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Localization of Light in Lossless Inhomogeneous Dielectrics

Alexander Figotin
Department of Mathematics
University of North Carolina at Charlotte
Charlotte, NC 28223
figotin@uncc.edu

Abel Klein
Department of Mathematics
University of California, Irvine
Irvine, CA 92697-3875
aklein@math.uci.edu

Abstract

The phenomenon of localization of electromagnetic waves in losless inhomogeneous dielectric media is studied. We consider a three dimensional lossless periodic medium (photonic crystal) having a gap in the frequency spectrum (photonic band gap). If such a medium is perturbed by either a single defect or a random array of defects, exponentially localized electromagnetic waves arise with frequencies in the gap. For a single defect we derive equations for these midgap frequencies, and estimate their number. For a random medium we show the existence of the phenomenon of Anderson localization for electromagnetic waves.

1. Introduction

Localization of classical electromagnetic waves has received much attention in recent years[2,40,21,22,32,35,36,10,12–14]. This phenomenon arises from coherent multiple scattering and interference and occurs when the scale of the coherent multiple scattering reduces to the wavelength itself. Numerous potential applications[6,22,35,36,42,20,41] and the fundamental significance of localization of classical waves motivate the interest in this phenomenon.

The basic scenario for localization we consider here is as follows. We start with is a perfectly periodic lossless dielectric medium which we will call *photonic crystal*[23]. The propagation of electromagnetic (EM) waves in photonic crystals has been the subject of intensive studies in recent years[6,23,42,34–36,43,42,15,16,41]. The most significant manifestation of coherent multiple scattering in the periodic medium is the rise of a gap in the spectrum called *photonic band gap*[23,42,20]. If a periodic dielectric medium with a band gap perturbed by either a single defect (impurity) or a random array of defects, localized EM waves can arise under some conditions. The frequencies of these localized waves lie in the gap. In the case of a single defect the localized eigenmodes are often called defect or impurity midgap eigenmodes. In the case of a random medium the phenomenon of localization has the same nature as the Anderson localization of electrons[1,37,29], which is now well understood in the mathematical literature[18,17,19,5,7,3,33,4,9,11].

The physical origin of photonic band gaps and the localization phenomenon is the same: multiple scattering and destructive wave interference. Hence, the propagation (or nonpropagation) of EM waves in photonic crystals and the EM localization are intimately related which is reflected in our mathematical studies of EM waves localization[10,12–14].

Localization of a wave caused by a single defect or a random array of defects in a perfectly periodic medium is a general wave phenomenon. In addition to electron and EM waves, the phenomenon holds also for acoustic waves[9–14,31,33], elastic waves[27,25], acoustic phonons[30], and more complicated excitations involving coupled waves such as polaritons[8].

The subject of this paper is the localization of classical electromagnetic waves in a lossless linear dielectric medium in full three dimensions. In this paper we intentionally select and deal only with those quantities which can be treated rigorously. The rigorous investigation of the propagation of EM waves in three dimensional inhomogeneous media poses challenging mathematical problems. Though the burden of mathematical rigor compels to limit the scope of investigated physical quantities to a few ones, some physically important quantities can be and have been studied in detail. For instance, (i) we found sufficient conditions for a defect in a periodic dielectric with a spectral gap to generate midgap defect eigenmodes, and gave a priori estimates on their radii of localization[13]; (ii) we derived equations for the midgap defect frequencies and obtained estimates of the number of those frequencies in the gap[14]; (ii) we proved the Anderson localization of EM waves in random media[12]. Our focus is on the mathematical concepts and methods that give a solid mathematical basis to the physical theory, as well as on the tools for reliable computational schemes for the quantities describing the localization of EM waves. The relevant statements are formulated in the form of theorems and have been rigorously proven[9–14].

2. Inhomogeneous dielectric media

We assume that the propagation of EM waves is described by the classical Maxwell equations@mx1

$$\begin{aligned} &\frac{\partial}{\partial t}\mathbf{B} = -\nabla \times \mathbf{E} \qquad && \nabla \cdot \mathbf{B} = 0 \\ &\frac{\partial}{\partial t}\mathbf{D} = \nabla \times \mathbf{H} \qquad && \nabla \cdot \mathbf{D} = 0, \end{aligned} \tag{1}$$

with the linear constitutive relationships @mx2

$$\begin{aligned} \mathbf{B}(\mathbf{x},t) &= \mu(\mathbf{x})\,\mathbf{H}(\mathbf{x},t) \\ \mathbf{D}(\mathbf{x},t) &= \varepsilon(\mathbf{x})\,\mathbf{E}(\mathbf{x},t). \end{aligned} \tag{2}$$

We use the Giorgi system of units. The vector fields $\mathbf{E}, \mathbf{D}, \mathbf{H}$ and $\mathbf{B}$ are respectively the position and time dependent electric field, electric induction, magnetic field and magnetic induction. Since we consider inhomogeneous media the dielectric constant $\varepsilon = \varepsilon(\mathbf{x})$ and the magnetic permeability $\mu = \mu(\mathbf{x})$ are in general position dependent. Later on we will consider media for which the magnetic permeability is approximately constant, so we will take it to be identically 1. As to the dielectric constant $\varepsilon(\mathbf{x})$, we neglect its frequency dependence and, since the medium is lossless, $\varepsilon(\mathbf{x})$ is real valued. We always assume that @bound

$$0 < \varepsilon_- \leq \varepsilon(\mathbf{x}) \leq \varepsilon_+ < \infty \tag{3}$$

for some constants ε_- and ε_+.

The energy density $\mathcal{E}(\mathbf{x},t) = \mathcal{E}_{\mathbf{H},\mathbf{E}}(\mathbf{x},t)$ and the (conserved) energy of a solution $(\mathbf{H},\mathbf{E})$ of the Maxwell's equations (1) are given by

$$\mathcal{E}(\mathbf{x},t) = \frac{1}{2}\left[\mu(\mathbf{x})|\mathbf{H}(\mathbf{x},t)|^2 + \varepsilon(\mathbf{x})|\mathbf{E}(\mathbf{x},t)|^2\right], \tag{4}$$
$$\mathcal{E} = \mathcal{E}_{\mathbf{H},\mathbf{E}} = \int_{\mathbb{R}^3} \mathcal{E}(\mathbf{x},t)\, d\mathbf{x},$$

where $\mathbb{R}^3$ is three dimensional space.

Maxwell's equations can be recast as a Schrödinger-like equation, i.e., as a first order conservative linear equation, of the form@mx3

$$-i\frac{\partial}{\partial t}\mathbf{\Psi}_t = \mathbb{M}\mathbf{\Psi}_t, \tag{5}$$

with@MPs

$$\mathbf{\Psi}_t = \begin{pmatrix} \mathbf{H}_t \\ \mathbf{E}_t \end{pmatrix}, \quad \mathbb{M} = \begin{bmatrix} 0 & \frac{i}{\mu}\nabla^\times \\ \frac{-i}{\varepsilon}\nabla^\times & 0 \end{bmatrix}, \tag{6}$$

where $\nabla^\times$ is the symbol for the curl operator, i.e. $[\nabla^\times \mathbf{H}](\mathbf{x}) = \operatorname{curl}\mathbf{H}(\mathbf{x}) = \nabla \times \mathbf{H}(\mathbf{x})$. It is convenient and appropriate to introduce scalar products for the fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{H}(\mathbf{x},t)$ as follows: @nH,nE

$$(\mathbf{H}_1,\mathbf{H}_2)_\mu = \int_{\mathbb{R}^3} \overline{\mathbf{H}_1(\mathbf{x})}\mathbf{H}_2(\mathbf{x})\,\mu(\mathbf{x})\,d\mathbf{x}, \tag{7}$$

$$(\mathbf{E}_1,\mathbf{E}_2)_\varepsilon = \int_{\mathbb{R}^3} \overline{\mathbf{E}_1(\mathbf{x})}\mathbf{E}_2(\mathbf{x})\,\varepsilon(\mathbf{x})\,d\mathbf{x}, \tag{8}$$

where $\overline{\mathbf{E}}$ is complex conjugate to $\mathbf{E}$. The corresponding norms are defined as usual: $\|\mathbf{H}\|_\mu = \sqrt{(\mathbf{H},\mathbf{H})_\mu}$ and $\|\mathbf{E}\|_\varepsilon = \sqrt{(\mathbf{E},\mathbf{E})_\varepsilon}$. Let $\mathbb{S}_\mu$ be the Hilbert space of solenoidal magnetic fields $\mathbf{H}(\mathbf{x})$: @H2

$$\|\mathbf{H}\|_\mu < \infty \text{ and } \nabla \cdot \mu(\mathbf{x})\mathbf{H}(\mathbf{x}) = 0. \tag{9}$$

Similarly we introduce $\mathbb{S}_\varepsilon$ as the Hilbert space of solenoidal electric fields $\mathbf{E}(\mathbf{x})$: @H3

$$\|\mathbf{E}\|_\varepsilon < \infty \text{ and } \nabla \cdot \varepsilon(\mathbf{x})\,\mathbf{E}(\mathbf{x}) = 0. \tag{10}$$

The Hilbert space $\mathbb{H}_{\mu,\varepsilon}$ of finite energy EM fields is defined as the set of pairs $\mathbf{\Psi} = (\mathbf{H},\mathbf{E})$ such that $\mathbf{H} \in \mathbb{S}_\mu$ and $\mathbf{E} \in \mathbb{S}_\varepsilon$. The scalar product in $\mathbb{H}_{\mu,\varepsilon}$ is set to be @H4

$$(\mathbf{\Psi}_1,\mathbf{\Psi}_2) = \frac{1}{2}\left[(\mathbf{H}_1,\mathbf{H}_2)_\mu + (\mathbf{E}_1,\mathbf{E}_2)_\varepsilon\right], \tag{11}$$

so the energy of an EM field $\mathbf{\Psi}$ is given by the square of its norm: @H5

$$\mathcal{E}_{\mathbf{\Psi}} = \|\mathbf{\Psi}\|^2 = (\mathbf{\Psi},\mathbf{\Psi}) = \frac{1}{2}\left[\|\mathbf{H}\|^2 + \|\mathbf{E}\|_\varepsilon^2\right]. \tag{12}$$

The operator $\mathbb{M}$ governing the dynamics of EM fields is a self-adjoint operator in the Hilbert space $\mathbb{H}_{\mu,\varepsilon}$, hence the solution to (5) is given by @Ps1

$$\mathbf{\Psi}_t = e^{it\mathbb{M}}\mathbf{\Psi}_0, \tag{13}$$

so we have energy conservation:

$$\mathcal{E}_{\mathbf{\Psi}_t} = \|\mathbf{\Psi}_t\|^2 = \|\mathbf{\Psi}_0\|^2 = \mathcal{E}_{\mathbf{\Psi}_0}. \tag{14}$$

If $\mathbf{\Psi}_t$ is a solution of the equation (5), it must also satisfy the second order equation $\frac{\partial^2}{\partial t^2}\mathbf{\Psi}_t = -\mathbb{M}^2\mathbf{\Psi}_t$, so the magnetic and electric fields satisfy the second order equations

$$\frac{\partial^2}{\partial t^2}\mathbf{H}_t = -\frac{1}{\mu}\nabla^\times \frac{1}{\varepsilon}\nabla^\times \mathbf{H}_t, \;\; \mathbf{H}_t \in \mathbb{S}_\mu, \tag{15}$$

$$\frac{\partial^2}{\partial t^2}\mathbf{E}_t = -\frac{1}{\varepsilon}\nabla^\times \frac{1}{\mu}\nabla^\times \mathbf{E}_t, \;\; \mathbf{E}_t \in \mathbb{S}_\varepsilon. \tag{16}$$

It is natural to introduce the Maxwell operators @MHE

$$\mathbf{M_H} = \frac{1}{\mu}\nabla^\times \frac{1}{\varepsilon}\nabla^\times, \quad \mathbf{M_E} = \frac{1}{\varepsilon}\nabla^\times \frac{1}{\mu}\nabla^\times, \tag{17}$$

which are nonnegative self-adjoint operators on $\mathbb{S}_\mu$ and $\mathbb{S}_\varepsilon$ respectively. The two Maxwell operators are unitarily equivalent, more precisely

$$\mathbf{M_E} = U\mathbf{M_H}U^*, \tag{18}$$

where U is the unitary operator mapping $\mathbb{S}_\mu$ onto $\mathbb{S}_\varepsilon$, given by

$$U\mathbf{H} = \frac{-i}{\varepsilon}\nabla^\times \frac{1}{\sqrt{\mathbf{M_H}}}\mathbf{H}\;, \tag{19}$$

for $\mathbf{H}$ in the range of the operator $\sqrt{\mathbf{M_H}}$. Thus if $\sigma(A)$ stands for the spectrum of the operator A, we have

$$\sigma(\mathbb{M}) = \sigma\left(\sqrt{\mathbf{M_H}}\right) \cup \left[-\sigma\left(\sqrt{\mathbf{M_H}}\right)\right].$$

In particular, if a frequency ω belongs to the spectrum of the operator $\mathbb{M}$, then $-\omega$ must also be in this spectrum. Moreover, to obtain solutions of (5) we may set @sol

$$\mathbf{\Psi}_{\pm,t} = \left(e^{\pm it\sqrt{\mathbf{M_H}}}\mathbf{H}_0, \pm U e^{\pm it\sqrt{\mathbf{M_H}}}\mathbf{H}_0\right), \; \mathbf{H}_0 \text{ in } \mathbb{S}_\mu. \tag{20}$$

Conversely, any solution of (5) can be written as a linear combination of at most four solutions of this form.

It follows that to find all eigenvalues and eigenmodes for $\mathbb{M}$, it is necessary and sufficient to find all eigenvalues and eigenmodes for $\mathbf{M_H}$. For if ω^2 $(\omega > 0)$ and $\mathbf{H}_\omega$ are respectively an eigenvalue and the corresponding eigenmode of $\mathbf{M_H}$, i.e., @mh1

$$\mathbf{M_H}\mathbf{H}_\omega = \omega^2\mathbf{H}_\omega, \;\mathbf{H}_\omega \text{ in } \mathbb{S}_\mu, \tag{21}$$

then we have

$$U\mathbf{H}_\omega = \frac{-i}{\omega\varepsilon}\nabla^\times\mathbf{H}_\omega, \tag{22}$$

so (6), (20) and (21) imply

$$\mathbb{M}\mathbf{\Psi}_{\pm\omega} = \pm\omega\mathbf{\Psi}_{\pm\omega}, \tag{23}$$

where @eg1

$$\mathbf{\Psi}_{\pm\omega} = (\mathbf{H}_{\pm\omega}, \mathbf{E}_{\pm\omega}), \tag{24}$$
$$\mathbf{E}_{\pm\omega} = \pm U\mathbf{H}_{\pm\omega} = \pm\frac{-i}{\omega\varepsilon}\nabla^\times\mathbf{H}_{\pm\omega}.$$

Conversely, if $\mathbb{M}(\mathbf{H}_{\pm\omega}, \mathbf{E}_{\pm\omega}) = \pm\omega(\mathbf{H}_{\pm\omega}, \mathbf{E}_{\pm\omega})$, with $\omega > 0$ and nonzero vector $(\mathbf{H}_{\pm\omega}, \mathbf{E}_{\pm\omega})$ in $\mathbb{H}_{\mu,\varepsilon}$, then it follows that $\mathbf{M_H}\mathbf{H}_{\pm\omega} = \omega^2\mathbf{H}_{\pm\omega}$ and $\mathbf{E}_{\pm\omega} = \pm U\mathbf{H}_{\pm\omega} = \pm\frac{-i}{\omega\varepsilon}\nabla^\times\mathbf{H}_{\pm\omega}$.

An important conclusion of the previous considerations is that to study the spectral properties of the medium described by the Maxwell equations (1)-(2), *it is sufficient to study the spectral properties of the self-adjoint operator* $\mathbf{M_H}$ *defined in (17).*

From now on we will work primarily with the operator $\mathbf{M_H}$. Later on we will assume

$$\mu(\mathbf{x}) = 1, \tag{25}$$

and we will simply write @Max

$$\mathbf{M} = \mathbf{M_H} = \nabla^\times \frac{1}{\varepsilon}\nabla^\times \tag{26}$$

acting on the Hilbert space $\mathbb{S} = \mathbb{S}_1$, which is a closed subspace of the Hilbert space of square integrable, vector valued functions (L^2). We will denote by $P_\mathbb{S}$ the orthogonal projection onto $\mathbb{S}$, so $I_\mathbb{S} = P_\mathbb{S}^*$ takes a function in $\mathbb{S}$ into the same function as an element of L^2. We will also need the unrestricted Maxwell operator:

$$M = M_\mathbf{H} = \nabla^\times \frac{1}{\varepsilon}\nabla^\times \text{ acting on } L^2. \tag{27}$$

We have $\mathbf{M} = P_\mathbb{S} M I_\mathbb{S}$; $\mathbf{M}$ is the restriction of M to the orthogonal complement of its kernel. Note that $\mathbf{M}$ and M have the same spectrum, and we can work with M to answer questions about the spectrum of $\mathbf{M}$.

3. Localized waves

A *localized electromagnetic wave* is a *finite energy solution* of Maxwell's equations with the property that *almost all of the wave's energy remains in a fixed bounded region of space at all times*, e.g.,

$$\lim_{R\to\infty} \inf_t \frac{1}{\mathcal{E}} \int_{|\mathbf{x}|\le R} \mathcal{E}(\mathbf{x}, t)\, d\mathbf{x} = 1. \tag{28}$$

In spectral terms the localization phenomenon can be characterized as follows. The dielectric medium in which the waves propagate is fully described by the linear Maxwell operator $\mathbb{M}$ defined by (6). The basic spectral attributes of $\mathbb{M}$ are its eigenvalues ω and eigenmodes

$\boldsymbol{\Psi}_\omega(\mathbf{x}) = (\mathbf{H}_\omega(\mathbf{x}), \mathbf{E}_\omega(\mathbf{x}))$, i.e., $\mathbb{M}\boldsymbol{\Psi}_\omega = \omega\boldsymbol{\Psi}_\omega$. If for some ω the eigenmode $\boldsymbol{\Psi}_\omega$ has finite energy, i.e., @lc1

$$\int_{\mathbb{R}^3} \frac{1}{2}\left[\mu(\mathbf{x})|\mathbf{H}_\omega(\mathbf{x})|^2 + \varepsilon(\mathbf{x})|\mathbf{E}_\omega(\mathbf{x})|^2\right] d\mathbf{x} < \infty, \tag{29}$$

we call it a *localized eigenmode.* Observe that for extended eigenmodes, similar to plane EM waves in a homogeneous medium, the condition (29) is clearly violated since the total energy of the wave will be infinite. It turns out that in many cases the inequality (29) follows from a stronger property: exponential decay of $\boldsymbol{\Psi}_\omega(\mathbf{x})$ as $|\mathbf{x}| \to \infty$[10,12–14], i.e., @lc2

$$|\boldsymbol{\Psi}_\omega(\mathbf{x})| \leq \text{const}\exp\left(-\frac{|\mathbf{x}|}{L_\omega}\right). \tag{30}$$

The smallest L_ω for which (30) still holds is called the radius of localization of the eigenmode $\boldsymbol{\Psi}_\omega$.

Having a localized eigenmode $\boldsymbol{\Psi}_\omega$, we can easily construct a localized EM wave in the sense of (28). Namely, observe that $\boldsymbol{\Psi}_{\omega,t} = e^{it\omega}\boldsymbol{\Psi}_\omega$ is a localized electromagnetic wave, i.e., it satisfies (5) and (28). Notice also that in this case $-\omega$ is also an eigenvalue of $\mathbb{M}$ with eigenmode $\overline{\boldsymbol{\Psi}}_\omega$, so $\overline{\boldsymbol{\Psi}}_{\omega,t} = e^{-it\omega}\overline{\boldsymbol{\Psi}}_\omega$ is also a localized wave, since if $\mathbf{J}$ denotes the antiunitary involution corresponding to complex conjugation on $\mathbb{H}_{\mu,\varepsilon}$, i.e., $\mathbf{J}\boldsymbol{\Psi} = \overline{\boldsymbol{\Psi}}$, we have $\mathbf{J}\mathbb{M}\mathbf{J} = -\mathbb{M}$. It also follows that the spectrum of $\mathbb{M}$ is symmetric, i.e., $\sigma(\mathbb{M}) = -\sigma(\mathbb{M})$, with $\mathbf{J}\mathbb{M}_+\mathbf{J} = \mathbb{M}_-$, $\mathbb{M}_\pm$ being the positive and negative parts of $\mathbb{M}$. In addition to that, using the same arguments we find that any linear combinations of localized eigenmodes of $\mathbb{M}$ give raise to localized EM waves.

4. Periodic dielectric media

There is a well known relationship between the localization phenomenon for a wave propagating in an inhomogeneous medium and the rise of spectral gaps (stop bands) in a periodic medium. Both phenomena are caused by multiple scattering and destructive wave interference. For the localization phenomenon studied here we always start with a periodic dielectric medium with a spectral gap, and then perturb it by either a single or a random array of defects. In this section we describe the basic properties of the periodic dielectric medium (*photonic crystal*). For many dielectric materials of interest the magnetic permeability is very close to unity, so we will assume fom now on that (25) holds.

The periodic dielectric medium is described by a periodic dielectric function $\varepsilon_0(\mathbf{x})$. If $\mathbb{L}$ is the lattice of periods, we have

$$\varepsilon_0(\mathbf{x}+\mathbf{n}) = \varepsilon_0(\mathbf{x}) \text{ for any } \mathbf{n} \text{ from } \mathbb{L}.$$

We assume for simplicity that the primitive cell (PC) of $\mathbb{L}$ is a cube. The corresponding Maxwell operator then takes the form @M0

$$\mathbf{M_0}\boldsymbol{\Psi}(\mathbf{x}) = \nabla \times \frac{1}{\varepsilon_0(\mathbf{x})}\nabla \times \boldsymbol{\Psi}(\mathbf{x}), \tag{31}$$

where $\nabla \cdot \boldsymbol{\Psi}(\mathbf{x}) = 0$. In view of the periodicity, the Floquet-Bloch theory can be applied, so the spectrum of $\mathbf{M_0}$ has band structure and the eigenmodes have Bloch form:

$$\begin{gathered}\mathbf{M_0}\boldsymbol{\Psi}_{\omega_n(\mathbf{k})}(\mathbf{x}) = \omega_n^2(\mathbf{k})\boldsymbol{\Psi}_{\omega_n(\mathbf{k})}(\mathbf{x}), \ n = 1, 2, \ldots, \\ \boldsymbol{\Psi}_{\omega_n(\mathbf{k})}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}\boldsymbol{\Phi}_{\omega_n(\mathbf{k})}(\mathbf{x}), \\ \boldsymbol{\Phi}_{\omega_n(\mathbf{k})}(\mathbf{x}+\mathbf{n}) = \boldsymbol{\Phi}_{\omega_n(\mathbf{k})}(\mathbf{x}), \mathbf{n} \text{ in } \mathbb{L},\end{gathered}$$

where the quasimomentum $\mathbf{k}$ belongs to the Brillouin zone (BZ), the primitive cell of the lattice reciprocal to the $\mathbb{L}$, and n is the index of a zone. The frequency $\omega_n(\mathbf{k})$ is called the dispersion relationship of the n-th zone, and $\boldsymbol{\Psi}_{\omega_n(\mathbf{k})}$ is the eigenmode of the n-th zone. We naturally order the frequencies $\omega_n(\mathbf{k})$ so that

$$\omega_1(\mathbf{k}) \leq \omega_2(\mathbf{k}) \leq \ldots \leq \omega_n(\mathbf{k}) \leq \ldots.$$

If we denote by I_n the interval of values of the function $\omega_n^2(\mathbf{k})$, where $\mathbf{k}$ runs the Brillouin zone, then the spectrum $\sigma(\mathbf{M}_0)$ of the periodic operator $\mathbf{M}_0$ coincides with the union of these intervals, i.e., @sp0

$$\sigma(\mathbf{M}_0) = \bigcup_{n=1,2,\ldots} I_n. \tag{32}$$

It can be easily verified that the spectrum $\sigma(\mathbf{M}_0)$ is a closed subset of the positive semiaxis $[0,\infty)$, and that 0 is in its spectrum. It may happen that the intervals I_n do not cover all the semiaxis $[0,\infty)$ and, hence, there will an interval (ω_a^2, ω_b^2), called a spectral gap, that does not belong to the spectrum $\sigma(\mathbf{M}_0)$. *The physical significance of the existence of a gap lies in the fact that a wave with frequency in the gap cannot propagate in the medium.*

The zone structure of the spectrum is a generic property due to the periodicity. *But the existence of spectral gaps is not a generic property.* It depends in a subtle way on the geometry and the distribution of the dielectric materials in the periodic medium. It is quite evident that the higher contrast periodic media favors the rise of spectral gaps, but this observation alone is far from sufficient to establish the existence of a gap. In particular, the mathematical clarification of the very concept of high contrast is, we believe, a nontrivial and important problem. The existence of gaps for some periodic dielectric and acoustic media has been rigorously proved[15,16].

The cited papers also give some constructive approach to what may be called a high contrast medium.

Since our interest is primarily in the localization phenomenon, we simply assume from now on that the background periodic medium has at least one spectral gap. @agap

Assumption 1 (A gap in the spectrum). *There exists frequencies $0 < \omega_a < \omega_b$ such that ω_a^2 and ω_b^2 are in the spectrum of $\sigma(\mathbf{M}_0)$ and the interval (ω_a^2, ω_b^2), called the spectral gap, does not belong to $\sigma(\mathbf{M}_0)$, in the sense that (ω_a^2, ω_b^2) has no intersection with $\sigma(\mathbf{M}_0)$.*

Why does a wave with frequency in a spectral gap not propagate in the medium? In fact, that kind of the medium response is a very general property of a linear conservative medium occupying infinite space. If we excite the medium locally at a frequency ω which is not an eigenfrequency, the amplitude of the forced oscillations will die out exponentially away from the location of the excitation. The quantity which describes the amplitude of these forced oscillations is the corresponding Green's function $\mathbf{G}_0(\omega; \mathbf{x}, \mathbf{y})$, defined by @G0

$$\mathbf{G}_0(\omega; \mathbf{x}, \mathbf{y}) = \left(\mathbf{M}_0 - \omega^2 \mathbf{I}\right)^{-1}(\mathbf{x}, \mathbf{y}), \tag{33}$$

where $\mathbf{y}$ is the location of the source of the harmonic excitation at frequency ω. The Green's function $\mathbf{G}_0(\omega; \mathbf{x}, \mathbf{y})$ estimates the amplitude of the medium response at an observation point $\mathbf{x}$. If the frequency of forced oscillation ω falls in a spectral gap we expect $\mathbf{G}_0(\omega; \mathbf{x}, \mathbf{y})$ to decay exponentially, i.e., for some constants C_ω and L_ω @G01

$$|\mathbf{G}_0(\omega; \mathbf{x}, \mathbf{y})| \le C_\omega \exp\left\{-\frac{|\mathbf{x}-\mathbf{y}|}{L_\omega}\right\}, \ |\mathbf{x}-\mathbf{y}| \to \infty. \tag{34}$$

In this case it is easy to see that we are in a nonpropagation regime. Indeed, let us take a sphere centered at $\mathbf{y}$ of a large radius R. Then in view of (34) the energy flow through the sphere can be estimated as follows @G02

$$\begin{aligned} &\pi R^2 \left|\mathbf{G}_0(\omega; \mathbf{x}, \mathbf{y})\right|^2 \Big|_{|\mathbf{x}-\mathbf{y}|=R} \\ &\le \pi R^2 C_\omega \exp\left\{-\frac{2R}{L_\omega}\right\} \underset{R\to\infty}{\longrightarrow} 0. \end{aligned} \tag{35}$$

This lack of energy flow can be interpreted as a nonpropagation regime. On the the other hand, if the frequency ω is in the spectrum then $|\mathbf{G}_0(\omega; \mathbf{x}, \mathbf{y})|$ is proportional to $|\mathbf{x}-\mathbf{y}|^{-1}$, which results in a nonzero energy flow.

The rate of the exponential decay of the Green's function $\mathbf{G}_0(\omega; \mathbf{x}, \mathbf{y})$, for ω in a spectral gap, can be estimated rigorously as follows. Let @ch1

$$\chi(\mathbf{y}) = \begin{cases} 1 \text{ if } \mathbf{y} \text{ is in } PC \\ 0 \quad \text{otherwise} \end{cases} \tag{36}$$

be the characteristic function of the primitive cell of our lattice of periods, and let @ch2

$$\chi_{\mathbf{x}}(\mathbf{y}) = \chi_{\mathbf{x}}(\mathbf{y}-\mathbf{x}). \tag{37}$$

Consider the operators@GG1

$$\mathbf{G}(\omega) = \left(\mathbf{M} - \omega^2 \mathbf{I}\right)^{-1}, \ \mathbf{G}_0(\omega) = \left(\mathbf{M}_0 - \omega^2 \mathbf{I}\right)^{-1} \tag{38}$$

acting in the Hilbert space $\mathbb{S}$. We recall that the norm of an operator $\mathbf{A}$ is defined by @GG2

$$\|\mathbf{A}\| = \sup_{\boldsymbol{\Psi}\neq \mathbf{0}} \frac{\|\mathbf{A}\boldsymbol{\Psi}\|}{\|\boldsymbol{\Psi}\|}. \tag{39}$$

The following statement holds[12]. @texG

Theorem 1 *Let $\varepsilon(\mathbf{x})$ satisfy (3), let $\mathbf{M}$ be an operator of the form (26) having a spectral gap, and let ω^2 fall in this gap. Then there exists a finite absolute constant C_0 such that for all $\mathbf{x}$ and $\mathbf{y}$ the following inequality hold: @GG3*

$$\|\chi_{\mathbf{x}} \mathbf{G}(\omega) \chi_{\mathbf{y}}\| \ \le \frac{C_0}{\eta} \exp\left\{-\frac{|\mathbf{x}-\mathbf{y}|}{L_\omega}\right\}, \tag{40}$$

where

$$L_\omega = \frac{4\left[2\varepsilon_-^{-1} + \omega^2 + \eta\right]}{\eta}, \tag{41}$$

η being the distance from ω^2 to the edges of the spectral gap. Moreover, @GG4

$$\|\chi_{\mathbf{x}} \nabla^\times \mathbf{G}(\omega) \chi_{\mathbf{y}}\| \ \le \frac{C\left(1+\omega^2\right)}{\eta} \exp\left\{-\frac{|\mathbf{x}-\mathbf{y}|}{L_\omega}\right\}, \tag{42}$$

where C is a constant depending on ε_- and ε_+. In particular, the inequalities (40) and (42) hold for $\mathbf{G}_0(\omega)$.

Another important property of the periodic medium is a certain regularity of the gap edges which we define as follows. Let us consider, for instance, the edge ω_a^2. Let $n_1 \le n \le n_0$ be the indices of all the bands such that ω_a^2 is their right edge, i.e.,

$$\max_{\mathbf{k} \text{ in } BZ} \omega_n^2(\mathbf{k}) = \omega_a^2.$$

A natural regularity condition at ω_a^2 is a *non-degeneracy condition*: for each $n_1 \le n \le n_0$ we have $\omega_n^2(\mathbf{k}) = \omega_a^2$

for only finitely many $\mathbf{k}$, say $\mathbf{k}_{n,1}, \ldots, \mathbf{k}_{n,s_n}$, and for any $i = 1, ..., s_n$ we have

$$a - \omega_n^2(\mathbf{k}) \geq c_i \left|\mathbf{k} - \mathbf{k}_{n,i}\right|^2 \tag{43}$$

for $|\mathbf{k} - \mathbf{k}_{n,i}|$ small, with $c_i > 0$.

This non-degeneracy condition is a common assumption in the physical literature[20]. It was also used in the study of Lifshitz' tails in spectral gaps of periodic Schrödinger operator[26]. In fact, the verification of the regularity of a gap edge is not a simple matter even for Schrödinger operators. For our purposes a weaker notion of regularity of a spectral edge suffices. To motivate the condition, note that for $\omega_a^2 < \omega^2 < \omega_b^2$ we always have[14] $\mathrm{Tr}\left[\left(\chi I_{\mathbb{S}} \left|G_0\left(\omega^2\right)\right| P_{\mathbb{S}} \chi\right)^2\right] < \infty$, where $\mathrm{Tr}A$ denotes the trace of the operator A. *Our definition of regularity at an edge of the gap requires that this quantity remains finite as we approach this edge.* Let us introduce the quantities

$$\xi\left(\omega_a^2\right) = \limsup_{\eta \downarrow 0} \mathrm{Tr}\left[\left(\chi I_{\mathbb{S}} \left|G_0\left(\omega_a^2 + \eta\right)\right| P_{\mathbb{S}} \chi\right)^2\right] \tag{44}$$

and

$$\xi\left(\omega_b^2\right) = \limsup_{\eta \downarrow 0} \mathrm{Tr}\left[\left(\chi I_{\mathbb{S}} \left|G_0\left(\omega_b^2 - \eta\right)\right| P_{\mathbb{S}} \chi\right)^2\right], \tag{45}$$

where χ is as in (36). These quantities, in terms of which we will give a rigorous definition of regularity of an edge, will appear in estimates of the number of eigenvalues a defect can create in the gap. *Note that with the following definition non-degenerate edges will always be regular*[14].
@dper2

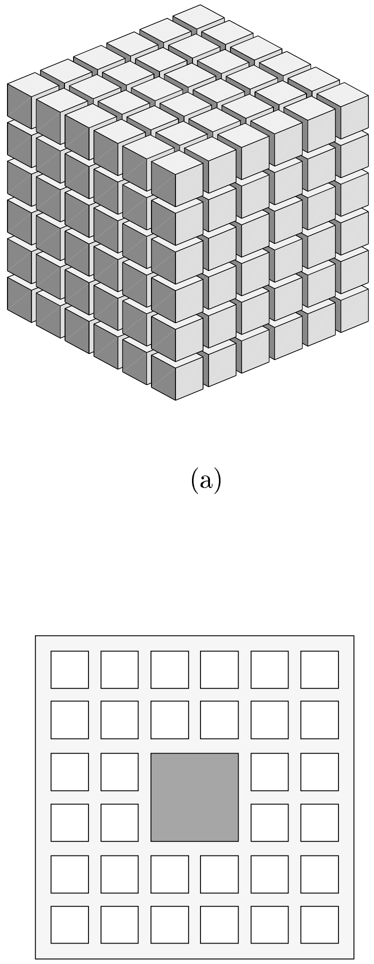
Definition 1 (Regularity at an edge). *The left edge ω_a^2 of the gap $\left(\omega_a^2, \omega_b^2\right)$ in the spectrum of the periodic operator $\mathbf{M}_0$ is regular if $\xi\left(\omega_a^2\right) < \infty$. Similarly, the right edge ω_b^2 is regular if $\xi\left(\omega_b^2\right) < \infty$.*

5. Midgap defect eigenmodes

It is a well known fact in solid state physics that, in three dimensions, a potential well of depth U and of radius a generates an exponentially localized state if

$$a^2 U > \frac{\pi^2 \hbar^2}{8m}, \tag{46}$$

where m is the mass of the quantum particle[39]. The question is whether an analogous condition can be found for classical electromagnetic waves.



(a)

(b)

Fig. 1. A slab of a photonic crystal (a) with an interior defect of higher dielctric constant. The defect is shown on the crossection of the slab (b) as a darker square.

In spite of the fundamental similarity between the creation of localized eigenmodes for classical and electron waves, there are some important differences. First of all, for the electron it suffices to perturb a homogeneous medium (i.e., a constant potential) locally to generate a localized eigenmode. For classical waves a local perturbation of a homogenous media (i.e., $\varepsilon_0(x)$ is constant) cannot generate a localized eigenmode. This can be easily seen from the consideration of a one dimensional model. Indeed, in that case we consider the eigenvalue problem $-\left(\frac{1}{\varepsilon(x)}u'(x)\right)' = \xi u'(x)$, $-\infty < x < \infty$, where $\varepsilon(x) = \text{const}$ if $|x| > R$ for some R, and ξ is a positive number. It is clear that this equation cannot have square integrable solutions. Since, in general, the one dimensional case is the most favorable for localization, we should not expect localization under this circumstances in the multidimensional case.

The reason for this difference between classical waves and electrons can be explained as follows. The motion of the electron in a homogenous medium is described by the Schrödinger operator $H_0 = -\Delta + V_0$ with a constant potential $V_0(x) \equiv v_0$. Clearly the spectrum $\sigma(H_0)$ of the operator H_0 is the interval $[v_0, \infty)$, so we may consider the infinite interval $(-\infty, v_0)$ as a gap in the spectrum of the operator H_0. Notice that the edge v_0 of the gap depends on the homogeneous medium. Hence, if we perturb this homogeneous medium by a defect, say by a potential well, the spectrum could expand in the interior of the gap $(-\infty, v_0)$, and if this happens the corresponding eigenmodes will be exponentially localized. For EM waves in a homogeneous medium, described by the Maxwell operator $\mathbf{M}$ with constant $\varepsilon(\mathbf{x})$, we always have $\sigma(\mathbf{M}) = [0, \infty)$, so, as for Schrödinger, we may consider the infinite interval $(-\infty, 0)$ as a gap in the spectrum. But for classical waves the bottom 0 of the spectrum does not depend on the coefficient $\varepsilon(\mathbf{x})$ of the medium at all. This is why a local perturbation of any medium by a defect does not expand the spectrum into the gap $(-\infty, 0)$, as we saw in the one-dimensional model.

Thus, in order to employ a mechanism for localization of classical waves similar to the one for electronic localization, we have to start with a medium described by a coefficient $\varepsilon_0(\mathbf{x})$, such that the corresponding acoustic or Maxwell operator has a gap inside its spectrum, and the edges of the gaps must depend on the medium, i.e., on the coefficient $\varepsilon_0(\mathbf{x})$. Such media with media dependent gaps can be perturbed locally by a defect and generate exponentially localized eigenmodes with corresponding eigenvalues in the interior of the gaps.

A *defect* is a perturbation of a given medium in a finite domain (see Fig. 1). Defects in the medium generate localized waves by creating localized eigenmodes of the operator $\mathbf{M}$.

We will say that the medium described by $\varepsilon(\mathbf{x})$ is *obtained from the background medium by the insertion of a defect*, if $\varepsilon(\mathbf{x})$ and $\varepsilon_0(\mathbf{x})$ differ only in a bounded domain Λ. In this case we will say that $\varepsilon(\mathbf{x})$ and $\varepsilon_0(\mathbf{x})$ *differ by a defect.* A simple way to tailor these defects is as follows. Let Λ be a bounded domain containing the origin 0. Typically, we take Λ to be the cube of side 1 centered at the origin, or the unit ball centered at the origin. Let set $\Lambda_\ell = \ell\Lambda$ for $\ell > 0$, so Λ_ℓ is the cube of side ℓ centered at the origin, etc. We insert a defect by changing the value of $\varepsilon_0(\mathbf{x})$ inside Λ_ℓ to a given constant $\epsilon > 0$, i.e.@def1

$$\varepsilon(\mathbf{x}) = \varepsilon_{\epsilon,\ell}(\mathbf{x}) = \begin{cases} \epsilon & \text{if} \quad \mathbf{x} \in \Lambda_\ell \\ \varepsilon_0(\mathbf{x}) & \text{otherwise.} \end{cases} \tag{47}$$

We recall that the essential spectrum $\sigma_{ess}(\mathbf{M})$ of an operator $\mathbf{M}$ consists of all the points of its spectrum $\sigma(\mathbf{M})$ which are not isolated eigenvalues with finite multiplicity.

@tess

Theorem 2 (Stability of essential spectrum). *Suppose $\varepsilon(\mathbf{x})$ and $\varepsilon_0(\mathbf{x})$ differ by a defect. Then*

$$\sigma_{\text{ess}}(\mathbf{M}) = \sigma_{\text{ess}}(\mathbf{M}_0).$$

If (ω_a^2, ω_b^2) is a gap in the spectrum of $\mathbf{M}_0$, the spectrum of $\mathbf{M}$ in (ω_a^2, ω_b^2) consists at most of isolated eigenvalues with finite multiplicity, with the corresponding eigenmodes decaying exponentially fast, with a rate depending on the distance from the eigenvalue to the edges of the gap.

Theorem 2 has been proven rigorously[13], and it says that a finite defect can create only isolated eigenvalues in the gap (ω_a^2, ω_b^2) with exponentially decaying eigenmodes. Here we consider just the basic arguments. Recall that the spectrum $\sigma(\mathbf{M})$ of a self-adjoint operator $\mathbf{M}$ can be defined as the set of real numbers ω^2 such that for any positive δ we can find a square integrable $\mathbf{\Psi}$ such that

$$\left\|\left(\mathbf{M} - \omega^2\mathbf{I}\right)\mathbf{\Psi}\right\| \le \delta\left\|\mathbf{\Psi}\right\|, \ \left\|\mathbf{\Psi}\right\|^2 = \int \left|\mathbf{\Psi}(\mathbf{x})\right|^2 d\mathbf{x}, \tag{48}$$

where $\mathbf{I}$ is the identity operator. In particular, for the periodic operator $\mathbf{M}_0$ we can always choose $\mathbf{\Psi}$ vanishing outside a ball B_δ of sufficiently large radius such that B_δ does not intersect the domain Λ of the defect and that (48) holds for $\mathbf{M}_0$. But for that ψ we clearly have $\mathbf{M\Psi} = \mathbf{M}_0\mathbf{\Psi}$ and, hence, (48) holds also for $\mathbf{M}$, so we should have $\sigma_{\text{ess}}(\mathbf{M}) \supset \sigma_{\text{ess}}(\mathbf{M}_0)$. On the hand, let $\mathbf{\Phi}_\omega$ be an eigenmode (extended or localized) of $\mathbf{M}$ corresponding to the eigenvalue ω^2 in the gap (ω_a^2, ω_b^2). Let us show that $\mathbf{\Phi}_\omega$ must be localized and, in addition to that, exponentially decaying function away from the location of the defect Λ. Indeed, from

$$\mathbf{M\Phi}_\omega = \omega^2\mathbf{\Phi}_\omega \tag{49}$$

we obtain

$$\mathbf{\Phi}_{\omega}(\mathbf{x}) = -\int \mathbf{G}_0(\omega;\mathbf{x},\mathbf{y})\left[(\mathbf{M}-\mathbf{M}_0)\mathbf{\Phi}_{\omega}\right](\mathbf{y})\,d\mathbf{y}, \tag{50}$$

where $\mathbf{G}_0(\omega;\mathbf{x},\mathbf{y}) = (\mathbf{M}_0 - \omega\mathbf{I})^{-1}(\mathbf{x},\mathbf{y})$ is the Green's function of the periodic operator $\mathbf{M}_0$. Observe that since $\mathbf{M}$ and $\mathbf{M}_0$ differ only on the domain Λ, (50) implies

$$\mathbf{\Phi}_{\omega}(\mathbf{x}) = -\int_{\Lambda} \mathbf{G}_0(\omega;\mathbf{x},\mathbf{y})\left[(\mathbf{M}-\mathbf{M}_0)\mathbf{\Phi}_{\omega}\right](\mathbf{y})\,d\mathbf{y}. \tag{51}$$

Recall now that since the eigenvalue ω^2 is in the gap of the operator $\mathbf{M}_0$ the Green's function $\mathbf{G}_0(\omega;\mathbf{x},\mathbf{y})$ must decay exponentially, i.e.,

$$|\mathbf{G}_0(\omega;\mathbf{x},\mathbf{y})| \le C_1 \exp(-C_2|\mathbf{x}-\mathbf{y}|). \tag{52}$$

It follows from general considerations that the eigenmode $\mathbf{\Phi}_{\omega}(\mathbf{x})$ must be effectively bounded in any finite domain regardless to whether it is extended or localized. This last comment, (51) and the bound (52) imply that $\mathbf{\Phi}_{\omega}(\mathbf{x})$ must be an exponentially decaying function, and, hence, all the eigenmodes of $\mathbf{M}$ with corresponding eigenvalues in the gap must be exponentially localized.

Theorem 2 states that if we have any spectrum generated by a defect in the spectral gap (ω_a^2,ω_b^2) of the original periodic operator $\mathbf{M}_0$, it must be associated with exponentially decaying eigenmodes. But the question remains if the defect creates any spectrum at all in the spectral gap of $\mathbf{M}_0$. Using simple space scaling we observe that a sufficient condition for the existence of localized eigenmodes generated by the defect should have the form

$$\ell^2\epsilon > C(\Lambda,\omega_a,\omega_b), \tag{53}$$

where the constant $C(\Lambda,\omega_a,\omega_b)$, in general, depends on the shape Λ of the defect and the location of the gap (ω_a^2,ω_b^2).

When Λ is a cube of side 1, and the defect is as in (47), a more delicate and rigorous analysis[13] leads to the following sufficient condition for the existence of localized eigenmodes. In particular, it gives (53) with

$$C(\Lambda,\omega_a,\omega_b) = \frac{158\left(\omega_a^2+\omega_b^2\right)}{\left(\omega_b^2-\omega_a^2\right)^2}. \tag{54}$$

@tceg

Theorem 3 (Creation of defect eigenmodes). *Let (ω_a^2,ω_b^2) be a gap in the spectrum of $\mathbf{M}_0$, select $\tau \in (\omega_a^2,\omega_b^2)$, and pick $0<\gamma<1$ such that the interval $[\tau(1-\gamma),\tau(1+\gamma)]$ is contained in the gap, i.e., $[\tau(1-\gamma),\tau(1+\gamma)] \subset (\omega_a^2,\omega_b^2)$. If $\varepsilon(x) \equiv \epsilon$ in a cube of side ℓ, with @ed*

$$\ell^2\epsilon > \frac{79}{\tau\gamma^2}, \tag{55}$$

the corresponding operator $\mathbf{M}$ has at least one defect eigenmode with corresponding eigenvalue inside the interval $[\tau(1-\gamma),\tau(1+\gamma)]$.

Note that condition (55) is analogous to condition (46).

We will call a defect *positive* if $\varepsilon(\mathbf{x}) \le \varepsilon_0(\mathbf{x})$, in which case we have $\mathbf{M}-\mathbf{M}_0 \ge 0$. Similarly, a defect is *negative* if $\varepsilon(\mathbf{x}) \ge \varepsilon_0(\mathbf{x})$, so $\mathbf{M}-\mathbf{M}_0 \le 0$. If the defect is either positive or negative we can say more about how the eigenvalues are distributed in the gap. To do so, we use a modified Birman-Schwinger method to reduce the problem to the study of the eigenvalues of a compact operator (in fact, a Hilbert-Schmidt operator)[14]. The usual Birman-Schwinger method[38,24] cannot be used directly, as $\mathbf{M}-\mathbf{M}_0$ is not relatively compact with respect to $\mathbf{M}_0$. The way out is to use the resolvents: we set $H = (\mathbf{M}+I)^{-1}$ and $H_0 = (\mathbf{M}_0+I)^{-1}$, prove that $V = H - H_0$ is a Hilbert-Schmidt operator in dimension 3 or less, and then use the Birman-Schwinger method for $H = H_0 + V$. This method gives equations for the defect eigenmodes and corresponding midgap eigenvalues in terms of the spectral attributes of an auxiliary Hilbert-Schmidt operator.

So let $0<\omega_a<\omega_b$ such that (ω_a^2,ω_b^2) is a gap in the spectrum of the operator $\mathbf{M}_0$, and let us insert a negative defect, so $\varepsilon(\mathbf{x}) \ge \varepsilon_0(\mathbf{x})$ and $V \ge 0$. Let us consider the eigenvalue problem for the operator $\mathbf{M}$ in the gap: @Weig

$$\mathbf{M\Psi} = \omega^2\mathbf{\Psi}, \quad \text{where } \omega \in (\omega_a,\omega_b). \tag{56}$$

This is clearly equivalent to the eigenvalue problem@HQp1

$$H\mathbf{\Psi} = (\mathbf{M}+I)^{-1}\mathbf{\Psi} = \left(\omega^2+1\right)^{-1}\mathbf{\Psi}, \quad \text{where } \omega \text{ is in } (\omega_a,\omega_b). \tag{57}$$

On the other hand, the eigenvalue problem @meg71

$$H\mathbf{\Psi} = H_0\mathbf{\Psi} + V\mathbf{\Psi} = \xi\mathbf{\Psi}, \quad \text{where } \xi \text{ is not in } \sigma(H_0), \tag{58}$$

can be rewritten as@meg81

$$\mathbf{\Psi} = -R_0(\xi)V\mathbf{\Psi}, \quad \text{with} \quad R_0(\xi) = (H_0-\xi I)^{-1}. \tag{59}$$

Setting @meg8b1

$$\mathcal{R}(\xi) = -\sqrt{V}R_0(\xi)\sqrt{V}, \tag{60}$$

we obtain the eigenvalue problem @meg8a1

$$\mathcal{R}\left(\xi\right)\mathbf{\Phi}=\mathbf{\Phi},\quad \text{where}\quad \mathbf{\Phi}=\sqrt{V}\mathbf{\Psi}, \tag{61}$$

which is equivalent to the eigenvalue problem (58) . The Birman-Schwinger operator $\mathcal{R}\left(\xi\right)$ is a self-adjoint Hilbert-Schmidt operator (V is Hilbert-Schmidt). The original eigenvalue problem (56) for $\mathbf{M}$ can now be rewritten as follows:@HQp4b

$$\mathcal{S}\left(\omega\right)\mathbf{\Phi}=\mathbf{\Phi},\quad \text{where}\quad \mathbf{\Phi}=\sqrt{V}\mathbf{\Psi},\ \omega\in\left(\omega_a,\omega_b\right), \tag{62}$$

and $\mathcal{S}\left(\omega\right)$ is the self-adjoint Hilbert-Schmidt operator given by @HQp4ba

$$\begin{aligned}\mathcal{S}\left(\omega\right)&=\mathcal{R}\left(\left(\omega^2+1\right)^{-1}\right)\\&=\left(\omega^2+1\right)\sqrt{V}\frac{\mathbf{M}_0+I}{\mathbf{M}_0-\omega^2 I}\sqrt{V}.\end{aligned} \tag{63}$$

In the case of a positive defect, so $\varepsilon\left(\mathbf{x}\right)\leq\varepsilon_0\left(\mathbf{x}\right)$ and $V\leq 0$, the analog of the equations (62) and (63) takes the form@HQp5b

$$\begin{aligned}&\mathcal{S}\left(\omega\right)\mathbf{\Phi}=-\mathbf{\Phi}\quad\text{where}\\&\quad\mathbf{\Phi}=\sqrt{-V}\mathbf{\Psi},\ \omega\text{ is in }\left(\omega_a,\omega_b\right),\end{aligned} \tag{64}$$

with @HQp5a

$$\mathcal{S}\left(\omega\right)=\left(\omega^2+1\right)\sqrt{-V}\frac{\mathbf{M}_0+I}{\mathbf{M}_0-\omega^2 I}\sqrt{-V}. \tag{65}$$

Observe now that, since $\left(\mathbf{M}_0-\omega^2 I\right)^{-1}$ is a monotonically increasing, norm continuous operator function of $\omega^2\in\left(\omega_a^2,\omega_b^2\right)$, the operator $\mathcal{S}\left(\omega\right)$ is also a monotonically increasing, norm continuous function of $\omega\in\left(\omega_a,\omega_b\right)$ for both negative and positive defects. Since $\mathcal{S}\left(\omega\right)$ is a self-adjoint Hilbert-Schmidt operator, its spectrum consists of eigenvalues of finite multiplicity with 0 being the only possible point of accumulation. Let $\rho_1^+\left(\omega\right)\geq\rho_2^+\left(\omega\right)\geq\ldots\geq 0$ and $\rho_1^-\left(\omega\right)\leq\rho_2^-\left(\omega\right)\leq\ldots\leq 0$ be the sequences of respectively the positive and negative eigenvalues of the operator $\mathcal{S}\left(\omega\right)$, repeated according to their multiplicity. If we have a finite number of either positive or negative eigenvalues, we complete the sequence by assigning the value 0. The functions $\rho_n^{\pm}\left(\omega\right)$ are monotonically increasing and continuous in $\omega\in\left(\omega_a,\omega_b\right)$.

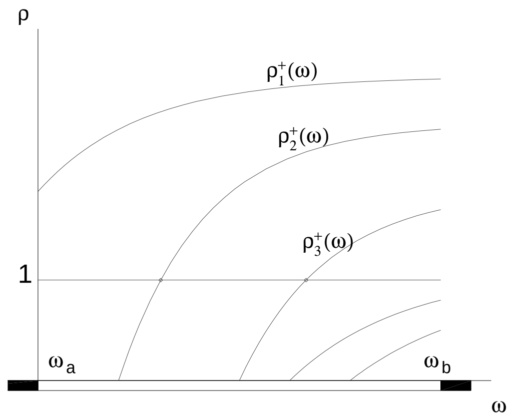


Fig. 2. The equations for the eigenvalues for the defect eigenmodes take the form $\rho_n^+(\omega)=1$ where the functions $\rho_n^+(\omega)$ are the eigenvalues of an auxiliary compact operator depending on the spectral parameter ω. @teeq

Theorem 4 (Equations for defect eigenvalues). *Let $\varepsilon_0(\mathbf{x})$ be a periodic function satisfying (3), with the interval (ω_a^2,ω_b^2) a gap in the spectrum of $\mathbf{M}_0$, and let $\varepsilon(\mathbf{x})$ be obtained from $\varepsilon_0(\mathbf{x})$ by the insertion of a defect. If the defect is negative, the only possible point of accumulation of the defect eigenvalues of $\mathbf{M}$ in (ω_a^2,ω_b^2) is ω_b^2. In this case the frequencies $\omega_1\leq\omega_2\leq\ldots$ in (ω_a,ω_b), such that $\omega_1^2\leq\omega_2^2\leq\ldots$ are the eigenvalues of the operator $\mathbf{M}$ in the gap (ω_a^2,ω_b^2), coincide with the set of the solutions of the equations @eeq1*

$$\rho_n^+\left(\omega\right)=1,\quad n=1,2,\ldots, \tag{66}$$

where $\rho_n^+\left(\omega\right)$ are the positive eigenvalues of the operator $\mathcal{S}\left(\omega\right)$ defined by (63). Moreover, if φ_i is an eigenmode of the operator $\mathcal{S}\left(\omega_i\right)$ with eigenvalue $\rho_{n_i}^+\left(\omega_i\right)=1$, then@eeq2

$$\mathbf{\Psi}_i=\frac{\mathbf{M}_0+I}{\mathbf{M}_0-\omega_i^2 I}\sqrt{V}\mathbf{\Phi}_i \tag{67}$$

is an exponentially localized eigenmode of the operator $\mathbf{M}$ with eigenvalue ω_i^2. Similarly, if the defect is positive, the only possible point of accumulation of of the defect eigenvalues of $\mathbf{M}$ in (ω_a^2,ω_b^2) is ω_a^2. In this case the eigenvalues the frequencies $\omega_1\geq\omega_2\geq\ldots$ in (ω_a,ω_b), such that $\omega_1^2\geq\omega_2^2\geq\ldots$ are the eigenvalues of the operator $\mathbf{M}$ in the gap (ω_a^2,ω_b^2), coincide with the set of the solutions of the equations@eeq3

$$\rho_n^-(\omega) = -1, \quad n = 1, 2, \ldots, \tag{68}$$

where $\rho_n^-(\omega)$ are the negative eigenvalues of the operator $\mathcal{S}(\omega)$ defined by (65). Moreover, if φ_i is an eigenmode of the operator $\mathcal{S}(\omega_i)$ with eigenvalue $\rho_{n_i}^-(\omega_i) = -1$, then@eeq4

$$\mathbf{\Psi}_i = -\frac{\mathbf{M}_0 + I}{\mathbf{M}_0 - \omega_i^2 I}\sqrt{-V}\mathbf{\Phi}_i \tag{69}$$

is an exponentially localized eigenmode of the operator $\mathbf{M}$ with eigenvalue ω_i^2.

It follows from Theorem 3 that, for defects as in (47), satisfying (55) with $\tau = \frac{\omega_a^2+\omega_b^2}{2}$ and $\gamma = \frac{\omega_b^2-\omega_a^2}{\omega_a^2+\omega_b^2}$, defect eigenmodes and midgap eigenvalues always exist, so in this case we can guarantee the existence of a solution for some of the equations (66) and (68).

Theorem 4 reduces the search for defect eigenmodes and midgap eigenvalues of the perturbed operator $\mathbf{M}$ to the investigation of the spectral attributes of the relevant Hilbert-Schmidt operator $\mathcal{S}(\omega)$. When it comes to numerical estimations, the reduction to the Hilbert-Schmidt operator $\mathcal{S}(\omega)$ is quite valuable, since a compact operators as $\mathcal{S}(\omega)$ is more suitable for truncations than the original unbounded differential operator $\mathbf{M}$ with nonsmooth coefficient $\varepsilon(\mathbf{x})$.

To estimate the number of defect eigenvalues we need a function counting eigenvalues. For a given a self-adjoint operator H and interval (α, β), we define the counting function by the formula

$$N_H(\alpha, \beta) = \mathrm{Tr}\ \chi_{(\alpha,\beta)}(H). \tag{70}$$

Note $N_H(\alpha, \beta)$ is always a nonnegative integer unless it is infinite. If H has discrete spectrum in (α, β), $N_H(\alpha, \beta)$ gives the number of eigenvalues of H in (α, β), counted according to their multiplicity.

It will be convenient to write

$$\varepsilon(\mathbf{x}) = \frac{\varepsilon_0(\mathbf{x})}{1 + \theta(\mathbf{x})}, \tag{71}$$

where $\theta(x)$ is a bounded measurable function with compact support satisfying

$$-1 < \theta_- \le \theta(x) \le \theta_+ < \infty \tag{72}$$

for some constants θ_- and θ_+. Note that a defect is *positive* if $\theta(\mathbf{x}) \ge 0$, in which case we have $\theta_- = 0$. Similarly, a defect is *negative* if $\theta(\mathbf{x}) \le 0$, so $\theta_+ = 0$.

Our estimate of the number of eigenvalues in a gap is given by the following theorem.

@maint

Theorem 5 . *Let $\varepsilon_0(\mathbf{x})$ be a periodic function satisfying (3), with the interval (ω_a^2, ω_b^2) a gap in the spectrum of $\mathbf{M}_0$. Let us insert a defect by taking $\varepsilon(\mathbf{x})$ be as in (71), where $\theta(\mathbf{x})$ satisfies (72) and vanishes outside a cube Λ_ℓ of side $\ell > 0$. Let $0 < \delta \le 1$, we have:*

Suppose the left edge ω_a^2 is regular and the defect is positive, then

$$N_{\mathbf{M}}(\omega_a^2, \omega_b^2) \le C_{\omega_a^2,\varepsilon_{0,\pm},\theta_+,\ \delta}\theta_+^2 (\ell + 3)^{9+\delta}\ \xi(\omega_a^2) < \infty, \tag{73}$$

with

$$C_{\omega_a^2,\varepsilon_{0,\pm},\theta_+,\delta} = \frac{(\omega_a^2+1)^2}{\delta} C\left[1 + \varepsilon_-^{-(6+\delta)}\right] C' \tag{74}$$

$$C' = \left[1 + 6\frac{\sqrt{\varepsilon_{0,+}}}{\varepsilon_{0,-}}\left(\sqrt{\varepsilon_-} + \frac{1}{\sqrt{\varepsilon_-}}\right)\right]^2, \tag{75}$$

where C is some constant, independent of δ and of the other parameters, $\varepsilon_- = \frac{\varepsilon_{0,-}}{1+\theta_+}$, and $\xi(\omega_a^2)$ is given in (44)

.

Suppose the right edge ω_b^2 is regular and the defect is negative, then

$$N_{\mathbf{M}}(\omega_a^2, \omega_b^2) \le C_{\omega_b^2,\varepsilon_{0,\pm},\theta_-,\ \delta}\theta_-^2 (\ell + 3)^{9+\delta}\, \xi(\omega_b^2) < \infty, \tag{76}$$

with

$$C_{\omega_b^2,\varepsilon_{0,\pm},\theta_-,\delta} = \frac{(b+1)^2}{\delta} C\left[1 + \varepsilon_{0,-}^{-(6+\delta)}\right] C' \tag{77}$$

$$C' = \left[1 + 6\frac{\sqrt{\varepsilon_+}}{\varepsilon_{0,-}}\left(\sqrt{\varepsilon_{0,-}} + \frac{1}{\sqrt{\varepsilon_{0,-}}}\right)\right]^2, \tag{78}$$

where C is some constant, independent δ and of the other parameters, $\varepsilon_+ = \frac{\varepsilon_{0,+}}{1+\theta_-}$, and $\xi(\omega_b^2)$ is given in (45).

Since the function $N_{\mathbf{M}}(\omega_a^2, \omega_b^2)$ is integer valued, $N_{\mathbf{M}}(\omega_a^2, \omega_b^2) < 1$ implies $N_{\mathbf{M}}(\omega_a^2, \omega_b^2) = 0$. Thus we have the following immediate corollary, which tells us that there are no midgap eigenvalues if the defect is small.

@maintcor

Corollary 1 . *Let $\varepsilon_0(\mathbf{x})$ be a periodic function satisfying (3), with the interval (ω_a^2, ω_b^2) a gap in the spectrum of $\mathbf{M}_0$. Let us insert a defect by taking $\varepsilon(\mathbf{x})$ be as in (71), where $\theta(\mathbf{x})$ satisfies (72) and vanishes outside a cube Λ_ℓ of side $\ell > 0$. Then:*

If the left edge ω_a^2 is regular and the defect is positive, we must have $N_{\mathbf{M}}(\omega_a^2, \omega_b^2) = 0$ for $\theta_+ > 0$ small, how small depending only on ω_a^2, $\varepsilon_{0,\pm}$ and ℓ. In other words, there are no eigenvalues in the gap for weak defects.

If the right edge ω_b^2 is regular and the defect is negative, we must have $N_{\mathbf{M}}(\omega_a^2, \omega_b^2) = 0$ for $\theta_- < 0$ small, how small depending only on ω_b^2, $\varepsilon_{0,\pm}$ and ℓ, so there are no eigenvalues in the gap for weak defects.

6. Localization in disordered media

As we have seen, a strong enough single defect in a periodic dielectric medium with a spectral gap creates exponentially localized electromagnetic waves. If we have a random array of such defects then, under some natural conditions, the localized waves created by individual defects do not couple (i.e., the electromagnetic wave tunneling becomes inefficient), so we get an infinite number of localized waves whose frequencies are dense in an interval contained in the spectral gap of the underlying periodic medium. This phenomenon is analogous to the Anderson localization of electron waves in random media, which has been studied intensively for the last four decades, both in the physics and mathematics literature[1,37,29,18,17,7,5,3,33].

The relevant mathematical problems led to studies of the spectral attributes of differential, partial differential, and matrix linear operators with random position dependent coefficients. For electron waves the coefficient describing the medium is a random potential $V(\mathbf{x})$, where $\mathbf{x}$ is the position in space. In our setting the medium is described by a dielectric constant $\varepsilon(\mathbf{x})$, which will be the random coefficient. The randomness means, first of all, that for any given $\mathbf{x}$ we have that $\varepsilon(\mathbf{x})$ is a random quantity and, secondly, that the random field $\varepsilon(\mathbf{x})$ is statistically homogeneous and ergodic[5,3,29,33]. These latter conditions can be formulated as follows. The random medium will be a randomization of an underlying periodic medium with a lattice of periods $\mathbb{L}$. For any set of positions $\mathbf{x}_1, \ldots, \mathbf{x}_N$ the joint probability distribution $P_{\mathbf{x}_1,\ldots,\mathbf{x}_N}$ of the random quantities $\varepsilon(\mathbf{x}_1), \ldots, \varepsilon(\mathbf{x}_N)$ is $\mathbb{L}$-invariant, namely for any $\mathbf{m}$ in the lattice $\mathbb{L}$ we have @Pr1

$$P_{\mathbf{x}_1,\ldots,\mathbf{x}_N} = P_{\mathbf{x}_1+\mathbf{m},\ldots,\mathbf{x}_N+\mathbf{m}}. \tag{79}$$

This statistical homogeneity condition is very natural from a physical point of view. The ergodicity of the probability measure $\mathbb{P}$ describing all the random quantities $\varepsilon(\mathbf{x})$, $\mathbf{x}$ running over three dimensional space, means that all $\mathbb{L}$ -invariant events must be trivial, i.e., their probability must either 0 or 1. Lack of statistical correlations between $\varepsilon(\mathbf{x})$ and $\varepsilon(\mathbf{y})$ for large $|\mathbf{x}-\mathbf{y}|$ is usually sufficient for ergodicity. *We always assume that a random dielectric constant* $\varepsilon(\mathbf{x})$ *is statistically homogeneous and ergodic.*

Let us turn now to the spectral properties of a random self-adjoint differential or matrix operator A. It is customary to classify a spectral point ω based on whether the corresponding eigenmode u_ω is square integrable or not. In first case, i.e., $\int |u_\omega(\mathbf{x})|^2 \, d\mathbf{x} < \infty$, we say that ω is an eigenvalue (true eigenvalue) and u_ω is a localized eigenmode. We denote by $\sigma_{\mathrm{pp}}(A)$ the set of eigenvalues of A. It is a well known basic fact that $\sigma_{\mathrm{pp}}(A)$ is at most countable, i.e.,

$$\sigma_{\mathrm{pp}}(A) = \{\omega_1, \omega_2, \ldots\}.$$

If $\int |u_\omega(\mathbf{x})|^2 \, d\mathbf{x} = \infty$, as, for instance, for a plane wave, we call u_ω an extended eigenmode and classify ω as a point of the continuous spectrum $\sigma_{\mathrm{c}}(A)$, a closed set. The entire spectrum $\sigma(A)$ is defined the union

$$\sigma(A) = \overline{\sigma_{\mathrm{pp}}(A)} \bigcup \sigma_{\mathrm{c}}(A),$$

where $\overline{\sigma_{\mathrm{pp}}(A)}$ is the closure of the set $\sigma_{\mathrm{pp}}(A)$.

For a random (differential or matrix) operator A, we list below some of its unusual spectral properties, which may seem peculiar and exotic, but are truly typical and always hold for a physically meaningful random medium. These spectral properties are due to ergodicity and self-averaging, and hold in large generality with probability one[5,3,29,33]:

- the *spectra* $\sigma(A)$, $\overline{\sigma_{\mathrm{pp}}(A)}$ and $\sigma_{\mathrm{c}}(A)$ are *nonrandom*;
- the *integrated density of states* (the number of states per unit volume) is *nonrandom*;
- any interval contains either zero and infinitely many points of the spectrum;
- the probability that a fixed number θ is an eigenvalue of A is zero; in other words, though the closure of the pure point spectrum $\overline{\sigma_{\mathrm{pp}}(A)}$ is not random, the countable set $\sigma_{\mathrm{pp}}(A)$ of eigenvalues ω_j is random, and, in addition to that, the eigenvalues ω_j are very sensitive to a particular sample, "easily movable", and because of that, with probability one never hit θ.

In view of these properties, random operators typically exhibit spectral behavior unusual in classical spectral theory. For instance, suppose an interval I belongs the spectrum $\sigma(A)$ of a random operator A (we can speak unambiguously of the spectrum, or of the closure of the pure point spectrum, of the random operator, since the spectrum, or the closure of the pure point spectrum, is the same for almost all realizations of the random operator), and suppose that it has only pure point spectrum there. Then with probability one the eigenvalues ω_j of the operator A form a dense countable subset in the interval I, i.e., any subinterval I' of I contains infinitely many eigenvalues ω_j. But there is zero probability that a fixed ω in I will be an eigenvalue. In many interesting cases one can prove that the eigenmodes of the pure point spectrum decay exponentially. One can look at these "unusual", but typical for random operators, spectral properties as merely a reflection of the complexity of the random environment.

Our strategy for proving Anderson localization of electromagnetic waves is the following: first we study the effect of random perturbations on a spectral gap of the underlying periodic medium; we obtain estimates on the size of the spectral gap of the perturbed medium, showing that the gap does not close for not too large random perturbations. Then the Maxwell operator $\mathbf{M_H}$ of the random medium is shown to have pure point spectrum in some closed subinterval I of the spectral gap of the underlying periodic medium, with all the corresponding eigenfunctions being exponentially decaying (in the sense of having exponentially decaying local L^2-norms[12]). For this operator we prove that the curl of an exponentially decaying eigenfunction is also exponentially decaying, so it follows from (18) and (22) that the corresponding operator $\mathbf{M_E}$ has also pure point spectrum in the closed interval I, with all the corresponding eigenfunctions being exponentially decaying. In addition, it ensues from (23) that the operator $\mathbb{M}$ has pure point spectrum in $\left\{\omega \in \mathbb{R};\ \ \omega^2 \in I\right\}$, with all the corresponding eigenfunctions being exponentially decaying, so the energy densities of the corresponding solutions of (1) are also exponentially decaying, uniformly in the time t, satisfying (28).

The localization of electromagnetic waves is thus a consequence of Anderson localization for operators $\mathbf{M} = \mathbf{M_H} = \nabla^{\times}\frac{1}{\varepsilon}\nabla^{\times}$ on $\mathbb{S}$, i.e., the existence of closed intervals where these operators have pure point spectrum with exponentially decaying eigenfunctions.

We model a random array of defects in a periodic medium by a random dielectric constant $\varepsilon(\mathbf{x})$. For simplicity we take the lattice $\mathbb{L}$ of periods to be $q\mathbb{Z}^3$, with q a positive integer and $\mathbb{Z}^3$ the usual cubic lattice.

@aeu

Assumption 2 (The Random Media). $\varepsilon_g(\mathbf{x}) = \varepsilon_{g,\zeta}(\mathbf{x})$ *is a random function of the form*

$$\varepsilon_{g,\zeta}(\mathbf{x}) = \varepsilon_0(\mathbf{x})\,\gamma_{g,\zeta}(\mathbf{x}) \tag{80}$$

$$\textit{with}\quad \gamma_{g,\zeta}(\mathbf{x}) = 1 + g\sum_{i\in\mathbb{Z}^3}\zeta_i u_i(\mathbf{x}), \tag{81}$$

where

(i) $\varepsilon_0(\mathbf{x})$ is a measurable real valued function which is q-periodic for some $q \in \mathbb{N}$, i.e., $\varepsilon_0(\mathbf{x}) = \varepsilon_0(\mathbf{x} + q\mathbf{i})$ for all $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{i} \in \mathbb{Z}^3$, with

$$0 < \varepsilon_{0,-} \leq \varepsilon_0(\mathbf{x}) \leq \varepsilon_{0,+} < \infty \tag{82}$$

for some constants $\varepsilon_{0,-}$ and $\varepsilon_{0,+}$.

(ii) $u_i(\mathbf{x}) = u(\mathbf{x} - \mathbf{i})$ for each $\mathbf{i} \in \mathbb{Z}^3$, u being a nonnegative measurable real valued function with compact support, say $u(\mathbf{x}) = 0$ if $\|x\|_\infty \leq r_u$ for some $r_u < \infty$, such that

$$0 < U_- \leq U(\mathbf{x}) \equiv \sum_{i\in\mathbb{Z}^3} u_i(\mathbf{x}) \leq U_+ < \infty \tag{83}$$

for some constants U_- and U_+.

(iii) $\zeta = \{\zeta_i;\ i \in \mathbb{Z}^3\}$ is a family of independent, identically distributed random variables taking values in the interval $[-1, 1]$, whose common probability distribution has a bounded density $\rho(t) = \rho_{\zeta_i}(t) > 0$ almost everywhere in $[-1, 1]$.

(iv) g, satisfying $0 \leq g < \frac{1}{U_+}$, is the disorder parameter.

Notice that Assumption 2 implies that each $\varepsilon_{g,\zeta}$ satisfies (3) with

$$\varepsilon_\pm = \varepsilon_{g,\pm} = \varepsilon_{0,\pm}(1 \pm gU_+)\,. \tag{84}$$

The periodic operators associated with the periodic dielectric constant $\varepsilon_0(\mathbf{x})$ will carry the subscript 0, i.e., $M_0 = M(\varepsilon_0)$, $\mathbf{M}_0 = \mathbf{M}(\varepsilon_0)$. We study the random operators

$$M_g = M_{g,\zeta} = M(\varepsilon_{g,\zeta}) \tag{85}$$

$$\mathbf{M}_g = \mathbf{M}_{g,\zeta} = \mathbf{M}(\varepsilon_{g,\zeta})\,. \tag{86}$$

It is a consequence of ergodicity that the spectrum of these operators is nonrandom, i.e., there exists a nonrandom set Σ_g, such that $\sigma(\mathbf{M}_{g,\zeta}) = \sigma(M_{g,\zeta}) = \Sigma_g$ with probability one. In addition, the decompositions of $\sigma(\mathbf{M}_{g,\zeta})$ and $\sigma(M_{g,\zeta})$ into pure point spectrum, absolutely continuous spectrum and singular continuous spectrum are also independent of the choice of ζ with probability one[5,3,33].

The following theorem gives information on the location of Σ_g, the (nonrandom) spectrum of the random Maxwell operator $\mathbf{M}_g$. It shows that for not too large values of the coupling constant g the random array of defects shrinks the gap, but does not close it. We recall that a function $f(g)$ is called Lipschitz continuous on an interval if $|f(g) - f(g')| \leq C\,|g - g'|$ for all g, g' in the interval.

@tlct

Theorem 6 (Location of the spectrum). *Let the random operator* $\mathbf{M}_g$ *defined by (85) satisfy Assumptions 2 and 1 . There exists* g_0*, with@g0*

$$\frac{1}{U_+}\left(1-\frac{\omega_a}{\omega_b}\right) \tag{87}$$

$$\leq g_0 \leq \frac{1}{U_+}\min\left\{1,\left(\left(\frac{\omega_b}{\omega_a}\right)^{\frac{U_+}{U_-}}-1\right)\right\}, \tag{88}$$

and strictly increasing, Lipschitz continuous real valued functions $\omega_a^2(g)$ *and* $-\omega_b^2(g)$ *on the interval* $\left[0,\frac{1}{U_+}\right)$, *with* $\omega_a^2(0)=\omega_a^2$, $\omega_b^2(0)=\omega_b^2$ *and* $\omega_a^2(g)\leq\omega_b^2(g)$, *such that:*

(i) *Under the random perturbation, the spectrum expands into the gap* $(\omega_a^2(g),\omega_b^2(g))$*:*

$$\Sigma_g \bigcap \left[\omega_a^2,\omega_b^2\right] = \left[\omega_a^2,\omega_a^2(g)\right] \bigcup \left[\omega_b^2(g),\omega_b^2\right]\ . \tag{89}$$

(ii) *For* $g<g_0$, *we have* $\omega_a^2(g)<\omega_b^2(g)$ *and* $(\omega_a^2(g),\omega_b^2(g))$ *is a gap in the spectrum of the random operator* $\mathbf{M}_g$, *located inside the gap* (ω_a^2,ω_b^2) *of the unperturbed periodic operator* $\mathbf{M}_0$. *Moreover, we have*

$$\omega_a^2 \leq \omega_a^2\left(1+gU_+\right)^{\frac{U_-}{U_+}} \leq \omega_a^2(g) \leq \frac{\omega_a^2}{1-gU_+} \tag{90}$$

and

$$\omega_b^2\left(1-gU_+\right) \leq \omega_b^2(g) \leq \frac{\omega_b^2}{\left(1+gU_+\right)^{\frac{U_-}{U_+}}} \leq \omega_b^2. \tag{91}$$

(iii) *If* $g_0<\frac{1}{U_+}$, *we have* $\omega_a^2(g)=\omega_b^2(g)$ *for all* $g\in\left[g_0,\frac{1}{U_+}\right)$, *and the random operator* $\mathbf{M}_g$ *has no gap inside the gap* (ω_a^2,ω_b^2) *of the unperturbed periodic operator* $\mathbf{M}_0$, *i.e.,* $\left[\omega_a^2,\omega_b^2\right]\subset\Sigma_g$.

The proof of this theorem[12] proceeds by first approximating the (nonrandom) spectrum of the random operator by spectra of (nonrandom) periodic operators, which are then approximated by spectra of operators on finite cubes with periodic boundary condition. The latter operators have compact resolvents (i.e., Green's functions), and bounds on their eigenvalues are obtained by the min-max principle.

To state our results on localization, we need the following definition.

Definition 2 (Exponential localization). *We say that the random operator* $\mathbf{M}_g$ *exhibits localization in an interval* $I\subset\Sigma_g$, *if* $\mathbf{M}_g$ *has only pure point spectrum in* I *with probability one. We have exponential localization in* I *if we have localization and, with probability one, all the eigenfunctions corresponding to eigenvalues in* I *are exponentially decaying (in the sense of having exponentially decaying local* L^2*-norms).*

Remark 1 . *The curls of exponentially decaying eigenfunctions of* $\mathbf{M}_g$ *always have exponentially decaying local* L^2*-norms*[12]. *Thus the corresponding energy densities (see (4)) also have exponentially decaying local* L^2*-norms, uniformly in the time* t.

We show that random perturbations create exponentially localized eigenfunctions near the edges of the gap. Our method of proof requires low probability of extremal values for the random variables. The results below are formulated for the left edge of the gap, with similar results holding at the right edge.
@loc1

Theorem 7 (Localization at the edge). *Let the random operator* $\mathbf{M}_g$ *defined by (85) satisfy Assumptions 2 and 1, with@ext*

$$\int_{1-\gamma}^{1}\rho(t)dt \leq K\gamma^\eta \quad \textit{for} \quad 0\leq\gamma\leq 1, \tag{92}$$

where $K<\infty$ *and* $\eta>\frac{3}{2}$. *For any* $g<g_0$ *there exists* $\delta(g)>0$, *such that the random operator* $\mathbf{M}_g$ *exhibits exponential localization in the interval* $[\omega_a^2(g)-\delta(g),\omega_a^2(g)]$.

Theorem 7 is proven by a multiscale analysis[7,11,12], which reduces the proof to the verification of exponential decay of the (random) Green's function in a given finite scale, with high probability. This decay with high (appropriate for the scale) probability is then shown to hold for larger and larger scales. Finally, the exponential decay of the Green's function in all scales is used to show exponential localization. We will discuss some of the key steps in the proof.

Given an open cube Λ in three dimensional space and M as in (27), we will denote by M_Λ the restriction of M to Λ with Dirichlet boundary condition, i.e., $M_\Lambda = \nabla^\times\frac{1}{\varepsilon}\nabla^\times$ acting on square integrable functions on Λ with zero tangential component at the boundary. The corresponding Green's function is given by

$$\mathbf{G}_\Lambda(\omega) = \left(M_\Lambda-\omega^2\mathbf{I}\right)^{-1}. \tag{93}$$

The multiscale analysis requires control of the norm of the Green's functions of the operators $M_{g,\zeta,\Lambda}$, with high probability. This is given by a Wegner-type estimate[12], which says that the probability that the corresponding Green's functions $\mathbf{G}_{g,\zeta,\Lambda}(\omega)$ are bigger than a given number $\frac{1}{\eta}$, $0 < \eta \leq \omega^2$, is no more than proportional to $\eta \left|\Lambda\right|^2$, in fact,

$$\mathbb{P}\left\{ \left\| \mathbf{G}_{g,\zeta,\Lambda}\left(\omega\right) \right\| \geq \frac{1}{\eta} \right\} \leq Q\omega\eta \left|\Lambda\right|^2 , \tag{94}$$

where Q is some constant. This estimate is typically used when Λ is a cube of size L and $\eta = L^{-s}$ for suitable $s > 6$, so $\frac{1}{\eta} = L^s$ is large and $\eta \left|\Lambda\right|^2 = L^{-s+6}$ is small. It already indicates that, with high probability, the eigenvalues of $\mathbf{M}_{g,\zeta,\Lambda}$ do not want to be too close to any given ω^2, a precursor of Andeson localization.

Now let $\Lambda_L(\mathbf{x})$ denote the cube of side L centered at $\mathbf{x}$. We write $\mathbf{G}_{g,\zeta,\mathbf{x},L}\left(\omega\right) = \mathbf{G}_{g,\zeta,\Lambda_L(\mathbf{x})}\left(\omega\right)$. Given $m > 0$ and $\omega > 0$, we say that the cube $\Lambda_L(\mathbf{x})$ is regular (for a fixed g, ζ), if

$$\left\| \Gamma_{\mathbf{x},L} \mathbf{G}_{g,\zeta,\mathbf{x},L}\left(\omega\right) \chi_{\mathbf{x}} \right\| \leq \mathrm{e}^{-m\frac{L}{2}}, \tag{95}$$

where $\chi_{\mathbf{x}}$ is as in (37) and $\Gamma_{\mathbf{x},L}$ is the characteristic function of $\Lambda_{L-q}(\mathbf{x}) \backslash \Lambda_{L-3q}(\mathbf{x})$, which plays the role of a thick boundary. Note that (95) says that the finite cube Green's function is localized in the sense that it decays exponentially from the center of the cube to its boundary, with the given rate m. It turns out that this regularity of the finite cube Green's function is an indication of the exponential localization of the random operator, and that to prove localization, it suffices to show that it occurs with high probability at a sufficiently large scale.

For a given scale L, let us call $\mathbf{P}(L)$ the probabilistic statement:

$$\mathbb{P}\left\{ \Lambda_L(\mathbf{x}) \text{ is regular} \right\} \geq 1 - \frac{1}{L^p}, \tag{96}$$

where $p > 3$ is some fixed number of our choice. The multiscale analysis states that if we can verify $\mathbf{P}(L_0)$ for some sufficiently large scale L_0, then $\mathbf{P}(L)$ is also true for all scales $L = L_k$, where $L_{k+1} = L_k^\alpha$ for some appropriate $\alpha > 1$, $k = 0, 1, 2, \ldots$.This is shown by an induction argument[7], using (94). In fact, one proves a stronger statement at all scales L_k, from which we get the conclusions of Theorem 7.

Thus, to prove Theorem 7 it suffices to verify $\mathbf{P}(L)$ for some sufficiently large scale $L = L_0$. This is done using the assumption (92) and a finite cube version of Theorem 1. *We call* $\mathbf{P}(L_0)$ *the starting hypothesis for the multiscale analysis.*

Theorem 7 can be extended to the situation when the gap is totally filled by the spectrum of the random operator[12]. In this case we establish the existence of an interval (inside the original gap) where the random Maxwell operator exhibits exponential localization. Under somewhat different assumptions we can arrange for localization in any fraction of the gap as we want.

@loc10

Theorem 8 (Localization at the meeting of the edges). *Let the random operator* $\mathbf{M}_g$ *defined by (85) satisfy Assumptions 2 and 1, with*

$$\int_{1-\gamma}^{1} \rho(t)dt, \ \int_{-1}^{-1+\gamma} \rho(t)dt \leq K\gamma^\eta \tag{97}$$

$$\text{for } 0 \leq \gamma \leq 1,$$

where $K < \infty$ *and* $\eta > 3$. *Suppose* $g_0 < \frac{1}{U_+}$ *(e.g., if* $\left(\frac{\omega_b}{\omega_a}\right)^{\frac{U_+}{U_-}} < 2$*), so the random operator* $\mathbf{M}_g$ *has no gap inside* (ω_a^2, ω_b^2) *for* $g \in [g_0, \frac{1}{U_+})$. *Then there exist* $0 < \epsilon < \frac{1}{U_+} - g_0$ *and* $\delta > 0$, *such that the random operator* $\mathbf{M}_g$ *exhibits exponential localization in the interval* $[\omega_a^2(g_0) - \delta, \omega_a^2(g_0) + \delta]$ *for all* $g_0 \leq g < g_0 + \epsilon$.

Theorem 8 is proved exactly as Theorem 7, taking into account both edges of the gap.

Remark 2 *Theorems 7 and 8 should be true without the extra hypotheses (92) and (97), at least if the edges of the spectral gap are non-degerate. They are used to obtain the starting hypothesis for the multiscale analysis, in the proof of localization. If the edge of the gap is non-degenerate, one may expect estimates similar to Lifshitz tails*[33] *for the density of states inside the gap, which would replace (92) and (97) in the proofs. This is how the starting hypothesis is obtained for random Schrödinger operators at the bottom of the spectrum*[19]. *Estimates of Lifshitz' tails in spectral gaps of periodic Schrödinger operator have been obtained at non-degenerate edges*[26].

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7. Conclusions

We discussed the phenomenon of localization of electromagnetic waves in losless inhomogeneous dielectric media. Our starting point was a three dimensional lossless periodic dielectric medium (photonic crystal), exhibiting a gap in the frequency spectrum (photonic band gap). If such a medium is perturbed by either a single defect or a random array of defects, exponentially localized electromagnetic waves may arise with frequencies in the gap.

For a single defect we gave a simple condition to ensure the rise of exponentially localized electromagnetic waves with frequency in a specified subinterval of the photonic band gap. We derived equations for these midgap frequencies, and estimated their number.

For a random array of defects we showed that, under some natural conditions, the gap shrinks, but does not close, and we get an infinite number of localized electromagnetic waves with frequencies dense in an interval contained in the spectral gap of the underlying periodic medium. This phenomenon is analogous to the Anderson localization of electron waves in random media.

An important technical achievement of our proofs is that no assumptions are made on the smoothness of the function $\varepsilon(\mathbf{x})$, which are so common in almost all classical results on partial differential elliptic operators. This was possible due to a variational approach to the problems and the treatment of the relevant operators as quadratic forms. *Such general conditions on* $\varepsilon(\mathbf{x})$, *i.e., the bounds in (3) and the lack of smoothness, are required on physical grounds.* In practice only a few materials are used in the fabrication of periodic and disordered media, in which case $\varepsilon(\mathbf{x})$ takes just a finite number of values, so $\varepsilon(\mathbf{x})$ is piecewise constant, hence discontinuous.

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