Bound–state asymptotic estimates for window–coupled Dirichlet strips and layers

Pavel Exner[a,b] and Simeon A. Vugalter[a]

a) Nuclear Physics Institute, Academy of Sciences, 25068 Řež near Prague,
b) Doppler Institute, Czech Technical University, Břehová 7, 11519 Prague,
exner@ujf.cas.cz, vugalter@ujf.cas.cz

We consider the discrete spectrum of the Dirichlet Laplacian on a manifold consisting of two adjacent parallel straight strips or planar layers coupled by a finite number N of windows in the common boundary. If the windows are small enough, there is just one isolated eigenvalue. We find upper and lower asymptotic bounds on the gap between the eigenvalue and the essential spectrum in the planar case, as well as for $N=1$ in three dimensions. Based on these results, we formulate a conjecture on the weak–coupling asymptotic behaviour of such bound states.

1 Introduction

There has been some interest recently to Laplacians on strips or layers. Such a system is trivial when the manifold is straight and the boundary conditions are translationally invariant, so there is a natural separation of variables. On the other hand, the spectral properties become nontrivial if the transverse modes are coupled, which can be achieved, *e.g.*, if the manifold is bent, locally deformed, or coupled to another one [EŠ, DE, BGRS, EŠTV, EV1, EV2].

The interest stems from two sources. On the physical side, such operators with Dirichlet boundary conditions are used as models of various mesoscopic semiconductor structures. The corresponding solid–state literature is rather rich — see [DE, EŠTV] for some references. On the other hand, bound states in systems with open geometries pose also mathematical questions such as the weak–coupling limit, validity of the semiclassical approximation, resonance scattering in such structures, *etc.* Some properties of them can be seen numerically [EŠTV] while analytical proofs are missing. Recall also that a closely related problem concerns Neumann Laplacians, namely the existence of trapped modes in acoustic waveguides [ELV, DE].

In a recent paper [EV1] we have studied a pair of parallel Dirichlet strips of widths d_1, d_2 coupled laterally through a window of a width $2a$ in the common boundary; we have shown that there are positive c_1, c_2 such that the gap between the ground state and the threshold of the continuous spectrum can be estimated as

$$-c_1 a^4 \;\le\; \epsilon(a) - \left(\frac{\pi}{d}\right)^2 \;\le\; -c_2 a^4 \tag{1.1}$$

for any a small enough. The numerical result of [EŠTV] suggests that the true asymptotics is of the same type, but proving this assertion and finding the coefficient in the leading term remains an open problem.

The aim of the present paper is to generalize the above inequalities to the case of a finite number of connecting windows and to a higher dimension. In the following section we shall prove the bounds for a pair of strips with N windows. In Section 3 we formulate the analogous problem for two layers and prove two–sided asymptotic bounds for a single window shrinking to a point. Proofs rely in both cases on variational estimates and follow the same basic strategy as in [EV1]. On the other hand, the existence of multiple windows or the change in dimension require numerous modifications, which prompts us to present the argument with enough details.

The upper and lower asymptotics bounds we are going to derive are in each case of the same type differing just by values of the constants. We are convinced that ground state has an asymptotic expansion and its lowest–order is given by functions analogous to our bounds. This conjecture is formulated in the concluding section. At the same time, our present method does not allow to squeeze the bounds, or even to come close to the true values as the Remark 2.2 below illustrates.

2 N windows in dimension two

Consider a straight planar strip $\Sigma := I\!\!R \times [-d_2, d_1]$. Given finite sequences $\mathcal{C} \equiv \{x_k\}_{k=1}^N$ of mutually distinct points of the x–axis and $A = \{a_k\}_{k=1}^N$ with $a_k > 0$, we denote $\mathcal{W}_k := [x_k - a_k, x_k + a_k]$ and set $\mathcal{W} := \bigcup_{k=1}^n \mathcal{W}_k$. Then we define $H(d_1, d_2; \mathcal{W})$ as the Laplacian on $L^2(\Sigma)$ subject to the Dirichlet condition at $y = -d_2, d_1$ as well as at the $I\!\!R \setminus \mathcal{W}$ part of the x–axis; this operator coincides with the Dirichlet Laplacian at the strip with the appropriate piecewise cut (see Fig. 1) defined in the standard way [RS4, Sec.XIII.15]. Following the notation introduced in [EV1] we put $d := \max\{d_1, d_2\}$ and $D := d_1 + d_2$. If $d_1 = d_2$, the operator decomposes into an orthogonal sum with respect to the y–parity; the nontrivial part is unitarily equivalent to the Laplacian on $L^2(\Sigma_+)$, where $\Sigma_+ := I\!\!R \times [0, d]$, with the Neumann condition at window part $\mathcal{W}$ of the x–axis and Dirichlet at the remaining part of the boundary; we denote it by $H(d; \mathcal{W})$. If the specification is clear from the context, we will often denote the operator in question simply as H.

We need a quantity to express the "smallness" of the window set. We define

$$I(\mathcal{W}) \;:=\; \sum_{k=1}^N a_k |\mathcal{W}_k| \;=\; 2\sum_{k=1}^N a_k^2\,; \tag{2.2}$$

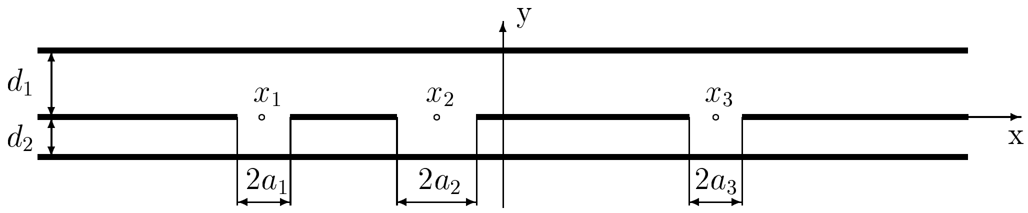

Figure 1: Window–coupled planar waveguides

then the result of [EV1] generalizes to the present situation as follows:

Theorem 2.1 $\sigma_{\mathrm{ess}}(H(d_1,d_2;\mathcal{W}))=[(\pi/d)^2,\infty)$. *The discrete spectrum is contained in* $((\pi/D)^2,(\pi/d)^2)$ *, finite, and nonempty provided* $\mathcal{W}\neq\emptyset$ *. If* $I(\mathcal{W})$ *is sufficiently small,* $\sigma_{\mathrm{disc}}(H(d_1,d_2;\mathcal{W}))$ *consists of just one simple eigenvalue* $\epsilon(\mathcal{W})\le(\pi/d)^2$ *and there are positive* $c_1,\,c_2$ *such that*

$$-c_1 I(\mathcal{W})^2\,\le\,\epsilon(a)-\left(\frac{\pi}{d}\right)^2\le\,-c_2 I(\mathcal{W})^2 \tag{2.3}$$

holds for any $I(\mathcal{W})$ *small enough.*

Proof: (a) The upper bound. In the symmetric case, $d_1=d_2$, the trial function will be chosen as $\psi=F+G$, where

$$F(x,y)\,:=\,f_1(x)\chi_1(y)\,, \tag{2.4}$$

with

$$f_1(x)\,:=\,\max\{\,\chi_{[x_1-a_1,x_N+a_N]}(x),\,e^{-\kappa|x-x_1+a_1|},\,e^{-\kappa|x-x_N-a_N|}\}\,,$$

and

$$G(x,y)\,:=\,\sum_{k=1}^{N}G_k(x,y) \tag{2.5}$$

with

$$G_k(x,y)\,:=\,\frac{2\eta_k a_k}{|\mathcal{W}|}\,\chi_{[x_k-a_k,x_k+a_k]}(x)\,\cos\left(\frac{\pi(x-x_k)}{2a_k}\right)\,R_k(y)\,, \tag{2.6}$$

where $|\mathcal{W}|:=2\sum_{k=1}^{N}a_k$, and

$$R_k(y)\,:=\,\begin{cases} e^{-\pi y/2a_k} & \dots \quad y\in\left[0,\frac{d}{2}\right] \\ \\ 2\left(1-\frac{y}{d}\right)e^{-\pi d/4a_k} & \dots \quad y\in\left[\frac{d}{2},d\right] \end{cases} \tag{2.7}$$

for $k=1,2,\dots,N$. As before $\chi_n(y)=\sqrt{\frac{2}{d}}\,\sin\left(\frac{\pi n y}{d}\right)$, $n=1,2,\dots,$ denote the "transverse" eigenfunctions — to be not confused with the indicator function χ_M of a set M. Notice that as long as we work with trial functions of $Q(H)$, the window smoothing employed in [EV1] is in fact not needed — *cf.* [RS4].

The functional $L(\psi) := (H\psi,\psi) - \left(\frac{\pi}{d}\right)^2 \|\psi\|^2$ can be expressed as

$$L(\psi) \;=\; \|\psi_x\|^2 + \|G_y\|^2 - \left(\frac{\pi}{d}\right)^2 \|G\|^2 - 2\,\frac{\pi}{d}\sqrt{\frac{2}{d}}\,\sum_{k=1}^N \int_{x_k-a_k}^{x_k+a_k} G_k(x,0)\,dx\,. \qquad (2.8)$$

Since f_x, G_x have disjoint supports, we have $\|\psi_x\|^2 = \|F_x\|^2 + \sum_{k=1}^N \|G_{k,x}\|^2$, where $G_{k,x} := \partial_x G_k$. The kth term of the last sum equals $\eta_k^2\pi^2 a_k|\mathcal{W}|^{-2}\,\|R_k\|^2_{L^2(0,d)}$, and

$$\|R_k\|^2_{L^2(0,d)} \;=\; \frac{a_k}{\pi} \;+\; \left(\frac{d}{6}-\frac{a_k}{\pi}\right)\, e^{-\pi d/2a_k} \;<\; \frac{a_k}{\pi}\,(1+\varepsilon_1)$$

for any $\varepsilon_1>0$ and a_k small enough. Obviously, $\int_{x_k-a_k}^{x_k+a_k} G_k(x,0)\,dx = \frac{8}{\pi}\,\eta_k a_k^2|\mathcal{W}|^{-1}$, and furthermore, a bound to $\|G_{k,y}\|^2$ follows from

$$\|R_k'\|^2_{L^2(0,d)} \;=\; \frac{\pi}{4a_k} \;+\; \left(\frac{2}{d}-\frac{\pi}{4a_k}\right)\, e^{-\pi d/2a_k} \;<\; \frac{\pi}{4a_k}$$

for $a_k<\pi d/8$, which means that $\|G_{k,y}\|^2 < \pi\sum_k \eta_k^2 a_k^2|\mathcal{W}|^{-2}$. Now we can put these estimates together using $\|F_x\|^2=\kappa$; neglecting the negative term $-\left(\frac{\pi}{d}\right)^2\|G\|^2$, we arrive at the inequality

$$L(\psi) \;<\; \kappa - \frac{16\sqrt{2}}{d^{3/2}}\sum_{k=1}^N \frac{\eta_k a_k^2}{|\mathcal{W}|} \;+\; \pi(2+\varepsilon_1)\sum_{k=1}^N \frac{\eta_k^2 a_k^2}{|\mathcal{W}|^2}\,.$$

The sum of the last two terms at the *rhs* is minimized by $-\frac{2^7}{\pi d^3(2+\varepsilon_1)}\sum_k a_k^2$. To conclude the argument, we have to estimate the trial function norm $\|\psi\|^2$ from below. The tail part is $\|\psi\|^2_{x\in I\!R\setminus\mathcal{W}} = \kappa^{-1}$, while the window contributes by

$$\|\psi\|^2_{x\in\mathcal{W}} \;\le\; 2\|F\|^2_{x\in\mathcal{W}} + 2\|G\|^2_{x\in\mathcal{W}} \;=\; |x_N\!-\!x_1\!+\!a_N\!+\!a_1| \;+4\,\sum_{k=1}^N \frac{\eta_k^2 a_k^3}{|\mathcal{W}|^2}\,\|R_k\|^2_{L^2(0,d)}\,,$$

so $\|\psi\|^2 > (1\!-\!\varepsilon_2)\kappa^{-1}$ holds for any $\varepsilon_2>0$ provided $|\mathcal{W}|$ is small enough. Minimizing the obtained estimate of $L(\psi)/\|\psi\|^2$ over κ, we find

$$\frac{L(\psi)}{\|\psi\|^2} \;<\; -\,(1-\varepsilon_2)^{-1}\left(\frac{2^6}{\pi d^3(2+\varepsilon_1)}\,\sum_{k=1}^N a_k^2\right)^2 \qquad (2.9)$$

which yields the upper bound in (2.3) for $d_1=d_2$. The extension to the nonsymmetric case proceeds as for $N=1$; the trial function is chosen in the above form for the wider channel, while in the narrower one it is given by (2.5) transversally rescaled.

Remark 2.2 The bound can be improved, for instance, by replacing the factorized form (2.6) by a series, whose terms will be products of the trigonometric basis in the window with the functions $R_{k,n}(y)$ decaying as $\exp\{-\frac{\pi n y}{2a_k}\}$ around $y=0$ (in the above estimate we used just the first term of such a series). However, the gain is not large. To illustrate this fact, take $N=1$ and $d=\pi$. The use of the series leads then to the upper bound $\left(\frac{2a}{\pi}\right)^4$ improving the coefficient by $(\pi^2/8)^2 \approx 1.52$. A comparison to the numerically determined ground state [EŠTV] shows that the true asymptotic behaviour should be $\approx (2.23\,a)^4$, so the obtained c_2 is still two orders of magnitude off mark. The reason is obviously that the wavefunction is affected by the window outside the transverse "window strip" as well.

Before proceeding to the lower bound, let us state some auxiliary results:

Lemma 2.3 *Let* $J[\phi] := \int_a^b (\phi'(t)^2 + m^2\phi(t)^2)\, dt$ *for* $\phi \in C^2(a,b)$ *with* $\phi(a) = c_a$ *(a fixed number). Given* $m_0 > 0$, *there is* $\alpha_0 > 0$ *such that*

$$J[\phi] \,\geq\, \alpha_0 m c_a^2 \tag{2.10}$$

holds for all $m \geq m_0$.

Proof: The mimimum is obviously reached with $\phi'(b) = 0$. The corresponding Euler's equation is solved by $\phi_0(t) = d_1 e^{-mt} + d_2 e^{mt}$, where $d_1 = c_a\left(e^{-ma} + e^{m(a-2b)}\right)^{-1}$ and $d_2 = d_1 e^{-2mb}$. Since $m^{-1}c_a^{-2}\inf J(\phi) > 0$ for any $m \geq m_0$, it is sufficient to check that (2.10) remains valid as $m \to \infty$; evaluating the functional for ϕ_0 we find $\lim_{m\to\infty} J(\phi) = mc_a^2$. ∎

Lemma 2.4 *Suppose that* ϕ *minimizes* $J[\phi] := \int_a^{2a} (\phi'(t)^2 + p^2\phi(t)^2)\, dt$ *for positive* $a,\, p$ *within* $C^2(a,2a)$ *with the boundary condition* $\phi(a) = c_a$; *then*

$$|\phi(2a)| \,\leq\, 2|c_a|\, e^{-pa}. \tag{2.11}$$

Proof: Assume for definiteness that $c_a > 0$. By the mentioned symmetry argument again, $\phi'(2a) = 0$, and its explicit form is $\phi(t) = c_a \cosh p(2a-t)/\cosh pa$, which yields $\phi(2a) \leq 2c_a e^{-pa}$. ∎

For the sake of completeness we reproduce also the following assertion the proof of which is given in [EV1]:

Lemma 2.5 *Let* $\phi \in C^2[0,d]$ *with* $\phi(0) = \beta$ *and* $\phi(d) = 0$. *If* $(\phi, \chi_1) = 0$, *then for every* $m > 0$ *there is* $d_0 > 0$ *such that*

$$\int_0^d \phi'(t)^2 dt \,+\, \left(\frac{m}{a}\right)^2 \int_0^a \phi(t)^2 dt \,-\, \left(\frac{\pi}{d}\right)^2 \int_0^d \phi(t)^2 dt \,\geq\, \frac{d_0\beta^2}{a} \tag{2.12}$$

holds for all a *small enough.*

(b) Proof of Theorem 2.1, continued: The lower bound is again the more difficult; however, we may restrict ourselves to the symmetric case only because inserting an

additional Neumann boundary into the window we get a lower bound, and therefore we consider in the following the spectrum of $H \equiv H(d;\mathcal{W})$.

We begin with a simple observation that it is sufficient to estimate $L(\psi) := (H\psi,\psi) - \left(\frac{\pi}{d}\right)^2 \|\psi\|^2$ from below for all *real* ψ of a core of H, say, all C^2–smooth $\psi \in L^2(\Sigma_+)$ satisfying the boundary conditions, since H commutes with complex conjugation. The main difficulty brought by the existence of multiple windows is that we are no longer allowed to restrict ourselves to trial functions symmetric with respect to the window centers. The strategy we employ is to split from the beginning a part of the kinetic–energy contribution to the functional, say, $\frac{1}{4}\|\psi_x\|^2$, which will be at the end used to mend the problems coming from the asymmetry, *i.e.*, we begin with estimating $L_0(\psi) := L(\psi) - \frac{1}{4}\|\psi_x\|^2$.

A trial function of the indicated set will be written in the form of a Fourier series,

$$\psi(x,y) = \sum_{n=1}^{\infty} c_n(x)\chi_n(y) \tag{2.13}$$

with smooth coefficients $c_n(x) = (\psi(x,\cdot),\chi_n)$, which is uniformly convergent outside the windows, $x \notin \mathcal{W}$. We split further the lowest transverse–mode coefficient by putting

$$f_1 := c_1 - \sum_{k=1}^{N} \hat f_k\,, \tag{2.14}$$

where

$$\hat f_k := \begin{cases} c_k(x) - \alpha_k & \dots \quad x \in [x_k - 2a_k, x_k + a_k] \\ 0 & \dots \quad \text{otherwise} \end{cases} \tag{2.15}$$

with $\alpha_k := c_1(x_k - 2a_k)$, *i.e.*, each one of the functions $\hat f_k$ vanishes at the left endpoint of the appropriate extended window; in contrast to [EV1] we double the left half of the window only. Writing the full trial function as

$$\psi(x,y) = F(x,y) + G(x,y)\,, \qquad F(x,y) := f_1(x)\chi_1(y)\,, \tag{2.16}$$

we can cast the reduced energy functional into the form

$$L_0(\psi) = \frac{3}{4}\|\psi_x\|^2 + \|G_y\|^2 - \left(\frac{\pi}{d}\right)^2 \|G\|^2 - \sum_{k=1}^{N} 2\alpha_k \frac{\pi}{d}\sqrt{\frac{2}{d}} \int_{\mathcal{W}_k} G(x,0)\,dx\,. \tag{2.17}$$

Contributions to (2.17) from different parts of the strip Σ_+ will be estimated separately. The out–of–window part consists of the sets

$$\begin{aligned} \omega_1 &= \{(x,y) : x \le x_1 - a_1\}\,, \\ \omega_k &= \{(x,y) : x_{k-1} + a_{k-1} \le x \le x_k - a_k\}\,, \quad k = 2,\dots,N\,, \\ \omega_{N+1} &= \{(x,y) : x \ge x_N + a_N\}\,. \end{aligned}$$

The expansion (2.13) yields

$$\frac{1}{4}\|\psi_x\|^2_{\omega_k} + \|G_y\|^2_{\omega_k} - \left(\frac{\pi}{d}\right)^2 \|G\|^2_{\omega_k}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\int_{\omega_k} c_n'(x)^2\,dx \;+\; \sum_{n=1}^{\infty}\left(\frac{\pi}{d}\right)^2\left(n^2-1\right)\int_{\omega_k} c_n(x)^2\,dx\,,$$

and therefore

$$\frac{1}{4}\,\|\psi_x\|^2_{\omega_k}+\|G_y\|^2_{\omega_k}-\left(\frac{\pi}{d}\right)^2\|G\|^2_{\omega_k} \;>\; \mu_0\sum_{n=2}^{\infty} nc_n(x_k-a_k)^2$$

with some $\mu_0>0$ follows from Lemma 2.3 (applied to $c_n(-x)$) for $k=2,\dots,N$. The same inequality for $k=1$ is derived as in [EV1]; for the right tail we use just the fact that the expression is positive so we can neglect it. Since $\psi_x=G_x$ inside the (left extended) windows, we arrive at the bound

$$\begin{aligned} L_0(\psi) \;&>\; \frac{1}{2}\,\|\psi_x\|^2_{x\notin\mathcal{W}}+\sum_{k=1}^{N}\Bigg\{\frac{3}{4}\,\|G_x\|^2_{x\in\mathcal{W}_k}+\|G_y\|^2_{x\in\mathcal{W}_k}-\left(\frac{\pi}{d}\right)^2\|G\|^2_{x\in\mathcal{W}_k} \\ &\quad+\;\mu_0\sum_{n=2}^{\infty} nc_n(x_k-a_k)^2-\;2\alpha_k\,\frac{\pi}{d}\sqrt{\frac{2}{d}}\int_{\mathcal{W}_k} G(x,0)\,dx\Bigg\}\,. \end{aligned} \tag{2.18}$$

Our next goal is to estimate the contribution to $\|G_x\|^2$ from the extended windows, $\mathcal{E}_k:=[x_k-2a_k,x_k+a_k]$. In distinction to the case $N=1$, however, even the lowest–mode projection of G may not vanish at the right endpoints of these intervals, so the inequality (5.6) of [EV1] has to be modified. Fortunately, it is sufficient to change the coefficient: if a function $\tilde G:\,\Sigma_+\to C^2(\Sigma_+)$ vanishes for $x=x_k-2a_k$, the inequality (4.2) of [EV1] in combination with a symmetry argument imply

$$\|\tilde G_x\|^2_{x\in\mathcal{E}_k}\;\ge\;\left(\frac{\pi}{6a_k}\right)^2\|\tilde G\|^2_{x\in\mathcal{E}_k}\,. \tag{2.19}$$

To use this result we split the function by singling out the projection of G onto the first transverse mode,

$$G(x,y)\;=\;G_1(x,y)+G_2(x,y)\,,\qquad G_1(x,y)\;=\;\sum_{k=1}^{N}\hat f_k(x)\chi_1(y)\,. \tag{2.20}$$

We have

$$\frac{1}{2}\,\|\psi_x\|^2_{x\in\mathcal{E}_k\setminus\mathcal{W}_k}+\frac{3}{4}\,\|G_x\|^2_{x\in\mathcal{W}_k}\;\ge\;\frac{1}{2}\,\|G_x\|^2_{x\in\mathcal{E}_k}\;=\;\frac{1}{2}\,\|G_{1,x}\|^2_{x\in\mathcal{E}_k}+\frac{1}{2}\,\|G_{2,x}\|^2_{x\in\mathcal{E}_k}\,,$$

and therefore

$$\begin{aligned} L_0(\psi) \;&>\; \frac{1}{2}\,\|\psi_x\|^2_{x\notin\mathcal{E}}+\sum_{k=1}^{N}\Bigg\{\frac{1}{2}\,\|G_{2,x}\|^2_{x\in\mathcal{E}_k}\;+\;\|G_y\|^2_{x\in\mathcal{W}_k}\;-\;\left(\frac{\pi}{d}\right)^2\|G\|^2_{x\in\mathcal{W}_k} \\ &\quad+\mu_0\sum_{n=2}^{\infty} nc_n(x_k-a_k)^2\;-\;2\alpha_k\,\frac{\pi}{d}\sqrt{\frac{2}{d}}\int_{\mathcal{W}_k} G(x,0)\,dx\;+\;\frac{1}{2}\left(\frac{\pi}{6a_k}\right)^2\|G_1\|^2_{x\in\mathcal{E}_k}\Bigg\}\,. \end{aligned} \tag{2.21}$$

with $\mathcal{E} := \bigcup_{k=1}^{N} \mathcal{E}_k$. To proceed further we split the function G_2 in the k–th extended window as $G_2(x,y) = \hat{G}(x,y) + \Gamma(x,y)$, where

$$\Gamma(x,y) := \sum_{n=2}^{\infty} c_n(x_k - 2a_k)\chi_n(y) .$$

The second part is independent of x while the first one vanishes at left endpoint, so $G_{2,x} = \hat{G}_x$ may be estimated by means of (2.19) and the Schwarz inequality as

$$\begin{aligned} \|G_{2,x}\|^2_{x\in\mathcal{E}_k} &\geq \left(\frac{\pi}{6a_k}\right)^2 \|\hat{G}\|^2_{x\in\mathcal{E}_k} \geq \left(\frac{\pi}{6a_k}\right)^2 \|\hat{G}\|^2_{\Omega_k} \\ &\geq \frac{1}{2}\left(\frac{\pi}{6a_k}\right)^2 \|G_2\|^2_{\Omega_k} - \left(\frac{\pi}{6a_k}\right)^2 \|\Gamma\|^2_{\Omega_k} , \end{aligned} \tag{2.22}$$

where we have denoted $\Omega_k := \mathcal{E}_k \times [0, a_k]$. To make use of the last estimate we have to find an upper bound to $\|\Gamma\|^2_{\Omega_k}$. To this end we notice that

(i) instead of assuming $c_n \in C^2$, the lower bound can be looked for in a wider class of ψ with piecewise continuous coefficients,

(ii) on the other hand, we may restrict ourselves to those ψ which satisfy for $x \in \mathcal{E}_k \setminus \mathcal{W}_k$ and $n \geq 2$ the inequality

$$|c_n(x)| \leq c_n^{ex}(x) := |c_n(a)| \frac{\cosh\left(\frac{\pi}{d}\sqrt{n^2-1}\,(x - x_k + 2a_k)\right)}{\cosh\left(\frac{\pi a_k}{d}\sqrt{n^2-1}\right)} . \tag{2.23}$$

To see that we split the trial function in analogy with [EV1],

$$\tilde{\psi}(x,y) := \begin{cases} \psi(x,y) - c_n(x)\chi_n(y) & \dots \quad x \in \mathcal{E}_k \setminus \mathcal{W}_k \\ \psi(x,y) & \dots \quad \text{otherwise} \end{cases}$$

The basic expression $L(\psi)/\|\psi\|^2$ can be then rewritten as

$$\frac{\tilde{L}(\tilde{\psi}) - \left(\frac{\pi}{d}\right)^2 \|\tilde{\psi}\|^2 + \sum_{k=1}^{N} \int_{\mathcal{E}_k\setminus\mathcal{W}_k} \left[c_n'(x)^2\, dx + \left(\frac{\pi}{d}\sqrt{n^2-1}\right)^2 c_n(x)^2 \right] dx}{\|\tilde{\psi}\|^2 + \sum_{k=1}^{N} \int_{\mathcal{E}_k\setminus\mathcal{W}_k} c_n(x)^2 dx ,}$$

where $\tilde{L}(\tilde{\psi}) := \int_{\Sigma_+} \left(|\tilde{\psi}_x|^2 + |\tilde{\psi}_y|^2\right)(x,y)\, dx\, dy$. We may assume only those ψ for which the numerator is negative; the part of its last term corresponding to the "window neighborhoods" is minimized by the hyperbolic function c_n^{ex} of (2.23) (see the proof of Lemma 2.4). It follows that replacing $c_n(x)^2$ by $\min\{c_n(x)^2, c_n^{ex}(x)^2\}$ we can only get a larger negative number, while the positive denominator can be only diminished.

To estimate the norm of Γ restricted to Ω_k, we adapt again the argument of [EV1] and divide the series into parts referring to small and large values of y, and employ, respectively, the smallness of $\|\chi_n\upharpoonright[0,a]\|$ and the bound (2.23). This yields

$$\|\Gamma\|^2_{\Omega_k} = \int_{\mathcal{E}_k} dx \int_0^{a_k} dy \left(\sum_{n=2}^{\infty} c_n[2a_k]\chi_n(y)\right)^2$$

$$\leq 6a_k \int_0^{a_k} \left(\sum_{n=2}^{[a_k^{-1}]+1} c_n[2a_k]\chi_n(y)\right)^2 dy + 6a_k \int_0^{a_k} \left(\sum_{2\leq n=[a_k^{-1}]+2}^{\infty} c_n[2a_k]\chi_n(y)\right)^2 dy$$

$$\leq 24a_k \left(\sum_{n=2}^{[a_k^{-1}]+1} n^{-1}c_n[a_k]^2 \int_0^{a_k} \chi_n(y)^2 dy\right)\left(\sum_{n=2}^{[a_k^{-1}]+1} n\right)$$

$$+ 24a_k \left(\sum_{2\leq n=[a_k^{-1}]+2}^{\infty} nc_n[a_k]^2 \int_0^{a_k} \chi_n(y)^2 dy\right)\left(\sum_{2\leq n=[a_k^{-1}]+2}^{\infty} n^{-1}e^{-(2\pi a_k/d)\sqrt{n^2-1}}\right),$$

where $c_n[ja_k] := c_n(x_k - ja_k)$ and $[\cdot]$ denotes the entire part; in the the last step we have used the bound $|c_n[2a_k]| < 2|c_n[a_k]| \exp\left\{-\frac{\pi a_k}{d}\sqrt{n^2-1}\right\}$ which follows from Lemma 2.4. In analogy with [EV1], this implies the existence of a positive C_k such that

$$\|\Gamma\|^2_{\Omega_k} \leq C_k a_k^2 \sum_{n=2}^{\infty} nc_n(x_k - a_k)^2 . \tag{2.24}$$

¿From now on we consider continuous coefficient functions again. By (2.22) we have

$$\frac{1}{2}\|G_{2,x}\|^2_{x\in\mathcal{E}_k} + \mu_0 \sum_{n=2}^{\infty} nc_n(x_k - a_k)^2$$

$$\geq \delta\left(\frac{\pi}{12a_k}\right)^2 \|G_2\|^2_{\Omega_k} - \frac{\delta}{2}\left(\frac{\pi}{6a_k}\right)^2 \|\Gamma\|^2_{\Omega_k} + \mu_0 \sum_{n=2}^{\infty} nc_n(x_k - a_k)^2$$

for an arbitrary $\delta \in (0,1]$; if we choose the latter sufficiently small, the sum of the last two terms is nonnegative for each $k = 1,\dots,N$ due to (2.24), so

$$L_0(\psi) > \frac{1}{2}\|\psi_x\|^2_{x\notin\mathcal{E}} + \sum_{k=1}^{N}\left\{\|G_y\|^2_{x\in\mathcal{W}_k} - \left(\frac{\pi}{d}\right)^2 \|G\|^2_{x\in\mathcal{W}_k} + \frac{m^2}{a_k^2}\|G_2\|^2_{\Omega_k}\right. \tag{2.25}$$

$$\left. - 2\alpha_k \frac{\pi}{d}\sqrt{\frac{2}{d}} \int_{\mathcal{W}_k} G(x,0)\,dx + \frac{1}{2}\left(\frac{\pi}{6a_k}\right)^2 \|G_1\|^2_{x\in\mathcal{E}_k}\right\},$$

where we have denoted $m := \frac{\pi}{12}\sqrt{\delta}$.

Next we express the first term in the curly bracket using the decomposition (2.20), properties of the transverse base, and an integration by parts,

$$\|G_y\|^2_{x\in\mathcal{W}_k} = \|G_{1,y}\|^2_{x\in\mathcal{W}_k} + \|G_{2,y}\|^2_{x\in\mathcal{W}_k} - 2\frac{\pi}{d}\sqrt{\frac{2}{d}} \int_{\mathcal{W}_k} \hat{f}_k(x)G(x,0)\,dx .$$

As in [EV1] we estimate the last term by the Schwarz inequality, substitute into (2.25), neglect $\|G_{1,y}\|^2_{x\in\mathcal{W}_k}$ as well as

$$\frac{\pi^2}{72a_k^2}\,\|G_1\|^2_{x\in\mathcal{E}_k} - \frac{\pi(\pi+\sqrt{2})}{d^2}\,\|G_{1,y}\|^2_{x\in\mathcal{W}_k}$$

which is positive for a_k small enough, obtaining

$$\begin{aligned} L_0(\psi) \;>\; & \frac{1}{2}\,\|\psi_x\|^2_{x\notin\mathcal{E}} + \sum_{k=1}^N \left\{ \|G_{2,y}\|^2_{x\in\mathcal{W}_k} + \frac{m^2}{a_k^2}\,\|G_2\|^2_{\Omega_k} - \left(\frac{\pi}{d}\right)^2 \|G\|^2_{x\in\mathcal{W}_k} \right. \\ & \left. - \frac{\pi\sqrt{2}}{d}\,\|G(\cdot,0)\|^2_{x\in\mathcal{W}_k} - 2\alpha_k\,\frac{\pi}{d}\sqrt{\frac{2}{d}}\int_{\mathcal{W}_k} G(x,0)\,dx \right\}. \end{aligned} \qquad (2.26)$$

By Lemma 2.5, the sum of the first three terms in the curly bracket is bounded from below by $\frac{d_k}{a_k}\,\|G(\cdot,0)\|^2_{x\in\mathcal{W}_k}$ for some $d_k>0$. Since $(d_k/2a_k)-(\pi\sqrt{2}/d)>0$ holds for a_k small enough, we have

$$L_0(\psi) \;>\; \frac{1}{2}\,\|\psi_x\|^2_{x\notin\mathcal{E}} + \sum_{k=1}^N \left\{ \frac{d_k}{2a_k}\,\|G_2(\cdot,0)\|^2_{x\in\mathcal{W}_k} - 4\alpha_k\,\frac{\pi}{d}\sqrt{\frac{a_k}{d}}\,\|G_2(\cdot,0)\|_{x\in\mathcal{W}_k} \right\},$$

where we have employed again the Schwarz inequality. The k–th term of the sum reaches its minimum w.r.t. the norm at $-\frac{8\pi^2}{d_k d^3}\,\alpha_k^2 a_k^2$. Returning to the original functional and neglecting in the first term of the last estimate all contributions except the one coming from the leftmost component of $\mathbb{R}\setminus\mathcal{E}$, we see that there is a positive γ such that

$$L(\psi) \;>\; \frac{1}{4}\,\|\psi_x\|^2 + \frac{1}{2}\,\|\psi_x\|^2_{x<x_k-2a_k} - \gamma\sum_{k=1}^N \alpha_k^2 a_k^2 \qquad (2.27)$$

holds provided $|\mathcal{W}|$ is small enough.

To conclude the proof, we denote $\ell_k := x_k-2a_k-x_1+2a_1$ and employ the identity

$$\sum_{k=1}^N \alpha_k^2 a_k^2 \;=\; \sum_{k=1}^N \alpha_1^2 a_k^2 + \sum_{k=1}^N (\alpha_k^2-\alpha_1^2)a_k^2 \qquad (2.28)$$

together with the estimate

$$\frac{1}{4}\,\|\psi_x\|^2 \;\ge\; \frac{1}{4}\,\|c_1'\|^2 \;\ge\; \frac{1}{4N}\sum_{k=1}^N \frac{(\alpha_k-\alpha_1)^2}{\ell_k}\,.$$

If $-\gamma(\alpha_k^2-\alpha_1^2)a_k^2 + \frac{1}{4N\ell_k}(\alpha_k-\alpha_1)^2 \ge 0$ holds for all $k=2\,\dots,N$, the bound (2.27) reduces to

$$L(\psi) \;>\; \frac{1}{2}\,\|\psi_x\|^2_{x<x_k-2a_k} - \gamma\alpha_1^2\sum_{k=1}^N a_k^2\,. \qquad (2.29)$$

On the other hand, suppose that the endpoint values satisfy $\alpha_k - \alpha_1 = \mathcal{O}(a_k)$ as $a_k \to 0$ for $k \in \mathcal{K} \subset \{2, \dots, N\}$. In view of (2.28) we have

$$L(\psi) > \frac{1}{2} \|\psi_x\|^2_{x<x_k-2a_k} - \gamma\alpha_1^2 \sum_{k=1}^{N} a_k^2 + \sum_{k\in\mathcal{K}} \left\{ \frac{1}{4N\ell_k}(\alpha_k - \alpha_1)^2 - \gamma(\alpha_k^2 - \alpha_1^2)a_k^2 \right\} ;$$

however, the last term is $\mathcal{O}\left(\sum_{k\in\mathcal{K}} a_k^2\right)$, so (2.29) is valid again with a smaller positive coefficient in the last term. Since $\|\psi\|^2 \ge 2\int_{-\infty}^{x_1-2a_1} c_1(x)^2 dx$, the quantity of interest is bounded from below by

$$\frac{L(\psi)}{\|\psi\|^2} > \frac{\int_{-\infty}^{x_1-2a_1} c_1'(x)^2 dx \; - \; \gamma\alpha_1^2 I(\mathcal{W})}{2\int_{-\infty}^{x_1-2a_1} c_1(x)^2 dx} \,.$$

The *rhs* is minimized by the function $c_1(x) = \alpha_1 e^{\kappa(x-x_1+2a_1)}$ which yields the value $(\kappa^2/2) - \gamma I(\mathcal{W})\kappa$; taking the minimum over κ we find

$$\frac{L(\psi)}{\|\psi\|^2} > -\gamma^2 I(\mathcal{W})^2 . \quad \blacksquare \tag{2.30}$$

3 Window–coupled layers

The setting of the three–dimensional problem is similar. We have a straight layer, $\Sigma := I\!\!R^2 \times [-d_2, d_1]$, and a set $\mathcal{W} \subset I\!\!R^2$ which can be written as a finite union, $\mathcal{W} := \cup_{k=1}^{N} \mathcal{W}_k$, whose components are open, connected sets of nonzero Lebesgue measure; without loss of generality we may suppose they are mutually disjoint. Then we define $H(d_1, d_2; \mathcal{W})$ as the Laplacian on $L^2(\Sigma)$ obeying the Dirichlet condition at the boundary of Σ, *i.e.*, $y = -d_2, d_1$, as well as at $I\!\!R^2 \setminus \mathcal{W}$. This operator coincides again with the Dirichlet Laplacian [RS4, Sec.XIII.15] for the sliced layer the two parts of which are connected through the window set $\mathcal{W}$. We use the same notation as above, $d := \max\{d_1, d_2\}$ and $D := d_1 + d_2$. The nontrivial part of the symmetric case, $d_1 = d_2$, reduces again to analysis of the Laplacian $L^2(\Sigma_+)$, where $\Sigma_+ := I\!\!R^2 \times [0, d]$, with the Neumann condition at window part of the plane $y = 0$ and Dirichlet at the remaining part of the boundary; this operator will be denoted as by $H(d; \mathcal{W})$.

Our main aim here is to prove a weak–coupling asymptotic estimate for a pair of layers connected by a single window.

Theorem 3.1 $\sigma_{\text{ess}}(H(d_1, d_2; \mathcal{W})) = [(\pi/d)^2, \infty)$. *The discrete spectrum is contained in* $((\pi/D)^2, (\pi/d)^2)$, *finite, and nonempty provided* $\mathcal{W} \ne \emptyset$. *Suppose further that* $N = 1$ *and* $\mathcal{W} = aM$ *for an nonempty open set* M *contained in the unit ball* $B_1 \subset I\!\!R^2$. *Then* $\sigma_{\text{disc}}(H(d_1, d_2; aM))$ *consists of just one simple eigenvalue* $\epsilon(aM) \le (\pi/d)^2$ *for all* a *small enough, and there are positive* c_1, c_2 *such that*

$$-\exp\left(-c_1 a^{-3}\right) \le \epsilon(a) - \left(\frac{\pi}{d}\right)^2 \le -\exp\left(-c_2 a^{-3}\right) . \tag{3.1}$$

Proof is based again on variational estimates. *The upper bound* in the symmetric case, $d_1 = d_2$, employs the trial function $\psi = F + \eta G$, where $F(x,y) := f_1(x)\chi_1(y)$ again with

$$f_1(x) \;:=\; \min\left\{ 1,\; \frac{K_0(\kappa|x|)}{K_0(\kappa a)} \right\}, \tag{3.2}$$

and

$$G(x,y) \;:=\; \chi_{aM}(x)\phi_1(x)R(y)\,, \tag{3.3}$$

where $\phi_1^{(a)}$ is the ground–state eigenfunction, $\|\phi_1^{(a)}\| = 1$, of the operator $-\Delta_D^{aM}$ corresponding to the positive eigenvalue $\mu_1(a) = \mu_1(1)a^{-2}$, and

$$R(y) \;:=\; \begin{cases} e^{-\sqrt{\mu_1(a)}\,y} & \dots \quad y \in \left[0, \frac{d}{2}\right] \\ 2\left(1-\frac{y}{d}\right)\exp\left(-\frac{d}{2}\sqrt{\mu_1(a)}\right) & \dots \quad y \in \left[\frac{d}{2}, d\right] \end{cases} \tag{3.4}$$

Using $-\chi_1'' = (\pi/d)^2\chi_1$, a simple integration by parts, and the fact that the vector functions ∇f_1 and $\nabla\phi_1^{(a)}$ have disjoint supports, we can express the reduced energy functional $L(\psi) := (H\psi,\psi) - \left(\frac{\pi}{d}\right)^2 \|\psi\|^2$ as

$$\begin{aligned} L(\psi) \;&=\; \|\nabla f_1\|^2_{L^2(I\!R^2)} + \eta^2\left(\mu_1(a) - \left(\frac{\pi}{d}\right)^2\right)\|R\|^2_{L^2(0,d)} \\ &-\; \eta^2\|R'\|^2_{L^2(0,d)} - 2\eta\chi_1'(0)\int_{aM}\phi_1^{(a)}(x)\,dx\,, \end{aligned} \tag{3.5}$$

where the negative term in the bracket can be, of course, neglected. The second and the third term at the *rhs* can be estimated in analogy with [EV1],

$$\mu_1(a)\|R\|^2_{L^2(0,d)} - \eta^2\|R'\|^2_{L^2(0,d)} \;<\; \frac{\sqrt{\mu_1(1)}}{2a}\,(2+\varepsilon_1)$$

for a fixed $\varepsilon_1 > 0$ and any a small enough. In a similar way, the last term equals $-2\eta\chi_1'(0)Ca$, where $C := \int_M \phi_1^{(1)}(x)\,dx$. Finally, the first one can be evaluated by means of [AS, 9.6.26],[PBM, 1.12.3.2],

$$K_0(\kappa a)^2\|\nabla f_1\|^2_{L^2(I\!R^2)} \;=\; 2\pi\left[\frac{1}{2}\kappa^2a^2K_1'(\kappa a)^2 - \frac{1}{2}\left(\kappa^2a^2+1\right)K_1(\kappa a)^2\right].$$

Using $-K_1'(\xi) = K_0(\xi) + \xi^{-1}K_1(\xi)$ in combination with the asymptotic expressions $K_0(\xi) = -\ln\xi + \mathcal{O}(1)$, $K_1(\xi) = \xi^{-1} + \mathcal{O}(\ln\xi)$, we find

$$\|\nabla f_1\|^2_{L^2(I\!R^2)} \;<\; -\,\frac{2\pi(1+\varepsilon_2)}{\ln\kappa a}$$

for a fixed ε_2 and a small enough. Substituting these estimates into (3.5) and taking a minimum over η we arrive at the bound

$$L(\psi) \;<\; -\,\frac{2\pi(1+\varepsilon_2)}{\ln\kappa a} \;-\; \frac{2\chi_1'(0)^2C^2}{(2+\varepsilon_1)\sqrt{\mu_1(1)}}\,a^3\,. \tag{3.6}$$

It remains to find a lower bound to

$$\|\psi\|^2 \;\geq\; \|\psi\|^2_{|x|\geq a} - 2\|F\|^2_{|x|\leq a} - 2\eta^2\|F\|^2_{|x|\leq a} \;=\; \|\psi\|^2_{|x|\geq a} - 2\pi a^2 - 2\eta^2\|R\|^2_{L^2(0,d)}\,.$$

The last term is $\mathcal{O}(a)$, while the first one can be expressed as

$$K_0(\kappa a)^2\|F\|^2_{|x|\geq a} \;=\; \pi a^2\left[K_1(\kappa a)^2 - K_0(\kappa a)^2\right] \;=\; \frac{\pi}{\kappa^2} \,+\, \mathcal{O}(a^2\ln\kappa a)\;;$$

using the asymptotic behaviour of K_0 we find $\|\psi\|^2 \geq \pi\kappa^{-2}(\ln\kappa a)^{-2}(1-\varepsilon_3)$ for a fixed $\varepsilon_3 > 0$ and a small enough. Hence

$$\frac{L(\psi)}{\|\psi\|^2} \;<\; -\,\frac{\kappa^2\ln\kappa a}{\pi(1-\varepsilon_3)}\,(Da^3\ln\kappa a + E)\,, \tag{3.7}$$

where $E := 2\pi(1+\varepsilon_2)$ and

$$D \;:=\; \frac{2\chi_1'(0)^2C^2}{(2+\varepsilon_1)\sqrt{\mu_1(1)}}\,.$$

Minimizing the *rhs* of (3.7) with respect to κ, we conclude that to fixed positive $\varepsilon_1,\, \varepsilon_2$ and $\varepsilon_3 \in (0,1)$ there is a function g such that

$$\frac{L(\psi)}{\|\psi\|^2} \;<\; g(a) \qquad \text{and} \qquad g(a) \;\approx\; -\,\frac{1+\varepsilon_2}{1-\varepsilon_3}\,\frac{1}{a^2}\,e^{-2E/Da^3} \tag{3.8}$$

as $a \to 0$. The upper bound in (3.1) follows readily from (3.8); the extension to the nonsymmetric case is obtained as in [EV1].

Remark 3.2 In fact, one could suppose $M = B_1$ because the eigenvalue is pushed up if we reduce the window to a circle contained in M, and the obtained bound is all the same not optimal as in Remark 2.2. In the rest of the proof we *embedd* M into a circle leaving the question about relations between the constants and the geometry of M to more sophisticated methods.

The lower bound can again be proven in the symmetric case only. We begin with auxiliary results. When constructing the trial function component (3.2), we have used implicitly the fact that the functional $F\,:\; F(\phi) = \int_a^\infty (\phi'(t)^2 + m^2\phi(t)^2)\;tdt$ on $C^2([a,\infty))$ with the condition $\phi(a) = \alpha$ and fixed positive a, m and is minimized by

$$\phi_0 \;:\; \phi_0(t) \;=\; \alpha\,\frac{K_0(mt)}{K_0(ma)}\,, \tag{3.9}$$

as can be easily seen from solution of the appropriate Euler's equation. Furthermore, a two–dimensional analogy of the bound (4.2) in [EV1] is given by the *Friedrichs inequality* [Ne, Thm. 1.9]: if $\Omega \subset I\!\!R^n$, $n \geq 2$, is a bounded domain with Lipschitz boundary, there is a positive c such that

$$\|\nabla f\|^2 \geq \; c\|f\|^2 \tag{3.10}$$

holds for every $f \in H_0^1(\Omega)$. The constant is, of course, easy to find for the circle $\Omega = B_a$ in terms of the appropriate Bessel zero, $c = j_{0,1}^2 a^{-2}$.

Repeating the argument of [EV1] and the previous section, we infer that one has to find a lower bound to $L(\psi)/\|\psi\|^2$ over all real $\psi \in L^2(\Sigma)$, which are C^2, radially symmetric, and vanish at the boundary except in the window. We can express such a ψ in the form of the series (2.13) again, where the convergence is uniform for $|x| \geq a$. The coefficients c_n depend in fact only of $r := |x|$. Moreover, in analogy with (2.23) we may restrict our attention to trial functions with

$$|c_n(r)| \,\leq\, |c_n(a)|\, \frac{K_0\left(\frac{\pi}{d}\sqrt{n^2-1}\,r\right)}{K_0\left(\frac{\pi}{d}\sqrt{n^2-1}\,a\right)} \tag{3.11}$$

for $n \geq 2$. As before we introduce

$$F(x,y) \,:=\, \begin{cases} \alpha\chi_1(y) & \dots \quad 0 \leq r \leq 2a \\ c_1(r)\chi_1(y) & \dots \quad r \geq 2a \end{cases} \tag{3.12}$$

with $\alpha := c_1(2a)$, and divide the rest $G(x,y) = \psi(x,y) - F(x,y)$ into

$$G_1(x,y) \,:=\, (c_1(r)-\alpha)\chi_1(y)$$

supported in the extended window region, $r \leq 2a$, and $G_2(x,y) = \hat{G}(x,y) + \Gamma(x,y)$ with

$$\Gamma(x,y) \,:=\, \sum_{n=2}^{\infty} c_n(2a)\chi_n(y)\,.$$

We start estimating the reduced energy functional

$$L(\psi) \,=\, \|\nabla_x\psi\|^2 + \|G_y\|^2 - \left(\frac{\pi}{d}\right)^2 \|G\|^2 - 2\alpha\chi_1'(0)\int_{B_a} G(x,0)\,dx \tag{3.13}$$

from the "external" contribution to the first "two and a half" terms,

$$\begin{aligned} L_1 \,&:=\, \frac{1}{2}\,\|\nabla_x\psi\|^2_{r\geq a} + \|G_y\|^2_{r\geq a} - \left(\frac{\pi}{d}\right)^2 \|G\|^2_{r\geq a} \\ &=\, \pi \sum_{n=1}^{\infty} \int_a^{\infty} \left(c_n'(r)^2 + 2\left(\frac{\pi}{d}\right)^2 (n^2-1)c_n(r)^2 \right) r\,dr \\ &\geq\, \pi \sum_{n=2}^{\infty} \int_a^{\infty} \left(c_n'(r)^2 + 2\left(\frac{\pi n}{d}\right)^2 c_n(r)^2 \right) r\,dr \\ &\geq\, \pi \sum_{n=2}^{\infty} c_n(a)^2\, \frac{\pi n}{d}\, a\, \frac{K_1\left(\frac{\pi n}{d}\,a\right)}{K_0\left(\frac{\pi n}{d}\,a\right)} \,\geq\, \frac{\pi^2 a}{d} \sum_{n=2}^{\infty} n\, c_n(a)^2\,, \end{aligned} \tag{3.14}$$

where in the last line we have used (3.9), evaluated the integral as in the first part of the proof, and employed the inequality $K_1(\xi) \geq K_0(\xi)$ which follows from the well–known integral representation [AS, 9.6.24]. Next we turn to

$$L_2 \,:=\, \|\nabla_x\psi\|^2_{r\leq 2a} \,=\, \|\nabla_x G_1\|^2_{r\leq 2a} + \|\nabla_x G_2\|^2_{r\leq 2a}\,. \tag{3.15}$$

By assumption, G_1 vanishes at $r=2a$, so the first term can be estimated from (3.10) as

$$\|\nabla_x G_1\|^2_{r\le 2a} \;\ge\; \frac{C_1}{4a^2}\,\|G_1\|^2_{r\le 2a} \;=\; \frac{C_1}{a^2}\,\|G_1\|^2\,, \tag{3.16}$$

where $4C_1 := j^2_{0,1}$. Furthermore, introducing the window neighbourhood $\Omega_a := B_{2a}\times[0,a]$, we have

$$\begin{aligned}\|\nabla_x G_2\|^2_{r\le 2a} \;&=\; \|\nabla_x \hat G\|^2_{r\le 2a} \;\ge\; \frac{C_1}{a^2}\,\|\hat G\|^2_{r\le 2a} \\ &\ge\; \frac{C_1}{a^2}\,\|\hat G\|^2_{\Omega_{2a}} \;\ge\; \frac{\delta C_1}{a^2}\,\|\hat G\|^2_{\Omega_{2a}} \;\ge\; \frac{\delta C_1}{2a^2}\,\|G_2\|^2_{\Omega_{2a}} - \frac{\delta C_1}{a^2}\,\|\Gamma\|^2_{\Omega_{2a}}\end{aligned} \tag{3.17}$$

for all $a\le d$ and $\delta\in(0,1]$. The last norm can be estimated as in the previous cases by combining the smallness of the χ_n norm restricted to $[0,a]$ with the dominated decay (3.11),

$$\begin{aligned}\|\Gamma\|^2_{\Omega_a} \;&=\; 4\pi a^2\int_0^a\left(\sum_{n=2}^{\infty} c_n(2a)\chi_n(y)\right)^2 dy \\ &\le\; 8\pi a^2\left(\sum_{n=2}^{[a^{-1}]+1} n^{-1}c_n(a)^2\int_0^a \chi_n(y)^2dy\right)\sum_{n=2}^{[a^{-1}]+1} n \\ &+\; 8\pi a^2\left(\sum_{2\le n=[a^{-1}]+2}^{\infty} nc_n(a)^2\int_0^a \chi_n(y)^2dy\right)\sum_{2\le n=[a^{-1}]+2}^{\infty}\frac{K_0^2\left(\frac{2\pi a}{d}\sqrt{n^2-1}\right)}{nK_0^2\left(\frac{\pi a}{d}\sqrt{n^2-1}\right)} \\ &\le\; \frac{16\pi a^3}{d}\left(\frac{2\pi^2}{3d^2}+\sum_{2\le n=[a^{-1}]+2}^{\infty}\frac{K_0^2\left(\frac{2\pi a}{d}\sqrt{n^2-1}\right)}{nK_0^2\left(\frac{\pi a}{d}\sqrt{n^2-1}\right)}\right)\sum_{n=2}^{\infty} nc_n(a)^2\,.\end{aligned}$$

The sum in the bracket can be estimated as

$$\sum_{2\le n=[a^{-1}]+2}^{\infty}\frac{K_0^2\left(\frac{2\pi a}{d}\sqrt{n^2-1}\right)}{nK_0^2\left(\frac{\pi a}{d}\sqrt{n^2-1}\right)} \;\le\; \int_{a^{-1}}^{\infty}\frac{K_0^2\left(\frac{2\pi a}{d}\sqrt{\xi^2-1}\right)}{\xi K_0^2\left(\frac{\pi a}{d}\sqrt{\xi^2-1}\right)}\,d\xi \;\le\; \int_1^{\infty}\frac{K_0^2\left(\frac{\pi\xi}{d}\right)}{\xi K_0^2\left(\frac{\pi\xi}{2d}\right)}$$

for $a<\frac12\sqrt3$, and the integral on the *rhs* is convergent, because $K_0(\xi)\approx\sqrt{\frac{\pi}{2\xi}}\,e^{-\xi}$ as $\xi\to\infty$. Hence there is a positive C_2 independent of ψ and a such that

$$\frac{C_1}{a^2}\,\|\Gamma\|^2_{\Omega_a} \;<\; C_2 a\sum_{n=2}^{\infty} nc_n(a)^2\,. \tag{3.18}$$

Combining the estimates (3.14)–(3.18), we arrive at

$$L_1+L_2 \;\ge\; a\left(\frac{\pi^2}{d}-\delta C_2\right)\sum_{n=2}^{\infty} nc_n(a)^2+\frac{C_1}{a^2}\,\|G_1\|^2+\frac{\delta C_1}{2a^2}\,\|G_2\|^2_{\Omega_a}\,,$$

which gives

$$L_1 + L_2 \;\geq\; \frac{C_1}{a^2}\,\|G_1\|^2 + \frac{m^2}{a^2}\,\|G_2\|^2_{\Omega_a} \tag{3.19}$$

for some $m>0$ and all sufficiently small a.

The norm of G_y is estimated as in the two–dimensional case [EV1],

$$\|G_y\|^2_{r\leq a} \;\geq\; \|G_{2,y}\|^2_{r\leq a} - \frac{2\pi}{d^2}\,\left(2\|G_1\|^2_{r\leq a} + d\|G_2(\cdot,0)\|^2_{r\leq a}\right)\,,$$

which together with (3.19) yields

$$\begin{aligned}
L_1 \;+\; L_2 \;+\; & \|G_y\|^2_{r\leq a} - \left(\frac{\pi}{d}\right)^2 \|G\|^2_{r\leq a} \\
\geq\; & \|G_{2,y}\|^2_{r\leq a} - \left(\frac{\pi}{d}\right)^2 \|G_2\|^2_{r\leq a} + \left(\frac{C_1}{a^2} - \frac{\pi(\pi+4)}{d^2}\right) \|G_1\|^2_{r\leq a} \\
+\; & \frac{m^2}{a^2}\,\|G_2\|^2_{\Omega_a} - \frac{2\pi}{d}\,\|G_2(\cdot,0)\|^2_{r\leq a} \\
\geq\; & \left(\frac{c_0}{a} - \frac{2\pi}{d}\right) \|G_2(\cdot,0)\|^2_{r\leq a} \;\geq\; \frac{c_0}{2a}\,\|G_2(\cdot,0)\|^2_{r\leq a}
\end{aligned}$$

for a positive c_0 and any a small enough; in the second step we have neglected a positive term and employed Lemma 2.5. Substituting from here to (3.13) and using the Schwarz inequality,

$$\int_{B_a} G(x,0)\,dx \;\leq\; \|G_2(\cdot,0)\|_{r\leq a}\sqrt{\pi}\,a$$

we get

$$\begin{aligned}
L(\psi) \;\geq\; & \frac{1}{2}\,\|\nabla_x\psi\|^2_{r\geq 2a} - 2\alpha a\chi_1'(0)\sqrt{\pi}\,\|G_2(\cdot,0)\|_{r\leq a} + \frac{c_0}{2a}\,\|G_2(\cdot,0)\|^2_{r\leq a} \\
\geq\; & \frac{1}{2}\,\|\nabla_x\psi\|^2_{r\geq 2a} - \frac{2\pi\alpha^2\chi_1'(0)^2}{c_0}\,a^3\,.
\end{aligned}$$

The first term on the *rhs* can be estimated from below by the first transverse–mode contribution. The same applies to $\|\psi\|^2$, so finally we find

$$\frac{L(\psi)}{\|\psi\|^2} \;\geq\; \frac{\int_{2a}^\infty c_1'(r)^2 r\,dr - \frac{\pi\chi_1'(0)^2}{c_0}\,a^3 c_1(2a)^2}{2\int_{2a}^\infty c_1(r)^2 r\,dr}\,. \tag{3.20}$$

In analogy with (3.9) one has to solve the appropriate Euler's equation to check that the *rhs* of (3.20) is minimized by $c_1=\phi_\kappa$ for some $\kappa>0$, where

$$\phi_\kappa(r) \;:=\; c_1(2a)\,\frac{K_0(\kappa r)}{K_0(2\kappa a)}\,.$$

Substituting into (3.20), evaluating the integrals, and taking the asymptotics for small a, we infer that

$$\frac{L(\psi)}{\|\psi\|^2} \geq -\kappa^2 \ln(2\kappa a) \left(\frac{\pi \chi_1'(0)^2}{c_0(1+\varepsilon_2)}\, a^3 \ln(2\kappa a) + \frac{1-\varepsilon_1}{1+\varepsilon_2} \right)$$

holds for any fixed $\varepsilon_1,\, \varepsilon_2 > 0$ and all sufficiently small a. It remains to find the minimum of the *rhs* with respect to κ. However, since it differs from (3.7) just by the values of the constants, the argument is concluded as in the first part of the proof. ∎

4 Conclusions

To make sense of the derived bounds one has to take into account two aspects of the problem. First of all, we have mentioned already that the discrete spectrum can also be found numerically by means of the mode–matching method; a detailed description of the two–dimensional case is given in [EŠTV]. Although the method converges rather slowly if the window is narrow, the results obtained for a single window clearly suggest that the true asymptotics exists and is of the same type as our asymptotic bounds.

Another insight can be obtained from comparing our result with the well-known weak–coupling asymptotics for Schrödinger operators in dimension one and two [BGS, Kl, Si]. The ground state of the coupled strips in the narrow–window case is dominated the lowest transverse–mode component with long exponentially decaying tails and a local modification in the coupling region. In a similar way, a link can be made between window–connected layers and a two–dimensional Schrödinger operator. The comparison shows that the attractive interaction due to opening a narrow window (in particular, by changing the Dirichlet b.c. to Neumann at a short segment of the boundary in the symmetric case) acts effectively as a potential well of a depth proportional to the size of the window.

Conjecture 4.1 *Let $H(d_1,d_2;\mathcal{W})$ be the operators described above. The ground–state eigenvalue behaves for small $|\mathcal{W}|$ as*

$$\epsilon(a) \approx \left(\frac{\pi}{d}\right)^2 - \frac{1}{d^2} \left(\sum_{k=1}^{N} c_{2,k}(\nu) a_k^2 \right)^2 \qquad \dots \quad \dim\Sigma = 2 \qquad (4.1)$$

$$\epsilon(a) \approx \left(\frac{\pi}{d}\right)^2 - \frac{1}{d^2} \exp\left\{ - \left(\sum_{k=1}^{N} c_{3,k}(\nu) a_k^3 \right)^{-1} \right\} \qquad \dots \quad \dim\Sigma = 3 \qquad (4.2)$$

where $\nu := d^{-1} \min\{d_1,d_2\}$, and a_k in the three-dimensional case is the scaling parameter of the k–th window.

The conjecture is based on the described analogy only, and therefore it is difficult to say more about the coefficients. It is not excluded that they depend on the geometry of the window–center set for $N > 1$; in the three-dimensional case the shapes of the scaled windows may also play role. We refrain from speculating about the nature of the error terms.

On the other hand, we are convinced that the open "constant cross–section" shape of our regions Σ is crucial for the asymptotics. For instance, if Σ is instead a bounded planar region with the Dirichlet boundary in which we open a window (to another bounded region the essential spectrum threshold of which is not lower) or a Neumann segment, we conjecture that leading term in the ground state shift is proportional to the *square* of the window width. Moreover, the same asymptotics is expected to be valid for higher eigenvalues provided the corresponding eigenfunctions are locally *symmetric* with respect to the window axis. In any case, proving of such asymptotic properties represents an intriguing mathematical problem.

Acknowledgments

Thanks are due to P. Lindovský for useful discussions. The research has been partially supported by the Grant GACR No. 202–0218.

References

[AS] M.S. Abramowitz, I.A. Stegun, eds.: Handbook of Mathematical Functions, Dover, New York 1965.

[BGS] R. Blanckenbecler, M.L. Goldberger, B. Simon: The bound states of weakly coupled long-range on-dimensional quantum Hamiltonians, Ann. Phys. **108** (1977), 89-78.

[BGRS] W. Bulla, F. Gesztesy, W. Renger, B. Simon: Weakly coupled bound states in quantum waveguides, mp_arc 95–422; Proc. Am. Math. Soc., to appear

[DP] E.B. Davies, L. Parnowski: Trapped modes in acoustic waveguides, mp_arc 96–46.

[DE] P. Duclos, P. Exner: Curvature–induced bound states in quantum waveguides in two and three dimensions, Rev. Math. Phys. **7**, 73–102 (1995).

[ELV] D.V. Evans, D.M. Levitin, D. Vassiliev: Existence theorem for trapped modes, J. Fluid Mech. **261**, 21–31 (1994).

[EŠ] P. Exner, P. Šeba: Bound states in curved quantum waveguides, J. Math. Phys. **30**, 2574–2580 (1989).

[EŠTV] P. Exner, P. Šeba, M. Tater, D. Vaněk: Bound states and scattering in quantum waveguides coupled laterally through a boundary window, J. Math. Phys. **37**, 4867–4887 (1996).

[EV1] P. Exner, S.A. Vugalter: Asymptotic estimates for bound states in quantum waveguides coupled laterally through a narrow window, Ann. Inst. H. Poincaré: Phys. théor. **65**, 109–123 (1996).

[EV2] P. Exner, S.A. Vugalter: Bound states in a locally deformed waveguide: the critical case, Lett. Math. Phys. **39** (1997), 59–68.

[Kl] M. Klaus: On the bound state of Schrödinger operators in one dimension, Ann. Phys. **108** (1977), 288-300.

[Ne] J. Nečas: Les méthodes directes en théorie des équations elliptiques, Academia, Prague 1967.

[PBM] A.P. Prudnikov, Yu.O. Brychkov, O.I. Marichev: Integrals and Series, II. Special Functions, Gordon and Breach, New York 1986.

[RS4] M. Reed, B. Simon: Methods of Modern Mathematical Physics, IV. Analysis of Operators, New York: Academic Press 1978.

[Si] B. Simon: The bound state of weakly coupled Schrödinger operators in one and two dimensions, Ann. Phys. **97** (1976), 279-288.