interference from small-scale fluctuations. Thus we are more interested in the Riesz means themselves than in the Riesz limits.

IV. THE RELATION BETWEEN MEANS WITH RESPECT TO DIFFERENT VARIABLES

Let ω be related to λ by

$$\lambda = \omega^k \qquad (k > 0,\ k \neq 1). \tag{41}$$

We are primarily interested in the cases $k = 2$ and $k = \frac{1}{2}$. (More generally, one could treat two variables related by any orientation-preserving diffeomorphism of $[0, \infty)$.) We write σ and τ as integration variables corresponding to λ and ω respectively — hence $\sigma = \tau^k$. We let

$$\tilde{f}(\omega) \equiv f(\omega^k) = f(\lambda) \tag{42}$$

and may omit tildes when no confusion seems likely.

The Riesz mean $R^\alpha_\lambda f$ can be expressed in terms of the mean $R^\alpha_\omega f$ [$\equiv R^\alpha_\omega \tilde{f}$] and vice versa. We call the following the *Hardy formula* because it is implicit in Hardy's 1916 proof of the "second theorem of consistency":

$$R^\alpha_\lambda f = k^\alpha R^\alpha_\omega f + \int_0^\lambda J_{k,\alpha}(\lambda, \sigma) R^\alpha_\tau f \, d\sigma, \tag{43}$$

where

$$J_{k,\alpha}(\lambda, \sigma) \equiv \sum_{j=0}^{\alpha-1} (-1)^{\alpha-j} \lambda^{-j-1} \frac{1}{j!\,(\alpha-1-j)!} \frac{\Gamma(kj+k)}{\Gamma(kj+k-\alpha)} \sigma^j. \tag{44}$$

In (43), $R^\alpha_\tau f$ means $R^\alpha_\tau \tilde{f}$ evaluated at $\tau = \sigma^{1/k}$. In (44), the ratio of Γ functions is interpreted as 0 if $kj + k - \alpha$ is a nonpositive integer.

Proof: Use successively the definitions (36), (33), (41), and (32):

$$\begin{aligned}
R^\alpha_\lambda f &= \alpha! \lambda^{-\alpha} \partial_\lambda^{-\alpha} f \\
&= \alpha \lambda^{-\alpha} \int_0^\lambda (\lambda - \sigma)^{\alpha-1} f(\sigma) \, d\sigma \\
&= \alpha \omega^{-\alpha k} \int_0^\omega (\omega^k - \tau^k)^{\alpha-1} k \tau^{k-1} \tilde{f}(\tau) \, d\tau \\
&= \alpha \omega^{-\alpha k} \int_0^\omega (\omega^k - \tau^k)^{\alpha-1} k \tau^{k-1} \partial_\tau^\alpha (\partial_\tau^{-\alpha} \tilde{f}) \, d\tau.
\end{aligned}$$

Integrate by parts α times. The lower-endpoint term always vanishes, because $\partial_\omega^{-\beta} f(0) = 0$ for all β. Until the last step, the upper-endpoint term contains positive powers of $\omega^k - \tau^k$, so it also vanishes. At the last step there is an upper-endpoint contribution

$$R_1 = (-1)^{\alpha-1} \alpha \omega^{-\alpha k} (\alpha - 1)!\, (-k\tau^{k-1})^\alpha (-1) \partial_\tau^{-\alpha} \tilde{f} \Big|_{\tau=\omega} = \alpha!\, k^\alpha \omega^{-\alpha} \partial_\omega^{-\alpha} f = k^\alpha R^\alpha_\omega f.$$

The remaining integral is

$$R_2 = (-1)^\alpha \alpha \omega^{-\alpha k} \int_0^\omega \partial_\tau^\alpha [(\omega^k - \tau^k)^{\alpha-1} k \tau^{k-1}] \partial_\tau^{-\alpha} \tilde{f} \, d\tau.$$

But

$$(\omega^k - \tau^k)^{\alpha-1} k \tau^{k-1} = k \sum_{j=0}^{\alpha-1} (-1)^j \binom{\alpha-1}{j} \omega^{k(\alpha-1-j)} \tau^{kj+k-1},$$

so

$$
\begin{aligned}
&(-1)^\alpha \alpha \omega^{-\alpha k} \partial_\tau^\alpha [(\omega^k - \tau^k)^{\alpha-1} k \tau^{k-1}] \\
&= \alpha k \sum_{j=0}^{\alpha-1} (-1)^{\alpha-j} \binom{\alpha-1}{j} \left[\prod_{i=0}^{\alpha-1} (kj + k - 1 - i) \right] \tau^{kj+k-1-\alpha} \omega^{-kj-k}.
\end{aligned}
$$

Thus, by (36) applied to $\tilde{f}(\tau)$,

$$
\begin{aligned}
R_2 &= \int_0^\omega \sum_{j=0}^{\alpha-1} \frac{(-1)^{\alpha-j} \alpha!}{j!\,(\alpha-1-j)!} \frac{\Gamma(kj+k)}{\Gamma(kj+k-\alpha)} \tau^{kj} \partial_\tau(\tau^k) \omega^{-kj-k} \frac{1}{\alpha!} R_\tau^\alpha \tilde{f}\, d\tau \\
&= \int_0^\lambda J_{k,\alpha}(\lambda,\sigma) R_\tau^\alpha f\, d\sigma.
\end{aligned}
$$

Adding R_1 and R_2, one obtains the formula (43) to be proved.

As a corollary we see:

> If an integral (29) is summable by αth-order Riesz means with respect to λ, then it is summable to the same value by αth means with respect to ω, and conversely. (45)

Proof: We show the converse; the direct statement follows when one replaces k by $1/k$ and interchanges λ and ω. It suffices, since Riesz means are linear, to consider the special cases (i) where the Riesz limit in question is 0, and (ii) where $f(\lambda) = C$, a constant, for all $\lambda \neq 0$. The latter case is trivial since all Riesz means and hence all Riesz limits are equal to C. In case (i), we are given that $R_\tau^\alpha f \to 0$ as $\sigma \to \infty$; that is, for every $\varepsilon > 0$ there is a K such that $|R_\tau^\alpha f(\sigma)| < \varepsilon$ when $\sigma > K$. For $\lambda > K$, write

$$
R_2 = \int_0^K J_{k,\alpha}(\lambda,\sigma) R_\tau^\alpha f\, d\sigma + \int_K^\lambda J_{k,\alpha}(\lambda,\sigma) R_\tau^\alpha f\, d\sigma \equiv R_{21} + R_{22}\,.
$$

From the form of $J_{k,\alpha}$ (44), R_{21} approaches 0 as $\lambda \to \infty$, and $|R_{22}|$ has the form

$$
|R_{22}| \le \varepsilon \sum_{j=0}^{\alpha-1} c_j \lambda^{-j-1} \int_K^\lambda \sigma^j\, d\sigma = \varepsilon O(\lambda^0),
$$

which can be made arbitrarily small. Thus $R_\lambda^\alpha f$ approaches 0, as asserted.

A generalization of this argument (due to Riesz and Hardy[6,7]) proves the *second theorem of consistency* (for α integer): Let $\lambda = g(\omega)$, where g is an increasing function in $C^\infty(\mathbb{R}^+)$, $g(0) = 0$, $g(\infty) = \infty$, and

$$
\partial_\omega^r g(\omega) = O(\omega^{-r} g(\omega)) \quad \text{for all } r = 1, 2, \ldots. \tag{46}
$$

Then (45) holds, except possibly for the clause "and conversely". Conditions sufficient together to guarantee (46) (given the other conditions on g) are

$$
g(\omega) = O(\omega^\Delta) \quad \text{for some } \Delta > 0 \tag{47}
$$

and

$g(\omega)$ is given (for sufficiently large ω) by an explicit, finite formula involving the logarithmic and exponential functions, real constants, and (real, finite) elementary algebraic operations. (48)

The significance of this theorem is clearer in the following imprecise paraphrase: (1) If ω increases to ∞, but *more slowly* than λ, then λ-summability implies ω-summability, *provided* that the increase of ω with λ is sufficiently "steady" — that is, derivatives of g must not oscillate so as to disrupt (46). (2) Under those conditions, ω-summability does not ensure λ-summability — e.g., an integral R^α-summable with respect to ω may not be R^α-summable with respect to $\lambda \sim e^\omega$. However, if the rates of increase of λ and ω differ only by a power, then the two types of summability are coextensive.

If k is an integer (necessarily ≥ 2), there is another relation between λ-means and ω-means, which we call the *Hörmander formula* [Ref. 10, Sec. 5]:

$$\begin{aligned} R^\alpha_\lambda f(\lambda) &= \int_0^\omega \left[\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}\left(1-\frac{\tau}{\omega}\right)^j\right]^\alpha d\tilde{f}(\tau) \\ &= \sum_{\beta=\alpha}^{\alpha k} b_\beta R^\beta_\omega f \quad \text{for certain numbers } b_\beta\,. \end{aligned} \tag{49}$$

This is obtained from (36) by expanding the factor $(1-\sigma/\lambda) = 1-[1-(1-\tau/\omega)]^k$ by the binomial theorem. In view of (40), (49) has the converse part of (45) as a corollary. More significantly, the absence of an integral in (49) as compared with (43) means that the asymptotic ($\lambda \to \infty$) behavior of $R^\alpha_\lambda f$ is entirely determined by the *asymptotic* behavior of $R^\beta_\omega f$, a conclusion which we shall reach independently below. We shall see also that the converse is false: The asymptotic behavior of $R^\alpha_\omega f$ is affected by the values of f at *small* λ, in a way that is not captured by the asymptotic behavior of $R^\beta_\lambda f$ for any β, no matter how large. Thus it is really essential in (49) that k is an integer (so that the binomial series terminates).

We now work out in each direction the relation between the Riesz means with respect to the spectral parameter and those with respect to its square root, taking note of a fundamental difference between the two calculations.

1. First assume that as $\lambda \to \infty$,

$$R^\alpha_\lambda \mu(\lambda) = \sum_{s=0}^{\alpha} a_{\alpha s}\lambda^{\frac{m}{2}-\frac{s}{2}} + O(\lambda^{\frac{m}{2}-\frac{\alpha}{2}-\frac{1}{2}}) \tag{50}$$

for some positive integer m, as suggested by Hörmander's results on the Riesz means of spectral functions of second-order operators (where m is the dimension of the underlying manifold). Let us attempt to determine the asymptotic behavior of $R^\alpha_\omega \tilde{\mu}$ ($\lambda \equiv \omega^2$) from (43) and (44) with $f=\mu$, $k=\frac{1}{2}$, and λ and ω interchanged:

$$R^\alpha_\omega \mu = 2^{-\alpha} R^\alpha_\lambda \mu + \int_0^\omega J_{\frac{1}{2},\alpha}(\omega,\tau) R^\alpha_\sigma \mu \, d\tau, \tag{51}$$

$$J_{\frac{1}{2},\alpha}(\omega,\tau) = \sum_{j=0}^{\alpha-1}(-1)^{\alpha-j}\omega^{-j-1}\frac{1}{j!\,(\alpha-1-j)!}\frac{\Gamma(\frac{j}{2}+\frac{1}{2})}{\Gamma(\frac{j}{2}+\frac{1}{2}-\alpha)}\,\tau^j\,. \tag{52}$$

Note that all terms in (52) with j odd are equal to zero, since $\frac{1}{2}(j+1)$ is an integer $\le \alpha$.

Since μ is of locally bounded variation, it is bounded as $\lambda \downarrow 0$. Hence $\partial_\lambda^{-\alpha}\mu = O(\lambda^\alpha)$ at small λ (see Remark at end of Sec. V), so R_λ^α is bounded there. Thus there is no problem with convergence at the lower limit of the integral in (51), as a whole. However, some of the individual terms in (50), and hence the remainder term by itself, will be singular as $\tau \downarrow 0$ if $\alpha \ge m$. Therefore, for *each* integral encountered when (50) is substituted into (51), one must choose an appropriate lower limit of integration, τ_0. Let R be the part of the total integral thereby omitted; thus the integrand of R equals $J_{\frac{1}{2},\alpha} R_\sigma^\alpha \mu(\tau^2)$ for small τ, and is redefined as τ increases — in such a way that R is convergent and depends on ω only through the factors ω^{-j-1}. Then R will be of the form

$$R = \sum_{\substack{j=0 \\ j \text{ even}}}^{\alpha-1} Z_j \omega^{-j-1}, \tag{53}$$

where the Z_j are constants, which cannot be determined from the information in (50) since they are affected by the behavior of $R_\lambda^\alpha \mu$ at small λ. Next consider the part of the integral in (51) associated with the series in (50):

$$\sum_{s=0}^{\alpha} a_{\alpha s} \int_{\tau_0}^{\omega} \sum_{\substack{j=0 \\ j \text{ even}}}^{\alpha-1} (-1)^{\alpha-j} \omega^{-j-1} \frac{1}{j!\,(\alpha-1-j)!} \frac{\Gamma(\frac{j}{2}+\frac{1}{2})}{\Gamma(\frac{j}{2}+\frac{1}{2}-\alpha)} \tau^{m-s+j}\, d\tau. \tag{54}$$

When each integral in (54) is evaluated, the contribution of the lower limit is of the form (53) and may be henceforth counted as a part of R. The contribution of the upper limit is proportional to ω^{m-s}, unless $m-s$ is odd and negative, in which case the term $j = -m+s-1$ yields something proportional to $\omega^{m-s}\ln\omega$ and the other values of j yield more terms to be absorbed into R. Finally, when integrating the remainder term in (50) we may assume that τ_0 and ω are so large that for some K, the integral is less than

$$\begin{aligned} K\int_{\tau_0}^{\omega} |J_{\frac{1}{2},\alpha}(\omega,\tau)|\tau^{m-\alpha-1}\, d\tau &\le \sum_{\substack{j=0 \\ j \text{ even}}}^{\alpha-1} K_j \omega^{-j-1} \int_{\tau_0}^{\omega} \tau^{m-\alpha+j-1}\, d\tau \\ &= \sum_{\substack{j=0 \\ j \text{ even}}}^{\alpha-1} K_j (m-\alpha+j)^{-1} (\omega^{m-\alpha-1} - \omega^{-j-1}\tau_0^{m-\alpha+j}), \end{aligned} \tag{55}$$

except that if $m-\alpha+j=0$, the corresponding term is

$$K_j \omega^{m-\alpha-1}(\ln\omega - \ln\tau_0). \tag{56}$$

Now τ_0 must be chosen differently for different j: If $m-\alpha+j<0$, take $\tau_0 \to \infty$, leaving in (55) a single contribution to the error of order $O(\omega^{m-\alpha-1})$. (An integral over a finite interval is thereby included in R, but (53) is still valid.) If $m-\alpha+j \ge 0$, we have $-j-1 \le m-\alpha-1$, so that *both* terms in (55) are $O(\omega^{m-\alpha-1})$ for finite τ_0, with a possible extra logarithmic factor in the worst case (56).

Therefore, adding all contributions in (51), one has that whenever $R_\lambda^\alpha\mu$ has the asymptotic behavior (50), $R_\omega^\alpha\mu$ has the behavior (as $\omega \to \infty$)

$$R_\omega^\alpha\mu = \sum_{s=0}^{\alpha} c_{\alpha s}\omega^{m-s} + \sum_{\substack{s=m+1\\ s-m \text{ odd}}}^{\alpha} d_{\alpha s}\omega^{m-s}\ln\omega + O(\omega^{m-\alpha-1}\ln\omega), \tag{57}$$

where

$$c_{\alpha s} \text{ is undetermined if } s > m \text{ and } s-m \text{ is odd;} \tag{58}$$

$$d_{\alpha s} = (-1)^{\alpha+m-s+1}\frac{1}{(s-m-1)!\,(m-s+\alpha)!}\frac{\Gamma(\frac{s}{2}-\frac{m}{2})}{\Gamma(\frac{s}{2}-\frac{m}{2}-\alpha)}\,a_{\alpha s} \quad \text{(if } s>m \text{ and } s-m \text{ is odd);} \tag{59}$$

$$c_{\alpha s} = \left[\sum_{\substack{j=0\\ j \text{ even}}}^{\alpha-1} (-1)^{\alpha-j}\frac{(m-s+j+1)^{-1}}{j!\,(\alpha-1-j)!}\frac{\Gamma(\frac{j}{2}+\frac{1}{2})}{\Gamma(\frac{j}{2}+\frac{1}{2}-\alpha)}\right] a_{\alpha s} + 2^{-\alpha}a_{\alpha s} \quad \text{if } s\le m \text{ or } s-m \text{ is even.} \tag{60}$$

Remark: Writing $\ln\omega$ in (57) instead of $\ln\kappa\omega$, κ some numerical constant, is arbitrary. In fact, in an application, κ is likely to have physical dimensions (such as length). Changing κ redefines the undetermined coefficient $c_{\alpha s}$ by adding a multiple of $d_{\alpha s}$.

2. Now we contrast the foregoing calculation with the parallel calculation of $R_\lambda^\alpha\mu$ from $R_\omega^\alpha\mu$, assuming the latter to have the form (57). We use (43) and (44) with $k=2$:

$$R_\lambda^\alpha\mu = 2^\alpha R_\omega^\alpha\mu + \int_0^\lambda J_{2,\alpha}(\lambda,\sigma)R_\tau^\alpha\mu\,d\sigma,$$

$$J_{2,\alpha}(\lambda,\sigma) = \sum_{j=0}^{\alpha-1}(-1)^{\alpha-j}\lambda^{-j-1}\frac{1}{j!\,(\alpha-1-j)!}\frac{\Gamma(2j+2)}{\Gamma(2j+2-\alpha)}\,\sigma^j.$$

This time the j term vanishes for all $j \le \frac{1}{2}\alpha - 1$ (equivalently, $j < [\alpha/2]$). Hence the contribution of small σ will be of the form (as $\lambda\to\infty$)

$$R = \sum_{j=[\frac{\alpha}{2}]}^{\alpha-1} Z_j\lambda^{-j-1} = O(\lambda^{-\alpha/2}). \tag{61}$$

The integral of the remainder in (57) (over large σ) is less than

$$\sum_{j=[\frac{\alpha}{2}]}^{\alpha-1} K_j\lambda^{-j-1}\int_{\sigma_0}^\lambda \sigma^{\frac{m}{2}-\frac{\alpha}{2}+j-\frac{1}{2}}\ln\sigma\,d\sigma = \text{ terms of form (61) } + O(\lambda^{\frac{m}{2}-\frac{\alpha}{2}-\frac{1}{2}}\ln\lambda), \tag{62}$$

except that the second term is $O(\lambda^{\frac{m}{2}-\frac{\alpha}{2}-\frac{1}{2}}(\ln\lambda)^2)$ if the value $j=\frac{1}{2}(\alpha-m-1)$ occurs. Here the relevant integral formulas are

$$\int \lambda^{p-1}\ln\lambda\,d\lambda = \lambda^p(p^{-1}\ln\lambda - p^{-2}) + C \quad \text{if } p\ne 0, \qquad \int\lambda^{-1}\ln\lambda\,d\lambda = \frac{1}{2}(\ln\lambda)^2 + C. \tag{63}$$

Since $m \geq 1$, it follows that R and the similar terms in (62) are of at least as high order as the last term in (62). Hence there are, in effect, *no* undetermined constants of integration when the calculation is done in this direction ($\omega \to \lambda$). The remaining, explicit integrals are

$$R_1 = \sum_{s=0}^{\alpha} c_{\alpha s} \int_{\sigma_0}^{\lambda} \sum_{j=\left[\frac{\alpha}{2}\right]}^{\alpha-1} (-1)^{\alpha-j} \lambda^{-j-1} \frac{1}{j!\,(\alpha-1-j)!} \frac{\Gamma(2j+2)}{\Gamma(2j+2-\alpha)}\, \sigma^{\frac{m}{2}-\frac{s}{2}+j}\, d\sigma$$
$$+ \sum_{\substack{s=m+1 \\ s-m \text{ odd}}}^{\alpha} \frac{1}{2} d_{\alpha s} \int_{\sigma_0}^{\lambda} \sum_{j=\left[\frac{\alpha}{2}\right]}^{\alpha-1} (-1)^{\alpha-j} \lambda^{-j-1} \frac{1}{j!\,(\alpha-j)!} \frac{\Gamma(2j+2)}{\Gamma(2j+2-\alpha)}\, \sigma^{\frac{m}{2}-\frac{s}{2}+j} \ln\sigma\, d\sigma.$$

Using (63), one sees that the values of s for which $d_{\alpha s} = 0$ yield terms proportional to $c_{\alpha s}\lambda^{(m-s)/2}$, but that when $d_{\alpha s} \neq 0$ there are, *a priori*, two types of term, proportional to

$$c_{\alpha s}\lambda^{\frac{m}{2}-\frac{s}{2}} + \frac{1}{2} d_{\alpha s} \lambda^{\frac{m}{2}-\frac{s}{2}} \ln\lambda \quad \text{and} \quad d_{\alpha s}\lambda^{\frac{m}{2}-\frac{s}{2}},$$

respectively. (Since

$$\frac{m}{2} - \frac{s}{2} + j > -\frac{\alpha}{2} + \left(\frac{\alpha}{2} - 1\right) = -1,$$

no logarithms are "created" by the integration as in (54).) Since it must be possible to recover the expansion (50) in this way, there must be a numerical coincidence which causes all terms proportional to $d_{\alpha s}\ln\lambda$ to cancel. By the observation just made, this same numerical identity will cause all terms involving $c_{\alpha s}$ to cancel, if $s - m$ is odd and positive. Thus these numbers $c_{\alpha s}$ do not affect at all the asymptotic behavior of $R^\alpha_\lambda \mu$ — as was to be expected from the fact that the latter does not determine them (58). Obviously, similar cancellations of logarithmic and $c_{\alpha s}$ terms must occur when $R^\alpha_\lambda \mu$ is calculated from $R^\alpha_\omega \mu, \ldots, R^{2\alpha}_\omega \mu$ by Hörmander's formula (49).

The conclusion therefore is that if $R^\alpha_\omega \mu$ has the asymptotic behavior (57), then $R^\alpha_\lambda \mu$ has the behavior (50) with error term $O(\lambda^{\frac{m}{2}-\frac{\alpha}{2}-\frac{1}{2}}(\ln\lambda)^2)$ in the worst case. The formulas for the coefficients are

$$a_{\alpha s} = \left[-\frac{1}{2} \sum_{j=\left[\frac{\alpha}{2}\right]}^{\alpha-1} (-1)^{\alpha-j} \frac{\left(\frac{m}{2}-\frac{s}{2}+j+1\right)^{-2}}{j!\,(\alpha-1-j)!} \frac{\Gamma(2j+2)}{\Gamma(2j+2-\alpha)} \right] d_{\alpha s}$$
$$\text{if } s > m \text{ and } s-m \text{ is odd;} \tag{64}$$

$$a_{\alpha s} = \left[\sum_{j=\left[\frac{\alpha}{2}\right]}^{\alpha-1} (-1)^{\alpha-j} \frac{\left(\frac{m}{2}-\frac{s}{2}+j+1\right)^{-1}}{j!\,(\alpha-1-j)!} \frac{\Gamma(2j+2)}{\Gamma(2j+2-\alpha)} \right] c_{\alpha s} + 2^\alpha c_{\alpha s}$$
$$\text{if } s \leq m \text{ or } s-m \text{ is even.} \tag{65}$$

The conditions of consistency between the two calculations are (for $0 \le s \le \alpha$, $m \ge 1$, $\alpha \ge 1$)

$$\left[-\frac{1}{2} \sum_{j=\left[\frac{\alpha}{2}\right]}^{\alpha-1} (-1)^{\alpha-j} \frac{\left(\frac{m}{2} - \frac{s}{2} + j + 1\right)^{-2}}{j!\,(\alpha-1-j)!} \frac{\Gamma(2j+2)}{\Gamma(2j+2-\alpha)} \right]$$
$$\times \left[(-1)^{\alpha+m-s+1} \frac{1}{(s-m-1)!\,(m-s+\alpha)!} \frac{\Gamma(\frac{s}{2}-\frac{m}{2})}{\Gamma(\frac{s}{2}-\frac{m}{2}-\alpha)} \right] = 1$$
$$\text{if } s > m \text{ and } s-m \text{ is odd} \tag{66}$$

(from (59) and (64));

$$\left[\sum_{j=\left[\frac{\alpha}{2}\right]}^{\alpha-1} (-1)^{\alpha-j} \frac{\left(\frac{m}{2} - \frac{s}{2} + j + 1\right)^{-1}}{j!\,(\alpha-1-j)!} \frac{\Gamma(2j+2)}{\Gamma(2j+2-\alpha)} + 2^{\alpha} \right]$$
$$\times \left[\sum_{\substack{j=0 \\ j \text{ even}}}^{\alpha-1} (-1)^{\alpha-j} \frac{(m-s+j+1)^{-1}}{j!\,(\alpha-1-j)!} \frac{\Gamma(\frac{j}{2}+\frac{1}{2})}{\Gamma(\frac{j}{2}+\frac{1}{2}-\alpha)} + 2^{-\alpha} \right] = 1$$
$$\text{if } s \le m \text{ or } s-m \text{ is even} \tag{67}$$

(from (60) and (65));

$$\sum_{j=\left[\frac{\alpha}{2}\right]}^{\alpha-1} (-1)^{\alpha-j} \frac{\left(\frac{m}{2} - \frac{s}{2} + j + 1\right)^{-1}}{j!\,(\alpha-1-j)!} \frac{\Gamma(2j+2)}{\Gamma(2j+2-\alpha)} + 2^{\alpha} = 0$$
$$\text{if } s > m \text{ and } s-m \text{ is odd} \tag{68}$$

(the "numerical coincidence"). These can easily be verified for small values of α.

In the Appendix it is shown that (67) is valid for all values of $s-m$ as a complex variable, except those where one of the factors is undefined. Then (68) is obtained (for any α) in the limit as $s-m$ approaches a pole of the second factor, and (66) is obtained similarly from the derivative of (67).

V. THE MEANS OF A STIELTJES INTEGRAL IN TERMS OF THE MEANS OF THE MEASURE

The calculation in this section is the central lemma relating the asymptotics of Green functions at small t to the asymptotics of spectral measures of various kinds at large λ.

Let μ and f be functions of locally bounded variation, vanishing at 0, related as in (29):

$$f(\lambda) = \int_0^{\lambda} a(\sigma)\, d\mu(\sigma).$$

Assume that $a(\lambda)$ is a C^{∞} function, and that it is well-behaved at the origin as described in the remark at the end of this section. Integrating by parts, one has

$$f(\lambda) = a(\lambda)\mu(\lambda) - \int_0^\lambda a'(\sigma)\mu(\sigma)\,d\sigma. \tag{69}$$

Continuing similarly, one can express $\partial_\lambda^{-\alpha} f$ in terms of integrals whose integrands involve only $\partial_\lambda^{-\alpha}\mu$ and derivatives of a:

$$\begin{aligned}\partial_\lambda^{-\alpha} f(\lambda) &= a(\lambda)\partial_\lambda^{-\alpha}\mu(\lambda) + \sum_{j=1}^{\alpha+1}(-1)^j\binom{\alpha+1}{j}\frac{1}{(j-1)!}\int_0^\lambda (\lambda-\sigma)^{j-1}\partial_\sigma^j a(\sigma)\partial_\sigma^{-\alpha}\mu(\sigma)\,d\sigma \\ &= \sum_{j=0}^{\alpha+1}\int_0^\lambda d\sigma_1\cdots\int_0^{\sigma_{j-1}} d\sigma\,(-1)^j\binom{\alpha+1}{j}\partial_\sigma^j a(\sigma)\partial_\sigma^{-\alpha}\mu(\sigma).\end{aligned} \tag{70}$$

(The integral in the last version is j-fold.) This relation can be written

$$R_\lambda^\alpha f(\lambda) = a(\lambda)R_\lambda^\alpha\mu(\lambda) + \lambda^{-\alpha}\sum_{j=1}^{\alpha+1}(-1)^j\binom{\alpha+1}{j}\frac{1}{(j-1)!}\int_0^\lambda (\lambda-\sigma)^{j-1}\sigma^\alpha\partial_\sigma^j a(\sigma)R_\sigma^\alpha\mu(\sigma)\,d\sigma. \tag{71}$$

Proof: For $\alpha=0$, (70) is (69). Assume (70) for α and prove it for $\alpha+1$: First,

$$\begin{aligned}\partial_\lambda^{-\alpha-1} f(\lambda) &= \int_0^\lambda \partial_\sigma^{-\alpha} f(\sigma)d\sigma \\ &= \int_0^\lambda a(\sigma)\partial_\sigma^{-\alpha}\mu(\sigma)d\sigma + \sum_{j=1}^{\alpha+1}(-1)^j\binom{\alpha+1}{j}\int_0^\lambda d\sigma\int_0^\sigma d\sigma_1\cdots\int_0^{\sigma_{j-1}} d\tau\,\partial_\tau^j a(\tau)\partial_\tau^{-\alpha}\mu(\tau).\end{aligned}$$

The last term can be rewritten by (33) as

$$\sum_{j=1}^{\alpha+1}(-1)^j\binom{\alpha+1}{j}\frac{1}{j!}\int_0^\lambda (\lambda-\sigma)^j\partial_\sigma^j a(\sigma)\partial_\sigma^{-\alpha}\mu(\sigma)\,d\sigma.$$

Now integrate by parts in both terms:

$$\begin{aligned}\partial_\lambda^{-\alpha-1} f(\lambda) = {}& a(\lambda)\partial_\lambda^{-\alpha-1}\mu(\lambda) - \int_0^\lambda \partial_\sigma a(\sigma)\partial_\sigma^{-\alpha-1}\mu(\sigma)\,d\sigma \\ &+ \sum_{j=1}^{\alpha+1}(-1)^{j+1}\binom{\alpha+1}{j}\frac{1}{j!}\int_0^\lambda \partial_\sigma[(\lambda-\sigma)^j\partial_\sigma^j a(\sigma)]\partial_\sigma^{-\alpha-1}\mu(\sigma)\,d\sigma.\end{aligned}$$

The last term equals

$$\begin{aligned}&\sum_{j=1}^{\alpha+1}(-1)^j\binom{\alpha+1}{j}\frac{1}{(j-1)!}\int_0^\lambda (\lambda-\sigma)^{j-1}\partial_\sigma^j a(\sigma)\partial_\sigma^{-\alpha-1}\mu(\sigma)\,d\sigma \\ &+ \sum_{j=2}^{\alpha+2}(-1)^j\binom{\alpha+1}{j-1}\frac{1}{(j-1)!}\int_0^\lambda (\lambda-\sigma)^{j-1}\partial_\sigma^j a(\sigma)\partial_\sigma^{-\alpha-1}\mu(\sigma)\,d\sigma.\end{aligned}$$

Using Pascal's triangle relation

$$\binom{\alpha+1}{j} + \binom{\alpha+1}{j-1} = \binom{\alpha+2}{j}, \tag{72}$$

one simplifies the expression to

$$\partial_\lambda^{-\alpha-1} f(\lambda) = a(\lambda)\partial_\lambda^{-\alpha-1}\mu(\lambda) + \sum_{j=1}^{\alpha+2}(-1)^j \binom{\alpha+2}{j} \frac{1}{(j-1)!}\int_0^\lambda (\lambda-\sigma)^{j-1}\partial_\sigma^j a(\sigma)\partial_\sigma^{-\alpha-1}\mu(\sigma)\,d\sigma,$$

as was to be proved.

Remark: In the foregoing it was tacitly assumed that $\partial_\sigma^j a(\sigma)$ was bounded as $\sigma \downarrow 0$, so that there were no lower-endpoint contributions in the integrations by parts. Actually, since

$$\partial_\sigma^{-\alpha}\mu(\sigma) = O(\sigma^\alpha) \quad \text{as } \sigma \downarrow 0 \tag{73}$$

(as follows from (33) and the boundedness of μ), it suffices to assume that

$$\partial_\sigma^\alpha a(\sigma) = o(\sigma^{-\alpha}) \quad \text{as } \sigma \downarrow 0. \tag{74}$$

VI. HEAT KERNELS (LAPLACE TRANSFORMS) AND RIESZ MEANS

The standard heat kernel of a second-order operator has a natural association with the variable we have called λ, while the cylinder kernel (introduced in Sec. II) is associated in the same way with the variable ω. The terms in the asymptotic expansions of these Green functions are in direct correspondence with those in the asymptotic expansions of the associated Riesz means. However, it is instructive to try to calculate each Green function in terms of the "wrong" variable, to observe how information gets lost, or needs to be resupplied, in passing from one quantity to another. Therefore, this section divides naturally into four parts.

1. Given $\mu(\lambda)$, a function of locally bounded variation on $[0,\infty)$, we consider

$$K(t) \equiv \int_0^\infty e^{-\lambda t} d\mu(\lambda) \qquad (t > 0). \tag{75}$$

As $t \downarrow 0$, we anticipate an expansion of the form

$$K(t) \sim \sum_{s=0}^\infty b_s t^{-\frac{m}{2}+\frac{s}{2}}; \tag{76}$$

we shall demonstrate detailed equivalence of (76) with (50).

In spectral theory, (75) has a number of possible interpretations. Let H be a positive, elliptic, second-order differential operator on an m-dimensional manifold. Then: (1) If the manifold is compact, K may be the integrated "trace" of the heat kernel of H, $\mu(\lambda)$ being the number of eigenvalues less than or equal to λ. (2) K may be the diagonal value of the heat kernel at a point x, $\mu(\lambda) \equiv E_\lambda(x,x)$ being the diagonal value of the spectral function (integral kernel of the spectral projection). In this case one knows that $b_s = 0$ if s is odd. (3) K may be the diagonal value of some spatial derivative of the heat kernel, μ being the corresponding derivative of the spectral function. (In this case, m in (76) depends on the order of the derivative as well as on the dimension.) If $m = 1$, μ is a Titchmarsh-Kodaira spectral measure. (4) $K = K(t,x,y)$ may be the full heat kernel (off-diagonal), μ being the full spectral function.

Let us assume (50),

$$R_\lambda^\alpha \mu(\lambda) = \sum_{s=0}^{\alpha} a_{\alpha s} \lambda^{\frac{m}{2}-\frac{s}{2}} + O(\lambda^{\frac{m}{2}-\frac{\alpha}{2}-\frac{1}{2}}),$$

and calculate $K(t)$ from (71), with $a(\lambda) = e^{-\lambda t}$,

$$\partial_\lambda^j a(\lambda) = (-t)^j e^{-\lambda t},$$

and $\lambda \to \infty$ in (71). (See the first theorem of consistency, (40).) In the present case all integrals in (71) converge as $\lambda \to \infty$, and hence the only term that survives in the limit is the one with $j = \alpha + 1$, $(\lambda - \sigma)^{j-1} = \lambda^\alpha + o(\lambda^\alpha)$. So

$$\begin{aligned} K(t) &= (-1)^{\alpha+1} \frac{1}{\alpha!} \int_0^\infty \sigma^\alpha (-t)^{\alpha+1} e^{-\sigma t} R_\sigma^\alpha \mu(\sigma)\, d\sigma \\ &= t^{\alpha+1} \frac{1}{\alpha!} \int_0^\infty \left[\sum_{s=0}^{\alpha} a_{\alpha s} \sigma^{\frac{m}{2}-\frac{s}{2}+\alpha} + O(\sigma^{\frac{m}{2}+\frac{\alpha}{2}-\frac{1}{2}}) \right] e^{-\sigma t} d\sigma. \end{aligned}$$

Using

$$\int_0^\infty \lambda^{p-1} e^{-\lambda t} d\lambda = \Gamma(p) t^{-p} \qquad (p > 0), \tag{77}$$

we have

$$K(t) = \sum_{s=0}^{\alpha} \frac{\Gamma(\frac{m}{2} - \frac{2}{2} + \alpha + 1)}{\Gamma(\alpha+1)} a_{\alpha s} t^{-\frac{m}{2}+\frac{s}{2}} + O(t^{-\frac{m}{2}+\frac{\alpha}{2}+\frac{1}{2}}). \tag{78}$$

(The undeterminable contribution from small σ is $O(t^{\alpha+1})$, which is of higher order than the remainder.) Thus

$$b_s = \frac{\Gamma(\frac{m}{2} - \frac{s}{2} + \alpha + 1)}{\Gamma(\alpha+1)}\, a_{\alpha s} \tag{79}$$

holds for $\alpha \geq s$, and formally for smaller α. (In the latter context a pole of $\Gamma(\frac{1}{2}(m-s)+\alpha+1)$ may be encountered. The significance of (79) then is that $a_{\alpha s} = 0$. Thus, when $m - s$ is even, α must be sufficiently large before $a_{\alpha s}$ will uniquely determine b_s, even formally.) For $a < s$, (79) may be taken as a definition of $a_{\alpha s}$.

To check that the right-hand side of (79) is independent of α, write (50) as

$$\partial_\lambda^{-\alpha} \mu(\lambda) \sim \frac{1}{\alpha!} \sum_{s=0}^{\alpha} a_{\alpha s} \lambda^{\frac{m}{2}-\frac{s}{2}+\alpha} \tag{80}$$

and differentiate formally:

$$\partial_\lambda^{-\alpha+1} \mu(\lambda) \sim \frac{1}{\alpha!} \sum_s \left(\frac{m}{2} - \frac{s}{2} + \alpha \right) a_{\alpha s} \lambda^{\frac{m}{2}-\frac{s}{2}+\alpha-1}.$$

By comparison with (80) for $\alpha - 1$, one has

$$\frac{1}{\alpha!}\left(\frac{m}{2}-\frac{s}{2}+\alpha\right)a_{\alpha s}=\frac{1}{(\alpha-1)!}a_{\alpha-1,s}\,,$$

which may be variously rewritten as

$$\frac{\Gamma(\frac{m}{2}-\frac{s}{2}+\alpha+1)}{\Gamma(\alpha+1)}a_{\alpha s}=\frac{\Gamma(\frac{m}{2}-\frac{s}{2}+\alpha)}{\Gamma(\alpha)}a_{\alpha-1,s} \tag{81}$$

(showing consistency of (79)), or as

$$a_{\alpha-1,s}=\frac{1}{\alpha}\left(\frac{m}{2}-\frac{s}{2}+\alpha\right)a_{\alpha s}\qquad(\alpha\geq 1). \tag{82}$$

In particular, we can stem the profusion of constants by choosing a_{ss} as the fiducial member of the family $\{a_{\alpha s}\}$. We have

$$b_s=\frac{\Gamma(\frac{m}{2}+\frac{s}{2}+1)}{\Gamma(s+1)}\,a_{ss}\,, \tag{83}$$

and

$$a_{\alpha s}=\frac{\Gamma(\alpha+1)\Gamma(\frac{m}{2}+\frac{s}{2}+1)}{\Gamma(\frac{m}{2}-\frac{s}{2}+\alpha+1)\Gamma(s+1)}\,a_{ss} \tag{84}$$

(meaning 0, of course, when the first factor in the denominator has a pole).

Remark: When $m-s$ is even, $a_{\alpha s}$ is zero for small α, and the nonzero value for large α (hence the nonzero value of b_s) may be regarded as a constant of integration encountered in the passage from a low-order Riesz mean to a higher-order one. A case of particular interest is the local diagonal value of the heat kernel, for which $b_s=0$ when s is odd but generally $b_s\neq 0$ for s even. Here there is an essential difference between m even and m odd. If m is odd, then a_{0s} is 0 for odd s and nonzero for even s, and the constants of integration are also 0. If m is even, then $a_{0s}=0$ for *all* $s>m$, and the nonzero values of b_s for $s>m$, s even, come entirely from constants of integration. This dimensional property of a_{0s} is reflected in the poles of the zeta function.[11,12]

2. It is of interest to consider how (76) could be calculated from the ω-means, (57). We have

$$\begin{aligned}
\partial_\omega(e^{-\omega^2t})&=e^{-\omega^2t}(-2\omega t),\\
\partial_\omega^2(e^{-\omega^2t})&=e^{-\omega^2t}(-2t+4\omega^2t^2),\\
\partial_\omega^3(e^{-\omega^2t})&=e^{-\omega^2t}(12\omega t^2-8\omega^3t^3),\\
\partial_\omega^4(e^{-\omega^2t})&=e^{-\omega^2t}(12t^2-48\omega^2t^3+16\omega^4t^4),
\end{aligned} \tag{85}$$

and in general the form

$$\partial_\omega^j(e^{-\omega^2t})=e^{-\omega^2t}\sum_{i=0}^{[\frac{j}{2}]}z_i\omega^{j-2i}t^{j-i}. \tag{86}$$

As before, when $\omega\to\infty$ the formula (71) reduces to one term,

$$K(t) = \lim_{\omega\to\infty} R^\alpha_\omega \left[\int_0^\omega e^{-\tau^2 t} d\mu(\tau)\right] = (-1)^{\alpha+1} \frac{1}{\alpha!} \int_0^\infty \omega^\alpha \partial_\omega^{\alpha+1}(e^{-\omega^2 t}) R^\alpha_\omega \mu \, d\omega,$$

where (57)

$$R^\alpha_\omega \mu = \sum_{s=0}^{\alpha} c_{\alpha s}\omega^{m-s} + \sum_{\substack{s=m+1 \\ s-m \text{ odd}}}^{\alpha} d_{\alpha s}\omega^{m-s}\ln\omega + O(\omega^{m-\alpha-1}\ln\omega).$$

The relevant integrals this time are ($p > 0$)

$$\int_0^\infty e^{-\omega^2 t}\omega^{2p-1} d\omega = \frac{1}{2}\Gamma(p)t^{-p}, \tag{87}$$

$$\int_0^\infty e^{-\omega^2 t}\omega^{2p-1}\ln\omega \, d\omega = \frac{1}{4}\Gamma(p)[\psi(p) - \ln t]t^{-p}, \tag{88}$$

where

$$\psi(p) \equiv \partial_p \ln\Gamma(p) \tag{89}$$

satisfies

$$\psi(p+1) = \psi(p) + \frac{1}{p}, \tag{90}$$

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k} \qquad (n = 1, 2, \ldots), \tag{91}$$

$$\psi\left(\frac{1}{2} + n\right) = -\gamma - 2\ln 2 + 2\sum_{k=1}^{n}\frac{1}{2k-1} \qquad (n = 0, 1, \ldots), \tag{92}$$

$$\gamma \equiv +0.57721566490 \qquad \text{(Euler's constant)}. \tag{93}$$

From (86) we see that the contribution of small ω is $O(t^{[(\alpha+2)/2]})$, and the integral of the error term out to $\omega = \infty$ is $O(t^{-\frac{m}{2}+\frac{\alpha}{2}+\frac{1}{2}}\ln t)$; the latter is dominant, as in (78). Therefore, the remaining, explicit terms should reproduce exactly the summation in (78). This requires cancellation of all $\ln t$ terms; comparison of (87) with (88) shows that this will entail cancellation of all $c_{\alpha s}$ terms with $s - m$ odd and positive, just as in the calculation at the end of Sec. IV (and also, coincidentally, of all terms containing $\gamma + 2\ln 2$). We illustrate by working out the case $m = 1$, $\alpha = 3$, using the last of Eqs. (85):

$$\begin{aligned}
K(t) &\sim \frac{1}{6}\int_0^\infty e^{-\omega^2 t}(12t^2 - 48\omega^2 t^3 + 16\omega^4 t^4) \\
&\quad \times [c_{30}\omega^4 + c_{31}\omega^3 + c_{32}\omega^2 + c_{33}\omega + d_{32}\omega^2\ln\omega] d\omega \\
&= c_{30}t^{-1/2}\left[\Gamma\left(\tfrac{5}{2}\right) - 4\Gamma\left(\tfrac{7}{2}\right) + \tfrac{4}{3}\Gamma\left(\tfrac{9}{2}\right)\right] + c_{31}\left[\Gamma(2) - 4\Gamma(3) + \tfrac{4}{3}\Gamma(4)\right] \\
&\quad + c_{32}t^{1/2}\left[\Gamma\left(\tfrac{3}{2}\right) - 4\Gamma\left(\tfrac{5}{2}\right) + \tfrac{4}{3}\Gamma\left(\tfrac{7}{2}\right)\right] + c_{33}t\left[\Gamma(1) - 4\Gamma(2) + \tfrac{4}{3}\Gamma(3)\right] \\
&\quad - \frac{1}{2}d_{32}(\gamma + 2\ln 2 + \ln t)t^{1/2}\left[\Gamma\left(\tfrac{3}{2}\right) - 4\Gamma\left(\tfrac{5}{2}\right) + \tfrac{4}{3}\Gamma\left(\tfrac{7}{2}\right)\right] \\
&\quad + d_{32}t^{1/2}\left[\Gamma\left(\tfrac{3}{2}\right) - \tfrac{16}{3}\Gamma\left(\tfrac{5}{2}\right) + \tfrac{92}{45}\Gamma\left(\tfrac{7}{2}\right)\right] \\
&= 2\Gamma\left(\tfrac{1}{2}\right)c_{30}t^{-1/2} + c_{31} + \tfrac{1}{3}\Gamma\left(\tfrac{1}{2}\right)d_{32}t^{1/2} - \tfrac{1}{3}c_{33}t.
\end{aligned}$$

Moreover, the numerical coefficients agree with those computed from (78)–(79) and (64)–(65).

3. Next consider the quantity

$$T(t) \equiv \int_0^\infty e^{-\omega t} d\mu \qquad (t > 0), \tag{94}$$

where $\mu = \mu(\lambda) = \mu(\omega^2) = \tilde{\mu}(\omega)$. $K(t)$ bears the same relation to the heat kernel of the operator H which $T(t)$ bears to the kernel of the operator $\exp(-\sqrt{H}\, t)$, the cylinder kernel. That operator solves the elliptic partial differential equation

$$\partial_t^2 \psi(t,x) - H\psi(t,x) = 0 \qquad (t > 0) \tag{95}$$

in a cylindrical manifold of dimension $m+1$, with inhomogeneous Dirichlet data on the m-dimensional boundary surface $t = 0$ and a decay condition as $t \to \infty$.

The calculation of the small-t expansion of $T(t)$ from the ω-mean expansion (57) starts off in precise analogy to the previous calculation of the expansion of $K(t)$ from the λ-mean expansion (50). One obtains from (71), with ω in the role of λ,

$$\begin{aligned} T(t) &= (-1)^{\alpha+1} \frac{1}{\alpha!} \int_0^\infty \tau^\alpha (-t)^{\alpha+1} e^{-\tau t} R_\tau^\alpha \tilde{\mu}\, d\tau \\ &= t^{\alpha+1} \frac{1}{\alpha!} \int_0^\infty \left[\sum_{s=0}^{\alpha} c_{\alpha s} \omega^{m-s+\alpha} + \sum_{\substack{s=m+1 \\ s-m \text{ odd}}}^{\alpha} d_{\alpha s} \omega^{m-s+\alpha} \ln\omega + O(\omega^{m-1}\ln\omega) \right] e^{-\omega t} d\omega. \end{aligned}$$

In addition to (77) in the form

$$\int_0^\infty \omega^{p-1} e^{-\omega t} d\omega = \Gamma(p) t^{-p}$$

we need a transformation of (88),

$$\int_0^\infty e^{-\omega t} \omega^{p-1} \ln\omega \, d\omega = \Gamma(p)[\psi(p) - \ln t] t^{-p} \qquad (p > 0). \tag{96}$$

The integral of the error term is of order $O(t^{-m+\alpha+1}\ln t)$, and the contribution from small ω is $O(t^{\alpha+1})$, hence of higher order than the error term. Evaluating the integrals of the summations, we get

$$\begin{aligned} T(t) = &\sum_{s=0}^{\alpha} \frac{\Gamma(m-s+\alpha+1)}{\Gamma(\alpha+1)} c_{\alpha s} t^{-m+s} \\ &+ \sum_{\substack{s=m+1 \\ s-m \text{ odd}}}^{\alpha} \frac{\Gamma(m-s+\alpha+1)}{\Gamma(\alpha+1)} [\psi(m-s+\alpha+1) - \ln t]\, d_{\alpha s} t^{-m+s} \\ &+ O(t^{-m+\alpha+1}\ln t). \end{aligned} \tag{97}$$

Thus, if we define notation by

$$T(t) \sim \sum_{s=0}^{\infty} e_s t^{-m+s} + \sum_{\substack{s=m+1 \\ s-m \text{ odd}}}^{\infty} f_s t^{-m+s} \ln t, \tag{98}$$

we have

$$e_s = \frac{\Gamma(m-s+\alpha+1)}{\Gamma(\alpha+1)}\, c_{\alpha s} \tag{99}$$

if $s-m$ is even or negative, and

$$e_s = \frac{\Gamma(m-s+\alpha+1)}{\Gamma(\alpha+1)}[c_{\alpha s} + \psi(m-s+\alpha+1)d_{\alpha s}], \tag{100}$$

$$f_s = -\frac{\Gamma(m-s+\alpha+1)}{\Gamma(\alpha+1)}\, d_{\alpha s} \tag{101}$$

for $s-m$ odd and positive. These equations are rigorously valid for $\alpha \ge s$, and hold formally for smaller α — that is, they can be used to *define* $c_{\alpha s}$ and $d_{\alpha s}$ for $\alpha < s$, with one exception: When

$$m - s + \alpha < 0, \tag{102}$$

the Γ function in the numerator has a pole. In the context of (99) or (101), this is understood to force $c_{\alpha s} = 0$ or $d_{\alpha s} = 0$. However, one may not conclude that the $c_{\alpha s}$ in (100) is zero in this situation, since ψ also has a pole; indeed, we know that $c_{\alpha s}$ is generally nonzero in the case $\alpha = 0$, m odd, where the relation of $R^\alpha_\omega \mu$ to $R^\alpha_\lambda \mu$ and the heat kernel $K(t)$ is trivial.

The right-hand sides of (99)–(101) must be independent of α. To verify this, rewrite (57) as

$$\partial_\omega^{-\alpha}\mu \sim \frac{1}{\alpha!}\sum_{s=0}^{\alpha} c_{\alpha s}\omega^{m-s+\alpha} + \frac{1}{\alpha!}\sum_{\substack{s=m+1\\ s-m \text{ odd}}}^{\alpha} d_{\alpha s}\omega^{m-s+\alpha}\ln\omega \tag{103}$$

and differentiate:

$$\begin{aligned}\partial_\omega^{-\alpha+1}\mu \sim{}& \frac{1}{\alpha!}\sum_s (m-s+\alpha)c_{\alpha s}\omega^{m-s+\alpha-1}\\ &+\frac{1}{\alpha!}\sum_{\substack{s\ge m+1\\ s-m \text{ odd}}} d_{\alpha s}\omega^{m-s+\alpha-1} + \frac{1}{\alpha!}\sum_{\substack{s\ge m+1\\ s-m \text{ odd}}}(m-s+\alpha)d_{\alpha s}\omega^{m-s+\alpha-1}\ln\omega.\end{aligned}$$

This yields the recursion relations

$$\frac{1}{\alpha!}(m-s+\alpha)c_{\alpha s} = \frac{1}{(\alpha-1)!}\, c_{\alpha-1,s} \tag{104}$$

for $s-m$ even or negative;

$$\frac{1}{\alpha!}(m-s+\alpha)c_{\alpha s} + \frac{1}{\alpha!}\, d_{\alpha s} = \frac{1}{(\alpha-1)!}\, c_{\alpha-1,s}\,, \tag{105}$$

$$\frac{1}{\alpha!}(m-s+\alpha)d_{\alpha s} = \frac{1}{(\alpha-1)!}\, d_{\alpha-1,s} \tag{106}$$

for $s-m$ odd and positive. Multiplying by $\Gamma(m-s+\alpha)$, we immediately notice consistency of (104) and (106) with (99) and (101) [cf. (81)], while (105) becomes

$$\frac{\Gamma(m-s+\alpha+1)}{\Gamma(\alpha+1)} c_{\alpha s} + \frac{\Gamma(m-s+\alpha)}{\Gamma(\alpha+1)} d_{\alpha s} = \frac{\Gamma(m-s+\alpha)}{\Gamma(\alpha)} c_{\alpha-1,s} \, .$$

This, with (90) and (106), implies

$$\frac{\Gamma(m-s+\alpha+1)}{\Gamma(\alpha+1)}[c_{\alpha s} + \psi(m-s+\alpha+1)d_{\alpha s}] = \frac{\Gamma(m-s+\alpha)}{\Gamma(\alpha)}[c_{\alpha-1,s} + \psi(m-s+\alpha)d_{\alpha-1,s}],$$

establishing consistency of (100).

The form of the recursion relations analogous to (82) is (for $\alpha \geq 1$)

$$c_{\alpha-1,s} = \frac{1}{\alpha}(m-s+\alpha)c_{\alpha s} \tag{107}$$

if $s-m$ is even or negative;

$$c_{\alpha-1,s} = \frac{1}{\alpha}(m-s+\alpha)c_{\alpha s} + \frac{1}{\alpha}d_{\alpha s} \, , \tag{108}$$

$$d_{\alpha-1,s} = \frac{1}{\alpha}(m-s+\alpha)d_{\alpha s} \tag{109}$$

if $s-m$ is odd and positive. Note the consistency with the remarks made in connection with (102): If $s-m$ is odd and positive, then $c_{\alpha s}$ for a value of α satisfying (102) will generally be nonzero and proportional to $d_{\alpha s}$ for values of α violating (102). Otherwise, $c_{\alpha s}$ and $d_{\alpha s}$ will vanish when (102) is satisfied, because the factor $(m-s+\alpha)$ in (107) or (109) will have vanished for some larger value of α.

Let us express all the coefficients in terms of c_{ss} and d_{ss}: In analogy to (83), we have

$$e_s = \frac{\Gamma(m+1)}{\Gamma(s+1)} c_{ss} \tag{110}$$

if $s-m$ is even or negative;

$$e_s = \frac{\Gamma(m+1)}{\Gamma(s+1)}[c_{ss} + \psi(m+1)d_{ss}], \tag{111}$$

$$f_s = -\frac{\Gamma(m+1)}{\Gamma(s+1)} d_{ss} \tag{112}$$

if $s-m$ is odd and positive. The analogues of (84) are

$$c_{\alpha s} = \frac{\Gamma(\alpha+1)\Gamma(m+1)}{\Gamma(m-s+\alpha+1)\Gamma(s+1)} c_{ss} \tag{113}$$

if $s-m$ is even or negative; and in the contrary case,

$$c_{\alpha s} = \frac{\Gamma(\alpha+1)\Gamma(m+1)}{\Gamma(m-s+\alpha+1)\Gamma(s+1)}\{c_{ss} + [\psi(m+1) - \psi(m-s+\alpha+1)]d_{ss}\}, \tag{114}$$

$$d_{\alpha s} = \frac{\Gamma(\alpha+1)\Gamma(m+1)}{\Gamma(m-s+\alpha+1)\Gamma(s+1)}\, d_{ss}\,. \tag{115}$$

If $m - s + \alpha \geq 0$, then (114) is unambiguous, and the difference of ψ functions is calculable from (90) or (91). But when (102) holds, extra work is needed to obtain a usable formula. In that case iteration of (108) leads to

$$c_{\alpha s} = (-1)^{s-m-\alpha-1}\frac{\Gamma(s-m-\alpha)\Gamma(\alpha+1)\Gamma(m+1)}{\Gamma(s+1)}\, d_{ss} \tag{116}$$

for $\alpha < s - m$, $s - m$ odd. This formula also can be obtained from (114) by comparing residues of Γ and ψ:

$$\frac{\psi(\varepsilon - n)}{\Gamma(\varepsilon - n)} = \frac{\Gamma'(\varepsilon - n)}{\Gamma(\varepsilon - n)^2} = (-1)^{n-1} n! + O(\varepsilon) \qquad (n = 0, 1, \ldots) \tag{117}$$

(see (89)).

4. Finally, we investigate the relation between $T(t)$ and the λ-means. For use in (71) we find

$$\begin{aligned}
\partial_\lambda(e^{-t\lambda^{1/2}}) &= e^{-t\lambda^{1/2}}\left(-\tfrac{1}{2}t\lambda^{-1/2}\right), \\
\partial_\lambda^2(e^{-t\lambda^{1/2}}) &= e^{-t\lambda^{1/2}}\left(\tfrac{1}{4}t^2\lambda^{-1} + \tfrac{1}{4}t\lambda^{-3/2}\right), \\
\partial_\lambda^3(e^{-t\lambda^{1/2}}) &= e^{-t\lambda^{1/2}}\left(-\tfrac{1}{8}t^3\lambda^{3/2} - \tfrac{3}{8}t^2\lambda^{-2} - \tfrac{3}{8}t\lambda^{-5/2}\right), \\
\partial_\lambda^4(e^{-t\lambda^{1/2}}) &= e^{-t\lambda^{1/2}}\left(\tfrac{1}{16}t^4\lambda^{-2} + \tfrac{3}{8}t^3\lambda^{-5/2} + \tfrac{15}{16}t^2\lambda^{-3} + \tfrac{15}{16}t\lambda^{-7/2}\right),
\end{aligned} \tag{118}$$

and in general the form

$$\partial_\lambda^j(e^{-t\lambda^{1/2}}) = e^{-t\lambda^{1/2}}\sum_{i=1}^{j} y_i t^i \lambda^{-j+\frac{i}{2}}. \tag{119}$$

Note that (74) is satisfied. As in the other cases, there is only one relevant term in (71):

$$\begin{aligned}
T(t) &= (-1)^{\alpha-1}\frac{1}{\alpha!}\int_0^\infty \partial_\lambda^{\alpha+1}(e^{-t\lambda^{1/2}})\lambda^\alpha R_\lambda^\alpha \mu \, d\lambda \\
&= (-1)^{\alpha-1}\frac{1}{\alpha!}\int_0^\infty \partial_\lambda^{\alpha+1}(e^{-t\lambda^{1/2}})\left[\sum_{s=0}^{\alpha} a_{\alpha s}\lambda^{\frac{m}{2}-\frac{s}{2}+\alpha} + O(\lambda^{\frac{m}{2}+\frac{\alpha}{2}-\frac{1}{2}})\right] d\lambda.
\end{aligned} \tag{120}$$

However, unlike the other three cases, this time the unknown contribution from small λ is $O(t)$ — its order does not increase with α. Furthermore, the integrals of the individual terms in (120) do not converge at the lower limit, although the integral as a whole is convergent. Consequently, we must deal with integrals of the form

$$\int_{\lambda_0}^\infty e^{-t\lambda^{1/2}}\lambda^{-p} d\lambda = 2\int_{\omega_0}^\infty e^{-t\omega}\omega^{-2p+1} d\omega. \tag{121}$$

If $p<1$, we may set $\omega_0=0$ and use (77); if $p\geq 1$, we may use

$$\int_{\omega_0}^{\infty} e^{-t\omega}\omega^{-n}d\omega = \omega_0^{1-n}\int_1^{\infty} e^{-\omega_0 tu}u^{-n}\,du, \tag{122}$$

$$\int_1^{\infty} e^{-t\omega}\omega^{-n}\,d\omega \equiv E_n(t), \tag{123}$$

$$E_1(t) = -\gamma - \ln t - \sum_{j=1}^{\infty}\frac{(-1)^j}{jj!}\,t^j, \tag{124a}$$

$$E_n(t) = \frac{(-t)^{n-1}}{(n-1)!}[-\ln t+\psi(n)] - \sum_{\substack{j=0\\ j\neq n-1}}^{\infty}\frac{(-1)^j}{(j-n+1)j!}\,t^j. \tag{124b}$$

It can be seen that the integrals give rise to terms of the form (98), but without the restriction that $s-m$ be odd in the logarithmic terms. Moreover, the contribution from $\lambda<1$ is analytic, hence can be expanded as a series of positive integral powers of t with unknown coefficients. From (58) and (99)–(100), we would expect the coefficient of t^{-m+s} to be determinable if $s-m$ is even but not if $s-m$ is odd. Therefore, we anticipate conspiracies among the coefficients in (119) which will eliminate both the logarithms and the undeterminable coefficients when $s-m$ is even. Let us verify this for the case $m=1$, $\alpha=3$:

$$\begin{aligned}
T(t) &= \frac{1}{6}\int_0^{\infty} e^{-t\lambda^{1/2}}\left(\frac{1}{16}t^4\lambda^{-2}+\frac{3}{8}t^3\lambda^{-5/2}+\frac{15}{16}t^2\lambda^{-3}+\frac{15}{16}t\lambda^{-7/2}\right)\\
&\quad\times[a_{30}\lambda^{7/2}+a_{31}\lambda^3+a_{32}\lambda^{5/2}+a_{33}\lambda^2+O(\lambda^{3/2})]\,d\lambda\\
&= \frac{1}{48}\int_0^{\infty} e^{-t\omega}(a_{30}t^4\omega^4+6a_{30}t^3\omega^3+a_{31}t^4\omega^3+15a_{30}t^2\omega^2+6a_{31}t^3\omega^2+a_{32}t^4\omega^2\\
&\quad+15a_{30}t\omega+15a_{31}t^2\omega+6a_{32}t^3\omega+a_{33}t^4\omega+15a_{31}t+15a_{32}t^2+6a_{33}t^3\\
&\quad+15a_{32}t\omega^{-1}+15a_{33}t^2\omega^{-1}+15a_{33}t\omega^{-2}+O(t\omega^{-3}))\,d\omega\\
&= \frac{1}{48}\Bigg\{105a_{30}t^{-1}+48a_{31}+23a_{32}t+7a_{33}t^2\\
&\quad+\int_0^{\omega_0} e^{-t\omega}(15t^2\omega^{-6})[O(\omega^6)-a_{32}\omega^5]\omega\,d\omega+\int_0^{\omega_0} e^{-t\omega}(15t\omega^{-7})O(\omega^6)\omega\,d\omega+O(t^3)\\
&\quad+15a_{32}tE_1(\omega_0 t)+15a_{33}t^2E_1(\omega_0 t)+15a_{33}\omega_0^{-1}tE_2(\omega_0 t)+\int_{\omega_0}^{\infty} e^{-t\omega}O(t\omega^{-s})d\omega\Bigg\}.
\end{aligned}$$

(Here the first four terms come from integrating all terms in the middle member involving ω to a nonnegative power. To represent accurately the integral over small ω of the remaining terms, it has been necessary to go back to the first member and to note that the quantity in square brackets there is $O(\omega^6)$ by virtue of (120) and (73). Of the terms involving $t\lambda^{-7/2}$, the first two have already been accounted for in the explicit integrations, and the remainder of the square bracket is still $O(\omega^6)$ as $\omega\downarrow 0$; of the terms involving $t^2\lambda^{-3}$, the first three have been accounted for, and the subtraction of the third of these needs to be represented explicitly in our formula; the terms associated with the rest of $\partial_\lambda^4(e^{-t\lambda^{1/2}})$ are $O(t^3)$.) Substituting from (122)–(124), and noting that the two quantities represented by "$O(\omega^6)$" in the foregoing expression are the *same*, we obtain

$$
\begin{aligned}
T(t) = \frac{1}{48}\Big\{ & 105a_{30}t^{-1} + 48a_{31} + 23a_{32}t + 7a_{33}t^2 - 15t^2 a_{32}\omega_0 \\
& + \int_0^{\omega_0} 15t\, O(\omega^0)[\omega t + O(t^2) + 1 - \omega t]\, d\omega + O(t^3) + tO(\omega_0^{-2}E_3(\omega_0 t)) \\
& + 15a_{32}t[-\gamma - \ln\omega_0 t + \omega_0 t + O(t^2)] + 15a_{33}t^2[-\gamma - \ln\omega_0 t + O(t)] \\
& + 15a_{33}\omega_0^{-1}t[-\omega_0 t(-\ln\omega_0 t + \psi(2)) + 1 + O(t^2)]\Big\} \\
= \frac{1}{48}\Big\{ & 105a_{30}t^{-1} + 48a_{31} - 15a_{32}t\ln\omega_0 t \\
& + t\Big[23a_{32} + 15\int_0^{\omega_0} O(\omega^0)d\omega - 15\gamma a_{32} + 15a_{33}\omega_0^{-1}\Big] \\
& + t^2[7a_{33} - 15a_{32}\omega_0 + 15a_{32}\omega_0 - 15\gamma a_{33} + 15\gamma a_{33} - 15a_{33}] \\
& + O(t^3) + \omega_0^{-2}t\left[O\left(\tfrac{1}{2} - \omega_0 t\right) + O(t^2\ln t)\right]\Big\} \\
= \frac{1}{48}\{ & 105a_{30}t^{-1} + 48a_{31} - 15a_{32}t\ln\omega_0 t + \text{const.}\times t - 8a_{33}t^2 \\
& + O(t^3\ln t) + O(\omega_0^{-2}t) + O(\omega_0^{-1}t^2)\}.
\end{aligned}
$$

The bothersome term at the end could be removed by taking $\omega_0 \to \infty$. Now this term and its predecessor can be traced to the first two terms in a Taylor expansion of $e^{-t\omega}$ in (123) with $n = 3$. In analogy to the argument leading from (55) to (57), we may transfer this part of the integral from the high-ω account to the low-ω account; this simply amounts to adding something to the "$O(\omega^6)$" term, and the same cancellation of $O(t^2)$ terms just observed for that term will occur for this other contribution as well. Thus we have derived the expected form,

$$T(t) = \tfrac{35}{16}a_{30}t^{-1} + a_{31} - \tfrac{5}{16}a_{32}t\ln t + e_2 t - \tfrac{1}{6}a_{33}t^2 + O(t^3\ln t),$$

where e_2 is undeterminable from the asymptotic expansion of $R^3_\lambda\mu$. Furthermore, the coefficients agree with those calculated from (98)–(101) and (58)–(60).

Summary: The coefficients b_s in the asymptotic expansion of the heat kernel, $K(t)$, are in one-to-one correspondence with the "diagonal" coefficients a_{ss} in the asymptotic expansions of the Riesz means with respect to λ, $R^s_\lambda\mu$. The nondiagonal coefficients, $a_{\alpha s}$, in the expansions of the $R^\alpha_\lambda\mu$ differ only by numerical factors from the a_{ss}; because these factors may vanish, it is not possible to express a_{ss} in terms of $a_{\alpha s}$ if α is too small. Similarly, the coefficients e_s and f_s in the expansion of the cylinder kernel, $T(t)$, are in one-to-one correspondence with the diagonal coefficients c_{ss} and d_{ss} in the expansions of the Riesz means with respect to ω, $R^s_\omega\mu$. The connection in this case involves a two-termed equation, (111). Again the $c_{\alpha s}$ and $d_{\alpha s}$ can be expressed in terms of the c_{ss} and d_{ss}, but not conversely if α is too small. Finally, and perhaps most significantly, the c_{ss} and d_{ss} (or the e_s and f_s) contain information which is not contained in the a_{ss} (or the b_s) (but not conversely). The c_{ss} for $s - m$ odd and positive are "new spectral invariants" independent of the a_{ss}. In the case of the heat and cylinder kernels, the c_{ss} contain nonlocal geometrical information, while the a_{ss} are strictly local.

VII. CONCLUSION

Motivated by applications in spectral asymptotics and quantum field theory, we have investigated Riesz means in this setting: Two functions, $K(t)$ and $T(t)$, are related to a function $\mu(\lambda)$ by the Stieltjes integrals

$$K(t) = \int_0^\infty e^{-\lambda t}\, d\mu, \qquad T(t) = \int_0^\infty e^{-\omega t}\, d\mu, \tag{125}$$

where $\omega = \sqrt{\lambda}$, and they have the asymptotic expansions

$$K(t) \sim \sum_{s=0}^\infty b_s t^{-\frac{m}{2}+\frac{s}{2}}, \qquad T(t) \sim \sum_{s=0}^\infty e_s t^{-m+s} + \sum_{\substack{s=m+1\\ s-m \text{ odd}}}^\infty f_s t^{-m+s} \ln t, \tag{126}$$

as $t \downarrow 0$, for some $m \in \mathbb{Z}^+$. The behavior of μ as $\lambda \to +\infty$ is characterized by the constants a_{ss} , c_{ss} , d_{ss} , where

$$R^\alpha_\lambda \mu(\lambda) \sim \sum_{s=0}^\alpha a_{\alpha s} \lambda^{\frac{m}{2}-\frac{s}{2}}, \qquad R^\alpha_\omega \mu \sim \sum_{s=0}^\alpha c_{\alpha s} \omega^{m-s} + \sum_{\substack{s=m+1\\ s-m \text{ odd}}}^\alpha d_{\alpha s} \omega^{m-s} \ln \omega, \tag{127}$$

$R^\alpha_\lambda \mu$ and $R^\alpha_\omega \mu$ being the Riesz means with respect to λ and ω, defined by, for example,

$$R^\alpha_\omega \mu(\omega^2) = \int_{\tau=0}^\omega \left(1 - \frac{\tau}{\omega}\right)^\alpha d\mu(\tau^2) \tag{128a}$$

$$= \alpha!\, \omega^{-\alpha} \int_0^\omega d\tau_1 \cdots \int_0^{\tau_{\alpha-1}} d\tau_\alpha\, \mu({\tau_\alpha}^2). \tag{128b}$$

We have shown that a_{ss} and b_s contain the same information, being related by (83), and that (c_{ss}, d_{ss}) and (e_s, f_s) contain the same information, being related by (110)–(112). The inverses of the cited formulas are

$$a_{ss} = \frac{\Gamma(s+1)}{\Gamma(\frac{m}{2}+\frac{s}{2}+1)}\, b_s\,; \tag{129}$$

$$d_{ss} = -\frac{\Gamma(s+1)}{\Gamma(m+1)}\, f_s \quad \text{if } s-m \text{ is odd and positive;} \tag{130}$$

$$c_{ss} = \frac{\Gamma(s+1)}{\Gamma(m+1)} [e_s + \psi(m+1) f_s] \quad \text{if } s-m \text{ is odd and positive,} \tag{131}$$

$$c_{ss} = \frac{\Gamma(s+1)}{\Gamma(m+1)}\, e_s \quad \text{if } s-m \text{ is even or negative.} \tag{132}$$

Finally, these two collections of quantities are related to each other, but in an asymmetrical way. Specializing (58)–(60) and (64)–(65), we have

$$c_{ss} \text{ is undetermined by } a_{ss} \text{ if } s-m \text{ is odd and positive;} \tag{133}$$

$$d_{ss} = (-1)^{m+1} \frac{1}{(s-m-1)!\,m!} \frac{\Gamma(\frac{s}{2}-\frac{m}{2})}{\Gamma(-\frac{m}{2}-\frac{s}{2})} a_{ss} \quad \text{if } s-m \text{ is odd and positive;} \tag{134}$$

$$c_{ss} = \left[2^{-s} + \sum_{\substack{j=0 \\ j \text{ even}}}^{s-1} (-1)^{s-j} \frac{(m-s+j+1)^{-1}}{j!\,(s-1-j)!} \frac{\Gamma(\frac{j}{2}+\frac{1}{2})}{\Gamma(\frac{j}{2}+\frac{1}{2}-s)} \right] a_{ss}$$
$$\text{if } s-m \text{ is even or negative;} \tag{135}$$

$$a_{ss} = \left[-\frac{1}{2} \sum_{j=[\frac{s}{2}]}^{s-1} (-1)^{s-j} \frac{(\frac{m}{2}-\frac{s}{2}+j+1)^{-2}}{j!\,(s-1-j)!} \frac{\Gamma(2j+2)}{\Gamma(2j+2-s)} \right] d_{ss}$$
$$\text{if } s-m \text{ is odd and positive;} \tag{136}$$

$$a_{ss} = \left[2^{s} + \sum_{j=[\frac{s}{2}]}^{s-1} (-1)^{s-j} \frac{(\frac{m}{2}-\frac{s}{2}+j+1)^{-1}}{j!\,(s-1-j)!} \frac{\Gamma(2j+2)}{\Gamma(2j+2-s)} \right] c_{ss}$$
$$\text{if } s-m \text{ is even or negative.} \tag{137}$$

These relations obviously induce similar relations between b_s and (e_s, f_s). The quantities c_{ss} ($s-m$ odd and positive) summarize the information contained in the asymptotics of T but not in that of K.

In the context of spectral theory of differential operators, the quantities b_s , e_s , f_s are more accessible to calculation, because $K(t)$ and $T(t)$ are Green functions or their traces, but a_{ss} , c_{ss} , d_{ss} are more fundamental, since $\mu(\lambda)$ represents a spectral decomposition (the integral kernel of the spectral projections in the local case, the density of eigenvalues in the traced case). We expect that further work will bring the technically more difficult case of the Wightman function (and other Green functions of wave equations) into the same picture.

This picture clarifies and unifies results that have appeared in the literature piecemeal or imprecisely. One sees that the small-t expansion of K is not related in a one-to-one way with the formal large-λ expansion of μ (i.e., the a_{0s}); rather, the heat coefficients b_s can involve constants of integration from the Riesz averaging of μ over small λ. This explains how the formal expansion of μ (and the associated zeta-function poles) can be strikingly different in even and odd dimensions, while the heat-kernel expansion is notoriously dimension-independent. (See Remark below (84).) Further constants of integration appear in the passage from λ-means to ω-means; these carry nonlocal geometrical information in the spectral context. They are intimately related to the potential appearance of logarithmic terms (f_s, d_{ss}). On the other hand, the numerical relationships among the various series are independent of the spectral application; in particular, they will be universal for all (positive, elliptic) operators of a given order and dimension. (Only order 2 has been treated here.)

To conclude, we apply some of these formulas to the simple examples studied in Sec. II and compare the results to the known spectral densities for those cases. In these examples, $m=1$.

Case $\mathcal{M} = \mathbb{R}$

Formula (10) for the heat kernel states that

$$b_s = (4\pi)^{-1/2}\delta_{s0} \quad \text{if } y = x, \tag{138a}$$

$$b_s = 0 \quad \text{for all } s \quad \text{if } y \neq x. \tag{138b}$$

From (129), then, one has for the Riesz means of the spectral function

$$a_{ss} = \frac{1}{\pi}\,\delta_{s0} \quad \text{if } y = x, \tag{139a}$$

$$a_{ss} = 0 \quad \text{for all } s \quad \text{if } y \neq x. \tag{139b}$$

The eigenfunction expansion in this case is the Fourier transform, so the exact spectral density is

$$dE_\lambda(x,y) = \frac{1}{2\pi}\sum_{\operatorname{sgn} k} e^{ik(x-y)}\,d|k| = \frac{1}{\pi}\,\cos[\omega(x-y)]\,d\omega \qquad (\sqrt{\lambda} = \omega = |k|). \tag{140}$$

Thus

$$E_\lambda(x,x) = \frac{\omega}{\pi} = \frac{\sqrt{\lambda}}{\pi}, \tag{141a}$$

$$E_\lambda(x,y) = \frac{1}{\pi}\,\frac{\sin[\omega(x-y)]}{x-y} \quad \text{if } y \neq x. \tag{141b}$$

(Constants of integration are fixed by (28).) Clearly (141a) is consistent with (139a). Equation (139b) indicates that (141b) is "distributionally small", or decays rapidly at infinity in the Cesàro sense.[13,14,1] To see this by elementary, classical methods, one would use (36),

$$R^s_\lambda E_\lambda(x,y) = \frac{1}{\pi}\int_0^\lambda \left(1-\frac{\sigma}{\lambda}\right)^s d\left\{\frac{\sin[\sqrt{\sigma}(x-y)]}{x-y}\right\}, \tag{142}$$

and verify that this object is $o(\lambda^{\frac{1-s}{2}})$ as $\lambda \to \infty$. This can be done by repeated integration by parts (most easily after changing variable from σ to $\tau = \sqrt{\sigma}$).

For the less familiar cylinder kernel, (12) gives

$$f_s = 0, \tag{143}$$

$$e_s = \frac{1}{\pi}\,\delta_{s0} \quad \text{if } y = x, \tag{144a}$$

and if $y \neq x$,

$$e_s = \begin{cases} 0 & \text{if } s = 0 \text{ or } s \text{ is odd,} \\ (-1)^{\frac{s}{2}+1} \dfrac{1}{\pi} \dfrac{1}{(x-y)^s} & \text{if } s \text{ is even and positive.} \end{cases} \tag{144b}$$

From (130)–(132), this corresponds to the ω-means

$$d_{ss} = 0, \tag{145}$$

$$c_{ss} = \frac{1}{\pi}\, \delta_{s0} \quad \text{if } y = x, \tag{146a}$$

and if $y \neq x$,

$$c_{ss} = \begin{cases} 0 & \text{if } s = 0 \text{ or } s \text{ is odd,} \\ (-1)^{\frac{s}{2}+1} \dfrac{s!}{\pi} \dfrac{1}{(x-y)^s} & \text{if } s \text{ is even and positive.} \end{cases} \tag{146b}$$

One observes that (139) and (145)–(146) are related precisely as prescribed in (133)–(137). Again, (145)–(146) could be established directly from (128a) and (141) by integration by parts. (The earnest student who actually attempts these tedious exercises will find that certain endpoint terms that vanished in the previous case will produce the nonvanishing c_{ss} in this case.) The present case, however, is more easily treated by (128b).

Case $\mathcal{M} = \mathbb{R}^+$

Let us consider only the Dirichlet boundary condition. The heat-kernel asymptotics, and hence the λ-means a_{ss}, are again trivial. The effect of the boundary is seen (as dictated by (133)) only in the even-order coefficients of the cylinder kernel and the associated ω-means: from (15), for $s > 0$,

$$e_s = \frac{(-1)^{s/2}}{\pi(2x)^s} = \frac{1}{s!}\, c_{ss} \quad \text{if } y = x \text{ and } s \text{ is even,} \tag{147a}$$

$$e_s = -\frac{(-1)^{s/2}}{\pi} \left[\frac{1}{(x-y)^s} - \frac{1}{(x+y)^s} \right] = \frac{1}{s!}\, c_{ss} \quad \text{if } y \neq x \text{ and } s \text{ is even,} \tag{147b}$$

$$e_s = 0 = c_{ss} \quad \text{if } s \text{ is odd.} \tag{148}$$

The eigenfunction expansion in this case is the Fourier sine transform, so

$$dE_\lambda(x,y) = \frac{2}{\pi} \sin(\omega x) \sin(\omega y)\, d\omega. \tag{149}$$

Let us check consistency only for the diagonal values. We have

$$E_\lambda(x,x) = \frac{\omega}{\pi} - \frac{\sin(2\omega x)}{2\pi x}\,. \tag{150}$$

The first term is the same as (141a) and need not be discussed further. The λ-means of the second term are shown to vanish exactly as for (142). The ω-means of the second term can be calculated by (128b) exactly as for (146b), and they reproduce (147a).

Case $\mathcal{M} = S^1$

The eigenfunction expansion is the full Fourier series, so

$$\begin{aligned} dE_\lambda(x,y) &= \frac{1}{2L} \sum_{\operatorname{sgn} k} e^{ik(x-y)} \sum_{n=0}^{\infty} \delta\left(|k| - \frac{n\pi}{L}\right) d|k| \\ &= \left[\frac{1}{2L}\,\delta(\omega) + \frac{1}{L}\cos\left[\omega(x-y)\right] \sum_{n=1}^{\infty} \delta\left(\omega - \frac{n\pi}{L}\right)\right] d\omega. \end{aligned} \tag{151}$$

The traced expansions of the Green functions correspond to the eigenvalue distribution function,

$$dN(\lambda) = \int_{-L}^{L} dE_\lambda(x,x)\,dx = \left[\delta(\omega) + 2\sum_{n=1}^{\infty} \delta\left(\omega - \frac{n\pi}{L}\right)\right] d\omega. \tag{152}$$

The calculation of Riesz means leads to sums that can be regarded as trapezoidal-rule approximations to integrals of the types already considered. Therefore, the new features of this case can be sought in the Euler–Maclaurin formula (Ref. 13, Sec. 1.8) for the difference between the sum and the integral. The terms in that formula involve only the odd-order derivatives of the integrand at the endpoints of the interval. In an integral such as (128a) or (142), the derivatives of the factor such as $(1-\tau/\omega)^s$ to all relevant orders will vanish at the upper limit, so only the lower endpoint can contribute. In the λ-means (142), where this Riesz factor is $(1-\tau^2/\lambda)^s$, the odd-order derivatives all vanish at 0 by virtue of factors of τ or $\sin[\tau(x-y)]$. This is necessary for consistency with the trivial heat-kernel expansion, (10) and (20). For the ω-means, we need to calculate

$$\frac{1}{L}\left(\frac{\pi}{L}\right)^p \frac{d^p}{d\tau^p}\left[\left(1-\frac{\tau}{\omega}\right)^s \cos\left[\tau(x-y)\right]\right]. \tag{153}$$

When $y = x$, this derivative is of order ω^{-p}; in fact, it equals

$$\frac{(-1)^p \pi^p s!}{L^{p+1}\omega^p (s-p)!}. \tag{154}$$

(In the off-diagonal case, the Euler–Maclaurin series does not yield an expansion of the desired type — at least, not without a resummation, which we shall not attempt here.) To read off c_{ss} (for $s \geq 2$) we need to look at $p = s-1$ (see (127)) and, according to Euler–Maclaurin, multiply by $(-1)^p B_s/s!$, where B_s is a Bernoulli number. Thus we arrive at

$$c_{ss} = \frac{\pi^{s-1} B_s}{L^s}, \qquad e_s = \frac{\pi^{s-1} B_s}{L^s s!}. \tag{155}$$

This is, in fact, the same as (17)–(18) with $y = x$, because

$$\frac{1}{2}\,\frac{\sinh z}{\cosh z - 1} = \coth\frac{z}{2} = \frac{1}{z}\left[1 + \sum_{s=2}^{\infty} \frac{B_s}{s!} z^s\right], \tag{156}$$

$$B_s = 0 \quad \text{if } s \text{ is odd and } s > 2. \tag{157}$$

Case $\mathcal{M} = I$

For Dirichlet boundary conditions the eigenfunction expansion is the Fourier sine series, so

$$dE_\lambda(x,y) = \frac{2}{L}\sin(\omega x)\sin(\omega y)\sum_{n=1}^{\infty}\delta\left(\omega - \frac{n\pi}{L}\right)d\omega \tag{158}$$

and

$$dN(\lambda) = \sum_{n=1}^{\infty}\delta\left(\omega - \frac{n\pi}{L}\right)d\omega. \tag{159}$$

The novel feature of this case is that the trace (26) of the heat kernel contains a nontrivial term, of order t^0, and a similar term appears in the trace (27) of the cylinder kernel. The corresponding Riesz means are

$$a_{11} = b_1 = -\tfrac{1}{2}\,, \qquad c_{11} = e_1 = -\tfrac{1}{2}\,. \tag{160}$$

In an Euler–Maclaurin calculation these terms arise from the innocent fact that the initial term of the trapezoidal rule,

$$\tfrac{1}{2}\delta(\omega - 0)\,d\omega, \tag{161}$$

is missing from (159) and needs to be subtracted "by hand".

The reader will have noted that in most of these instances it is easier to obtain the Riesz means from the Green functions via (129)–(132) and Section II than to calculate the Riesz means directly, even though the spectral functions are known exactly. (Since the direct calculations are merely consistency checks for us, we have carried out some of the calculations only far enough to demonstrate the matching of the first few terms, rather than constructing complete proofs. The claims in this section are to be understood in that spirit.)

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APPENDIX: PROOF OF (67)

BY R. A. GUSTAFSON

We shall prove the following identity:

$$1=\left[\sum_{j=[\frac{\alpha}{2}]}^{\alpha-1}(-1)^{\alpha-j}\frac{(\frac{1}{2}z+j+1)^{-1}}{j!\,(\alpha-1-j)!}\frac{\Gamma(2j+2)}{\Gamma(2j+2-\alpha)}+2^{\alpha}\right]$$
$$\times\left[\sum_{\substack{j=0\\ j\text{ even}}}^{\alpha-1}(-1)^{\alpha-j}\frac{(z+j+1)^{-1}}{j!\,(\alpha-1-j)!}\frac{\Gamma(\frac{1}{2}(j+1))}{\Gamma(\frac{1}{2}(j+1)-\alpha)}+2^{-\alpha}\right],\tag{A1}$$

where $\alpha\geq 1$ is an integer and $z\in\mathbb{C}$ is not a pole of either factor. The key fact used is the ${}_3F_2$ transformation formula[16],

$${}_3F_2\left(\begin{matrix}a,b,-n\\ \\ e,f\end{matrix};1\right)=\frac{(e-a)_n(f-a)_n}{(e)_n(f)_n}\,{}_3F_2\left(\begin{matrix}1-s,a,-n\\ \\ 1+a-f-n,1+a-e-n\end{matrix};1\right)\tag{A2}$$

where $s=e+f-a-b+n$.

First we compute the sum in the first factor of (A1). With a little work one finds

$$\sum_{j=[\frac{\alpha}{2}]}^{\alpha-1}(-1)^{\alpha-j}\frac{(\frac{1}{2}z+j+1)^{-1}}{j!\,(\alpha-1-j)!}\frac{\Gamma(2j+2)}{\Gamma(2j+2-\alpha)}$$
$$=\frac{(-1)(2\alpha-1)!}{(\frac{z}{2}+\alpha)[(\alpha-1)!]^2}\,{}_3F_2\left(\begin{matrix}-\frac{z}{2}-\alpha,-\frac{\alpha}{2}+\frac{1}{2},-\frac{\alpha}{2}+1\\ \\ -\frac{z}{2}-\alpha+1,-\alpha+\frac{1}{2}\end{matrix};1\right).\tag{A3}$$

Case 1: α is even. In (A2) set $a=-\frac{\alpha}{2}+\frac{1}{2}$, $b=-\frac{z}{2}-\alpha$, and $-n=-\frac{\alpha}{2}+1$; then the expression (A3) becomes

$$\frac{(-1)(2\alpha-1)!}{(\frac{z}{2}+\alpha)[(\alpha-1)!]^2}\frac{(-\frac{z}{2}-\frac{\alpha}{2}+\frac{1}{2})_{\alpha/2-1}(-\frac{\alpha}{2})_{\alpha/2-1}}{(-\frac{z}{2}-\alpha+1)_{\alpha/2-1}(-\alpha+\frac{1}{2})_{\alpha/2-1}}\,{}_3F_2\left(\begin{matrix}1,-\frac{\alpha}{2}+\frac{1}{2},-\frac{\alpha}{2}+1\\ \\ \frac{z}{2}+\frac{3}{2},2\end{matrix};1\right).\tag{A4}$$

Now observe that

$${}_3F_2\left(\begin{matrix}1,-\frac{\alpha}{2}+\frac{1}{2},-\frac{\alpha}{2}+1\\ \\ \frac{z}{2}+\frac{3}{2},2\end{matrix};1\right)=\frac{(z+\frac{1}{2})}{(-\frac{\alpha}{2}-\frac{1}{2})(-\frac{\alpha}{2})}\left[{}_3F_2\left(\begin{matrix}1,-\frac{\alpha}{2}-\frac{1}{2},-\frac{\alpha}{2}\\ \\ \frac{z}{2}+\frac{1}{2},1\end{matrix};1\right)-1\right]$$
$$=\frac{(z+\frac{1}{2})}{(-\frac{\alpha}{2}-\frac{1}{2})(-\frac{\alpha}{2})}\left[{}_2F_1\left(\begin{matrix}-\frac{\alpha}{2}-\frac{1}{2},-\frac{\alpha}{2}\\ \\ \frac{z}{2}+\frac{1}{2}\end{matrix};1\right)-1\right].\tag{A5}$$

Hence the expression in (A3) becomes

$$\frac{(-1)(2\alpha-1)!}{(\frac{z}{2}+\alpha)[(\alpha-1)!]^2}\frac{(z+\frac{1}{2})}{(-\frac{\alpha}{2}-\frac{1}{2})(-\frac{\alpha}{2})}\left[{}_2F_1\left(\begin{matrix}-\frac{\alpha}{2}-\frac{1}{2},-\frac{\alpha}{2}\\ \\ \frac{z}{2}+\frac{1}{2}\end{matrix};1\right)-1\right]$$
$$\times\frac{(-\frac{z}{2}-\frac{\alpha}{2}+\frac{1}{2})_{\alpha/2-1}(-\frac{\alpha}{2})_{\alpha/2-1}}{(-\frac{z}{2}-\alpha+1)_{\alpha/2-1}(-\alpha+\frac{1}{2})_{\alpha/2-1}}. \tag{A6}$$

By the Gauss summation theorem for ${}_2F_1$, this is

$$\frac{(-1)(2\alpha-1)!\,(z+\frac{1}{2})(-\frac{z}{2}-\frac{\alpha}{2}+\frac{1}{2})_{\alpha/2-1}(-\frac{\alpha}{2})_{\alpha/2-1}}{(\frac{z}{2}+\alpha)[(\alpha-1)!]^2(-\frac{\alpha}{2}-\frac{1}{2})(-\frac{\alpha}{2})(-\frac{z}{2}-\alpha+1)_{\alpha/2-1}(-\alpha+\frac{1}{2})_{\alpha/2-1}}$$
$$\times\left[\frac{\Gamma(\frac{z}{2}+\frac{1}{2})\Gamma(\frac{z}{2}+\alpha+1)}{\Gamma(\frac{z}{2}+\frac{\alpha}{2}+1)\Gamma(\frac{z}{2}+\frac{\alpha}{2}+\frac{1}{2})}-1\right]. \tag{A7}$$

After carrying out a long, easy calculation we obtain:

$$\sum_{j=0}^{\alpha-1}(-1)^{\alpha-j}\frac{(\frac{1}{2}z+j-1)^{-1}}{j!\,(\alpha-1-j)!}\frac{\Gamma(2j+2)}{\Gamma(2j+2-\alpha)}=2^{\alpha}\frac{(\frac{z}{2}+\frac{1}{2})_{\alpha/2}}{(\frac{z}{2}+\frac{\alpha}{2}+1)_{\alpha/2}}-2^{\alpha}, \tag{A8}$$

recalling that α is assumed to be even.

Case 2: α is odd. We use (A3), now setting $a=-\frac{\alpha}{2}+1$, $b=-\frac{z}{2}-\alpha$ and $-n=-\frac{\alpha}{2}+\frac{1}{2}$. Doing a calculation very similar to the previous case, we obtain

$$\sum_{j=0}^{\alpha-1}(-1)^{\alpha-j}\frac{(\frac{1}{2}z+j+1)^{-1}\Gamma(2j+2)}{j!\,(\alpha-1-j)!\Gamma(2j+2-\alpha)}=2^{\alpha}\frac{(\frac{z}{2}+\frac{1}{2})_{\alpha/2+\frac{1}{2}}}{(\frac{z}{2}+\frac{\alpha}{2}+\frac{1}{2})_{\alpha/2+\frac{1}{2}}}-2^{\alpha}. \tag{A9}$$

We now turn to the sum in the second factor of (A1). With some work one obtains

$$\sum_{\substack{j=0\\ j\text{ even}}}^{\alpha-1}(-1)^{\alpha-j}\frac{(z+j+1)^{-1}}{j!\,(\alpha-1-j)!}\frac{\Gamma(\frac{1}{2}(j+1))}{\Gamma(\frac{1}{2}(j+1)-\alpha)}$$
$$=\frac{(-1)^{\alpha}(\frac{1}{2}-\alpha)_{\alpha}}{(z+1)(\alpha-1)!}\,{}_3F_2\left(\begin{matrix}\frac{z}{2}+\frac{1}{2},-\frac{\alpha}{2}+\frac{1}{2},-\frac{\alpha}{2}+1\\ \\ \frac{z}{2}+\frac{3}{2},-\alpha+\frac{1}{2}\end{matrix};1\right). \tag{A10}$$

Case 1: α is even. Again we apply (A2), setting $a=-\frac{\alpha}{2}+\frac{1}{2}$, $b=\frac{z}{2}+\frac{1}{2}$ and $-n=-\frac{\alpha}{2}+1$. After a computation quite similar to the above we obtain:

$$\sum_{\substack{j=0\\ j\text{ even}}}^{\alpha-1}(-1)^{\alpha-j}\frac{(z+j+1)^{-1}\Gamma(\frac{1}{2}(j+1))}{j!\,(\alpha-1-j)!\,\Gamma(\frac{1}{2}(j+1)-\alpha)}=2^{-\alpha}\frac{(\frac{z}{2}+\frac{\alpha}{2}+1)_{\alpha/2}}{(\frac{z}{2}+\frac{1}{2})_{\alpha/2}}-2^{-\alpha}. \tag{A11}$$

Case 2: α is odd. Set $a=-\frac{\alpha}{2}+1$, $b=\frac{z}{2}+\frac{1}{2}$ and $-n=-\frac{\alpha}{2}+\frac{1}{2}$ in (A2). After another computation very similar to the previous ones we find

$$\sum_{\substack{j=0\\ j\text{ even}}}^{\alpha-1}(-1)^{\alpha-j}\frac{(z+j+1)^{-1}\Gamma(\frac{1}{2}(j+1))}{j!\,(\alpha-1-j)!\,\Gamma(\frac{1}{2}(j+1)-\alpha)}=2^{-\alpha}\frac{(\frac{z}{2}+\frac{\alpha}{2}+\frac{1}{2})_{\alpha/2+\frac{1}{2}}}{(\frac{z}{2}+\frac{1}{2})_{\alpha/2+\frac{1}{2}}}-2^{-\alpha}. \tag{A12}$$

Substituting (A8)–(A12) back into (A1) yields the desired identity.

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