Taming Griffiths' Singularities in Long Range Random Ising Models

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Dedicated to the memory of Roland Dobrushin

Abstract

We study long range random Ising models and develop modified high temperature and strong magnetic field expansions that give decay of truncated correlation functions and uniqueness of Gibbs states, in spite of the presence of Griffiths' singularities.

1 Introduction

In 1969 Griffiths [G] considered the statistical mechanics of a random ferromagnetic Ising model; he showed that for the site diluted model, the quenched magnetization displayed non-analytic behavior for values of the inverse temperature at which the system has neither long-range order nor spontaneous magnetization (see also [F],[Su]). His arguments are believed to apply to a large class of random models. At the origin of this behavior is the fact that even if, with probability one, the infinite system is not ordered as whole, there are, also with probability one, infinitely many arbitrarily large regions inside which the system is strongly correlated.

This phenomenon of *Griffiths' singularities* is now recognized to be a regular feature in the statistical mechanics of disordered systems. It has the unpleasant consequence that the usual high temperature or strong magnetic field expansions, the standard tools for obtaining decay of correlation functions (and also existence and uniqueness of the thermodynamical limit), fail to converge.

*This author was supported in part by the NSF Grant DMS-9500720 .
†This author was supported in part by the NSF Grant DMS-9500720 .

In this article we consider long range random Ising models; we develop modified high temperature and strong magnetic field expansions that give decay of truncated correlation functions and uniqueness of Gibbs states, in spite of the presence of Griffiths' singularities. Our expansions are based on the work of von Dreifus, Klein and Perez [DKP].

The short range random Ising model was studied by Berretti [B], Fröhlich and Imbrie [FI], Bassalygo and Dobrushin [BD], von Dreifus, Klein and Perez [DKP] and Gielis and Maes [GM]. The high temperature behavior of long range spin glasses has been studied by Fröhlich and Zegarlinski [FZ] and Zegarlinski [Z].

The subject of this paper is the long range random Ising model whose Hamiltonian in a finite volume $\Lambda \subset \mathbb{Z}^d$ *is given by*

$$H_\Lambda(\sigma) = -\sum_{\{x,y\}\in\Lambda^*} J_{xy}\sigma_x\sigma_y + \mathcal{B}\sum_{x\in\Lambda} h_x\sigma_x, \tag{1}$$

with exchange couplings $\mathbf{J} = \left\{J_{xy}; \{x,y\} \in \mathbb{Z}^{d^*}\right\}$ *and magnetic field* $\mathbf{h} = \left\{h_x; x \in \mathbb{Z}^d\right\}$*; we use the notation* $\Gamma^* = \{\{x,y\}; x,y \in \Gamma, x \neq y\}$ *for* $\Gamma \subset \mathbb{Z}^d$, $\sigma = \left\{\sigma_x; x \in \mathbb{Z}^d\right\} \in \{-1,1\}^{\mathbb{Z}^d}$*. We assume* $J_{xy} = \frac{X_{xy}}{|x-y|^{\alpha d}}$ *for each* $\{x,y\} \in \mathbb{Z}^{d^*}$*, where* $\alpha > 1$*. We take* $\mathbf{X} = \left\{X_{xy}; \{x,y\} \in \mathbb{Z}^{d^*}\right\}$ *and* $\mathbf{h}$ *to be independent families of independent identically distributed (within each family) random variables, with* $\widetilde{X} \equiv \mathbb{E}(|X_{xy}|) < \infty$.

Since $\alpha > 1$, for any $x \in \mathbb{Z}^d$ we have $\sum_{y\in\mathbb{Z}^d} |J_{xy}| < \infty$ for a.e. $\mathbf{J}$, so boundary conditions can be introduced in the usual way. A boundary condition is a map $\chi\colon \mathbb{Z}^d \to [-1,1]$. It is an external boundary condition if it is a configuration on $\mathbb{Z}^d$, i.e., a map $\chi\colon \mathbb{Z}^d \to \{-1,1\}$. If $\chi \equiv 0$ we have free boundary conditions, in which case we omit χ in the notation below. For a given boundary condition χ we set

$$H_\Lambda^\chi(\sigma) = H_\Lambda(\sigma) - \sum_{\substack{x\in\Lambda \\ y\notin\Lambda}} J_{xy}\sigma_x\chi_y. \tag{2}$$

We now define thermal averages for local observables; A is a local observable if it depends nontrivially on only a finite number of sites which we call the support of A. We set $\|A\| = \sup_\sigma |A(\sigma)|$. If A and B are local observables we will write $d(A,B)$ for the distance between the supports of A and B. Given an inverse temperature β, a local observable A, a finite volume Λ containing the support of A and a boundary condition χ, we define

$$\langle A\rangle_\Lambda^\chi = \frac{1}{Z_\Lambda^\chi} \sum_\sigma A(\sigma)\, e^{-\beta H_\Lambda^\chi}(\sigma), \quad \text{with} \quad Z_\Lambda^\chi = \sum_\sigma e^{-\beta H_\Lambda^\chi(\sigma)}. \tag{3}$$

The truncated finite volume correlation function of two local observables A and B, with boundary condition χ, is defined by

$$\langle A;B\rangle_\Lambda^\chi = \langle AB\rangle_\Lambda^\chi - \langle A\rangle_\Lambda^\chi \langle B\rangle_\Lambda^\chi. \tag{4}$$

We start with the high temperature case. *In this case we fix arbitrary* $\mathcal{B} \in \mathbb{R}$ *and* $\mathbf{h} \in \mathbb{R}^{\mathbb{Z}^d}$ *in (1); only* $\mathbf{J}$ *is random (all our estimates will be uniform in* $\mathcal{B}$ *and* $\mathbf{h}$*).*

Theorem 1 (High Temperature Regime) *There exists $\beta_1 = \beta_1(\alpha, d) > 0$, such that for all $0 < \beta < \beta_1$ we can find $C = C(\beta) < \infty$, such that for any two local observables A and B and any finite Λ containing their supports, we have*

$$\mathbb{E}\left(\left|\langle A; B\rangle_\Lambda^\chi\right|\right) \le C \left\|A\right\| \left\|B\right\| \left|supp\ A\right| \left|\ supp\ B\right| d\left(A,B\right)^{-\alpha d}. \tag{5}$$

for all $\mathcal{B} \in \mathbb{R}$, $h \in \mathbb{R}^{\mathbb{Z}^d}$, and any boundary condition χ.

If in addition $\mathbb{E}\left(\left|X_{xy}\right|^{\frac{\alpha}{\alpha-1}+\epsilon}\right) \le C_1 < \infty$ for some $\epsilon > 0$, then for each $0 < \beta < \beta_1$ we have, with probability one (in $\mathbf{J}$), that there exists a unique Gibbs State and for every local observable A

$$\langle A\rangle \equiv \lim_{\Lambda \to \mathbb{Z}^d} \langle A\rangle_\Lambda^{\chi_\Lambda} \tag{6}$$

exists and is independent of the choice of the boundary conditions χ_Λ.

We now turn to the strong magnetic field case. *In this case both $\mathbf{J}$ and $\mathbf{h}$ are random, and we assume $\mathbb{P}(h_x = 0) = 0$.*

Theorem 2 (Strong Magnetic Field Regime) *Let $1 < \alpha' < \alpha$. Then for all $\beta > 0$ there exists $\mathcal{B}_1 = \mathcal{B}_1(\beta, \alpha', d) > 0$, such that for all $\mathcal{B}$ with $|\mathcal{B}| > \mathcal{B}_1$ we can find $C = C(\mathcal{B}) < \infty$ such that for any two local observables A and B and any finite Λ containing their supports, we have*

$$\mathbb{E}\left(\left|\langle A; B\rangle_\Lambda^\chi\right|\right) \le C \left\|A\right\| \left\|B\right\| \left|supp\ A\right| \left|\ supp\ B\right| d\left(A,B\right)^{-\alpha' d}. \tag{7}$$

If in addition $\mathbb{E}\left(\left|X_{xy}\right|^{\frac{\alpha'}{\alpha'-1}+\epsilon}\right) \le C_1 < \infty$ for some $\epsilon > 0$, then for all $|\mathcal{B}| > \mathcal{B}_1$ we have, with probability one (in $\mathbf{J}$ and $\mathbf{h}$), that there exists a unique Gibbs State and for every local observable A

$$\langle A\rangle \equiv \lim_{\Lambda \to \mathbb{Z}^d} \langle A\rangle_\Lambda^{\chi_\Lambda} \tag{8}$$

exists and is independent of the choice of the boundary conditions χ_Λ.

2 The Expansions

To deal with truncated correlation functions we use the duplication trick; we consider two non-interacting copies of the system, i.e., a new spin system with configurations $\widetilde{\sigma} = \left\{\widetilde{\sigma}_x = (\sigma_x, \sigma'_x); x \in \mathbb{Z}^d\right\}$, $\sigma_x, \sigma'_x \in \{-1, +1\}$, and finite volume Hamiltonian $\widetilde{H}_\Lambda^\chi(\sigma)$, where for any function $F(\sigma)$ we set $\widetilde{F}(\widetilde{\sigma}) = F(\sigma) + F(\sigma')$. Finite volume thermal averages of a local observable $C(\widetilde{\sigma})$ in the duplicated system, with boundary condition χ (same for both copies), are given by

$$\langle\langle C\rangle\rangle_\Lambda^\chi = \frac{1}{\widetilde{Z}_\Lambda^\chi} \sum_{\widetilde{\sigma}} C(\widetilde{\sigma})\, e^{-\beta \widetilde{H}_\Lambda^\chi(\widetilde{\sigma})}, \text{ where } \widetilde{Z}_\Lambda^\chi = \sum_{\widetilde{\sigma}} e^{-\beta \widetilde{H}_\Lambda^\chi(\widetilde{\sigma})}. \tag{9}$$

Truncated correlation functions of the original system may be expressed as ordinary correlation functions of the duplicated system through the identity

$$\langle A;B\rangle_{\Lambda}^{\chi}=\frac{1}{2}\left\langle\left\langle\widehat{A}\widehat{B}\right\rangle\right\rangle_{\Lambda}^{\chi} \tag{10}$$

where to every local observable A of the original system we associate an observable $\widehat{A}$ of the duplicated system by setting $\widehat{A}(\widetilde{\sigma})=A(\sigma)-A(\sigma')$.

2.1 The High Temperature Expansion

We start with an infinite range version of [DKP, Theorem 2.1]; the main difference being the use of long range self avoiding bond walks. This deterministic result holds for an arbitrary set $\mathcal{S}\subset\mathbb{Z}^{d^*}$; it will be later applied to the case of random exchange couplings $\mathbf{J}$, with the "singular" set $\mathcal{S}$ chosen appropriately for each realization of $\mathbf{J}$.

A self avoiding (bond) walk from a site x to a site y, written $\omega : x\to y$, is a finite sequence of bonds $\{x_1,y_1\},\{x_2,y_2\},\ldots,\{x_n\ ,y_n\}$, such that $x_1=x$ and $y_n=y$, $x_{i+1}=y_i$ for $i=1,\ldots,n-1$, and $x_i\neq x_j$ if $i\neq j$. For such a walk we set $|\omega|=n$. We define $\mathcal{W}_{xy}=\{\omega:x\to y\}$; if A and B are local observables we set $\mathcal{W}_{AB}=\{\omega:x\to y;\ x\in supp\ A,\ y\in supp\ B\}$.

Theorem 3 *Given $\mathcal{S}\subset\mathbb{Z}^{d^*}$ let*

$$\rho_{xy}=\rho_{xy}(\mathcal{S},\mathbf{J},\beta)=\begin{cases}\xi_{xy}\equiv e^{4\beta|J_{xy}|}-1 & \text{if } \{x,y\}\notin\mathcal{S}\\ 1 & \text{if } \{x,y\}\in\mathcal{S}\end{cases} \tag{11}$$

Then for any two local observables A and B, any finite Λ containing their supports, and any boundary condition χ we have

$$|\langle A;B\rangle_{\Lambda}^{\chi}|\leq 2\,\|A\|\,\|B\|\sum_{\omega\in\mathcal{W}_{AB}}\prod_{\{x,y\}\in\omega}\rho_{xy}. \tag{12}$$

Proof. We start by redefining the Hamiltonian as

$$H_{\Lambda}(\sigma)=-\sum_{\{x,y\}\in\Lambda^*}\left(J_{xy}\sigma_x\sigma_y+|J_{xy}|\right)+\mathcal{B}\sum_{x\in\Lambda}h_x\sigma_x \tag{13}$$

which differs from (1) by an overall constant. We then perform an expansion only in $\Lambda^*\backslash\mathcal{S}$. Let

$$T_{\Lambda^*\backslash\mathcal{S}}=-\sum_{\{x,y\}\in\Lambda^*\backslash\mathcal{S}}\left(J_{xy}\sigma_x\sigma_y+|J_{xy}|\right) \tag{14}$$

and

$$T_{\Lambda^*\cap\mathcal{S}}^{\chi}=H_{\Lambda}^{\chi}-T_{\Lambda^*\backslash\mathcal{S}}\ . \tag{15}$$

Also let

$$E_{xy}=\exp\left\{\beta\left(J_{xy}\left(\sigma_x\sigma_y+\sigma_x'\sigma_y'\right)+2\,|J_{xy}|\right)\right\}-1\ . \tag{16}$$

Then

$$\left\langle\left\langle \widehat{A}\widehat{B}\right\rangle\right\rangle_\Lambda^\chi = \frac{1}{\tilde{Z}_\Lambda^\chi}\sum_{\tilde{\sigma}} \widehat{A}\widehat{B}\, e^{-\beta \tilde{T}^\chi_{\Lambda^*\cap\mathcal{S}}}\, e^{-\beta \tilde{T}_{\Lambda^*\backslash\mathcal{S}}} \tag{17}$$

$$= \frac{1}{\tilde{Z}_\Lambda^\chi}\sum_{\tilde{\sigma}} \widehat{A}\widehat{B}\, e^{-\beta \tilde{T}^\chi_{\Lambda^*\cap\mathcal{S}}}\, e^{-\beta\left(-\sum_{\{x,y\}\in\Lambda^*\backslash\mathcal{S}}\left[J_{xy}\left(\sigma_x\sigma_y+\sigma_x'\sigma_y'\right)+2|J_{xy}|\right]\right)} \tag{18}$$

$$= \frac{1}{\tilde{Z}_\Lambda^\chi}\sum_{\tilde{\sigma}} \widehat{A}\widehat{B}\, e^{-\beta \tilde{T}^\chi_{\Lambda^*\cap\mathcal{S}}} \prod_{\{x,y\}\in\Lambda^*\backslash\mathcal{S}} \left(E_{xy}+1\right) \tag{19}$$

$$= \frac{1}{\tilde{Z}_\Lambda^\chi}\sum_{\tilde{\sigma}} \widehat{A}\widehat{B}\, e^{-\beta \tilde{T}^\chi_{\Lambda^*\cap\mathcal{S}}} \sum_{G\subset\Lambda^*\backslash\mathcal{S}}\ \prod_{\{x,y\}\in G} E_{xy}\ . \tag{20}$$

Now suppose G is such that $G\cup\mathcal{S}$ does not contain a walk $\omega\in\mathcal{W}_{AB}$. Due to the invariance of the Hamiltonian under the exchange $\sigma \longleftrightarrow \sigma'$, such a set would contribute zero to the above sum. We may therefore restrict the sum to those G of the form $G=\omega_{\mathcal{S}}\cup G'$, where $\omega_{\mathcal{S}}=\omega\backslash\mathcal{S}$ and $G'\subset(\Lambda^*\backslash\mathcal{S})\backslash\omega_{\mathcal{S}}$.

Thus

$$\left\langle\left\langle \widehat{A}\widehat{B}\right\rangle\right\rangle_\Lambda^\chi = \frac{1}{\tilde{Z}_\Lambda^\chi}\sum_{\tilde{\sigma}} \widehat{A}\widehat{B}\, e^{-\beta \tilde{T}^\chi_{\Lambda^*\cap\mathcal{S}}} \sum_{\omega\in\mathcal{W}_{AB}}\ \prod_{\{x,y\}\in\omega_{\mathcal{S}}} E_{xy} \sum_{G'\subset(\Lambda^*\backslash\mathcal{S})\backslash\omega_{\mathcal{S}}}\ \prod_{\{x',y'\}\in G'} E_{x'y'} \tag{21}$$

$$\leq \frac{1}{\tilde{Z}_\Lambda^\chi}\sum_{\omega\in\mathcal{W}_{AB}}\ \prod_{\{x,y\}\in\omega_{\mathcal{S}}} E_{xy} \sum_{\tilde{\sigma}} \widehat{A}\widehat{B}\, e^{-\beta \tilde{T}^\chi_{\Lambda^*\cap\mathcal{S}}} \sum_{G'\subset(\Lambda^*\backslash\mathcal{S})\backslash\omega_{\mathcal{S}}}\ \prod_{\{x',y'\}\in G'} E_{x'y'} \tag{22}$$

$$\leq 4\left\|A\right\|\,\left\|B\right\| \frac{1}{\tilde{Z}_\Lambda^\chi}\sum_{\omega\in\mathcal{W}_{AB}}\ \prod_{\{x,y\}\in\omega_{\mathcal{S}}} \xi_{xy} \sum_{\tilde{\sigma}}\ e^{-\beta \tilde{T}^\chi_{\Lambda^*\cap\mathcal{S}}} \prod_{\{x,y\}\in\Lambda^*\backslash\mathcal{S}} \left(E_{xy}+1\right) \tag{23}$$

$$= 4\left\|A\right\|\,\left\|B\right\| \frac{1}{\tilde{Z}_\Lambda^\chi}\left(\sum_{\omega\in\mathcal{W}_{AB}}\ \prod_{\{x,y\}\in\omega_{\mathcal{S}}} \xi_{xy}\right)\tilde{Z}_\Lambda^\chi \tag{24}$$

$$\leq 4\left\|A\right\|\,\left\|B\right\| \sum_{\omega\in\mathcal{W}_{AB}}\ \prod_{\{x,y\}\in\omega} \rho_{xy}\ , \tag{25}$$

which, in view of (10), is the same as (12). □

We can now prove the first part of Theorem 1. Taking expectation of both sides of (12) gives

$$\mathbb{E}\left(\left|\langle A;B\rangle_\Lambda^\chi\right|\right) \leq 2\left\|A\right\|\,\left\|B\right\| \sum_{\omega\in\mathcal{W}_{AB}}\ \prod_{\{x,y\}\in\omega} \overline{\rho_{xy}} \leq 2\left\|A\right\|\,\left\|B\right\| \sum_{\substack{x_0\in\operatorname{supp}A\\ y_0\in\operatorname{supp}B}} \Gamma_{x_0y_0}\,, \tag{26}$$

where $\Gamma_{x_0y_0}\equiv\sum_{\omega\in\mathcal{W}_{x_0y_0}}\prod_{\{x,y\}\in\omega}\overline{\rho_{xy}}$; we used the fact that the ρ_{xy}'s are independent random variables. Here

$$\overline{\rho_{xy}}\equiv\mathbb{E}\left(\rho_{xy}\right)\leq\mathbb{E}\left(\xi_{xy};\{x,y\}\notin\mathcal{S}\right)+\mathbb{P}\left(\{x,y\}\in\mathcal{S}\right). \tag{27}$$

Let us now choose $\mathcal{S}=\left\{\{x,y\}\in\mathbb{Z}^{d^*};\ \xi_{xy}>1\right\}$, i.e., $\mathcal{S}=\left\{\{x,y\}\in\mathbb{Z}^{d^*};|J_{xy}|>\delta\right\}$ with $\delta=\frac{\ln 2}{4\beta}$. It follows that

$$\mathbb{P}\left(\{x,y\}\in\mathcal{S}\right)\leq\frac{1}{\delta}\mathbb{E}\left(|J_{xy}|\right)=\frac{1}{\delta}\frac{\widetilde{X}}{|x-y|^{\alpha d}}\ . \tag{28}$$

Setting $J^{(\delta)}_{xy} = J_{xy}$ if $|J_{xy}| \le \delta$ and $J^{(\delta)}_{xy} = 0$ otherwise, it follows from (11) and (27) that for β small enough we have

$$\overline{\rho_{xy}} \le \mathbb{E}\ \left(5\beta \left|J^{(\delta)}_{xy}\right|\right) + \frac{1}{\delta}\frac{\widetilde{X}}{|x-y|^{\alpha d}} \le \left(5 + \frac{4}{\ln 2}\right)\frac{\beta \widetilde{X}}{|x-y|^{\alpha d}}. \tag{29}$$

¿From (29) we can see that given any $\theta > 0$ we can choose β small enough such that $\overline{\rho_{xy}} \le \dfrac{\theta}{|x-y|^{\alpha d}}$. This along with (26) gives

$$\begin{aligned}
\Gamma_{0y_0} &\le \sum_{\omega:0\to y_0} \prod_{\{x,y\}\in\omega} \frac{\theta}{|x-y|^{\alpha d}} && (30)\\
&= \sum_{n=1}^{\infty} \theta^n \sum_{x_1,\ldots,x_n} \frac{1}{|x_1|^{\alpha d}} \frac{1}{|x_2 - x_1|^{\alpha d}} \cdots \frac{1}{|y_0 - x_{n-1}|^{\alpha d}} && (31)\\
&= \sum_{n=1}^{\infty} \theta^n \underbrace{(\varphi * \varphi * \ldots * \varphi)}_{\text{n times}} (y_0)\,, && (32)
\end{aligned}$$

where

$$\varphi(x) = \begin{cases} \frac{1}{|x|^{\alpha d}} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{33}$$

The following Lemma gives an upper bound for the n-fold convolution in (32).

Lemma 4 *Let $\varphi, \psi : \mathbb{Z}^d \to \mathbb{R}$ and suppose there exist finite constants C_φ and C_ψ such that $|\varphi(x)| \le \dfrac{C_\varphi}{1+|x|^{\alpha d}}$ and $|\psi(x)| \le \dfrac{C_\psi}{1+|x|^{\alpha d}}$ for all $x \in \mathbb{Z}^d$. Then*

$$|(\varphi * \psi)(x)| \le \frac{2^{\alpha d+1} K C_\varphi C_\psi}{1+|x|^{\alpha d}} \tag{34}$$

for all $x \in \mathbb{Z}^d$, where $K \equiv \sum_{x\in\mathbb{Z}^d} \dfrac{1}{1+|x|^{\alpha d}}$.

Proof. We have

$$\begin{aligned}
|(\varphi * \psi)(x)| &= \left|\sum_{y\in\mathbb{Z}^d} \varphi(x-y)\psi(y)\right| \le \sum_{y\in\mathbb{Z}^d} \frac{C_\varphi}{1+|x-y|^{\alpha d}} \frac{C_\psi}{1+|y|^{\alpha d}} \\
&= \sum_{\substack{y\in\mathbb{Z}^d \\ |x-y|\le\frac{|x|}{2}}} \frac{C_\varphi}{1+|x-y|^{\alpha d}} \frac{C_\psi}{1+|y|^{\alpha d}} + \sum_{\substack{y\in\mathbb{Z}^d \\ |x-y|>\frac{|x|}{2}}} \frac{C_\varphi}{1+|x-y|^{\alpha d}} \frac{C_\psi}{1+|y|^{\alpha d}}. && (35)
\end{aligned}$$

We treat each sum in (35) separately:

$$\begin{aligned}
\sum_{\substack{y\in\mathbb{Z}^d\\ |x-y|\le\frac{|x|}{2}}} \frac{C_\varphi}{1+|x-y|^{\alpha d}}\frac{C_\psi}{1+|y|^{\alpha d}} &\le \frac{C_\varphi C_\psi}{1+\left(\frac{|x|}{2}\right)^{\alpha d}} \sum_{\substack{y\in\mathbb{Z}^d\\ |x-y|\le\frac{|x|}{2}}} \frac{1}{1+|x-y|^{\alpha d}} \\
&\le \frac{C_\varphi C_\psi K}{1+\left(\frac{|x|}{2}\right)^{\alpha d}} \le \frac{2^{\alpha d} C_\varphi C_\psi K}{1+|x|^{\alpha d}},
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
\sum_{\substack{y\in\mathbb{Z}^d\\ |x-y|>\frac{|x|}{2}}} \frac{C_\varphi}{1+|x-y|^{\alpha d}}\frac{C_\psi}{1+|y|^{\alpha d}} &\le C_\varphi C_\psi \sum_{\substack{y\in\mathbb{Z}^d\\ |x-y|>\frac{|x|}{2}}} \frac{1}{1+\left(\frac{|x|}{2}\right)^{\alpha d}}\frac{1}{1+|y|^{\alpha d}} \\
&\le \frac{2^{\alpha d} C_\varphi C_\psi K}{1+|x|^{\alpha d}}.
\end{aligned} \tag{37}$$

Combining (35), (36), and (37) we get (34) . □

Let φ be as in (33), using (34) and an induction we conclude that

$$\underbrace{(\varphi * \ldots * \varphi)}_{\text{n times}}(x) \le \frac{2\cdot\left(2^{\alpha d+2}K\right)^{n-1}}{1+|x|^{\alpha d}} \equiv K_1 K_2^n \frac{1}{1+|x|^{\alpha d}} \tag{38}$$

The above inequality and (32) lead to

$$\Gamma_{0y_0} \le 2\sum_{n=1}^{\infty} \theta^n K_1 K_2^n \frac{1}{1+|x|^{\alpha d}} = \frac{2K_1K_2\theta}{1-K_2\theta}\frac{1}{1+|x|^{\alpha d}} \equiv \frac{K_3}{1+|x|^{\alpha d}}, \tag{39}$$

where the equality holds if we choose $\theta < \dfrac{1}{K_2}$.

From (26), (39) and the translation invariance of expectations we get

$$\mathbb{E}\left(|\langle A;B\rangle_\Lambda^\chi|\right) \le 2\,\|A\|\,\|B\|\,K_3 \sum_{\substack{x_0\in\operatorname{supp}A\\ y_0\in\operatorname{supp}B}} \frac{1}{1+|x_0-y_0|^{\alpha d}}, \tag{40}$$

from which (5) follows, concluding the proof of the first part of Theorem 1.

2.2 The Strong Magnetic Field Expansion

The long range strong magnetic field expansion cannot be done as in the short range case [DKP]. The reason is that the events of sites being singular would not be independent for any two given sites. To circumvent this problem we rewrite the Hamiltonian given in (2) as the sum of "short range" and "long range" Hamiltonians; we then perform a short range strong magnetic field expansion on the short range part and a long range high temperature expansion on the long range part, the range being chosen large enough so that the latter expansion converges. Without loss of generality we assume $\mathcal{B} \geq 0$.

So let us fix a range $R > 0$, to be chosen later. We write $H_{\Lambda}^{\chi} = H_{\Lambda,s.r.}^{\chi} + H_{\Lambda,l.r.}^{\chi}$, where the short range Hamiltonian is

$$H_{\Lambda,s.r.}^{\chi}(\sigma) = -\sum_{\substack{\{x,y\} \in \Lambda^* \\ |x-y| \leq R}} J_{xy}\sigma_x\sigma_y + \mathcal{B}\sum_{x\in\Lambda} h_x\sigma_x - \sum_{\substack{x\in\Lambda \\ y\notin\Lambda \\ |x-y|\leq R}} J_{xy}\sigma_x\chi_y \ , \tag{41}$$

and the long range Hamiltonian is

$$H_{\Lambda,l.r.}^{\chi}(\sigma) = -\sum_{\substack{\{x,y\} \in \Lambda^* \\ |x-y| > R}} J_{xy}\sigma_x\sigma_y - \sum_{\substack{x\in\Lambda \\ y\notin\Lambda \\ |x-y| > R}} J_{xy}\sigma_x\chi_y. \tag{42}$$

We redefine $H_{\Lambda,l.r.}^{\chi}$ by adding an overall constant as follows:

$$H_{\Lambda,l.r.}^{\chi}(\sigma) = -\sum_{\substack{\{x,y\} \in \Lambda^* \\ |x-y| > R}} \left(J_{xy}\sigma_x\sigma_y + |J_{xy}|\right) - \sum_{\substack{x\in\Lambda \\ y\notin\Lambda \\ |x-y| > R}} J_{xy}\sigma_x\chi_y. \tag{43}$$

For a subset K of Λ^* we set

$$T_K(\sigma) = -\sum_{\substack{\{x,y\} \in K \\ |x-y| > R}} \left(J_{xy}\sigma_x\sigma_y + |J_{xy}|\right) \tag{44}$$

and

$$T_K^{\chi}(\sigma) = T_K(\sigma) - \sum_{\substack{x\in\Lambda \\ y\notin\Lambda \\ |x-y| > R}} J_{xy}\sigma_x\chi_y. \tag{45}$$

Note that

$$H_{\Lambda,l.r.}^{\chi} = T_{\Lambda^*}^{\chi} = T_{\Lambda^*\cap\mathcal{S}_b}^{\chi} + T_{\Lambda^*\setminus\ \mathcal{S}_b} \tag{46}$$

for any subset $\mathcal{S}_b$ of $\mathbb{Z}^{d^*}$.

We also define "long range bonds" and "short range sites" for a self avoiding walk $\omega = \{\{x_1,y_1\},\{x_2,y_2\},\ldots,\{x_n\ ,y_n\}\}$ (as defined in the previous subsection):

$$\begin{aligned} \omega_l &= \{\{x,y\} \in \omega; |x-y| > R\}, \\ \omega_s &= \left\{x \in \mathbb{Z}^d; x = x_i \text{ for some } i \text{ and either } |x - x_{i-1}| \leq R \text{ or } |x - x_{i+1}| \leq R\right\}. \end{aligned}$$

Theorem 5 *Given $\mathcal{S}_b \subset \mathbb{Z}^{d^*}$ and $\mathcal{S}_s \subset \mathbb{Z}^d$, let $\rho_{xy} = \rho_{xy}(\mathcal{S}_b, \mathbf{J}, \beta)$ be as in (11) and*

$$\theta_x = \theta_x(\mathcal{S}_s, \mathbf{J}, \mathcal{B}, \mathbf{h}, \beta) = \begin{cases} \zeta_x & \text{if } x \notin \mathcal{S}_s \\ 1 & \text{if } x \in \mathcal{S}_s \end{cases} \tag{47}$$

where

$$\zeta_x = \zeta_x(\mathbf{J}, \mathcal{B}, \mathbf{h}, \beta) = \exp\left\{-2\beta\left(2\mathcal{B}|h_x| - 6\sum_{\substack{y\in\mathbb{Z}^d\\|x-y|\le R}}|J_{xy}|\right)\right\}. \tag{48}$$

Then for any two local observables A and B, any finite Λ containing their supports, and any boundary condition χ we have

$$|\langle A;B\rangle_\Lambda^\chi| \le 2\|A\|\,\|B\|\sum_{\omega\in\mathcal{W}_{AB}}\prod_{x\in\omega_s}3\theta_x\prod_{\{x,y\}\in\omega_l}\rho_{xy}\ . \tag{49}$$

Proof. For $H^\chi_{\Lambda,s.r.}$ we proceed as in [DKP], we introduce variables $\eta = \left\{\eta_x; x\in\mathbb{Z}^d\right\}$, where

$$\eta_x = \frac{(\,\mathrm{sgn}\,h_x)\,\sigma_x + 1}{2} \in \{0,1\}, \tag{50}$$

with $\mathrm{sgn}\,u = 1$ if $u \ge 0$ and $\mathrm{sgn}\,u = -1$ otherwise. We set $K_{xy} = (\mathrm{sgn}\,h_x)(\mathrm{sgn}\,h_y)J_{xy}$. We redefine $H^\chi_{\Lambda,s.r.}$ by subtracting an overall constant and rewrite it in the new variables:

$$\begin{aligned} H^\chi_{\Lambda,s.r.}(\eta) = {} & -4\sum_{\substack{\{x,y\}\in\Lambda^*\\|x-y|\le R}}K_{xy}\eta_x\eta_y \\ & +2\sum_{x\in\Lambda}\left(\mathcal{B}|h_x| + \sum_{\substack{y\in\Lambda\\|x-y|\le R}}K_{xy} - \sum_{\substack{y\notin\Lambda\\|x-y|\le R}}K_{xy}(\mathrm{sgn}\,h_y)\chi_y\right)\eta_x. \end{aligned} \tag{51}$$

Given a configuration $\tilde\eta = (\eta,\eta')$ of the duplicated system we define

$$G_{\tilde\eta} = \left\{x\in\mathbb{Z}^d : \eta_x + \eta'_x > 0\right\},$$

and say that a configuration $\tilde\eta$ is compatible with $G\subset\mathbb{Z}^d$, and write $\tilde\eta \prec G$, if $G_{\tilde\eta} = G$. In what follows we consider $\tilde\sigma$ as a function of $\tilde\eta$ by (50), so we only need to sum over $\tilde\eta$. We have

$$\begin{aligned} \left\langle\left\langle\hat A\hat B\right\rangle\right\rangle^\chi_\Lambda &= \frac{1}{\tilde Z^\chi_\Lambda}\sum_{\tilde\eta}\hat A\hat B e^{-\beta\left(\tilde H^\chi_{\Lambda,s.r.}(\tilde\eta)+\tilde H^\chi_{\Lambda,l.r.}(\tilde\sigma)\right)} && (52)\\ &= \frac{1}{\tilde Z^\chi_\Lambda}\sum_{G\subset\Lambda}\sum_{\tilde\eta\prec G}\hat A\hat B e^{-\beta\tilde H^\chi_{\Lambda,s.r.}(\tilde\eta)}e^{-\beta\tilde H^\chi_{\Lambda,l.r.}(\tilde\sigma)} && (53)\\ &= \frac{1}{\tilde Z^\chi_\Lambda}\sum_{G\subset\Lambda}\sum_{\tilde\eta\prec G}\hat A\hat B e^{-\beta\tilde H^{\Lambda,\chi}_{G,s.r.}(\tilde\eta)}e^{-\beta\tilde H^\chi_{\Lambda,l.r.}(\tilde\sigma)} && (54)\\ &= \frac{1}{\tilde Z^\chi_\Lambda}\sum_{G\subset\Lambda}\sum_{\tilde\eta\prec G}\hat A\hat B\ e^{-\beta\tilde H^{\Lambda,\chi}_{G,s.r.}(\tilde\eta)}\ e^{-\beta\tilde T^\chi_{\Lambda^*\cap\mathcal{S}_b}(\tilde\sigma)}\ e^{-\beta\tilde T_{\Lambda^*\setminus\mathcal{S}_b}(\tilde\sigma)}, && (55) \end{aligned}$$

where

$$H^{\Lambda,\chi}_{G,s.r.}(\eta) = -4 \sum_{\substack{\{x,y\}\in G^* \\ |x-y|\le R}} K_{xy}\eta_x\eta_y \tag{56}$$

$$+2\sum_{x\in G}\left(\mathcal{B}\,|h_x| + \sum_{\substack{y\in\Lambda \\ |x-y|\le R}} K_{xy} - \sum_{\substack{y\notin\Lambda \\ |x-y|\le R}} K_{xy}\,(\operatorname{sgn} h_y)\,\chi_y\right)\eta_x.$$

The equality in (54) follows from the fact that for $\tilde{\eta} \prec G$ and $x \notin G$ we have $\eta_x = \eta'_x = 0$; (55) comes from (46).

To deal with the long range part of the above equation we follow the same method as in [DKP] using $E_{xy}(\tilde{\sigma}) = \exp\left\{\beta\left(J_{xy}\left(\sigma_x\sigma_y + \sigma'_x\sigma'_y\right) + 2\,|J_{xy}|\right)\right\} - 1$:

$$e^{-\beta\tilde{T}_{\Lambda^*\backslash\mathcal{S}_b}(\tilde{\sigma})} = \exp\left\{-\beta\left(-\sum_{\substack{\{x,y\}\in\Lambda^*\backslash\mathcal{S}_b \\ |x-y|>R}}\left(J_{xy}\left(\sigma_x\sigma_y+\sigma'_x\sigma'_y\right)+2\,|J_{xy}|\right)\right)\right\} \tag{57}$$

$$= \prod_{\substack{\{x,y\}\in\Lambda^*\backslash\mathcal{S}_b \\ |x-y|>R}}\left(E_{xy}(\tilde{\sigma})+1\right) = \sum_{G_b\subset\Lambda^*\backslash\mathcal{S}_b}\ \prod_{\substack{\{x,y\}\in G_b \\ |x-y|>R}} E_{xy}(\tilde{\sigma}), \tag{58}$$

so we have

$$\left\langle\left\langle\hat{A}\hat{B}\right\rangle\right\rangle^{\chi}_{\Lambda} = \frac{1}{\tilde{Z}^{\chi}_{\Lambda}}\sum_{G\subset\Lambda}\ \sum_{G_b\subset\Lambda^*\backslash\mathcal{S}_b}\ \sum_{\tilde{\eta}\prec G}\hat{A}\hat{B}\ e^{-\beta\tilde{H}^{\Lambda,\chi}_{G,s.r.}(\tilde{\eta})}\ e^{-\beta\tilde{T}^{\chi}_{\Lambda^*\cap\mathcal{S}_b}(\tilde{\sigma})}\prod_{\substack{\{x,y\}\in G_b \\ |x-y|>R}} E_{xy}(\tilde{\sigma}). \tag{59}$$

Now we notice that if $G\subset\Lambda$ and $G_b\subset\Lambda^*\backslash\mathcal{S}_b$, if there is no self avoiding walk ω from A to B with $\omega_s\subset G\cup\mathcal{S}_s$ and $\omega_l\subset G_b\cup\mathcal{S}_b$, then the term with G and G_b contribute zero to the sum in (59). So we can restrict the sum to pairs G, G_b for which there exist $\omega\in\mathcal{W}_{AB}$ such that $G = G'\cup(\omega_s\backslash\mathcal{S}_s)$, where $G'\subset\Lambda\backslash\left(\omega_s\backslash\mathcal{S}_s\right)$, and $G_b = G'_b\cup(\omega_l\backslash\mathcal{S}_b)$, where $G'_b\subset\left(\Lambda^*\backslash\mathcal{S}_b\right)\backslash\omega_l$. Thus

$$\left\langle\left\langle\hat{A}\hat{B}\right\rangle\right\rangle^{\chi}_{\Lambda} = \frac{1}{\tilde{Z}^{\chi}_{\Lambda}}\sum_{\omega\in\mathcal{W}_{AB}}\ \sum_{G'\subset\Lambda\backslash(\omega_s\backslash\mathcal{S}_s)}\ \sum_{\tilde{\eta}\prec G'\cup(\omega_s\backslash\mathcal{S}_s)}\hat{A}\hat{B}\ e^{-\beta\tilde{H}^{\Lambda,\chi}_{G'\cup(\omega_s\backslash\mathcal{S}_s),s.r.}(\tilde{\eta})} \tag{60}$$

$$\times e^{-\beta\tilde{T}^{\chi}_{\Lambda^*\cap\mathcal{S}_b}(\tilde{\sigma})}\left(\sum_{G'_b\subset(\Lambda^*\backslash\mathcal{S}_b)\backslash\omega_l}\ \prod_{\substack{\{x',y'\}\in G'_b \\ |x'-y'|>R}} E_{x'y'}\right)\left(\prod_{\{x,y\}\in\omega_l\backslash\mathcal{S}_b} E_{xy}\right),$$

and so

$$\left|\left\langle\left\langle\hat{A}\hat{B}\right\rangle\right\rangle^{\chi}_{\Lambda}\right| \le \frac{4}{\tilde{Z}^{\chi}_{\Lambda}}\,\|A\|\,\|B\|\sum_{\omega\in\mathcal{W}_{AB}}\ \sum_{G'\subset\Lambda\backslash(\omega_s\backslash\mathcal{S}_s)}\ \sum_{\tilde{\eta}\prec G'\cup(\omega_s\backslash\mathcal{S}_s)} e^{-\beta\tilde{H}^{\Lambda,\chi}_{G'\cup(\omega_s\backslash\mathcal{S}_s),s.r.}(\tilde{\eta})}$$

$$\times e^{-\beta \tilde{T}^{\chi}_{\Lambda^* \cap \mathcal{S}_b}} \left(\prod_{\substack{\{x',y'\} \in (\Lambda^* \backslash \mathcal{S}_b) \backslash \omega_l \\ |x'-y'|>R}} (E_{x'y'}+1) \right) \left(\prod_{\{x,y\} \in \omega_l \backslash \mathcal{S}_b} E_{xy} \right)$$

$$\times \prod_{\{x,y\} \in \omega_l \backslash \mathcal{S}_b} (E_{xy}+1) \ \text{(this expression is bounded below by 1!)} \quad (61)$$

$$= \frac{4}{\tilde{Z}^{\chi}_{\Lambda}} \|A\| \, \|B\| \sum_{\omega \in \mathcal{W}_{AB}} \sum_{G' \subset \Lambda \backslash (\omega_s \backslash \mathcal{S}_s)} \sum_{\tilde{\eta} \prec G' \cup (\omega_s \backslash \mathcal{S}_s)} e^{-\beta \tilde{H}^{\Lambda,\chi}_{G' \cup (\omega_s \backslash \mathcal{S}_s), s.r.}(\tilde{\eta})}$$

$$\underbrace{\times e^{-\beta \tilde{T}^{\chi}_{\Lambda^* \cap \mathcal{S}_b}} \left(\prod_{\substack{\{x',y'\} \in \Lambda^* \backslash \mathcal{S}_b \\ |x'-y'|>R}} (E_{x'y'}+1) \right)}_{\text{this is exactly } e^{-\beta \tilde{H}^{\chi}_{\Lambda, l.r.}(\tilde{\sigma})}} \left(\prod_{\{x,y\} \in \omega_l \backslash \mathcal{S}_b} E_{xy} \right) . \quad (62)$$

We will now look at the expression $H^{\Lambda,\chi}_{G' \cup (\omega_s \backslash \mathcal{S}_s), s.r.}(\eta)$ (given in (56)). Let us examine the first term of this expression :

$$\begin{aligned} -4 \sum_{\substack{\{x,y\} \in (G' \cup (\omega_s \backslash \mathcal{S}_s))^* \\ |x-y| \le R}} K_{xy} \eta_x \eta_y &= -4 \sum_{\substack{\{x,y\} \in (G')^* \\ |x-y| \le R}} K_{xy} \eta_x \eta_y - 4 \sum_{x \in (\omega_s \backslash \mathcal{S}_s)} \sum_{\substack{y \in G' \cup (\omega_s \backslash \mathcal{S}_s) \\ |x-y| \le R}} K_{xy} \eta_x \eta_y \\ &= -4 \sum_{\substack{\{x,y\} \in (G')^* \\ |x-y| \le R}} K_{xy} \eta_x \eta_y - 4 \sum_{x \in (\omega_s \backslash \mathcal{S}_s)} \sum_{\substack{y \in \Lambda \\ |x-y| \le R}} K_{xy} \eta_x \eta_y, \quad (63) \end{aligned}$$

where the last equality holds if $\tilde{\eta} \prec G' \cup (\omega_s \backslash \mathcal{S}_s)$. So now we have

$$\begin{aligned} H^{\Lambda,\chi}_{G' \cup (\omega_s \backslash \mathcal{S}_s), s.r.}(\eta) &= -4 \sum_{\substack{\{x,y\} \in (G')^* \\ |x-y| \le R}} K_{xy} \eta_x \eta_y - 4 \sum_{x \in (\omega_s \backslash \mathcal{S}_s)} \sum_{\substack{y \in \Lambda \\ |x-y| \le R}} K_{xy} \eta_x \eta_y \\ &\quad + 2 \sum_{x \in G'} \left(\mathcal{B} |h_x| + \sum_{\substack{y \in \Lambda \\ |x-y| \le R}} K_{xy} - \sum_{\substack{y \notin \Lambda \\ |x-y| \le R}} K_{xy} (\operatorname{sgn} h_y) \chi_y \right) \eta_x \\ &\quad + 2 \sum_{x \in \ (\omega_s \backslash \mathcal{S}_s)} \left(\mathcal{B} |h_x| + \sum_{\substack{y \in \Lambda \\ |x-y| \le R}} K_{xy} - \sum_{\substack{y \notin \Lambda \\ |x-y| \le R}} K_{xy} (\operatorname{sgn} h_y) \chi_y \right) \eta_x \\ &= H^{\Lambda,\chi}_{G', s.r.}(\eta) + \sum_{x \in (\omega_s \backslash \mathcal{S}_s)} \Gamma^{\chi}_{x,\Lambda}(\eta), \quad (64) \end{aligned}$$

where

$$\Gamma^{\chi}_{x,\Lambda}(\eta) = -4 \sum_{\substack{y \in \Lambda \\ |x-y| \le R}} K_{xy} \eta_x \eta_y + 2 \left(\mathcal{B} |h_x| + \sum_{\substack{y \in \Lambda \\ |x-y| \le R}} K_{xy} - \sum_{\substack{y \notin \Lambda \\ |x-y| \le R}} K_{xy} (\operatorname{sgn} h_y) \chi_y \right) \eta_x$$

$$\geq \quad 2\left(2\mathcal{B}\left|h_x\right| - 6\sum_{\substack{y\in\ \mathbb{Z}^d\\|x-y|\leq R}}\left|J_{xy}\right|\right)\equiv Y_{\mathcal{B},x}=Y_{\mathcal{B},x}\left(\mathbf{J}\ ,\mathbf{h}\right). \tag{65}$$

Thus

$$\begin{aligned}
\left|\left\langle\left\langle \hat{A}\hat{B}\right\rangle\right\rangle_{\Lambda}^{\chi}\right| &\leq \frac{4}{\tilde{Z}_{\Lambda}^{\chi}}\left\|A\right\|\left\|B\right\|\sum_{\omega\in\mathcal{W}_{AB}}\ \sum_{G'\subset\Lambda\backslash(\omega_s\backslash\mathcal{S}_s)}\ \sum_{\tilde{\eta}\prec G'\cup(\omega_s\backslash\mathcal{S}_s)} e^{-\beta\tilde{H}_{\Lambda,l.r.}^{\chi}(\tilde{\sigma})} \\
&\qquad \times e^{-\beta\tilde{H}_{G'}^{\Lambda,\chi}{}_{,s.r.}(\tilde{\eta})}\prod_{x\in(\omega_s\backslash\mathcal{S}_s)} e^{-\beta\tilde{\Gamma}_{x,\Lambda}^{\chi}(\tilde{\eta})}\left(\prod_{\{x,y\}\in\omega_l\backslash\mathcal{S}_b} E_{xy}\left(\tilde{\sigma}\right)\right) &(66)\\
&\leq \frac{4}{\tilde{Z}_{\Lambda}^{\chi}}\left\|A\right\|\left\|B\right\|\sum_{\omega\in\mathcal{W}_{AB}}\ \sum_{G'\subset\Lambda\backslash(\omega_s\backslash\mathcal{S}_s)}\ \sum_{\tilde{\eta}\prec G'\cup(\omega_s\backslash\mathcal{S}_s)} e^{-\beta\tilde{H}_{\Lambda,l.r.}^{\chi}(\tilde{\sigma})} \\
&\qquad \times e^{-\beta\tilde{H}_{G'}^{\Lambda,\chi}{}_{,s.r.}(\tilde{\eta})}\prod_{x\in(\omega_s\backslash\mathcal{S}_s)} e^{-\beta Y_{\mathcal{B},x}}\left(\prod_{\{x,y\}\in\omega_l\backslash\mathcal{S}_b} \xi_{xy}\right) &(67)\\
&= \frac{4}{\tilde{Z}_{\Lambda}^{\chi}}\left\|A\right\|\left\|B\right\|\sum_{\omega\in\mathcal{W}_{AB}}\ \prod_{x\in(\omega_s\backslash\mathcal{S}_s)} e^{-\beta Y_{\mathcal{B},x}}\left(\prod_{\{x,y\}\in\omega_l\backslash\mathcal{S}_b} \xi_{xy}\right) \\
&\qquad \times \sum_{G'\subset\Lambda\backslash(\omega_s\backslash\mathcal{S}_s)}\ \sum_{\tilde{\eta}\prec G'\cup(\omega_s\backslash\mathcal{S}_s)} e^{-\beta\tilde{H}_{\Lambda,l.r.}^{\chi}(\tilde{\sigma})}e^{-\beta\tilde{H}_{G'}^{\Lambda,\chi}{}_{,s.r.}(\tilde{\eta})}. &(68)
\end{aligned}$$

Since $\#\left\{\tilde{\eta}:\tilde{\eta}\prec G'\cup(\omega_s\backslash\mathcal{S}_s)\right\}=3^{|\omega_s\backslash\mathcal{S}_s|}\times\#\left\{\tilde{\eta}:\tilde{\eta}\prec G'\right\}$, we get

$$\begin{aligned}
\left|\left\langle\left\langle \hat{A}\hat{B}\right\rangle\right\rangle_{\Lambda}^{\chi}\right| &\leq \frac{4}{\tilde{Z}_{\Lambda}^{\chi}}\left\|A\right\|\left\|B\right\|\sum_{\omega\in\mathcal{W}_{AB}}\ \prod_{x\in(\omega_s\backslash\mathcal{S}_s)} 3e^{-\beta Y_{\mathcal{B},x}}\left(\prod_{\{x,y\}\in\omega_l\backslash\mathcal{S}_b} \xi_{xy}\right) \\
&\qquad \times \sum_{G'\subset\Lambda\backslash(\omega_s\backslash\mathcal{S}_s)}\ \sum_{\tilde{\eta}\prec G'} e^{-\beta\tilde{H}_{\Lambda,l.r.}^{\chi}(\tilde{\sigma})}e^{-\beta\tilde{H}_{G'}^{\Lambda,\chi}{}_{,s.r.}(\tilde{\eta})} &(69)\\
&\leq \frac{4}{\tilde{Z}_{\Lambda}^{\chi}}\left\|A\right\|\left\|B\right\|\sum_{\omega\in\mathcal{W}_{AB}}\ \prod_{x\in(\omega_s\backslash\mathcal{S}_s)} 3e^{-\beta Y_{\mathcal{B},x}}\left(\prod_{\{x,y\}\in\omega_l\backslash\mathcal{S}_b} \xi_{xy}\right) \\
&\qquad \times \underbrace{\sum_{G'\subset\Lambda}\ \sum_{\tilde{\eta}\prec G'} e^{-\beta\tilde{H}_{\Lambda,l.r.}^{\chi}(\tilde{\sigma})}e^{-\beta\tilde{H}_{G'}^{\Lambda,\chi}{}_{,s.r.}(\tilde{\eta})}}_{\text{this is exactly } \tilde{Z}_{\Lambda}^{\chi}} &(70)\\
&= 4\left\|A\right\|\left\|B\right\|\sum_{\omega\in\mathcal{W}_{AB}}\ \prod_{x\in(\omega_s\backslash\mathcal{S}_s)} 3e^{-\beta Y_{\mathcal{B},x}}\left(\prod_{\{x,y\}\in\omega_l\backslash\mathcal{S}_b} \xi_{xy}\right) &(71)\\
&= 4\left\|A\right\|\left\|B\right\|\sum_{\omega\in\mathcal{W}_{AB}}\ \prod_{x\in\omega_s} 3\theta_x \prod_{\{x,y\}\in\omega_l} \rho_{xy}, &(72)
\end{aligned}$$

so (49) now follows from (10). □

We will now prove the first part of Theorem 2. As before we will take expectations of both sides of (49). Notice that the random variables $\left\{\theta_x;\ x \in \mathbb{Z}^d\right\}$ are not independent. We set

$$\Xi_{0y_0} = \sum_{\omega:0\to y_0} \mathbb{E}\left(\prod_{x\in\omega_s} 3\theta_x\right) \prod_{\{x,y\}\in\omega_l} \overline{\rho_{xy}}. \tag{73}$$

We pick $\mathcal{S}_b = \left\{\{x,y\} \in \mathbb{Z}^{d^*}; |x-y| > R \text{ and } \xi_{xy} > 1\right\}$, so proceeding as in (29), we get, with R sufficiently large, that

$$\begin{aligned} \overline{\rho_{xy}} &\leq \left(5 + \frac{4}{\ln 2}\right) \frac{\beta\widetilde{X}}{|x-y|^{\alpha d}} && (74) \\ &< 13\beta\tilde{X}\frac{1}{|x-y|^{\alpha' d}}\frac{1}{|x-y|^{(\alpha-\alpha')d}} && \text{(for any } 1<\alpha'<\alpha \text{)} \\ &< \frac{13\beta\tilde{X}}{R^{(\alpha-a')d}}\frac{1}{|x-y|^{\alpha' d}} && \text{(since } |x-y|>R \text{).} \quad (75) \end{aligned}$$

Hence for any given $\theta > 0$ and any $1 < \alpha' < \alpha$ we can find R large enough so that $\overline{\rho_{xy}} < \frac{\theta}{|x-y|^{\alpha' d}}$.

We now turn to the θ_x. Let ω be a self avoiding walk with $\omega_s = \{x_1, \ldots x_N\}$, the $x's$ being ordered as they appear in ω, so $\prod_{x\in\omega_s}\theta_x = \prod_{i=1}^N \theta_{x_i}$. We obtain a subset of ω_s inductively as follows: First step is to consider those $x_i \in \omega_s$ for which $|x_i - x_1| \leq R$ and to throw them away. From the remaining x_i's $(i \neq 1)$ we now pick the one with the lowest index, say x_{i_2}, and throw away all remaining x_i's such that $|x_i - x_{i_2}| \leq R$. The procedure is repeated until we exhaust the x_i's. After this elimination process, what is left is a subset of ω_s , say $\{y_1\,(=x_1), y_2, \ldots, y_M\}$ with the property that $|y_i - y_j| > R$, for all $1 \leq i < j \leq M$ and so the θ_{y_i} are independent random variables. Note that at each step at most $R^d - 1$ sites are thrown out, so at the end at most $M\left(R^d-1\right)$ sites have been thrown out. Therefore $N \leq M + M\left(R^d - 1\right)$ and thus $M \geq \frac{N}{R^d}$.

Let us now choose $\mathcal{S}_s = \left\{x : 2\mathcal{B}\,|h_x| - 6\sum_{\substack{y:\\|x-y|\leq R}} |J_{xy}| \leq 0\right\}$, then, since $\mathbb{P}\,(h_x = 0) = 0$,

$$\mathbb{P}\,(x \in \mathcal{S}_s) \leq \mathbb{P}\left(\mathcal{B} \leq \frac{6\sum_{\substack{y:\\|x-y|\leq R}} |J_{xy}|}{2\,|h_x|}\right) \longrightarrow 0 \quad \text{as } \mathcal{B}\to\infty\ . \tag{76}$$

Thus $\overline{\theta} \equiv \mathbb{E}\,(\theta_x) \to 0$ as $\mathcal{B} \to \infty$.

Since the above definition for $\mathcal{S}_s$ implies that $\theta_x \leq 1$, we have

$$\mathbb{E}\left(\prod_{x\in\omega_s}\theta_x\right) \ \leq\ \mathbb{E}\left(\prod_{i=1}^{M}\theta_{y_i}\right) = \overline{\theta}^M \leq \overline{\theta}^{\frac{N}{R^d}} \equiv \widetilde{\theta}^N. \tag{77}$$

Thus given $\theta > 0$, $1 < \alpha' < \alpha$ and $R > 0$, if $\mathcal{B}$ is large enough we have $3\widetilde{\theta} < \frac{\theta}{R^{\alpha' d}}$. In this case it follows from (73) that

$$\Xi_{0y_0} \leq \sum_{\omega:0\to y_o} \prod_{\{x,y\}\in\omega_l} \frac{\theta}{|x-y|^{\alpha' d}} \prod_{x\in\omega_s} \frac{\theta}{R^{\alpha' d}} \tag{78}$$

$$\leq \sum_{\omega:0\to y_o} \prod_{\{x,y\}\in\omega} \frac{\theta}{|x-y|^{\alpha' d}} \ . \tag{79}$$

The inequality (7) now follows from (79) and (49), as in the proof of (5), concluding the proof of the first part of Theorem 2.

3 Uniqueness of Gibbs States

In this section we consider a long range random Ising model as in (1), except that we do not require independence of the random variables. (We still require $\mathbf{X}$ and $\mathbf{h}$ to be families of identically distributed (within each family) random variables.)

Lemma 6 *Suppose* (5) *(the expectation being over all the random variables) holds with* $\mathbb{E}\left(|X_{xy}|^{\frac{\alpha}{\alpha-1}+\epsilon}\right) \leq C_1 < \infty$ *for some* $\epsilon > 0$*. Then*

$$\lim_{L\to\infty} \mathbb{E}\left(\sup_{\chi}\left|\langle\sigma_\Delta\rangle_{B_L} - \langle\sigma_\Delta\rangle_{B_L}^{\chi}\right|\right) = 0, \tag{80}$$

where the supremum is taken over all external boundary conditions χ*,* B_L *is a box of side* $2L$ *centered at the origin in* $\mathbb{Z}^d$*,* Δ *is a finite subset of* $\mathbb{Z}^d$*, and* $\sigma_\Delta = \prod_{x\in\Delta}\sigma_x$*.*

Proof. It follows from the Fundamental Theorem of Calculus that

$$\left|\langle\sigma_\Delta\rangle_{B_L} - \langle\sigma_\Delta\rangle_{B_L}^{\chi}\right| = \left|\int_0^1 \frac{d}{ds}\langle\sigma_\Delta\rangle_{B_L}^{s\chi}\right| \leq \int_0^1 \beta \sum_{\substack{x\in B_L\\ y\notin B_L}} |J_{xy}| \ \left|\langle\sigma_\Delta;\sigma_x\rangle_{B_L}^{s\chi}\right| \ ds, \tag{81}$$

so

$$\sup_{\chi}\left|\langle\sigma_\Delta\rangle_{B_L} - \langle\sigma_\Delta\rangle_{B_L}^{\chi}\right| \leq \beta \sum_{\substack{x\in B_L\\ y\notin B_L}} |J_{xy}| \ G_L(\Delta, x), \tag{82}$$

where

$$G_L(\ \Delta, x) = \sup_{\substack{0\leq s\leq 1\\ \chi}} \left|\langle\sigma_\Delta;\sigma_x\rangle_{B_L}^{s\chi}\right|. \tag{83}$$

Clearly $G_L(\ \Delta, x) \leq 2$, so by hypothesis we have

$$\mathbb{E}\left(G_L(\Delta, x)\right) \leq \ \min\left\{\frac{C\,|\Delta|}{d\,(\Delta, x)^{\alpha d}}\,, 2\right\} \ \text{for all } L. \tag{84}$$

Taking the expectation of both sides in (82) gives

$$\mathbb{E}\left(\sup_{\chi}\left|\langle\sigma_\Delta\rangle_{B_L}-\langle\sigma_\Delta\rangle^{\chi}_{B_L}\right|\right)\leq\beta\sum_{\substack{x\in B_L\\ y\notin B_L}}\mathbb{E}\left(|J_{xy}|\ \ G_L(\Delta,x)\right). \tag{85}$$

Let us fix a number $\gamma:0<\gamma<\min\{1,\alpha-1\}$, and expand the last sum by dividing B_L into two parts so that

$$\sum_{\substack{x\in B_L\\ y\notin B_L}}\mathbb{E}\left(|J_{xy}|\ \ G_L(\Delta,x)\right)=S_1+S_2,$$

where $S_1=\sum_{\substack{|x|\leq L^\gamma\\ |y|>L}}\mathbb{E}\left(|J_{xy}|\ \ G_L(\Delta,x)\right)$ and $S_2=\sum_{\substack{L^\gamma<|x|\leq L\\ |y|>L}}\mathbb{E}\left(|J_{xy}|\ \ G_L(\Delta,x)\right)$.

We will first show convergence of S_1 to 0:

$$\begin{aligned}S_1&\leq\sum_{|x|\leq L^\gamma}\ \sum_{\substack{y:\\ |x-y|>L-L^\gamma}}\mathbb{E}\left(|J_{xy}|\ \ G_L(\Delta,x)\right) &(86)\\ &\leq 2\tilde{X}\sum_{|x|\leq L^\gamma}\ \sum_{\substack{y:\\ |x-y|>L-L^\gamma}}\frac{1}{|x-y|^{\alpha d}}\leq 2\tilde{X}C_d\sum_{|x|\leq L^\gamma}\ \sum_{r=L-L^\gamma+1}^{\infty}\frac{r^{d-1}}{r^{\alpha d}}, &(87)\end{aligned}$$

where C_d is some finite constant depending only on d. The series $\sum_{r=L-L^\gamma+1}^{\infty}\ r^{d-\alpha d-1}$ converges since $\alpha>1$. In fact it behaves like the integral $\int_{L-L^\gamma}^{\infty}t^{d-\alpha d-1}dt$ which converges to a constant multiple of $(L-L^\gamma)^{d-\alpha d}$. Therefore S_1 is bounded by a constant multiple of $L^{\gamma d}(L-L^\gamma)^{d-\alpha d}$, which (for large L) behaves like $L^{\gamma d}L^{d-\alpha d}=L^{\gamma d+d-\alpha d}$ which converges to 0 as $L\to\infty$.(Recall that we chose $0<\gamma<\alpha-1$.) Thus $S_1\to 0$ as $L\to\infty$.

We now apply Hölder's inequality to the expectation in S_2 with $q=\frac{\alpha}{\alpha-1}+\varepsilon$, $\frac{1}{p}+\frac{1}{q}=1$:

$$\begin{aligned}S_2\ &\leq\ \sum_{\substack{L^\gamma<|x|\leq L\\ |y|>L}}\left(\mathbb{E}\,|J_{xy}|^q\right)^{\frac{1}{q}}\ \left(\mathbb{E}\left(G_L(\Delta,x)\right)^p\right)^{\frac{1}{p}} &(88)\\ &\leq\ \sum_{\substack{L^\gamma<|x|\leq L\\ |y|>L}}\left(\frac{C_1}{|x-y|^{q\alpha d}}\right)^{\frac{1}{q}}\ \left(2^{p-1}\frac{C\,|\Delta|}{d(\Delta,x)^{\alpha d}}\right)^{\frac{1}{p}} &(89)\\ &=\ C_2\ |\Delta|^{\frac{1}{p}}\sum_{L^\gamma<|x|\leq L}\frac{1}{d(\Delta,x)^{\frac{\alpha d}{p}}}\sum_{|y|>L}\frac{1}{|x-y|^{\alpha d}} &(90)\\ &\leq\ C_3|\Delta|^{\frac{1}{p}}\sum_{L^\gamma<|x|\leq L}\frac{1}{\left(\frac{|x|}{2}\right)^{\frac{\alpha d}{p}}}\qquad\text{(for } L \text{ large enough)} &(91)\\ &\leq\ C_3|\Delta|^{\frac{1}{p}}C_d\ 2^{\frac{\alpha d}{p}}\sum_{r=L^\gamma+1}^{L}\frac{r^{d-1}}{r^{\frac{\alpha d}{p}}}, &(92)\end{aligned}$$

where C_2 and C_3 are finite constants. But

$$\sum_{r=\ L^\gamma+1}^{L} r^{d-1-\frac{\alpha d}{p}} \approx \int_{L^\gamma+1}^{L} t^{d-1-\frac{\alpha d}{p}} dt \to 0 \text{ as } L \to \infty.$$

(Recall that $\frac{\alpha}{p} > 1$ so that $d - \frac{\alpha d}{p}$ is a negative number.) Hence we also have $S_2 \to 0$ as $L \to \infty$ and that completes the proof. $\square$

Lemma 7 *Suppose (80) holds. Then there is a unique Gibbs state with probability one.*

Proof. It follows from (80) that, with probability one, for each finite $\Delta \subset \mathbb{Z}^d$ we can find a sequence $L_{\Delta,k} \to \infty$ (depending on Δ) such that

$$\lim_{k\to\infty} \sup_{\chi} \left| \langle \sigma_\Delta \rangle_{B_{L_{\Delta,k}}} - \langle \sigma_\Delta \rangle^{\chi}_{B_{L_{\Delta,k}}} \right| = 0. \tag{93}$$

Now suppose Φ is an arbitrary Gibbs state for a given $\mathbf{X}$ and $\mathbf{h}$. It follows from the DLR equations that for any finite $\Delta \subset \mathbb{Z}^d$ we have

$$|\Phi(\sigma_\Delta) - \langle \sigma_\Delta \rangle_\Lambda| \leq \sup_{\chi} |\langle \sigma_\Delta \rangle^{\chi}_\Lambda - \langle \sigma_\Delta \rangle_\Lambda| \tag{94}$$

for all finite subsets Λ of $\mathbb{Z}^d$. Thus it follows from (93) that, with probability one, we have

$$\lim_{k\to\infty} \left| \Phi(\sigma_\Delta) - \langle \sigma_\Delta \rangle_{B_{L_{\Delta,k}}} \right| \leq \lim_{k\to\infty} \sup_{\chi} \left| \langle \sigma_\Delta \rangle^{\chi}_{B_{L_{\Delta,k}}} - \langle \sigma_\Delta \rangle_{B_{L_{\Delta,k}}} \right| = 0 \tag{95}$$

for all finite $\Delta \subset \mathbb{Z}^d$. Thus Φ is unique with probability one. $\square$

The second parts of Theorems 1 and 2 now follows from the respective first parts and Lemmas 6 and 7 .

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