Scaling Variables and Asymptotic Expansions in Damped Wave Equations

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Dedicated to Professor K. Kirchgässner on the occasion of his sixty-fifth birthday

Abstract. We study the long time behavior of small solutions to the nonlinear damped wave equation

$$\varepsilon u_{\tau\tau} + u_\tau \;=\; (a(\xi)u_\xi)_\xi + \mathcal{N}(u, u_\xi, u_\tau) \;, \quad \xi \in \mathbf{R} \;, \quad \tau \geq 0 \;,$$

where ε is a positive, not necessarily small parameter. We assume that the diffusion coefficient $a(\xi)$ converges to positive limits $a_\pm$ as $\xi \to \pm\infty$, and that the nonlinearity $\mathcal{N}(u, u_\xi, u_\tau)$ vanishes sufficiently fast as $u \to 0$. Introducing scaling variables and using various energy estimates, we compute an asymptotic expansion of the solution $u(\xi,\tau)$ in powers of $\tau^{-1/2}$ as $\tau \to +\infty$, and we show that this expansion is entirely determined, up to second order, by a linear parabolic equation which depends only on the limiting values $a_\pm$. In particular, this implies that the small solutions of the damped wave equation behave for large τ like those of the parabolic equation obtained by setting $\varepsilon = 0$.

Keywords : damped wave equation, asymptotic behavior, scaling variables

AMS classification codes (1991) : 35B40, 35B35, 35L05, 35C20

1. Introduction

In this paper, we study the long time behavior of small solutions to the following damped hyperbolic equation

$$\varepsilon u_{\tau\tau} + u_\tau \;=\; (a(\xi)u_\xi)_\xi + \mathcal{N}(u, u_\xi, u_\tau) \;, \tag{1.1}$$

where $\xi \in \mathbf{R}$, $\tau \geq 0$, $u(\xi,\tau) \in \mathbf{R}$, and ε is a positive, not necessarily small parameter. We assume that the diffusion coefficient $a(\xi)$ is positive and converges to positive limits $a_\pm$ as $\xi \to \pm\infty$. In addition, we suppose that the nonlinearity $\mathcal{N}$ satisfies $\mathcal{N}(0,0,0) = 0$, and we shall make below specific hypotheses on $\mathcal{N}$ which ensure that $u = 0$ is a stable equilibrium point of Eq.(1.1).

Our main purpose is to describe precisely the long time behavior of small solutions $u(\xi,\tau)$ to Eq.(1.1) by computing an asymptotic expansion in powers of $\tau^{-1/2}$ as $\tau \to +\infty$. To keep this paper within a reasonable length, we shall limit ourselves here to a second order expansion, but our techniques are general and can be applied to higher orders as well. We shall show that, under appropriate hypotheses on the diffusion a and the nonlinearity $\mathcal{N}$, the asymptotic expansion of the solutions $u(\xi,\tau)$ of Eq.(1.1) up to second order is entirely determined by a linear parabolic equation (Eq.(1.13) below), which depends only on the limiting values $a_\pm$. In particular, the parameter ε enters in this expansion only through the coefficients α^*, β^* (see Theorems 1.1 and 1.2). As a consequence, the solutions of the damped hyperbolic equation (1.1) behave for large τ like those of the parabolic equation obtained by setting $\varepsilon = 0$ in (1.1).

Equations of the form (1.1) often arise as mathematical models in biology, especially in genetics and population dynamics ([DO], [Ha], [Za]). In this context, the spatial spread of individuals is usually modelled by Brownian motion, so that the population densities are solutions of reaction-diffusion systems [Mu]. If the motion takes place in an inhomogeneous medium, the diffusion coefficients in these equations may depend on the space variable ξ ([CK], [Ok]). However, the models of spatial spread based on a diffusion process are not realistic for short times, and several authors proposed to improve them by replacing the usual random walk in Brownian motion with a *correlated random walk* ([Ta], [Go], [Kac]). In one space dimension, this procedure leads to damped hyperbolic equations instead of reaction-diffusion equations, and in the case of an inhomogeneous medium, the equation (1.1) is obtained, possibly with additional terms [Zau, p. 21-26].

Evolution equations with a non constant diffusion term also arise very naturally in the modelling of various physical phenomena in thin, non uniform structures. For example, in the theory of nonlinear transmission lines, one encounters a damped Sine-Gordon equation defined on a long thin rod of cross section $\delta a(\xi)$, where $\delta \ll 1$ ([LFDFH], [LDLH]). When $\delta \to 0$, the Laplace operator (with Neumann conditions at the boundary of the rod) converges to $a^{-1}(\xi)\partial_\xi(a(\xi)\partial_\xi)$ [Ya], and the limiting one-dimensional Sine-Gordon equation takes the form of Eq.(1.1) ([HR3], [Ra]).

In Eq.(1.1), the diffusion operator has been written as $\partial_\xi(a(\xi)\partial_\xi)$ for convenience, but our results hold as well for the operators $a(\xi)\partial_{\xi\xi}$ or $a^{-1}(\xi)\partial_\xi(a(\xi)\partial_\xi)$, since these different forms are obtained from each other by a change of variables. For instance, if $v(y,\tau)$ is a solution of $\varepsilon v_{\tau\tau} + v_\tau = b^{-1}(bv_y)_y$, then defining $B(y) = \int_0^y b(z)\,dz$ and $u(\xi,\tau) = v(B^{-1}(\xi),\tau)$, we see that $u(\xi,\tau)$ satisfies $\varepsilon u_{\tau\tau} + u_\tau = (au_\xi)_\xi$ with $a(\xi) = b^2(B^{-1}(\xi))$. Other forms like $b^{-1}(\xi)\partial_\xi(a(\xi)\partial_\xi)$ or $\partial_\xi(a(\xi)\partial_\xi)+c(\xi)\partial_\xi$ (with c integrable) are also equivalent.

That the solutions of Eq.(1.1) behave, for ε small, like those of the corresponding parabolic equation, is a well-known fact. In a series of articles, Zlámal ([Zl1], [Zl2], [Zl3]) compared the solutions of the telegrapher's equation $\varepsilon u_{\tau\tau} + u_\tau = u_{\xi\xi}$ with those of the heat equation $U_\tau = U_{\xi\xi}$, under rather strong regularity assumptions on the initial data. In particular, if $u(\cdot,0) = U(\cdot,0)$, he showed that $u(\cdot,\tau) = U(\cdot,\tau) + \mathcal{O}(\varepsilon)$ uniformly in $\tau \geq 0$. Similar results were subsequently obtained by Fulks and Guenther [FG] for more general initial data. Finally, on a bounded domain, the global attractor of Eq.(1.1) can be compared, when ε is sufficiently small, with the attractor of the corresponding parabolic equation, see for instance Babin and Vishik [BV], Mora and Sola-Moralès [MSM], Hale and Raugel ([HR1],[HR2]).

Although the smallness of ε is essential to compare finite time trajectories or special sets of solutions of both equations, our results indicate that this assumption is no longer needed if we are interested only in qualitative properties of the long time behavior of the solutions, like the shape of an asymptotic expansion. Again such an idea is not completely new: some results in this direction have already been obtained by Hsiao and Liu [HL1], [HL2] and Nishihara [Ni], in the context of hyperbolic conservation laws with damping. More precisely, in analyzing the long time behavior of a damped hyperbolic system, these authors are led to show that the solutions of a conservative quasilinear wave equation similar to Eq.(1.1) converge as $\tau \to +\infty$ to self-similar solutions of the corresponding quasilinear heat equation, and they provide precise estimates of this convergence in various function spaces. In their proof, however, the conservation laws of the system play quite an important role.

On the other hand, in [GR1], we studied the properties of uniformly translating front solutions of Eq.(1.1) with $a \equiv 1$ and $\mathcal{N}(u, u_\xi, u_\tau) = u - u^2$, for any $\varepsilon > 0$. In particular, we proved various stability results which are similar to those of the corresponding parabolic equation. In a forthcoming paper [GR2], we shall describe more precisely the long time behavior of the perturbations around these fronts, and show that their asymptotic expansion in powers of $\tau^{-1/2}$ coincides to the leading order with the corresponding expansion in the parabolic case. These perturbations turn out to satisfy a damped hyperbolic equation with variable coefficients of a slightly more general form than Eq.(1.1), and this was one of our main motivations for considering here the simpler problem (1.1).

To obtain an asymptotic expansion of the solution $u(\xi,\tau)$ of Eq.(1.1) in powers of $\tau^{-1/2}$ as $\tau \to +\infty$, we use the *scaling variables* (or similarity variables) defined by

$$x = \frac{\xi}{\sqrt{\tau+\tau_0}} \ , \quad t = \log(\tau+\tau_0) \ , \tag{1.2}$$

where τ_0 is some positive time. These variables have been widely used to study the long time behavior of solutions to parabolic equations, in particular to prove convergence to self-similar solutions. Various results of this kind have been obtained for the nonlinear heat equation $u_\tau = \Delta u \pm |u|^{p-1}u$ ([GKS], [Ka], [BK2], [EKM]) and for the convection-diffusion equation $u_\tau = \Delta u + a\cdot\nabla(|u|^{q-1}u)$, $a \in \mathbf{R}^N$ ([EZ], [Zu]). However, as far as we know, the change of variables (1.2) has not been applied yet to hyperbolic equations, nor to parabolic equations with non-constant coefficients. We should also recall here that slightly different similarity variables have found use in the study of blow-up problems for parabolic equations (see [BK1], [GK], [HV1], [HV2], [MZ]), but these variables are not adapted to the study of blow-up phenomena for damped hyperbolic equations.

In a recent paper, Wayne [Wa] used the scaling variables (1.2) to construct finite-dimensional invariant manifolds for a nonlinear heat equation. His construction gives a nice geometric insight of the long time behavior of the solutions, and, like our method, allows to compute in a systematic way an asymptotic expansion to arbitrary order in $\tau^{-1/2}$. This method was subsequently applied in [EWW] to prove the stability of time-independent spatially periodic solutions of the Swift-Hohenberg equation, following the ideas of Schneider [Sch]. It is not clear, however, whether this construction can be extended to more general systems, in particular to equations with non-constant coefficients. For this reason, we shall use here a simpler, more robust method based on energy estimates only.

Finally, we point out that using the scaling variables (1.2) is like a continuous time counterpart of the so-called "renormalization group" method, which has been developed from a mathematical point of view by Bricmont and Kupiainen (see [BKL]). In this method, the problem of computing the long time asymptotics is reduced to analyzing the behavior of a rescaled equation on a *finite* time interval, in the limit where the scaling parameter L goes to infinity. In the case of Eq.(1.1) with $\mathcal{N} = 0$, the rescaled equation is $\varepsilon L^{-2}u_{\tau\tau} + u_\tau = (a(L\xi)u_\xi)_\xi$, which converges for any $\varepsilon > 0$ to the same parabolic limit as $L \to +\infty$. Therefore, the renormalization group method explains why our results are valid for all $\varepsilon > 0$, and shows that in Eq.(1.1) the long time behavior for fixed ε is closely related, through a scaling transformation, to the limit $\varepsilon \to 0$ for fixed time.

We now give the precise assumptions on the diffusion $a(\xi)$ and the nonlinearity $\mathcal{N}$ in Eq.(1.1). We suppose that $a : \mathbf{R} \to \mathbf{R}$ is a Lipschitz function satisfying

$$\text{(H1)} \qquad a(\xi) \,\ge\, \underline{a} \,>\, 0 \quad \text{for all } \xi \in \mathbf{R}\ , \quad \text{and} \lim_{\xi \to \pm\infty} a(\xi) \,=\, a_\pm\ ,$$

where a_+, a_- are positive constants. We set $a(\xi) = \tilde{a}(\xi) + b(\xi)$, where

$$\tilde{a}(\xi) \;=\; \begin{cases} a_+ & \text{if } \xi > 0\ , \\ a_- & \text{if } \xi < 0\ , \end{cases} \tag{1.3}$$

and assume that

$$\text{(H2)} \qquad (1 + |\xi|)^\mu b \in \mathrm{L}^2(\mathbf{R}) \cap \mathrm{L}^\infty(\mathbf{R})\ ,$$

for some $\mu > -1/2$.

Regarding the nonlinearity $\mathcal{N} : \mathbf{R}^3 \to \mathbf{R}$, we suppose that $\mathcal{N}(0,0,0) = 0$ and that $\mathcal{N}(x,y,z)$ can be written as a sum of n terms $\mathcal{N}_i(x,y,z)$, $i = 1, \ldots, n$, each of which satisfies the bound

$$\text{(N1)} \qquad |\mathcal{N}_i(x,y,z)| \,\le\, C_N |x|^{p_1(i)} |y|^{p_2(i)} |z|^{p_3(i)}\ , \quad \text{for all } x,y,z \in \mathbf{R}\ ,$$

where $C_N, p_1(i), p_2(i), p_3(i)$ are nonnegative constants satisfying $p_2(i) + p_3(i) \le 1$ and $p_1(i) + 2p_2(i) + 3p_3(i) > 3$. The last condition ensures that $u = 0$ is a stable equilibrium point of Eq.(1.1). For later use, we define

$$\begin{aligned} p(i) \;&=\; \frac{1}{2}(p_1(i) + 2p_2(i) + 3p_3(i) - 3) > 0\ , \quad p \;=\; \min_{1 \le i \le n} p(i)\ , \\ \nu_i \;&=\; p_1(i) + p_2(i) + p_3(i) - 1 > 0\ . \end{aligned} \tag{1.4}$$

To ensure local existence and uniqueness of solutions, we also assume that the following Lipschitz condition is satisfied. For all $r \geq 0$, there exists $L(r) \geq 0$ such that

$$\text{(N2)} \qquad \begin{aligned} |\mathcal{N}(x_1,y_1,z_1) - \mathcal{N}(x_2,y_2,z_2)| \;\le\; L(r)\big(&|x_1-x_2|(1+|y_1|+|y_2|+|z_1|+|z_2|) \\ &+ |y_1-y_2| + |z_1-z_2|\big) \;, \end{aligned}$$

for all $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbf{R}$ such that $|x_1| \le r$, $|x_2| \le r$. This assumption implies that the mapping $(u,v) \mapsto \mathcal{N}(u, u_\xi, v)$ is locally Lipschitz from $\mathrm{H}^1(\mathbf{R}) \times \mathrm{L}^2(\mathbf{R})$ to $\mathrm{L}^2(\mathbf{R})$.

Finally, for part of our results, we restrict ourselves to nonlinearities of the form

$$\text{(N3)} \qquad \mathcal{N}(u, u_\xi, u_\tau) \;=\; \mathcal{N}_1(u) + \mathcal{N}_2^*(u) u_\xi \;,$$

where $\mathcal{N}_1 \in \mathcal{C}^1(\mathbf{R})$ satisfies $|\mathcal{N}_1(u)| \le C_N |u|^{1+\nu_1}$, $|\mathcal{N}_1'(u)| \le C_N |u|^{\nu_1}$ for some $\nu_1 > 2$, and $\mathcal{N}_2^*$ is a Lipschitz function such that $|\mathcal{N}_2^*(u)| \le C_N |u|^{\nu_2}$ for some $\nu_2 > 1$. In this case, we have $p(1) = (\nu_1 - 2)/2$, $p(2) = (\nu_2 - 1)/2$ and $p = \min(p(1), p(2)) > 0$.

We now transform Eq.(1.1) using the scaling variables (1.2). Let $\tau_0 \ge 1$ be a fixed, sufficiently large real number, and let $x = \xi(\tau + \tau_0)^{-1/2}$, $t = \log(\tau + \tau_0)$. If $u(\xi, \tau)$ is a solution of (1.1), we define

$$v(x,t) \;=\; e^{t/2} u(x e^{t/2}, e^t - \tau_0) \;, \quad w(x,t) \;=\; e^{3t/2} u_\tau(x e^{t/2}, e^t - \tau_0) \;, \qquad (1.5)$$

or equivalently

$$\begin{aligned} u(\xi,\tau) \;&=\; \frac{1}{(\tau+\tau_0)^{1/2}}\, v\left(\frac{\xi}{\sqrt{\tau+\tau_0}} \,,\, \log(\tau+\tau_0)\right) \;, \\ u_\tau(\xi,\tau) \;&=\; \frac{1}{(\tau+\tau_0)^{3/2}}\, w\left(\frac{\xi}{\sqrt{\tau+\tau_0}} \,,\, \log(\tau+\tau_0)\right) \;. \end{aligned} \qquad (1.6)$$

Then the new functions $v(x,t), w(x,t)$ satisfy the system

$$\begin{aligned} v_t - \frac{x}{2} v_x - \frac{1}{2} v \;&=\; w \;, \\ \varepsilon e^{-t}\big(w_t - \frac{x}{2} w_x - \frac{3}{2} w\big) + w \;&=\; (a(x e^{t/2}) v_x)_x + e^{-pt} N(v, v_x, w, t) \;, \end{aligned} \qquad (1.7)$$

where $x \in \mathbf{R}$, $t \ge t_0 = \log(\tau_0)$, and

$$N(v, v_x, w, t) \;=\; e^{(p+3/2)t} \mathcal{N}(e^{-t/2} v, e^{-t} v_x, e^{-3t/2} w) \;. \qquad (1.8)$$

The initial data for v, w at time $t = t_0$ are related to those of u at time $\tau = 0$ by

$$v(x, t_0) \;=\; e^{t_0/2} u(x e^{t_0/2}, 0) \;, \quad w(x, t_0) \;=\; e^{3t_0/2} u_\tau(x e^{t_0/2}, 0) \;. \qquad (1.9)$$

We next introduce the function spaces in which we shall study the solutions of Eq.(1.7). For $k,\ell \in \mathbf{N}$, we denote by $\mathrm{H}^{k,\ell}$ the Hilbert space $\mathrm{H}^k(\mathbf{R},(1+x^2)^\ell dx)$ defined by the norm

$$\|u\|^2_{H^{k,\ell}} \;=\; \int_{\mathbf{R}} \Bigl(\sum_{i=0}^k |\partial_x^i u|^2\Bigr)(1+x^2)^\ell \, dx \;. \tag{1.10}$$

In particular, we set $X^\ell = \mathrm{H}^{1,\ell}$, $Y^\ell = \mathrm{H}^{0,\ell}$. We define the product spaces $Z^\ell = X^\ell \times Y^\ell$ and $\mathcal{D}^\ell = \mathrm{H}^{2,\ell} \times \mathrm{H}^{1,\ell}$ equipped with the standard norms

$$\|(v,w)\|_{Z^\ell} \;=\; (\|v\|^2_{X^\ell} + \|w\|^2_{Y^\ell})^{1/2}, \quad \|(v,w)\|_{\mathcal{D}^\ell} \;=\; (\|v\|^2_{\mathrm{H}^{2,\ell}} + \|w\|^2_{\mathrm{H}^{1,\ell}})^{1/2} \;. \tag{1.11}$$

Also, given $\varepsilon > 0$, we shall often endow the space Z^ℓ with the ε-dependent norm associated to the quadratic form

$$\Phi_\ell(\varepsilon,v,w) \;=\; \|v\|^2_{X^\ell} + \varepsilon\|w\|^2_{Y^\ell} \;. \tag{1.12}$$

This ε-dependent norm will be useful to state existence results and estimates which are uniform in ε, as $\varepsilon \to 0$ (see Theorems 1.1 and 1.2 below). In particular, this will allow us to obtain, without any additional work, analogous results for the corresponding parabolic equation (Theorem 1.3).

Before stating our results, we explain their content in a heuristic way. We first observe that Eq.(1.7) is a non-autonomous system, the coefficients of which converge to limits as $t \to +\infty$. Therefore, we expect that the long time behavior of the solutions (v,w) of Eq.(1.7) will be determined by the limiting system, namely

$$v_t \;=\; \tilde{\mathcal{L}}v \;\stackrel{\mathrm{def}}{=}\; (\tilde a(x)v_x)_x + \frac{x}{2}v_x + \frac{1}{2}v \;, \quad w \;=\; (\tilde a(x)v_x)_x \;. \tag{1.13}$$

Remark that Eq.(1.13) is a linear parabolic system with piecewise constant coefficients depending only on the limits $a_\pm$ of $a(\xi)$. In Appendix A below, we study the spectral properties of the limiting operator $\tilde{\mathcal{L}}$ in the function space Y^ℓ, $\ell \in \mathbf{N}$. We show that the spectrum of $\tilde{\mathcal{L}}$ consists of ℓ simple isolated eigenvalues $\lambda_n = -n/2$, $n \in \{0,\dots,\ell-1\}$, and of "continuous" spectrum filling the half-plane $\{z \in \mathbf{C}\,|\,\mathrm{Re}\,z \le 1/4 - \ell/2\}$. In particular, if $\ell \ge 2$, the eigenfunctions corresponding to $\lambda_0 = 0$ and $\lambda_1 = -1/2$ are given by

$$\begin{aligned} \varphi_0(x) \;&=\; \frac{1}{\sqrt{4\pi}}\,\frac{2}{\sqrt{a_+}+\sqrt{a_-}}\,\exp\Bigl(-\frac{x^2}{4\tilde a(x)}\Bigr) \;, \\ \varphi_1(x) \;&=\; \frac{1}{\sqrt{4\pi}}\,\frac{\sqrt{a_+a_-}}{\sqrt{a_+}+\sqrt{a_-}}\,\frac{x}{\tilde a(x)}\,\exp\Bigl(-\frac{x^2}{4\tilde a(x)}\Bigr) \;. \end{aligned} \tag{1.14}$$

These functions are normalized so that $\int \varphi_0(x)\,dx = \int \tilde{a}(x)^{-1} x\varphi_1(x)\,dx = 1$. According to these remarks, if (v,w) is any solution of Eq.(1.13) in Y^2, there exist $\alpha, \beta \in \mathbf{R}$ such that

$$\begin{aligned} v(x,t) &= \alpha\varphi_0(x) + \beta e^{-t/2}\varphi_1(x) + \mathcal{O}(e^{-3t/4})\ , \\ w(x,t) &= \alpha\psi_0(x) + \beta e^{-t/2}\psi_1(x) + \mathcal{O}(e^{-3t/4})\ , \end{aligned} \tag{1.15}$$

as $t \to +\infty$, where

$$\psi_i(x) = (\tilde{a}(x)\varphi_i'(x))' = -\frac{x}{2}\varphi_i'(x) - \frac{1+i}{2}\varphi_i(x)\ , \quad i = 0, 1\ . \tag{1.16}$$

The question we address in this paper is whether the small solutions of the full system (1.7) still have an asymptotic expansion of the form (1.15). The results below show that the answer is positive provided the coefficients in Eq.(1.7) converge sufficiently fast to their limits as $t \to +\infty$, *i.e.* provided the exponents μ in (H2) and p in (1.4) are sufficiently large. Our first result gives sufficient conditions for the existence of a first order asymptotic expansion.

Theorem 1.1. (First order asymptotics)
i) Assume that the hypotheses (H1), (H2), (N1), (N2) above are satisfied, and let $\varepsilon_0 > 0$. There exist $t_0 \ge 0$, $\delta_0 > 0$ and $C > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, the following holds. For all $(v_0, w_0) \in Z^1$ such that $\Phi_1(\varepsilon e^{-t_0}, v_0, w_0) \le \delta_0^2$, where Φ_1 is defined in (1.12), Eq.(1.7) has a unique solution $(v,w) \in \mathcal{C}([t_0, +\infty), Z^1)$ satisfying $(v(t_0), w(t_0)) = (v_0, w_0)$. In addition, there exists $\alpha^* \in \mathbf{R}$ such that, for all $t \ge t_0$,

$$\begin{aligned} \|v(t) - \alpha^*\varphi_0\|_{X^1}^2 + \varepsilon e^{-t}\|w(t) - \alpha^*\psi_0\|_{Y^1}^2 + \int_{t_0}^t e^{-2\lambda(t-s)}\|w(s) - \alpha^*\psi_0\|_{Y^1}^2\,ds \\ \le C(1+t)^2 e^{-2\lambda t}\Phi_1(\varepsilon e^{-t_0}, v_0, w_0)\ , \end{aligned} \tag{1.17}$$

where $\lambda = \min(1/4, p, 1/4 + \mu/2)$.
ii) If in addition the stronger assumption (N3) is satisfied, then we also have

$$\varepsilon\|w(t) - \alpha^*\psi_0\|_{L^2}^2 \le C(1+t)^2 e^{-2\lambda t}\|(v_0, w_0)\|_{Z^1}^2\ . \tag{1.18}$$

Remarks.

1) In the proof of Theorem 1.1 (and Theorem 1.2 below), it is convenient for technical reasons to take the parameter $t_0 = \log(\tau_0)$ sufficiently large, but this choice is irrelevant since the results for the original equation (1.1) are not affected.

2) In the second part of the theorem, we need the additional assumption (N3) to bound the quantity $\|w(t) - \alpha^*\psi_0\|_{L^2}$ because, in the course of the proof, we want to estimate the time derivative of Eq.(2.15) below. This is, however, not the only possibility. In the first remark after Theorem 1.2, we shall see that the hypotheses (H1), (H2), (N1), (N2) with $\mu > 1/2$ and $p > 1/2$ also imply an exponential decay for $\|w(t) - \alpha^*\psi_0\|_{L^2}$, provided the initial data are sufficiently small in Z^2. For instance, if $\mu \geq 1$ and $p \geq 3/4$, (1.18) and (1.19) give the same estimate $\|w(t) - \alpha^*\psi_0\|_{L^2} = \mathcal{O}(te^{-t/4})$ as $t \to +\infty$.

3) In terms of the original variables, Theorem 1.1 shows that, if the initial data $(u(0), u_\tau(0)) \in Z^1$ are sufficiently small, then Eq.(1.1) has a (unique) global solution $u \in \mathcal{C}([0,+\infty), X^1) \cap \mathcal{C}^1([0,+\infty), Y^1)$ which becomes asymptotically self-similar as $\tau \to +\infty$. In particular, (1.17) implies

$$\sup_{\xi\in\mathbf{R}} \left| u(\xi,\tau) - \frac{\alpha^*}{\sqrt{\tau}}\,\varphi_0\left(\frac{\xi}{\sqrt{\tau}}\right)\right| = \mathcal{O}\left(\frac{\log(\tau)}{\tau^{\lambda+1/2}}\right) , \quad \tau \to +\infty .$$

Moreover, if the additional assumption (N3) is satisfied, then by (1.18)

$$\left\| u_\tau(\cdot,\tau) - \frac{\alpha^*}{\tau^{3/2}}\,\psi_0\left(\frac{\cdot}{\sqrt{\tau}}\right)\right\|_{L^2} = \mathcal{O}\left(\frac{\log(\tau)}{\tau^{\lambda+5/4}}\right) , \quad \tau \to +\infty .$$

Our next result shows the existence of a second order asymptotic expansion for the solutions of Eq.(1.7) in Z^2, provided $\mu > 1/2$ and $p > 1/2$. For simplicity, we shall state (and prove) it in the particular case where $\mathcal{N} = 0$ and $a_+ = a_- = 1$, but these additional assumptions are not essential (see the remarks below).

Theorem 1.2. (Second order asymptotics)
Assume that $\mathcal{N} = 0$, and that the hypotheses (H1), (H2) are satisfied with $a_+ = a_- = 1$ and $\mu > 1/2$. Given $\varepsilon_0 > 0$, there exist $t_0 \geq 0$ and $C > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0]$ and all $(v_0, w_0) \in Z^2$, the unique solution $(v,w) \in \mathcal{C}([t_0,+\infty), Z^2)$ of Eq.(1.7) with $(v(t_0), w(t_0)) = (v_0, w_0)$ satisfies, for all $t \geq t_0$,

$$\begin{aligned}\|v(t)-\alpha^*\varphi_0-e^{-t/2}\beta^*\varphi_1\|_{X^2}^2 + \int_{t_0}^t e^{-2\lambda'(t-s)}\|w(s)-\alpha^*\psi_0-e^{-s/2}\beta^*\psi_1\|_{Y^2}^2\,ds \\ + \varepsilon e^{-t}\|w(t)-\alpha^*\psi_0\|_{Y^2}^2 \leq C(1+t)^2 e^{-2\lambda' t}\Phi_2(\varepsilon e^{-t_0}, v_0, w_0) ,\end{aligned} \tag{1.19}$$

where $\lambda' = \min(3/4, 1/4 + \mu/2)$ and α^*, β^* are given by (3.21), (3.26) below.

Remarks.

1) More generally, if the hypotheses (H1), (H2), (N1), (N2) are satisfied with $\mu > 1/2$,

$p > 1/2$, one can show that the estimate (1.19) holds with $\lambda' = \min(3/4, p, 1/4 + \mu/2)$, provided the initial data (v_0, w_0) are sufficiently small in Z^2. The proof follows essentially the same lines as in Section 3, but the computations are more involved for two main reasons. First, due to the presence of the nonlinearity $\mathcal{N}$, the a priori estimates of Section 3.2 are not available any more. Next, if $a_+ \neq a_-$, the second eigenfunction $\varphi_1(x)$ is not smooth at the origin, and we have to replace it in our asymptotic expansions by some approximate eigenfunction $\varphi_1^*(x,t)$ (see the remark at the end of Section 3).

2) In analogy with (1.18), we expect that, under the hypotheses of Theorem 1.2, the quantity $\|w(t) - \alpha^*\psi_0 - e^{-t/2}\beta^*\psi_1\|_{Y^1}$ decays like $(1+t)e^{-\lambda' t}$ as $t \to +\infty$. However, arguing as in the proof of Theorem 1.1 and assuming in addition that $b' \in \mathrm{L}^2(\mathbf{R})$, we could only obtain the decay rate $(1+t)e^{-(\lambda'-1/4)t}$ if b is not identically zero. Since this estimate is probably not optimal, the corresponding calculations will not be presented here.

3) In terms of the original variables, Theorem 1.2 implies that, for any initial data $(u(0), u_\tau(0)) \in Z^2$, the solution of Eq.(1.1) with $\mathcal{N} = 0$ satisfies

$$\sup_{\xi\in\mathbf{R}} \left| u(\xi,\tau) - \frac{\alpha^*}{\sqrt{\tau}}\varphi_0\left(\frac{\xi}{\sqrt{\tau}}\right) - \frac{\beta^*}{\tau}\varphi_1\left(\frac{\xi}{\sqrt{\tau}}\right)\right| = \mathcal{O}\left(\frac{\log(\tau)}{\tau^{\lambda'+1/2}}\right) , \quad \tau \to +\infty .$$

In particular, remark that the coefficients α^*, β^* in (1.19) are independent of the choice of t_0.

4) Theorems 1.1 and 1.2 show that the (small) solutions of Eq.(1.7) satisfy an asymptotic expansion of the form (1.15), which is entirely determined by the limiting system (1.13). A natural question is whether this property still holds for higher orders. In particular, is it true that $v(x,t) = \alpha\varphi_0(x) + \beta e^{-t/2}\varphi_1(x) + \gamma e^{-t}\varphi_2(x) + o(e^{-t})$, where φ_0, φ_1, φ_2 are the first three eigenfunctions of the operator $\tilde{\mathcal{L}}$? The answer is negative if $\varepsilon \neq 0$, as it can be seen from the simple example $a \equiv 1$, $\mathcal{N} = 0$. Indeed, in this case a direct calculation shows that

$$v(t) = \alpha^*\varphi_0 + e^{-t/2}\beta^*\varphi_1 + e^{-t}\gamma^*\varphi_2 - \varepsilon e^{-t}\alpha^*\varphi_0^{(4)} + \mathcal{O}(e^{-5t/4}) , \quad t \to +\infty ,$$

for some $\gamma^* \in \mathbf{R}$ (depending on v_0, w_0 and t_0).

We conclude this section with a brief discussion of the parabolic case. If $\varepsilon = 0$, Eq.(1.1) reduces to

$$u_\tau = (a(\xi)u_\xi)_\xi + \mathcal{N}(u, u_\xi, u_\tau) . \tag{1.20}$$

As before, defining $v(x,t), w(x,t)$ by (1.5), we obtain the scaled equation

$$v_t \;=\; (a(xe^{t/2})v_x)_x + \frac{x}{2}v_x + \frac{1}{2}v + e^{-pt}N(v,v_x,w,t)\;. \tag{1.21}$$

Since all the estimates in Theorems 1.1 and 1.2 and in their proofs are uniform in ε for $0 < \varepsilon \le \varepsilon_0$, we have following result:

Theorem 1.3. (Parabolic case)
i) Assume that the hypotheses (H1), (H2), (N1), (N2) are satisfied. There exist $t_0 \ge 0$, $\delta_0 > 0$ and $C > 0$ such that, for all $v_0 \in X^1$ with $\|v_0\|_{X^1} \le \delta_0$, Eq.(1.21) has a unique solution $v \in \mathcal{C}([t_0,+\infty), X^1)$ satisfying $v(t_0) = v_0$. In addition, there exists $\alpha^* \in \mathbf{R}$ such that, for all $t \ge t_0$,

$$\|v(t) - \alpha^*\varphi_0\|_{X^1}^2 \;\le\; C(1+t)^2 e^{-2\lambda t}\|v_0\|_{X^1}^2\;, \tag{1.22}$$

where $\lambda = \min(1/4, p, 1/4 + \mu/2)$.
ii) Assume that $\mathcal{N} = 0$, and that the hypotheses (H1), (H2) are satisfied with $a_+ = a_- = 1$ and $\mu > 1/2$. Then there exist $t_0 \ge 0$ and $C > 0$ such that, for all $v_0 \in X^2$, the unique solution $v \in \mathcal{C}([t_0,+\infty), X^2)$ of Eq.(1.21) with $v(t_0) = v_0$ satisfies, for all $t \ge t_0$,

$$\|v(t) - \alpha_0^*\varphi_0 - e^{-t/2}\beta_0^*\varphi_1\|_{X^2}^2 \le\; C(1+t)^2 e^{-2\lambda' t}\|v_0\|_{X^2}^2\;, \tag{1.23}$$

where $\lambda' = \min(3/4, 1/4 + \mu/2)$ and α_0^*, β_0^* are given by (3.21), (3.26) with $\varepsilon = 0$.

The rest of this paper is devoted to the proof of Theorem 1.1 (Section 2) and Theorem 1.2 (Section 3). The spectral properties of the operator $\tilde{\mathcal{L}}$ defined in (1.13) are studied in Appendix A.

2. First Order Asymptotics

This section is devoted to the proof of Theorem 1.1, which describes the first order asymptotics of the small solutions (v,w) to (1.7) as $t \to +\infty$. First (section 2.1), we prove a local existence result for the solutions of (1.7) in the energy space Z^ℓ. Next (section 2.2), we decompose these solutions using the first spectral projection of the operator $\tilde{\mathcal{L}}$ defined in (1.13). Then (section 2.3) we control their behavior in the space Z^1 using various energy functionals. In particular, we show that small solutions exist for all times and we obtain the decay estimate (1.17). Finally (section 2.4), we prove the bound (1.18) using refined energy estimates.

2.1. Local Existence of Solutions

Since the system (1.7) is obtained from the simpler equation (1.1) by the change of variables (1.5), (1.6), it is natural to use this mapping to show that the Cauchy problem for (1.7) is locally well-posed in the space $Z^\ell = X^\ell \times Y^ell$, see (1.11). Therefore, we start from the following local existence result in the original variables.

Lemma 2.1. *Let $\varepsilon > 0$, $\ell \in \mathbf{N}$, $\delta > 0$. There exists $\hat\tau > 0$ such that, for all initial data $(u_0, u_{0\tau}) \in Z^\ell$ with $\|(u_0, u_{0\tau})\|_{Z^\ell} \le \delta$, Eq.(1.1) has a unique (mild) solution $u \in \mathcal{C}([0,\hat\tau], X^\ell) \cap \mathcal{C}^1([0,\hat\tau], Y^\ell)$ satisfying $(u(0), u_\tau(0)) = (u_0, u_{0\tau})$. The solution $(u(\tau), u_\tau(\tau))$ depends continuously on the initial data in Z^ℓ, uniformly in $\tau \in [0,\hat\tau]$. In addition, if $(u_0, u_{0\tau}) \in \mathcal{D}^\ell$, then $u \in \mathcal{C}([0,\hat\tau], \mathrm{H}^{2,\ell}) \cap \mathcal{C}^1([0,\hat\tau], X^\ell) \cap \mathcal{C}^2([0,\hat\tau], Y^\ell)$ is a classical solution of Eq.(1.1) in Y^ℓ.*

Remark. Following Pazy [Pa], we call *mild solution* of a differential equation a continuous solution of the corresponding integral equation. In particular, every mild solution is a *weak* solution of the differential equation, in the sense of Ball [Ba].

Proof. Let $q(\xi) = (1+\xi^2)^{\ell/2}$. Then $u \in \mathcal{C}([0,\hat\tau], X^\ell) \cap \mathcal{C}^1([0,\hat\tau], Y^\ell)$ is a solution of (1.1) if and only if the function $s \in \mathcal{C}([0,\hat\tau], X^0) \cap \mathcal{C}^1([0,\hat\tau], Y^0)$ defined by $s(\xi,\tau) = q(\xi)u(\xi,\tau)$ is a solution of the equation

$$\varepsilon s_{\tau\tau} \;=\; (a(\xi)s_\xi)_\xi + \mathcal{M}(s, s_\xi, s_\tau)\;, \tag{2.1}$$

where

$$\mathcal{M}(s, s_\xi, s_\tau) \;=\; -s_\tau - \frac{2aq'}{q}s_\xi + \left(\frac{2aq'^2}{q^2} - \frac{(aq')'}{q}\right)s + q\mathcal{N}\left(\frac{s}{q}, \frac{s_\xi}{q} - \frac{q's}{q^2}, \frac{s_\tau}{q}\right)\;. \tag{2.2}$$

In addition, there exists $C \ge 1$ such that $C^{-1}\|(s,s_\tau)\|_{Z^0} \le \|(u,u_\tau)\|_{Z^\ell} \le C\|(s,s_\tau)\|_{Z^0}$. Therefore, to prove Lemma 2.1, it is sufficient to show that, for all initial data $(s_0, s_{0\tau}) \in Z^0$ satisfying $\|(s_0, s_{0\tau})\|_{Z^0} \le C\delta$, Eq.(2.1) has a unique solution $s \in \mathcal{C}([0,\hat\tau], X^0) \cap \mathcal{C}^1([0,\hat\tau], Y^0)$, which depends continuously on $(s_0, s_{0\tau})$, and that s is a classical solution of (2.1) whenever $(s_0, s_{0\tau}) \in \mathcal{D}^0 = \mathrm{H}^2(\mathbf{R}) \times \mathrm{H}^1(\mathbf{R})$.

To prove this, we set $S = (s, s_\tau)$ and we rewrite (2.1) as a first order system

$$\dot S(\tau) \;=\; A_\varepsilon S(\tau) + M_\varepsilon(S(\tau))\;, \tag{2.3}$$

where the linear operator A_ε and the nonlinearity M_ε are defined by

$$A_\varepsilon = \begin{pmatrix} 0 & 1 \\ \varepsilon^{-1}\partial_\xi(a(\xi)\partial_\xi) & 0 \end{pmatrix} , \quad M_\varepsilon(S) = \begin{pmatrix} 0 \\ \varepsilon^{-1}\mathcal{M}(s, s_\xi, s_\tau) \end{pmatrix} .$$

The assumption (H1) on the function $a(\xi)$ implies that $-\partial_\xi(a(\xi)\partial_\xi)$ is a nonnegative self-adjoint operator in $\mathrm{L}^2(\mathbf{R})$ with domain $\mathrm{H}^2(\mathbf{R})$. Therefore, $A_\varepsilon : \mathcal{D}^0 \subset Z^0 \to Z^0$ is the generator of a strongly continuous group $e^{\tau A_\varepsilon}$ of bounded linear operators in Z^0 [CH]. On the other hand, using (2.2) and the assumption (N2) on the nonlinearity $\mathcal{N}$, it is straightforward to verify that the mapping $M_\varepsilon : Z^0 \to Z^0$ is locally Lipschitz, uniformly on any bounded subset of Z^0. Therefore, by a classical result ([Pa], Theorem 6.1.4), there exists a time $\hat\tau > 0$ such that, for all $S_0 \in Z^0$ satisfying $\|S_0\|_{Z^0} \le C\delta$, the integral equation

$$S(\tau) = e^{\tau A_\varepsilon} S_0 + \int_0^\tau e^{A_\varepsilon(\tau-\sigma)} M_\varepsilon(S(\sigma))\, d\sigma , \quad 0 \le \tau \le \hat\tau ,$$

has a unique solution $S \in \mathcal{C}([0,\hat\tau], Z^0)$, which is called the mild solution of (2.3) with initial data S_0. This solution depends continuously on S_0, uniformly in $\tau \in [0,\hat\tau]$. In addition, if $S_0 \in \mathcal{D}^0$, then ([Pa], Theorem 6.1.6) $S \in \mathcal{C}([0,\hat\tau], \mathcal{D}^0) \cap \mathcal{C}^1([0,\hat\tau], Z^0)$ is a classical solution of (2.3) in Z^0. This concludes the proof of Lemma 2.1. □

Now, let $t_2 > t_1 \ge t_0$ and let $\tau_i = e^{t_i} - \tau_0$, $i = 1, 2$. We say that a function $(v, w) \in \mathcal{C}([t_1, t_2], Z^\ell)$ is a *solution* of the system (1.7) in Z^ℓ if there exists a (mild) solution $u \in \mathcal{C}([\tau_1, \tau_2], X^\ell) \cap \mathcal{C}^1([\tau_1, \tau_2], Y^\ell)$ of (1.1) such that (1.5) holds for all $t \in [t_1, t_2]$. If in addition $\ell \ge 1$ and $u \in \mathcal{C}([\tau_1, \tau_2], \mathrm{H}^{2,\ell}) \cap \mathcal{C}^1([\tau_1, \tau_2], X^\ell) \cap \mathcal{C}^2([\tau_1, \tau_2], Y^\ell)$ is a classical solution of (1.1) in Y^ℓ, then $(v, w) \in \mathcal{C}([t_1, t_2], \mathcal{D}^\ell) \cap \mathcal{C}^1([t_1, t_2], Z^{\ell-1})$ is a classical solution of (1.7) in $Z^{\ell-1}$. With these definitions, we have the following local existence result for the solutions of (1.7) in Z^ℓ:

Proposition 2.2. *Let $\varepsilon > 0$, $\ell \in \mathbf{N}$, $\delta_1 > 0$, $t_2 > t_0$. There exists $T > 0$ such that, for all $t_1 \in [t_0, t_2]$ and all $(v_1, w_1) \in Z^\ell$ with $\Phi_\ell(\varepsilon e^{-t_1}, v_1, w_1) \le \delta_1^2$, the system (1.7) has a unique solution $(v, w) \in \mathcal{C}([t_1, t_1 + T], Z^\ell)$ satisfying $(v(t_1), w(t_1)) = (v_1, w_1)$. This solution depends continuously on the initial data (v_1, w_1) in Z^ℓ, uniformly in $t \in [t_1, t_1 + T]$. In addition, if $\ell \ge 1$ and $(v_1, w_1) \in \mathcal{D}^\ell$, then $(v, w) \in \mathcal{C}([t_1, t_1 + T], \mathcal{D}^\ell) \cap \mathcal{C}^1([t_1, t_1 + T], Z^{\ell-1})$ is a classical solution of (1.7) in $Z^{\ell-1}$.*

Remark. In particular, Proposition 2.2 implies that, if $(v, w) \in \mathcal{C}([t_1, t_2), Z^\ell)$ is a maximal solution of (1.7) in Z^ℓ, and if $\Phi_\ell(\varepsilon e^{-t}, v(t), w(t)) \le \delta_1^2$ for all $t \in [t_1, t_2)$, then $t_2 = +\infty$, *i.e.* the solution can be continued to the whole interval $[t_1, +\infty)$.

Proof. Given $t_1 \in [t_0, t_2]$ and $(v_1, w_1) \in Z^\ell$ satisfying $\Phi_\ell(\varepsilon e^{-t_1}, v_1, w_1) \le \delta_1^2$, we define

$$u_1(\xi) \;=\; e^{-t_1/2} v_1(\xi e^{-t_1/2}) \;, \quad u_{1\tau}(\xi) \;=\; e^{-3t_1/2} w_1(\xi e^{-t_1/2}) \;, \quad \xi \in \mathbf{R} \;.$$

Then $(u_1, u_{1\tau}) \in Z^\ell$, and there exists a constant $C > 0$ (depending on ε, ℓ and t_2) such that $\|(u_1, u_{1\tau})\|_{Z^\ell} \le C\delta_1$. Since Eq.(1.1) is autonomous, it follows from Lemma 2.1 that there exists a time $\hat\tau > 0$, depending on ε, ℓ, $C\delta_1$ but not on $(u_1, u_{1\tau})$, such that (1.1) has a unique (mild) solution $u \in \mathcal{C}([e^{t_1}, e^{t_1} + \hat\tau], X^\ell) \cap \mathcal{C}^1([e^{t_1}, e^{t_1} + \hat\tau], Y^\ell)$ satisfying $u(\xi, e^{t_1}) = u_1(\xi)$, $u_\tau(\xi, e^{t_1}) = u_{1\tau}(\xi)$. Now, we set $T = \log(1 + \hat\tau e^{-t_2})$, and for all $t \in [t_1, t_1 + T] \subset [t_1, \log(e^{t_1} + \hat\tau)]$ we define

$$v(x,t) \;=\; e^{t/2} u(xe^{t/2}, e^t) \;, \quad w(x,t) \;=\; e^{3t/2} u_\tau(xe^{t/2}, e^t) \;. \tag{2.4}$$

By construction, $(v, w) \in \mathcal{C}([t_1, t_1 + T], Z^\ell)$ is a solution of (1.7) in Z^ℓ satisfying $(v(t_1), w(t_1)) = (v_1, w_1)$, and the uniqueness of this solution follows from the uniqueness of u as a mild solution of (1.1). Moreover, since u depends continuously on the initial data uniformly in time, so does (v, w). Finally, if $\ell \ge 1$ and $(v_1, w_1) \in \mathcal{D}^\ell$, then $(u_1, u_{1\tau}) \in \mathcal{D}^\ell$, hence $u \in \mathcal{C}([e^{t_1}, e^{t_1} + \hat\tau], \mathrm{H}^{2,\ell}) \cap \mathcal{C}^1([e^{t_1}, e^{t_1} + \hat\tau], X^\ell) \cap \mathcal{C}^2([e^{t_1}, e^{t_1} + \hat\tau], Y^\ell)$ is a classical solution of (1.1). By (2.4), we have $(v, w) \in \mathcal{C}([t_1, t_1 + T], \mathcal{D}^\ell) \cap \mathcal{C}^1([t_1, t_1 + T], Z^{\ell-1})$, and (v, w) is a classical solution of (1.7) in $Z^{\ell-1}$. This concludes the proof of Proposition 2.2. $\square$

2.2. Spectral Decomposition

From now on, we fix $\varepsilon_0 > 0$ and we assume that, for some $\varepsilon \in (0, \varepsilon_0]$, some $t_0 \ge 0$ and some $T > 0$, we are given a solution $(v, w) \in \mathcal{C}([t_0, t_0 + T], Z^1)$ of (1.7) satisfying

$$\|v(t)\|_{X^1} \;\le\; 1 \;, \quad t \in [t_0, t_0 + T] \;. \tag{2.5}$$

We set $(v_0, w_0) = (v(t_0), w(t_0))$. Our aim is to control the behavior of the solution (v, w) on the time interval $[t_0, t_0 + T]$ using energy functionals. To do that, we first decompose the solution (v, w) using the spectral projections of the operator $\tilde{\mathcal{L}}$ introduced in (1.13). For $t \in [t_0, t_0 + T]$, we define

$$v(x,t) \;=\; \alpha(t)\varphi_0(x) + f(x,t) \;, \quad w(x,t) \;=\; \dot\alpha(t)\varphi_0(x) + \alpha(t)\psi_0(x) + g(x,t) \;, \tag{2.6}$$

where φ_0, ψ_0 are given by (1.14), (1.16) and

$$\alpha(t) \;=\; \int_{\mathbf{R}} v(x,t)\, dx \;. \tag{2.7}$$

The following properties of the function α will be useful:

Lemma 2.3. *If $(v,w) \in \mathcal{C}([t_0,t_0+T],Z^1)$ is a solution of (1.7), then $\alpha \in \mathcal{C}^2([t_0,t_0+T])$ and the following relations hold*

$$\dot\alpha(t) \;=\; \int_{\mathbf{R}} w(x,t)\,dx\;, \quad \frac{d}{dt}\left(\varepsilon e^{-t}\dot\alpha(t)+\alpha(t)\right) \;=\; e^{-pt}n(t)\;, \tag{2.8}$$

where

$$n(t) \;=\; \int_{\mathbf{R}} N(v(x,t),v_x(x,t),w(x,t),t)\,dx\;. \tag{2.9}$$

Proof. We first note that $\alpha(t) = \int u(\xi,e^t-\tau_0)\,d\xi$, where $u(\xi,\tau)$ is defined by (1.6) and $\tau_0 = e^{t_0}$. Since $u \in \mathcal{C}([0,\tau_1],X^1)\cap\mathcal{C}^1([0,\tau_1],Y^1)$, where $\tau_1 = \tau_0(e^T-1)$, it follows that $\alpha \in \mathcal{C}^1([t_0,t_0+T])$ and $\dot\alpha(t) = e^t\int u_\tau(\xi,e^t-\tau_0)\,d\xi = \int w(x,t)\,dx$. To prove that $\alpha \in \mathcal{C}^2([t_0,t_0+T])$, we first assume that $(v_0,w_0) \in \mathcal{D}^1$. Then $u \in \mathcal{C}([0,\tau_1],\mathrm{H}^{2,1})\cap\mathcal{C}^1([0,\tau_1],X^1)\cap\mathcal{C}^2([0,\tau_1],Y^1)$ is a classical solution of (1.1), hence $\alpha \in \mathcal{C}^2([t_0,t_0+T])$ and

$$\begin{aligned}
&\frac{d}{dt}\left(\varepsilon e^{-t}\dot\alpha(t)+\alpha(t)\right) \;=\; e^t\int_{\mathbf{R}}(\varepsilon u_{\tau\tau}+u_\tau)(\xi,e^t-\tau_0)\,d\xi\\
&\quad= e^t\int_{\mathbf{R}}\mathcal{N}(u,u_\xi,u_\tau)(\xi,e^t-\tau_0)\,d\xi\\
&\quad= e^{3t/2}\int_{\mathbf{R}}\mathcal{N}\left(e^{-t/2}v(x,t),e^{-t}v_x(x,t),e^{-3t/2}w(x,t)\right)\,dx \;=\; e^{-pt}n(t)\;,
\end{aligned}$$

by (1.8), (2.9). Therefore, we have for any $t \in [t_0,t_0+T]$

$$\varepsilon e^{-t}\dot\alpha(t)+\alpha(t) \;=\; \varepsilon e^{-t_0}\dot\alpha(t_0)+\alpha(t_0)+\int_{t_0}^t e^{-ps}n(s)\,ds\;. \tag{2.10}$$

Now, by Proposition 2.2, both sides of (2.10) are continuous functions of the initial data (v_0,w_0) in Z^1. Since (2.10) is satisfied for all (v_0,w_0) in the dense subset $\mathcal{D}^1$, the equality must hold for all $(v_0,w_0) \in Z^1$. This shows that $\varepsilon e^{-t}\dot\alpha+\alpha \in \mathcal{C}^1([t_0,t_0+T])$ and that (2.8) holds. This concludes the proof of Lemma 2.3. $\square$

Using Lemma 2.3 and the properties of φ_0, ψ_0, it is straightforward to verify that the function $(f,g) \in \mathcal{C}([t_0,t_0+T],Z^1)$ defined in (2.6) is a solution of the system

$$\begin{aligned}
f_t-\frac{x}{2}f_x-\frac{1}{2}f \;&=\; g\;,\\
\varepsilon e^{-t}\left(g_t-\frac{x}{2}g_x-\frac{3}{2}g\right)+g \;&=\; (a(xe^{t/2})f_x)_x+\alpha(t)(b(xe^{t/2})\varphi_0')_x\\
&\qquad -\varepsilon e^{-t}h(x,t)+e^{-pt}r(x,t)\;,
\end{aligned} \tag{2.11}$$

where $b(\xi) = a(\xi) - \tilde{a}(\xi)$ and

$$\begin{aligned} h(x,t) &= 2\dot{\alpha}(t)\psi_0(x) + \alpha(t)\chi_0(x) \ , \\ \chi_0(x) &= (\tilde{a}(x)\psi_0'(x))_x \ = \ -\frac{x}{2}\psi_0'(x) - \frac{3}{2}\psi_0(x) \ , \\ r(x,t) &= N(v(x,t), v_x(x,t), w(x,t), t) - \varphi_0(x) n(t) \ . \end{aligned} \tag{2.12}$$

In addition, since $\int \varphi_0(x)\,dx = 1$ and $\int \psi_0(x)\,dx = 0$, it follows from (2.6), (2.8) that $\int f(x,t)\,dx = \int g(x,t)\,dx = 0$ for all $t \in [t_0, t_0+T]$. Therefore, we can define the primitive functions

$$F(x,t) = \int_{-\infty}^{x} f(y,t)\,dy \ , \quad G(x,t) = \int_{-\infty}^{x} g(y,t)\,dy \ , \quad t \in [t_0, t_0+T] \ . \tag{2.13}$$

By a classical inequality [HLP, section 9.9], we have

$$\|F\|_{L^2} \le 2\|xf\|_{L^2} \le 2\|f\|_{Y^1} \ , \quad \|G\|_{L^2} \le 2\|xg\|_{L^2} \le 2\|g\|_{Y^1} \ , \tag{2.14}$$

hence $(F,G) \in \mathcal{C}([t_0, t_0+T], \mathcal{D}^0)$, where $\mathcal{D}^0 = \mathrm{H}^2(\mathbf{R}) \times \mathrm{H}^1(\mathbf{R})$. Using a density argument as in the proof of Lemma 2.3, one verifies that $(F,G) \in \mathcal{C}^1([t_0, t_0+T], Z^0)$ is a classical solution in $Z^0 = \mathrm{H}^1(\mathbf{R}) \times \mathrm{L}^2(\mathbf{R})$ of the system

$$\begin{aligned} F_t - \frac{x}{2}F_x &= G \ , \\ \varepsilon e^{-t}\big(G_t - \frac{x}{2}G_x - G\big) + G &= a(xe^{t/2})F_{xx} + \alpha(t) b(xe^{t/2})\varphi_0' \\ &\quad - \varepsilon e^{-t} H(x,t) + e^{-pt} R(x,t) \ , \end{aligned} \tag{2.15}$$

where

$$H(x,t) = \int_{-\infty}^{x} h(y,t)\,dy \ , \quad R(x,t) = \int_{-\infty}^{x} r(y,t)\,dy \ . \tag{2.16}$$

For later use, we also note that, if $\tilde{F}(\xi,t) = F(\xi e^{-t/2}, t)$ and $\tilde{G}(\xi,t) = G(\xi e^{-t/2}, t)$, then $(\tilde{F}, \tilde{G}) \in \mathcal{C}([t_0, t_0+T], \mathcal{D}^0) \cap \mathcal{C}^1([t_0, t_0+T], Z^0)$ and

$$\tilde{F}_t(\xi,t) = \Big(F_t - \frac{x}{2}F_x\Big)(\xi e^{-t/2}, t) \ , \quad \tilde{G}_t(\xi,t) = \Big(G_t - \frac{x}{2}G_x\Big)(\xi e^{-t/2}, t) \ . \tag{2.17}$$

To bound the nonlinear terms in (2.11), (2.15), we introduce the additional notations

$$\begin{aligned} \sigma(t)^2 &= \int_{\mathbf{R}} (1+x^2)(f(x,t)^2 + f_x(x,t)^2)\,dx \ , \\ \theta(t)^2 &= \dot{\alpha}(t)^2 + \int_{\mathbf{R}} (1+x^2) g(x,t)^2\,dx \ , \\ \rho(t)^2 &= \alpha(t)^2 + \sigma(t)^2 \ . \end{aligned} \tag{2.18}$$

Then there exists $C_0 \geq 1$ such that

$$\begin{aligned} C_0^{-1}\|v(t)\|_{X^1} \;&\leq\; \rho(t) \;\leq\; C_0\|v(t)\|_{X^1}\;, \\ C_0^{-1}\|w(t)\|_{Y^1} \;&\leq\; \rho(t)+\theta(t) \;\leq\; C_0(\|v(t)\|_{X^1}+\|w(t)\|_{Y^1})\;, \end{aligned} \tag{2.19}$$

for all $t \in [t_0, t_0+T]$. In addition, we have the following estimates:

Lemma 2.4. *Assume that $(v, w) \in \mathcal{C}([t_0, t_0+T], Z^1)$ satisfies the bound (2.5). Then there exists $C_1 > 0$ such that*

$$\|h(t)\|_{Y^1} \;\leq\; C_1(\rho(t)+\theta(t))\;, \quad \|r(t)\|_{Y^1} \;\leq\; C_1(\rho(t)+\theta(t))\;. \tag{2.20}$$

Proof. Since $\psi_0, \chi_0 \in Y^1$, the first inequality follows immediately from (2.12), (2.18). To prove the second one, it is sufficient to show that $\|N(v, v_x, w, t)\|_{Y^1} \leq C(\rho+\theta)$ for some $C > 0$, since $\varphi_0 \in Y^1$ and $|n(t)| \leq \|N(v, v_x, w, t)\|_{L^1} \leq C\,\|N(v, v_x, w, t)\|_{Y^1}$. Using (1.4), (1.8) and the assumption (N1) on the nonlinearity, we have for $i = 1, \ldots, n$

$$\begin{aligned} (1+x^2)N_i(v, v_x, w, t)^2 \;&\equiv\; (1+x^2)e^{(2p+3)t}\left(\mathcal{N}_i(e^{-t/2}v, e^{-t}v_x, e^{-3t/2}w)\right)^2 \\ &\leq\; C_N^2 e^{2(p-p(i))t}(1+x^2)|v|^{2p_1(i)}|v_x|^{2p_2(i)}|w|^{2p_3(i)} \\ &\leq\; C_N^2|v|^{2\nu_i}\big((1+x^2)v^2\big)^{1-p_2(i)-p_3(i)}\big((1+x^2)v_x^2\big)^{p_2(i)}\big((1+x^2)w^2\big)^{p_3(i)}\;. \end{aligned}$$

Integrating both sides over $x \in \mathbf{R}$ and using Hölder's inequality, we obtain

$$\begin{aligned} \|N_i(v, v_x, w, t)\|_{Y^1} \;&\leq\; C_N\|v\|_{L^\infty}^{\nu_i}\|v\|_{Y^1}^{1-p_2(i)-p_3(i)}\|v_x\|_{Y^1}^{p_2(i)}\|w\|_{Y^1}^{p_3(i)} \\ &\leq\; C_N\|v\|_{X^1}^{1-p_3(i)}\|w\|_{Y^1}^{p_3(i)} \;\leq\; C_N(\|v\|_{X^1}+\|w\|_{Y^1})\;, \end{aligned}$$

since $\|v\|_{L^\infty} \leq \|v\|_{X^1} \leq 1$. Summing over i and using (2.19), the result follows. □

2.3. Energy Estimates

As before, let $(v, w) \in \mathcal{C}([t_0, t_0+T], Z^1)$ be a solution of (1.7) satisfying the bound (2.5). To control the time behavior of the functions f, g defined in (2.6), we introduce three sets of energy functionals. We first define

$$\begin{aligned} E_0(t) \;&=\; \frac{1}{2}\int_{\mathbf{R}}\left(F_x^2+\varepsilon e^{-t}\frac{G^2}{a(xe^{t/2})}\right)\,dx\;, \\ E_1(t) \;&=\; \int_{\mathbf{R}}\frac{1}{a(xe^{t/2})}\left(\frac{1}{2}F^2+\varepsilon e^{-t}FG\right)\,dx\;, \\ E_2(t) \;&=\; E_0(t)+E_1(t)\;, \end{aligned}$$

where F, G are given by (2.13).

Lemma 2.5. *Assume that $(v,w) \in \mathcal{C}([t_0, t_0+T], Z^1)$ is a solution of (1.7) satisfying the bound (2.5). Then $E_2 \in \mathcal{C}^1([t_0,t_0+T])$ and there exists a constant $C_2(\varepsilon_0) > 0$ such that, for all $t \in [t_0, t_0+T]$,*

$$\dot{E}_2(t) \;\le\; -\frac{1}{2}E_2(t) - \frac{1}{2}\int_{\mathbf{R}} f^2\, dx + C_2 e^{-\lambda t}(\rho+\theta)(\sigma+\theta)\;, \tag{2.21}$$

where ρ, σ, θ are defined in (2.18) and $\lambda = \min(1/4, p, 1/4+\mu/2)$.

Proof. We first note that

$$E_0(t) \;=\; \frac{1}{2}\int_{\mathbf{R}} F_x(x,t)^2\, dx + \frac{\varepsilon}{2}\, e^{-3t/2}\int_{\mathbf{R}} \frac{\tilde{G}(\xi,t)^2}{a(\xi)}\, d\xi\;,$$

where $\tilde{G}(\xi,t) = G(\xi e^{-t/2}, t)$. Since $F \in \mathcal{C}^1([t_0,t_0+T], X^0)$ and $\tilde{G} \in \mathcal{C}^1([t_0,t_0+T], Y^0)$ (see (2.17)), it follows that $E_0 \in \mathcal{C}^1([t_0,t_0+T])$ and a direct calculation yields

$$\dot{E}_0(t) \;=\; \int_{\mathbf{R}} \Bigl(-F_{xx}F_t + \frac{\varepsilon e^{-t}}{a(xe^{t/2})}\bigl(GG_t - \frac{x}{2}GG_x - \frac{3}{4}G^2\bigr)\Bigr)\, dx\;.$$

Using the identity

$$\begin{aligned}
&\varepsilon e^{-t}\bigl(GG_t - \frac{x}{2}GG_x - G^2\bigr) - a(xe^{t/2})F_{xx}F_t \;= \\
&\quad = -G^2 - a(xe^{t/2})\frac{x}{2}F_xF_{xx} + \alpha b(xe^{t/2})\varphi_0' G - \varepsilon e^{-t}HG + e^{-pt}RG\;,
\end{aligned}$$

which follows from (2.15), we thus find

$$\dot{E}_0(t) \;=\; \frac{1}{2}E_0(t) + \int_{\mathbf{R}} \frac{1}{a(xe^{t/2})}\Bigl(-G^2 + \alpha b(xe^{t/2})\varphi_0' G - \varepsilon e^{-t}HG + e^{-pt}RG\Bigr)\, dx\;. \tag{2.22}$$

In a similar way, $E_1 \in \mathcal{C}^1([t_0,t_0+T])$ and

$$\dot{E}_1 \;=\; \int_{\mathbf{R}} \frac{1}{a(xe^{t/2})}\Bigl(FF_t - \frac{x}{2}FF_x - \frac{1}{4}F^2 + \varepsilon e^{-t}\bigl((FG)_t - \frac{x}{2}(FG)_x - \frac{3}{2}FG\bigr)\Bigr)\, dx\;.$$

Using the identity

$$\begin{aligned}
&\varepsilon e^{-t}\bigl((FG)_t - \frac{x}{2}(FG)_x - FG - G^2\bigr) + FF_t \;= \\
&\quad = a(xe^{t/2})FF_{xx} + \frac{x}{2}FF_x + \alpha b(xe^{t/2})\varphi_0' F - \varepsilon e^{-t}HF + e^{-pt}RF\;,
\end{aligned}$$

we thus find

$$
\begin{aligned}
\dot{E}_1(t) &= -\frac{1}{2}E_1(t) - \int_{\mathbf{R}} F_x^2\,dx \\
&\quad + \int_{\mathbf{R}} \frac{1}{a(xe^{t/2})}\Big(\alpha b(xe^{t/2})\varphi_0' F + \varepsilon e^{-t}(G^2 - FH) + e^{-pt}RF\Big)\,dx\ .
\end{aligned} \tag{2.23}
$$

Combining (2.22), (2.23), we conclude that $E_2 \in \mathcal{C}^1([t_0, t_0+T])$ and

$$
\dot{E}_2(t) = -\frac{1}{2}E_2(t) - \int_{\mathbf{R}} \Big(\frac{1}{2}F_x^2 + \frac{G^2}{a(xe^{t/2})}\Big)\,dx + I_2(t)\ , \tag{2.24}
$$

where

$$
I_2(t) = \int_{\mathbf{R}} \frac{1}{a(xe^{t/2})}\Big((F+G)(\alpha b(xe^{t/2})\varphi_0' - \varepsilon e^{-t}H + e^{-pt}R) + \frac{3}{2}\varepsilon e^{-t}G^2\Big)\,dx\ .
$$

Now, we recall that $a(\xi) \ge \underline{a} > 0$ for all $\xi \in \mathbf{R}$ and $(1+|\xi|)^\mu b \in \mathrm{L}^2(\mathbf{R}) \cap \mathrm{L}^\infty(\mathbf{R})$ for some $\mu > -1/2$. Defining $\mu_0 = \min(\mu, 0)$, we thus have $|\xi|^{\mu_0} b \in \mathrm{L}^2(\mathbf{R})$ and $|x|^{-\mu_0}\varphi_0' \in \mathrm{L}^\infty(\mathbf{R})$. Therefore,

$$
\begin{aligned}
|I_2| &\le \frac{1}{\underline{a}}(\|F\|_{L^2} + \|G\|_{L^2})\Big(|\alpha| e^{-(1/4+\mu_0/2)t}\||\xi|^{\mu_0} b\|_{L^2}\||x|^{-\mu_0}\varphi_0'\|_{L^\infty} \\
&\qquad + \varepsilon e^{-t}\|H\|_{L^2} + e^{-pt}\|R\|_{L^2}\Big) + \frac{3}{2\underline{a}}\varepsilon e^{-t}\|G\|_{L^2}^2\ .
\end{aligned} \tag{2.25}
$$

By (2.14), (2.18), (2.20), we have $\|F\|_{L^2} \le 2\sigma$, $\|G\|_{L^2} \le 2\theta$, $\|H\|_{L^2} \le 2\|h\|_{Y^1} \le 2C_1(\rho+\theta)$ and $\|R\|_{L^2} \le 2\|r\|_{Y^1} \le 2C_1(\rho+\theta)$, hence

$$
|I_2| \le C(\sigma+\theta)\Big(e^{-(1/4+\mu_0/2)t}\rho + \varepsilon_0 e^{-t}(\rho+\theta) + e^{-pt}(\rho+\theta)\Big) + C\varepsilon_0 e^{-t}\theta^2\ , \tag{2.26}
$$

for some $C > 0$. Therefore, if $\lambda = \min(1/4, p, 1/4+\mu/2)$, it follows from (2.24), (2.26) that there exists $C_2 > 0$ (depending on ε_0) such that

$$
\dot{E}_2(t) \le -\frac{1}{2}E_2(t) - \frac{1}{2}\int_{\mathbf{R}} F_x^2\,dx + C_2 e^{-\lambda t}(\rho+\theta)(\sigma+\theta)\ ,
$$

for all $t \in [t_0, t_0+T]$. This concludes the proof of Lemma 2.5. □

Our next functionals are

$$
\begin{aligned}
E_3(t) &= \frac{1}{2}\int_{\mathbf{R}} \Big(a(xe^{t/2})f_x^2 + \varepsilon e^{-t}g^2\Big)\,dx + \alpha(t)\int_{\mathbf{R}} b(xe^{t/2})\varphi_0' f_x\,dx\ , \\
E_4(t) &= \int_{\mathbf{R}} \Big(\frac{1}{2}f^2 + \varepsilon e^{-t} fg\Big)\,dx\ , \\
E_5(t) &= E_3(t) + 2E_4(t)\ .
\end{aligned}
$$

Lemma 2.6. *Assume that $(v,w) \in \mathcal{C}([t_0,t_0+T],Z^1)$ is a solution of (1.7) satisfying the bound (2.5). Then $E_5 \in \mathcal{C}^1([t_0,t_0+T])$ and there exists a constant $C_3(\varepsilon_0) > 0$ such that, for all $t \in [t_0,t_0+T]$,*

$$\dot{E}_5(t) \;\le\; -\frac{1}{2}E_5(t) - \int_{\mathbf{R}} (a(xe^{t/2})f_x^2 + g^2)\,dx + \int_{\mathbf{R}} f^2\,dx + C_3 e^{-\lambda t}(\rho+\theta)(\sigma+\theta)\;. \quad (2.27)$$

Proof. As in the proof of Lemma 2.5, we first note that

$$E_3(t) \;=\; e^{-t/2}\int_{\mathbf{R}} \Bigl(\frac{1}{2}a(\xi)\tilde{f}_\xi(\xi,t)^2 + \alpha(t)b(\xi)\varphi_0'(\xi e^{-t/2})\tilde{f}_\xi(\xi,t)\Bigr)\,d\xi + \frac{1}{2}\varepsilon e^{-t}\int_{\mathbf{R}} g^2\,dx\;,$$

where $\tilde{f}(\xi,t) = e^{t/2}f(\xi e^{-t/2},t)$. If we assume that $(v_0,w_0) \in \mathcal{D}^1$, then $\tilde{f} \in \mathcal{C}^1([t_0,t_0+T],X^0)$ and $g \in \mathcal{C}^1([t_0,t_0+T],Y^0)$, hence $E_3 \in \mathcal{C}^1([t_0,t_0+T])$ and a direct calculation yields

$$\begin{aligned}\dot{E}_3(t) \;&=\; \int_{\mathbf{R}} \Bigl(a(xe^{t/2})\bigl(f_x f_{xt} - \frac{x}{2}f_x f_{xx} - \frac{1}{4}f_x^2\bigr) + \varepsilon e^{-t}\bigl(g g_t - \frac{1}{2}g^2\bigr)\Bigr)\,dx \\ &\quad + \alpha\int_{\mathbf{R}} b(xe^{t/2})\Bigl(\varphi_0'\bigl(f_{xt} - \frac{x}{2}f_{xx} - \frac{1}{2}f_x\bigr) - \frac{x}{2}\varphi_0'' f_x\Bigr)\,dx + \dot{\alpha}\int_{\mathbf{R}} b(xe^{t/2})\varphi_0' f_x\,dx\;.\end{aligned}$$

Next, we use the identities $f_{xt} - \frac{x}{2}f_{xx} - f_x = g_x$ and

$$\begin{aligned}&\varepsilon e^{-t}\bigl(g g_t - \frac{x}{2}g g_x - \frac{3}{2}g^2\bigr) - (a(xe^{t/2})f_x)_x f_t \;= \\ &\qquad = -g^2 - (a(xe^{t/2})f_x)_x\bigl(\frac{x}{2}f_x + \frac{1}{2}f\bigr) + \alpha(b(xe^{t/2})\varphi_0')_x g - \varepsilon e^{-t}hg + e^{-pt}rg\;,\end{aligned}$$

which follow from (2.11). After some integrations by parts, we obtain

$$\begin{aligned}\dot{E}_3(t) \;&=\frac{3}{2}E_3(t) - \int_{\mathbf{R}} g^2\,dx + (\dot{\alpha}-\alpha)\int_{\mathbf{R}} b(xe^{t/2})\varphi_0' f_x\,dx \\ &\quad - \alpha\int_{\mathbf{R}} b(xe^{t/2})\frac{x}{2}\varphi_0'' f_x\,dx + \int_{\mathbf{R}} \bigl(-\varepsilon e^{-t}hg + e^{-pt}rg\bigr)\,dx\;.\end{aligned} \quad (2.28)$$

Now, we observe that the right-hand side is a continuous function of the initial data (v_0,w_0) in the norm of Z^1, uniformly in $t \in [t_0,t_0+T]$. Therefore, using a density argument as in the proof of Lemma 2.3, we conclude that, for all $(v_0,w_0) \in Z^1$, $E_3 \in \mathcal{C}^1([t_0,t_0+T])$ and (2.28) holds.

In a similar way, $E_4 \in \mathcal{C}^1([t_0,t_0+T])$ and, using the identity

$$\begin{aligned}&\varepsilon e^{-t}\bigl((fg)_t - \frac{x}{2}(fg)_x - 2fg - g^2\bigr) + f f_t \;= \\ &\qquad = (a(xe^{t/2})f_x)_x f + \frac{x}{2}f f_x + \frac{1}{2}f^2 + \alpha(b(xe^{t/2})\varphi_0')_x f - \varepsilon e^{-t}hf + e^{-pt}rf\;,\end{aligned}$$

we obtain

$$\begin{aligned}\dot{E}_4(t) &= \int_{\mathbf{R}} \left(f f_t + \varepsilon e^{-t}\big((fg)_t - fg\big)\right) dx = \frac{1}{2}E_4(t) - \int_{\mathbf{R}} a(xe^{t/2}) f_x^2 \, dx \\ &\quad + \int_{\mathbf{R}} \Big(-\alpha b(xe^{t/2})\varphi_0' f_x + \varepsilon e^{-t}(g^2 - fh) + e^{-pt} rf\Big) \, dx \ .\end{aligned} \tag{2.29}$$

Combining (2.28), (2.29), we conclude that $E_5 \in \mathcal{C}^1([t_0, t_0+T])$ and

$$\dot{E}_5(t) = -\frac{1}{2}E_5(t) - \int_{\mathbf{R}} (a(xe^{t/2})f_x^2 + g^2) \, dx + \int_{\mathbf{R}} f^2 \, dx + I_5(t) \ ,$$

where

$$\begin{aligned} I_5(t) &= (\dot{\alpha} - \alpha) \int_{\mathbf{R}} b(xe^{t/2})\varphi_0' f_x \, dx - \alpha \int_{\mathbf{R}} \frac{x}{2} b(xe^{t/2})\varphi_0'' f_x \, dx \\ &\quad + \int_{\mathbf{R}} \left(\varepsilon e^{-t}(3g^2 + 2fg - 2fh - hg) + e^{-pt}(rg + 2rf)\right) dx \ . \end{aligned}$$

Proceeding exactly as in (2.25) above, we obtain

$$\begin{aligned} |I_5| &\le C\left(e^{-(1/4+\mu_0/2)t}(\rho+\theta)\sigma + \varepsilon_0 e^{-t}(\rho+\theta)(\sigma+\theta) + e^{-pt}(\rho+\theta)(\sigma+\theta)\right) \\ &\le C_3 e^{-\lambda t}(\rho+\theta)(\sigma+\theta) \ , \end{aligned}$$

for some $C_3 > 0$. This concludes the proof of Lemma 2.6. □

Finally, we define the weighted functionals

$$\begin{aligned} E_6(t) &= \frac{1}{2}\int_{\mathbf{R}} \left(x^2 a(xe^{t/2}) f_x^2 + \varepsilon e^{-t} x^2 g^2\right) dx + \alpha(t) \int_{\mathbf{R}} x^2 b(xe^{t/2})\varphi_0' f_x \, dx \ , \\ E_7(t) &= \int_{\mathbf{R}} \left(\frac{x^2}{2} f^2 + \varepsilon e^{-t} x^2 fg\right) dx \ , \\ E_8(t) &= E_6(t) + E_7(t) \ . \end{aligned}$$

Lemma 2.7. *Assume that $(v,w) \in \mathcal{C}([t_0, t_0+T], Z^1)$ is a solution of (1.7) satisfying the bound (2.5). Then $E_8 \in \mathcal{C}^1([t_0, t_0+T])$ and there exists a constant $C_4(\varepsilon_0) > 0$ such that, for all $t \in [t_0, t_0+T]$,*

$$\begin{aligned} \dot{E}_8(t) &\le -\frac{1}{2}E_8(t) - \int_{\mathbf{R}} \left(\frac{x^2}{2} a(xe^{t/2}) f_x^2 + x^2 g^2\right) dx \\ &\quad - 2\int_{\mathbf{R}} x a(xe^{t/2}) f_x (f+g) \, dx + C_4 e^{-\lambda t}(\rho+\theta)(\sigma+\theta) \ . \end{aligned} \tag{2.30}$$

Proof. Proceeding as in the proof of Lemma 2.6, we show that $E_8 \in \mathcal{C}^1([t_0, t_0+T])$ and

$$\dot{E}_8(t) \;=\; -\frac{1}{2}E_8(t) - \int_{\mathbf{R}} \Big(\frac{x^2}{2}a(xe^{t/2})f_x^2 + x^2 g^2\Big)\,dx - 2\int_{\mathbf{R}} xa(xe^{t/2})f_x(f+g)\,dx + I_8(t)\;,$$

where

$$\begin{aligned} I_8(t) \;&=(\dot\alpha - \alpha)\int_{\mathbf{R}} x^2 b(xe^{t/2})f_x\varphi_0'\,dx - 2\alpha\int_{\mathbf{R}} xb(xe^{t/2})\varphi_0'(f+g)\,dx \\ &-\alpha\int_{\mathbf{R}}\frac{x^3}{2}b(xe^{t/2})f_x\varphi_0''\,dx + \varepsilon e^{-t}\int_{\mathbf{R}} x^2\Big(\frac{3}{2}g^2 - (f+g)h\Big)\,dx \\ &+ e^{-pt}\int_{\mathbf{R}} x^2(f+g)r\,dx\;. \end{aligned}$$

As in (2.25) above, it is easy to verify that $|I_8| \le C_4 e^{-\lambda t}(\rho+\theta)(\sigma+\theta)$ for some $C_4 > 0$. This concludes the proof of Lemma 2.7. $\square$

The results of the three preceding lemmas can be summarized as follows. Let

$$E_9(t) \;=\; C_5(4E_2(t) + E_5(t)) + E_8(t) + \frac{1}{2}\varepsilon e^{-t}\dot\alpha(t)^2 + C_6 e^{-2\lambda t}\alpha(t)^2\;,$$

where $C_5 = 8\|a\|_{L^\infty} + 1$ and C_6 is a sufficiently large constant which will be determined below. Then the following holds:

Lemma 2.8. *Assume that $\varepsilon_0 e^{-t_0} \le 1/8$, and that $(v,w) \in \mathcal{C}([t_0, t_0+T], Z^1)$ is a solution of (1.7) satisfying the bound (2.5). Then $E_9 \in \mathcal{C}^1([t_0, t_0+T])$ and there exist constants $C_7 \ge 1$, $C_8(\varepsilon_0) > 0$ such that, for all $t \in [t_0, t_0+T]$,*

$$C_7^{-1}E_9(t) \;\le\; \sigma(t)^2 + \varepsilon e^{-t}\theta(t)^2 + e^{-2\lambda t}\alpha(t)^2 \;\le\; C_7 E_9(t)\;, \tag{2.31}$$

and

$$\dot{E}_9(t) \;\le\; -\frac{1}{2}E_9(t) - \theta(t)^2 - \int_{\mathbf{R}} f^2\,dx + C_8 e^{-\lambda t}(\rho+\theta)(\sigma+\theta)\;. \tag{2.32}$$

Proof. Since $\varepsilon e^{-t} \le \varepsilon_0 e^{-t_0} \le 1/8$, it is easy to verify that

$$\begin{aligned} E_2(t) \;&\ge\; \int_{\mathbf{R}}\Big(\frac{1}{2}F_x^2 + \frac{1}{4a(xe^{t/2})}(F^2 + \varepsilon e^{-t}G^2)\Big)\,dx \;\ge\; 0\;, \\ E_5(t) \;&\ge\; \int_{\mathbf{R}}\Big(\frac{1}{2}a(xe^{t/2})f_x^2 + \frac{1}{2}f^2 + \frac{1}{4}\varepsilon e^{-t}g^2\Big)\,dx - |\alpha|\int_{\mathbf{R}}|b(xe^{t/2})\varphi_0' f_x|\,dx \\ &\ge\; \int_{\mathbf{R}}\Big(\frac{1}{4}a(xe^{t/2})f_x^2 + \frac{1}{2}f^2 + \frac{1}{4}\varepsilon e^{-t}g^2\Big)\,dx - A\alpha^2 e^{-2\lambda t}\;, \\ E_8(t) \;&\ge\; \int_{\mathbf{R}}\Big(\frac{x^2}{4}a(xe^{t/2})f_x^2 + \frac{x^2}{4}f^2 + \frac{x^2}{4}\varepsilon e^{-t}g^2\Big)\,dx - B\alpha^2 e^{-2\lambda t}\;, \end{aligned}$$

where $A = \underline{a}^{-1}\||\xi|^{\mu_0}b\|_{L^2}^2\||x|^{-\mu_0}\varphi_0'\|_{L^\infty}^2$ and $B = \underline{a}^{-1}\||\xi|^{\mu_0}b\|_{L^2}^2\||x|^{1-\mu_0}\varphi_0'\|_{L^\infty}^2$, see (2.25) above. Therefore, if $C_6 \ge 2(C_5 A + B)$, then

$$E_9(t) \ \ge\ \frac{1}{4}\min(1,\underline{a})\sigma(t)^2 + \frac{1}{4}\varepsilon e^{-t}\theta(t)^2 + \frac{1}{2}C_6 e^{-2\lambda t}\alpha(t)^2 \ .$$

This proves the lower bound on E_9 in (2.31), and the upper bound follows using similar estimates. On the other hand, we have by (2.8)

$$\frac{d}{dt}\left(\frac{1}{2}\varepsilon e^{-t}\dot\alpha^2\right) \ =\ -\dot\alpha^2 + \frac{1}{2}\varepsilon e^{-t}\dot\alpha^2 + e^{-pt}\dot\alpha n(t)\ . \tag{2.33}$$

Combining (2.21), (2.27), (2.30), (2.33), we thus find

$$\begin{aligned}\dot E_9 \ \le\ & -\frac{1}{2}E_9 - C_5\int_{\mathbf{R}}\left(a(xe^{t/2})f_x^2 + f^2 + g^2\right)dx - \int_{\mathbf{R}}\left(\frac{x^2}{2}a(xe^{t/2})f_x^2 + x^2 g^2\right)dx \\ & - 2\int_{\mathbf{R}} xa(xe^{t/2})f_x(f+g)\,dx - \dot\alpha^2 + C_8 e^{-\lambda t}(\rho+\theta)(\sigma+\theta)\ ,\end{aligned}$$

for some $C_8(\varepsilon_0) > 0$. Since $C_5 = 8\|a\|_{L^\infty} + 1$ and

$$2\left|\int_{\mathbf{R}} xa(xe^{t/2})f_x(f+g)\,dx\right| \ \le\ \frac{1}{4}\int_{\mathbf{R}} x^2 a(xe^{t/2})f_x^2\,dx + 8\|a\|_{L^\infty}\int_{\mathbf{R}}(f^2+g^2)\,dx\ ,$$

we obtain

$$\begin{aligned}\dot E_9 \ \le\ & -\frac{1}{2}E_9 - \frac{1}{4}\int_{\mathbf{R}} a(xe^{t/2})(1+x^2)f_x^2\,dx - \int_{\mathbf{R}}(f^2 + (1+x^2)g^2)\,dx - \dot\alpha^2 \\ & + C_8 e^{-\lambda t}(\rho+\theta)(\sigma+\theta)\ ,\end{aligned}$$

which implies (2.32). This concludes the proof of Lemma 2.8. $\square$

Corollary 2.9. *Let $\varepsilon_0 > 0$. There exist constants $t_0(\varepsilon_0) \ge 0$ and $C_9 \ge 1$ such that, if $T > 0$ and $(v,w) \in \mathcal{C}([t_0, t_0+T], Z^1)$ is a solution of (1.7) satisfying the bound (2.5), then*

$$\Phi_1(\varepsilon e^{-t}, v(t), w(t)) \ \equiv\ \|v(t)\|_{X^1}^2 + \varepsilon e^{-t}\|w(t)\|_{Y^1}^2 \ \le\ C_9\Phi_1(\varepsilon e^{-t_0}, v_0, w_0)\ , \tag{2.34}$$

for all $t \in [t_0, t_0+T]$.

Proof. Let $E_{10}(t) = \frac{1}{2}\alpha(t)^2 + \varepsilon e^{-t}\alpha(t)\dot\alpha(t) + E_9(t)$. In view of (2.31), if $\varepsilon_0 e^{-t_0} \le 1/8$ and $\varepsilon_0 e^{-t_0} \le C_7^{-1}$, there exists $K_1 > 0$ such that $K_1^{-1}E_{10}(t) \le \rho(t)^2 + \varepsilon e^{-t}\theta(t)^2 \le K_1 E_{10}(t)$ for all $t \in [t_0, t_0+T]$. In addition, we have by (2.8), (2.32)

$$\begin{aligned}\dot E_{10}(t) \ &=\ \varepsilon e^{-t}\dot\alpha(t)^2 + e^{-pt}\alpha n(t) + \dot E_9(t) \ \le\ -\theta^2 + K_2 e^{-\lambda t}(\rho^2+\theta^2) \\ &\le\ -\theta^2(1 - K_2 e^{-\lambda t}) + K_1 K_2 e^{-\lambda t}E_{10}(t)\ ,\end{aligned}$$

for some $K_2(\varepsilon_0) > 0$. Therefore, if $K_2 e^{-\lambda t_0} \le 1$, then $\dot{E}_{10}(t) \le K_1 K_2 e^{-\lambda t} E_{10}(t)$, hence $E_{10}(t) \le K_3 E_{10}(t_0)$ for all $t \in [t_0, t_0 + T]$, where $K_3 = \exp(K_1 \lambda^{-1})$. Finally, it follows from (2.19) that

$$K_4^{-1}(\rho(t)^2 + \varepsilon e^{-t}\theta(t)^2) \le \|v(t)\|_{X^1}^2 + \varepsilon e^{-t}\|w(t)\|_{Y^1}^2 \le K_4(\rho(t)^2 + \varepsilon e^{-t}\theta(t)^2) ,$$

where $K_4 = 2C_0^2$. Combining these inequalities, we obtain (2.34), with $C_9 = K_1^2 K_3 K_4^2$. This concludes the proof of Corollary 2.9. □

Proof of Theorem 1.1 (first part). Assume that the hypotheses (H1), (H2), (N1), (N2) hold, and fix $\varepsilon_0 > 0$. Let $t_0 \ge 0$, $C_9 \ge 1$ be as in Corollary 2.9, and let $\delta_0 = (2C_9)^{-1/2}$, $\varepsilon \in (0, \varepsilon_0]$. If $(v_0, w_0) \in Z^1$ satisfies $\Phi_1(\varepsilon e^{-t_0}, v_0, w_0) \le \delta_0^2$, then (1.7) has a unique global solution $(v, w) \in \mathcal{C}([t_0, +\infty), Z^1)$ such that $(v(t_0), w(t_0)) = (v_0, w_0)$. Indeed, the local existence and uniqueness follow from Proposition 2.2, and Corollary 2.9 shows that the solution (v, w) satisfies $\Phi_1(\varepsilon e^{-t}, v(t), w(t)) \le 1/2$ whenever it exists. Then Proposition 2.2 (with $\delta_1 = 1/\sqrt{2}$) implies that the solution can be continued to $[t_0, +\infty)$. To prove (1.17), we first note that (2.32) implies

$$\begin{aligned}
\dot{E}_9 &\le -\frac{1}{2}E_9 - \theta^2 - \|f\|_{L^2}^2 + C_8 e^{-\lambda t}(\rho\sigma + 2\rho\theta + \theta^2) \\
&\le -\frac{1}{2}E_9 - \theta^2\big(1 - \frac{1}{4} - C_8 e^{-\lambda t}\big) - \|f\|_{L^2}^2 + C_8 e^{-\lambda t}\rho\sigma + 4C_8^2\rho^2 e^{-2\lambda t} ,
\end{aligned}$$

for all $t \in [t_0, +\infty)$. By (2.19), (2.31), (2.34), we have $\sigma(t)^2 \le C_7 E_9(t)$ and $\rho(t)^2 \le \rho_1^2 \equiv C_0^2 C_9 \Phi_1(\varepsilon e^{-t_0}, v_0, w_0)$ for all $t \ge t_0$. Assuming (without loss of generality) that $C_8 e^{-\lambda t_0} \le 1/4$, we thus obtain

$$\dot{E}_9 \le -\frac{1}{2}E_9 - \frac{1}{2}\theta^2 - \|f\|_{L^2}^2 + K_5\left(E_9^{1/2}\rho_1 e^{-\lambda t} + \rho_1^2 e^{-2\lambda t}\right) , \quad t \ge t_0 ,$$

for some $K_5 > 0$. Integrating this differential inequality for $E_9(t)$, it is not difficult to show that

$$E_9(t) + \frac{1}{2}\int_{t_0}^{t} e^{-2\lambda(t-s)}(\theta^2(s) + 2\|f(s)\|_{L^2}^2)\,ds \le K_6\rho_1^2(1+t)^2 e^{-2\lambda t} , \quad t \ge t_0 , \tag{2.35}$$

for some $K_6 > 0$. In view of (2.31), this implies, for all $t \ge t_0$,

$$\|f(t)\|_{X^1}^2 + \varepsilon e^{-t}(\dot{\alpha}(t)^2 + \|g(t)\|_{Y^1}^2) \equiv \sigma(t)^2 + \varepsilon e^{-t}\theta(t)^2 \le C_7 K_6 \rho_1^2 (1+t)^2 e^{-2\lambda t} , \tag{2.36}$$

and

$$\int_{t_0}^{t}(\dot{\alpha}(s)^2 + \|g(s)\|_{Y^1}^2)e^{2\lambda s}\,ds \equiv \int_{t_0}^{t}\theta(s)^2 e^{2\lambda s}\,ds \le 2K_6\rho_1^2(1+t)^2 . \tag{2.37}$$

In particular, it follows from (2.37) that the function $\alpha(t)$ converges to some $\alpha^* \in \mathbf{R}$ as $t \to +\infty$, and that

$$|\alpha(t) - \alpha^*|^2 + \int_{t_0}^{t} e^{-2\lambda(t-s)}|\alpha(s) - \alpha^*|^2\, ds \;\le\; K_7\rho_1^2(1+t)^2 e^{-2\lambda t}\ , \quad t \ge t_0\ , \qquad (2.38)$$

for some $K_7 > 0$. Now, we have by (2.6)

$$\begin{aligned} \|v(t) - \alpha^*\varphi_0\|_{X^1} \;&\le\; \|f(t)\|_{X^1} + |\alpha(t) - \alpha^*|\|\varphi_0\|_{X^1}\ , \\ \|w(t) - \alpha^*\psi_0\|_{Y^1} \;&\le\; \|g(t)\|_{Y^1} + |\dot\alpha(t)|\|\varphi_0\|_{Y^1} + |\alpha(t) - \alpha^*|\|\psi_0\|_{Y^1}\ , \end{aligned}$$

hence (1.17) follows from (2.36), (2.37) and (2.38). This concludes the first part of the proof of Theorem 1.1. □

2.4. Further Estimates on the Time Derivative

Throughout this section, we assume that the nonlinearity $\mathcal{N}$ in (1.1) satisfies the additional hypothesis (N3). In this case, if $(v,w) \in \mathcal{C}([t_0, t_0+T], Z^1)$ is a solution of (1.7) satisfying the bound (2.5), the functions H, R defined in (2.16) are differentiable with respect to t. Defining

$$\tilde H(x,t) \;=\; H_t(x,t) - \frac{x}{2}H_x(x,t) - H(x,t)\ , \quad \tilde R(x,t) \;=\; R_t(x,t) - \frac{x}{2}R_x(x,t) - pR(x,t)\ ,$$

we have the following result:

Lemma 2.10. *Assume that $(v,w) \in \mathcal{C}([t_0, t_0+T], Z^1)$ is a solution of (1.7) satisfying the bound (2.5). Then $H, R \in \mathcal{C}^1([t_0, t_0+T], \mathrm{L}^2)$ and there exists a constant $C_{10}(\varepsilon_0) > 0$ such that, for all $t \in [t_0, t_0+T]$,*

$$\varepsilon e^{-t}\|\tilde H(t)\|_{L^2} \;\le\; C_{10}(e^{-\lambda t}\rho(t) + \theta(t))\ , \quad \|\tilde R(t)\|_{L^2} \;\le\; C_{10}(\rho(t) + \theta(t))\ ,$$

where ρ, θ are defined in (2.18) and $\lambda = \min(1/4, p, 1/4 + \mu/2)$.

Proof. From (2.16), (2.12), we have

$$\tilde H(x,t) \;=\; (-\ddot\alpha + 3\dot\alpha/2 - \alpha/2)x\varphi_0(x) + \frac{1}{2}(\alpha - 3\dot\alpha)x\psi_0(x) - \frac{\alpha}{2}x\chi_0(x)\ ,$$

where φ_0, ψ_0 are defined in (1.14), (1.16). Since $|\alpha| \le \rho$, $|\dot\alpha| \le \theta$ and $\varepsilon e^{-t}|\ddot\alpha| \le |\dot\alpha| + e^{-pt}|n| \le \theta + Ce^{-pt}(\rho+\theta)$ by (2.8), (2.18) and Lemma 2.4, it follows that $\varepsilon e^{-t}\|\tilde H\|_{L^2} \le C_{10}(e^{-\lambda t}\rho + \theta)$ for some $C_{10} > 0$.

To bound $\tilde{R}(x,t)$, we note that $\mathcal{N}_2^*(u)u_\xi = (\Phi^*(u))_\xi$, where $\Phi^*(u) = \int_0^u \mathcal{N}_2^*(z)\,dz$. By (2.9), (2.12), (2.16), we thus have

$$R(x,t) \;=\; \int_{-\infty}^x (N_1(y,t) - \varphi_0(y)n(t))\;dy + e^{(p+1)t}\Phi^*(e^{-t/2}v(x,t))\;,$$

where $N_1(x,t) = e^{(p+3/2)t}\mathcal{N}_1(e^{-t/2}v(x,t))$ and $n(t) = \int N_1(x,t)dx$. Since $\mathcal{N}_1, \Phi^* \in \mathcal{C}^1(\mathbf{R})$, it is not difficult to show that $R \in \mathcal{C}^1([t_0, t_0+T], \mathrm{L}^2)$ and that

$$\tilde{R}(x,t) \;=\; R(x,t) + \int_{-\infty}^x \Big(\hat{N}_1(y,t) - \varphi_0(y)\hat{n}(t)\Big)\;dy + \frac{x}{2}\varphi_0(x)n(t) + \hat{N}_2(x,t)\;,$$

where $\hat{N}_1(x,t) = e^{(p+1)t}\mathcal{N}_1'(e^{-t/2}v(x,t))w(x,t)$, $\hat{n}(t) = \int \hat{N}_1(x,t)dx$ and $\hat{N}_2(x,t) = e^{(p+1/2)t}\mathcal{N}_2^*(e^{-t/2}v(x,t))w(x,t)$. By the hypothesis (N3), we have $|\hat{N}_1| \le C_N|v|^{\nu_1}|w|$ and $|\hat{N}_2| \le C_N|v|^{\nu_2}|w|$. Thus, proceeding as in the proof of Lemma 2.4, we easily obtain

$$\|\tilde{R}\|_{L^2} \;\le\; \|R\|_{L^2} + C\|\hat{N}_1\|_{Y^1} + \frac{1}{2}|n|\|x\varphi_0\|_{L^2} + \|\hat{N}_2\|_{L^2} \;\le\; C_{10}(\rho+\theta)\;.$$

This concludes the proof of Lemma 2.10. □

As before, let $(v,w) \in \mathcal{C}([t_0, t_0+T], Z^1)$ be a solution of (1.7) satisfying the bound (2.5). We define the new function $M(x,t) = G_t(x,t) - \frac{x}{2}G_x(x,t)$, where G is given by (2.13). Using a density argument as in the proof of Lemma 2.3, it is straightforward to show that $(G,M) \in \mathcal{C}^1([t_0, t_0+T], Z^0)$ is a solution of the system

$$\begin{aligned} G_t - \frac{x}{2}G_x \;&=\; M\;, \\ \varepsilon e^{-t}\big(M_t - \frac{x}{2}M_x - 2M\big) + M \;&=\; a(xe^{t/2})(G_{xx} + F_{xx}) + J(x,t)\;, \end{aligned} \tag{2.39}$$

where

$$J(x,t) \;=\; \dot{\alpha}(t)b(xe^{t/2})\varphi_0' - \alpha(t)b(xe^{t/2})\frac{x}{2}\varphi_0'' - \varepsilon e^{-t}(\tilde{H} + G) + e^{-pt}\tilde{R}\;.$$

In analogy with the preceding subsection, we introduce the energy functionals

$$\begin{aligned} \tilde{E}_0(t) \;&=\; \frac{1}{2}\int_{\mathbf{R}} \left(G_x^2 + \varepsilon e^{-t}\frac{M^2}{a(xe^{t/2})}\right)\;dx\;, \\ \tilde{E}_1(t) \;&=\; \int_{\mathbf{R}} \frac{1}{a(xe^{t/2})}\left(\frac{1}{2}G^2 + \varepsilon e^{-t}GM\right)\;dx\;, \\ \tilde{E}_2(t) \;&=\; \tilde{E}_0(t) + \tilde{E}_1(t)\;. \end{aligned}$$

Lemma 2.11. *Assume that $\varepsilon_0 e^{-t_0} \le 1/4$ and that $(v,w) \in \mathcal{C}([t_0, t_0+T], Z^1)$ is a solution of (1.7) satisfying the bound (2.5). Then $\tilde E_2 \in \mathcal{C}^1([t_0,t_0+T])$ and there exists a constant $C_{11}(\varepsilon_0) > 0$ such that, for all $t \in [t_0, t_0+T]$,*

$$\frac{d}{dt}\tilde E_2(t) \;\le\; -\frac{1}{2}\tilde E_2(t) + \int_{\mathbf{R}} f^2\,dx + C_{11}(e^{-2\lambda t}\rho^2 + \theta^2)\;. \tag{2.40}$$

Proof. Proceeding as in the proof of Lemma 2.5 and using a density argument as in Lemma 2.3, we show that $\tilde E_2 \in \mathcal{C}^1([t_0,t_0+T])$ and

$$\frac{d}{dt}\tilde E_2(t) \;=\; -\frac{1}{2}\tilde E_2(t) - \int_{\mathbf{R}}\left(\frac{1}{2}G_x^2 + \frac{M^2}{a(xe^{t/2})}\right)dx + \tilde I_2(t)\;, \tag{2.41}$$

where

$$\tilde I_2(t) \;=\; -\int_{\mathbf{R}} G_x F_x\,dx + \int_{\mathbf{R}} \frac{1}{a(xe^{t/2})}\left((G+M)J + \frac{1}{2}\varepsilon e^{-t}(5M^2+2GM)\right)dx\;.$$

In view of Lemma 2.10, we have as in (2.25)

$$\begin{aligned}\|J\|_{L^2} &\le e^{-(1/4+\mu_0/2)t}\||\xi|^{\mu_0} b\|_{L^2}\left(|\dot\alpha|\||x|^{-\mu_0}\varphi_0'\|_{L^\infty} + \frac{1}{2}|\alpha|\||x|^{1-\mu_0}\varphi_0''\|_{L^\infty}\right)\\ &\quad + 2C_{10}(e^{-\lambda t}\rho + \theta) + 2\varepsilon e^{-t}\theta \;\le\; K_1(e^{-\lambda t}\rho+\theta)\;,\end{aligned}$$

for some $K_1 > 0$. Therefore, we have

$$|\tilde I_2| \;\le\; \|G_x\|_{L^2}\|F_x\|_{L^2} + K_2(e^{-\lambda t}\rho+\theta)(\|G\|_{L^2}+\|M\|_{L^2}) + \varepsilon e^{-t}\int_{\mathbf{R}}\frac{3M^2+G^2}{a(xe^{t/2})}\,dx\;,$$

for some $K_2 > 0$. As a consequence, since $\varepsilon e^{-t} \le \varepsilon_0 e^{-t_0} \le 1/4$, there exists $K_3 > 0$ such that

$$|\tilde I_2| \;\le\; \int_{\mathbf{R}}\left(\frac{1}{4}G_x^2 + \frac{M^2}{a(xe^{t/2})}\right)dx + \|F_x\|_{L^2}^2 + K_3(e^{-2\lambda t}\rho^2 + \theta^2 + \|G\|_{L^2}^2)\;. \tag{2.42}$$

Since $\|F_x\|_{L^2} = \|f\|_{L^2}$ and $\|G\|_{L^2} \le 2\|xg\|_{L^2} \le 2\theta$, (2.40) follows from (2.41), (2.42). This concludes the proof of Lemma 2.11. □

Proof of Theorem 1.1 (second part). Assume that the hypothesis (N3) holds, and let $\varepsilon_0, t_0, \delta_0$ be as in the first part of the proof. Given $\varepsilon \in (0, \varepsilon_0]$ and $(v_0,w_0) \in Z^1$ such that $\Phi_1(\varepsilon e^{-t_0}, v_0, w_0) \le \delta_0^2$, let $(v,w) \in \mathcal{C}([t_0,+\infty), Z^1)$ be the unique solution of (1.7) satisfying $(v(t_0), w(t_0)) = (v_0, w_0)$. It remains to prove the decay estimate (1.18). First, since $\varepsilon e^{-t} \le \varepsilon_0 e^{-t_0} \le 1$, we have

$$\frac{1}{2}\int_{\mathbf{R}} g^2\,dx \;\le\; \tilde{E}_2(t) \;\le\; \frac{1}{2}\int_{\mathbf{R}} g^2\,dx + \int_{\mathbf{R}} \frac{1}{a(x,t)}(G^2 + \varepsilon e^{-t}M^2)\,dx\;. \tag{2.43}$$

To bound the right-hand side, we recall that

$$M \;=\; G_t - \frac{x}{2}G_x \;=\; G - H + \varepsilon^{-1}e^t\left(-G + a(xe^{t/2})f_x + \alpha b(xe^{t/2})\varphi_0' + e^{-pt}R\right)\;,$$

by (2.15). In particular, there exists $K_4 > 0$ such that $\|M(t_0)\|_{L^2} \le K_4(1+\varepsilon^{-1})\Lambda_0$, where $\Lambda_0 = \|(v_0, w_0)\|_{Z^1}$. By (2.43), it follows that $\varepsilon\tilde{E}_2(t_0) \le K_5\Lambda_0^2$ for some $K_5 > 0$ (depending on ε_0.) Next, we have by (2.34), (2.40)

$$\frac{d}{dt}\tilde{E}_2(t) \;\le\; -\frac{1}{2}\tilde{E}_2(t) + \|f(t)\|_{L^2}^2 + C_{11}(e^{-2\lambda t}\rho_1^2 + \theta(t)^2)\;,\quad t \ge t_0\;,$$

where $\rho_1^2 = C_0^2 C_9 \Phi_1(\varepsilon e^{-t_0}, v_0, w_0) \le K_6\Lambda_0^2$. Integrating this differential inequality and using (2.35), we easily obtain

$$\begin{aligned}\tilde{E}_2(t) \;&\le\; e^{-2\lambda(t-t_0)}\tilde{E}_2(t_0) + C_{11}\rho_1^2 e^{-2\lambda t}\left((t-t_0) + 2K_5(1+t)^2\right)\\ &\le\; K_7 e^{-2\lambda t}(\tilde{E}_2(t_0) + \rho_1^2(1+t)^2)\;,\quad t \ge t_0\;,\end{aligned}$$

for some $K_7 > 0$. Since $\varepsilon\tilde{E}_2(t_0) \le K_5\Lambda_0^2$ and $\rho_1^2 \le K_6\Lambda_0^2$, we conclude by (2.43) that there exists $K_8 > 0$ such that

$$\varepsilon\|g(t)\|_{L^2}^2 \;\le\; K_8\Lambda_0^2(1+t)^2 e^{-2\lambda t}\;,\quad t \ge t_0\;. \tag{2.44}$$

Finally, integrating the differential equation (2.8) for α and using the fact that $|n(t)| \le \|N_1\|_{L^1} \le C\rho_1$ for all $t \ge t_0$, it is straightforward to verify that

$$|\dot{\alpha}(t)| \;\le\; K_9(|\dot{\alpha}(t_0)| + \rho_1)e^{-\lambda t} \;\le\; K_{10}\Lambda_0 e^{-\lambda t}\;,\quad t \ge t_0\;, \tag{2.45}$$

for some $K_9, K_{10} > 0$. Now, since

$$\|w(t) - \alpha^*\psi_0\|_{L^2} \;\le\; \|g(t)\|_{L^2} + |\dot{\alpha}(t)|\|\varphi_0\|_{L^2} + |\alpha(t) - \alpha^*|\|\psi_0\|_{L^2}\;,$$

we conclude from (2.38), (2.44), (2.45) that $\varepsilon\|w(t) - \alpha^*\psi_0\|_{L^2}^2 \le K_{11}(\varepsilon_0)\Lambda_0^2(1+t)^2 e^{-2\lambda t}$, which is (1.18). This concludes the proof of Theorem 1.1. $\square$

3. Second Order Asymptotics

In this section we prove Theorem 1.2, *i.e.* we perform a second order asymptotic expansion of the solution (v, w) of (1.7) as $t \to +\infty$. As was already explained in the introduction, we restrict our analysis for simplicity to the particular case where $\mathcal{N} = 0$ and $a_+ = a_- = 1$. Thus we set $a(\xi) = 1 + b(\xi)$ and we assume that $b(\xi)$ satisfies the hypothesis (H2) for some $\mu > 1/2$. In a first step (section 3.1), using the spectral projections of the operator $\tilde{\mathcal{L}}$, we decompose the solutions (v, w) of (1.7) into the sum of two explicit terms and a remainder which decays faster than $e^{-t/2}$. Then (section 3.2) we exploit the invariance properties of Eq.(1.1) in the linear case to obtain a priori estimates for the first two terms in our expansion. Finally (section 3.3) we control the remainder with the help of various energy functionals.

3.1. Spectral Decomposition

We fix $\varepsilon_0 > 0$, $t_0 > 0$ and choose initial data $(v_0, w_0) = (v(t_0), w(t_0))$ in the space Z^2. Then the solution $(v(t), w(t))$ of (1.7) exists for all time $t \geq t_0$, is unique and belongs to $\mathcal{C}([t_0, +\infty), Z^2)$. We recall that the first two eigenvalues of the operator $\tilde{\mathcal{L}}$ defined in (1.13) are 0 and $-1/2$, with eigenfunctions

$$\varphi_0(x) \;=\; \frac{1}{\sqrt{4\pi}} \exp(-x^2/4) \;, \quad \varphi_1(x) \;=\; -\varphi_0'(x) \;. \tag{3.1}$$

The associated projections are given by

$$P_0 v \;=\; \varphi_0 \int_{\mathbf{R}} v \, dx \;, \quad P_1 v \;=\; \varphi_1 \int_{\mathbf{R}} x v \, dx \;.$$

Using these spectral projections, we decompose the solution (v, w) of (1.7) as

$$\begin{aligned} v(x,t) \;&=\; \alpha(t)\varphi_0(x) + \beta(t) e^{-t/2} \varphi_1(x) + f(x,t) \;, \\ w(x,t) \;&=\; (\dot{\alpha}(t)\varphi_0(x) + \alpha(t)\psi_0(x)) + e^{-t/2}(\dot{\beta}(t)\varphi_1(x) + \beta(t)\psi_1(x)) + g(x,t) \;, \end{aligned} \tag{3.2}$$

where $\psi_i = \varphi_i''$ $(i = 0, 1)$, and

$$\alpha(t) \;=\; \int_{\mathbf{R}} v(x,t) \, dx \;, \quad \beta(t) \;=\; e^{t/2} \int_{\mathbf{R}} x v(x,t) \, dx \;. \tag{3.3}$$

The following lemma states elementary properties of the functions $\alpha(t)$ and $\beta(t)$. Its proof is very similar to the one of Lemma 2.3, and will be omitted.

Lemma 3.1. *If $(v_0, w_0) \in Z^2$, then α and β belong to $\mathcal{C}^2([t_0, +\infty))$. The function α satisfies (2.8) with $n = 0$, so that*

$$\alpha(t) \;=\; \alpha(t_0) + \varepsilon\dot{\alpha}(t_0)e^{-t_0}\left(1 - \exp(-\frac{1}{\varepsilon}(e^t - e^{t_0}))\right) \;, \tag{3.4}$$

and

$$\alpha^* \;\equiv\; \lim_{t\to+\infty} \alpha(t) \;=\; \int_{\mathbf{R}} v_0(x,t)\,dx + \varepsilon e^{-t_0}\int_{\mathbf{R}} w_0(x,t)\,dx \;. \tag{3.5}$$

The function β satisfies the equations

$$\dot{\beta}(t) \;=\; e^{t/2}\int_{\mathbf{R}} xw(x,t)\,dx \;, \tag{3.6}$$

$$\frac{d}{dt}\left(\varepsilon e^{-t}\dot{\beta}(t) + \beta(t)\right) \;=\; -\int_{\mathbf{R}} b(xe^{t/2})e^{t/2}\left(\alpha(t)\varphi_0'(x) + \beta(t)e^{-t/2}\varphi_1'(x) + f_x(x,t)\right)\,dx \;. \tag{3.7}$$

Remark. From (3.2), (3.3) and (3.6), it follows that, for $t \ge t_0$,

$$\int_{\mathbf{R}} f(x,t)\,dx \;=\; \int_{\mathbf{R}} xf(x,t)\,dx \;=\; 0 \;, \quad \int_{\mathbf{R}} g(x,t)\,dx \;=\; \int_{\mathbf{R}} xg(x,t)\,dx \;=\; 0 \;. \tag{3.8}$$

Using Lemma 3.1 and the properties of φ_0 and φ_1, it is straightforward to verify that $(f,g) \in \mathcal{C}([t_0,+\infty], Z^2)$ is a solution of the system

$$\begin{aligned} f_t - \frac{x}{2}f_x - \frac{1}{2}f \;&= g \;, \\ \varepsilon e^{-t}\big(g_t - \frac{x}{2}g_x - \frac{3}{2}g\big) + g \;&= (a(xe^{t/2})f_x)_x + A_x(x,t) + \varphi_1\int_{\mathbf{R}} B(x,t)\,dx \\ &\qquad - \varepsilon e^{-t}h_0(x,t) - \varepsilon e^{-3t/2}h_1(x,t) \;, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} h_0(x,t) \;&=\; 2\dot{\alpha}(t)\psi_0(x) + \alpha(t)\psi_0''(x) \;, \quad h_1(x,t) \;=\; 2\dot{\beta}(t)\psi_1(x) + \beta(t)\psi_1''(x) \;, \\ A(x,t) \;&=\; \alpha(t)b(xe^{t/2})\varphi_0'(x) + e^{-t/2}\beta(t)b(xe^{t/2})\varphi_1'(x) \;, \\ B(x,t) \;&=\; A(x,t) + b(xe^{t/2})f_x(x,t) \;. \end{aligned} \tag{3.10}$$

Due to the relations (3.8), we can define the primitive functions $F(x,t)$, $G(x,t)$ by (2.13) as well as the double primitives

$$\mathcal{F}(x,t) \;=\; \int_{-\infty}^{x} F(y,t)\,dy \;, \quad \mathcal{G}(x,t) \;=\; \int_{-\infty}^{x} G(y,t)\,dy \;. \tag{3.11}$$

Using a generalized Hardy inequality [HLP, section 9.9], we obtain

$$\|\mathcal{F}\|_{L^2} + \|xF\|_{L^2} \le C_0\|f\|_{Y^2} , \quad \|\mathcal{G}\|_{L^2} + \|xG\|_{L^2} \le C_0\|g\|_{Y^2} , \tag{3.12}$$

hence (F,G) belongs to $\mathcal{C}([t_0,+\infty),\mathcal{D}^1)$. Arguing as in the previous section, one verifies that $(F,G) \in \mathcal{C}^1([t_0,+\infty),Z^1)$ is a classical solution in Z^1 of the system

$$\begin{aligned} F_t - \frac{x}{2}F_x &= G , \\ \varepsilon e^{-t}\big(G_t - \frac{x}{2}G_x - G\big) + G &= a(xe^{t/2})F_{xx} - \varphi_0(x)\int_{\mathbf{R}} b(xe^{t/2})f_x(x,t)\,dx \\ &\quad + (Q_0A)(x,t) - \varepsilon e^{-t}H_0(x,t) - \varepsilon e^{-3t/2}H_1(x,t) , \end{aligned} \tag{3.13}$$

where $Q_0 = Id - P_0$ and

$$H_0(x,t) = 2\dot\alpha(t)\varphi_0'(x) + \alpha(t)\psi_0'(x) , \quad H_1(x,t) = 2\dot\beta(t)\varphi_1'(x) + \beta(t)\psi_1'(x) . \tag{3.14}$$

Likewise, one checks that $(\mathcal{F},\mathcal{G}) \in \mathcal{C}^2([t_0,+\infty),Z^0)$ is a classical solution of the system

$$\begin{aligned} \mathcal{F}_t - \frac{x}{2}\mathcal{F}_x + \frac{1}{2}\mathcal{F} &= \mathcal{G} , \\ \varepsilon e^{-t}\big(\mathcal{G}_t - \frac{x}{2}\mathcal{G}_x - \frac{1}{2}\mathcal{G}\big) + \mathcal{G} &= \mathcal{F}_{xx} + \int_{-\infty}^{x} (Q_0B)(y,t)\,dy \\ &\quad - \varepsilon e^{-t}\mathcal{H}_0(x,t) - \varepsilon e^{-3t/2}\mathcal{H}_1(x,t) , \end{aligned} \tag{3.15}$$

where

$$\mathcal{H}_0(x,t) = 2\dot\alpha(t)\varphi_0(x) + \alpha(t)\psi_0(x) , \quad \mathcal{H}_1(x,t) = 2\dot\beta(t)\varphi_1(x) + \beta(t)\psi_1(x) . \tag{3.16}$$

To prove Theorem 1.2, we need to control the time behavior of the functions $\alpha(t)$, $\beta(t)$, $f(x,t)$ and $g(x,t)$ defined in (3.2). From (3.4), we already know that $\alpha(t)$ converges to α^* like $\exp(-\varepsilon^{-1}e^t)$ as $t \to +\infty$. As in Section 2, we could study the behavior of $\beta(t)$ by including it in some energy functionals, but here instead we shall exploit the invariance properties of (1.1) in the linear case to obtain a priori estimates on $\beta(t)$, $\dot\beta(t)$. This is the purpose of the next subsection, and the rest of the proof will be devoted to the control of $f(x,t)$, $g(x,t)$ (section 3.3).

3.2. A Priori Estimates

We first recall a decay estimate which follows from the proofs in Section 2. Let

$$
\begin{aligned}
f^*(x,t) &= \beta(t)e^{-t/2}\varphi_1(x) + f(x,t) \;, \\
g^*(x,t) &= e^{-t/2}(\dot\beta(t)\varphi_1(x) + \beta(t)\psi_1(x)) + g(x,t) \;.
\end{aligned} \tag{3.17}
$$

Then, following the lines of the proofs of Lemma 2.5 to Lemma 2.8 in the linear case, and using the hypothesis (H2) with $\mu > 1/2$ as well as the explicit expression (3.4) of $\alpha(t)$, it is straightforward to verify that, if $\varepsilon_0 e^{-t_0} \le 1/8$ and $(v_0, w_0) \in Z^1$, then

$$
\begin{aligned}
\|f^*(t)\|^2_{X^1} + \varepsilon e^{-t}\|g^*(t)\|^2_{Y^1} + &\int_{t_0}^t e^{-(t-s)/2}(\|f^*(s)\|^2_{X^1} + \|g^*(s)\|^2_{Y^1})\,ds \\
&\le C_1^2(b)e^{-(t-t_0)/2}\Phi_1(\varepsilon e^{-t_0}, v_0, w_0) \;,
\end{aligned} \tag{3.18}
$$

for all $t \ge t_0$, where $C_1(b)$ is a positive constant depending only on ε_0 and b. In other words, the factor $(1+t)^2$ in the estimate (1.17) can be removed if $\mathcal{N} = 0$ and $\mu > 1/2$.

Next, if $u \in \mathcal{C}^1([0,+\infty), Z^2)$ is the solution of (1.1) with initial data $u(\xi,0) = e^{-t_0/2}v_0(\xi e^{-t_0/2})$, $u_\tau(\xi,0) = e^{-3t_0/2}w_0(\xi e^{-t_0/2})$, we introduce the "invariant functions"

$$
I_0(\tau) = \int_{\mathbf{R}} u(\xi,\tau)\,d\xi \;, \quad I_1(\tau) = \int_{\mathbf{R}} p_1(\xi)u(\xi,\tau)\,d\xi \;,
$$

where $p_1(\xi) = \int_0^\xi (1+b(y))^{-1}dy$. As is easily verified, the equations

$$
\frac{d}{d\tau}(\varepsilon \dot I_j + I_j) = 0 \;, \quad j = 0,1 \;, \tag{3.19}
$$

hold, the solutions of which are

$$
I_j(\tau) = I_j(0) + \varepsilon \dot I_j(0)(1 - \exp(-\tau/\varepsilon)) \;, \quad j = 0,1 \;. \tag{3.20}
$$

The invariants I_j are closely related to the functions α, β defined in (3.3). Indeed, from (2.7), (1.5), we immediately see

$$
\alpha(t) = I_0(e^t - e^{t_0}) \;, \quad \alpha^* = I_0(0) + \varepsilon \dot I_0(0) \;. \tag{3.21}
$$

Moreover, defining

$$
q_1(\xi) = p_1(\xi) - \xi = -\int_0^\xi b(y)(1+b(y))^{-1}\,dy \;, \quad q_1^\pm = \lim_{\xi\to\pm\infty} q_1(\xi) \;,
$$

and using (3.3), (3.17), we find

$$\begin{aligned}\beta(t) &= I_1(e^t - e^{t_0}) - \int_{\mathbf{R}} q_1(xe^{t/2}) v(x,t)\,dx \\ &= I_1(e^t - e^{t_0}) - \alpha(t)\int_{\mathbf{R}} q_1(xe^{t/2})\varphi_0(x)\,dx - \int_{\mathbf{R}} q_1(xe^{t/2}) f^*(x,t)\,dx\ .\end{aligned} \tag{3.22}$$

Finally, differentiating (3.22), we obtain $\dot\beta(t) = e^t \dot I_1(e^t - e^{t_0}) - \int_{\mathbf{R}} q_1(x,t)w(x,t)\,dx$, hence

$$\begin{aligned}\dot\beta(t) &= e^t \dot I_1(e^t - e^{t_0}) - \dot\alpha(t)\int_{\mathbf{R}} q_1(xe^{t/2})\varphi_0(x)\,dx \\ &+ \frac{\alpha(t)}{2}\int_{\mathbf{R}} xe^{t/2} b(xe^{t/2}) a^{-1}(xe^{t/2})\varphi_0(x)\,dx - \int_{\mathbf{R}} q_1(xe^{t/2}) g^*(x,t)\,dx\ .\end{aligned} \tag{3.23}$$

These relations allow to show the following results.

Lemma 3.2. *If $\varepsilon_0 e^{-t_0} \le 1/8$ and $(v_0, w_0) \in Z^2$, the following estimates hold, for $t \ge t_0$,*

$$|\beta(t)|^2 + \int_{t_0}^t \dot\beta^2(s)\,ds \ \le\ C_2 e^{t_0}\Phi_2(\varepsilon e^{-t_0}, v_0, w_0)\ , \tag{3.24}$$

and

$$\varepsilon e^{-t}\dot\beta^2(t) \ \le\ C_3 e^{t_0} e^{-(t-t_0)/2}\Phi_2(\varepsilon e^{-t_0}, v_0, w_0)\ . \tag{3.25}$$

Moreover,

$$\lim_{t\to+\infty}\beta(t) \ =\ \beta^* \ =\ I_1(0) + \varepsilon\dot I_1(0) - \frac{1}{2}(I_0(0) + \varepsilon\dot I_0(0))(q_1^+ + q_1^-)\ , \tag{3.26}$$

and

$$|\beta(t) - \beta^*| \ \le\ C_4 e^{(1/2-\mu_1)(t-t_0)/2}(\Phi_2(\varepsilon e^{-t_0}, v_0, w_0))^{1/2}\ , \tag{3.27}$$

where $\mu_1 = \min(\mu, 1)$.

Remark. Here and in the sequel, $C_2, C_3, \dots$ denote positive constants depending only on b and ε_0. In particular, these constants are independent of the choice of t_0.

Proof. From (3.22) and (3.23), we deduce that, for $t \ge t_0$,

$$|\beta(t)| \ \le\ |I_1(e^t - e^{t_0})| + \|b\|_{L^1}\|a^{-1}\|_{L^\infty}(|\alpha(t)| + \|f^*\|_{L^1})\ , \tag{3.28}$$

and

$$\begin{aligned}|\dot\beta(t)| \leq & e^t|\dot I_1(e^t-e^{t_0})| + \|b\|_{L^1}\|a^{-1}\|_{L^\infty}(|\dot\alpha(t)|+\|g^*\|_{L^1}) \\ & + \frac{1}{2}|\alpha(t)|e^{-(\mu_1-1/2)t/2}\||x|^{\mu_1}b\|_{L^2}\|a^{-1}\|_{L^\infty}\||x|^{1-\mu_1}\varphi_0\|_{L^2} \ .\end{aligned} \tag{3.29}$$

Since

$$\begin{aligned}I_1(0) &= e^{t_0/2}\int_{\mathbf{R}} xv_0(x)\,dx + \int_{\mathbf{R}} q_1(xe^{t_0/2})v_0(x)\,dx \ , \\ \dot I_1(0) &= e^{-t_0/2}\int_{\mathbf{R}} xw_0(x)\,dx + e^{-t_0}\int_{\mathbf{R}} q_1(xe^{t_0/2})w_0(x)\,dx \ ,\end{aligned} \tag{3.30}$$

the estimates (3.24), (3.25) easily follow from (3.4), (3.18), and (3.20). Next, we remark that $\lim_{t\to+\infty} q_1(e^{t/2}x) = q_1^{\pm}$ if $x>0$ (respectively $x<0$), and thus, by the theorem of dominated convergence, $\int_{\mathbf{R}} q_1(xe^{t/2})\varphi_0(x)\,dx$ converges to $\frac{1}{2}(q_1^+ + q_1^-)$. Then (3.26) directly follows from (3.20), (3.21) and (3.18). Finally, we have

$$\begin{aligned}|\beta(t)-\beta^*| \leq & |I_1(e^t-e^{t_0}) - (I_1(0)+\varepsilon\dot I_1(0))| + |\alpha(t)-\alpha^*|\|b\|_{L^1}\|a^{-1}\|_{L^\infty} + |\alpha^*|\mathcal{I}^* \\ & + C_0(b)e^{-(t-t_0)/4}\|b\|_{L^1}\|a^{-1}\|_{L^\infty}(\Phi_1(\varepsilon e^{-t_0},v_0,w_0))^{1/2} \ ,\end{aligned} \tag{3.31}$$

where

$$\mathcal{I}^* = \Big|\int_{x<0}\varphi_0(x)(\int_{-\infty}^{xe^{t/2}} b(y)a^{-1}(y)\,dy)\,dx\Big| + \Big|\int_{x>0}\varphi_0(x)(\int_{xe^{t/2}}^{+\infty} b(y)a^{-1}(y)\,dy)\,dx\Big| \ .$$

Since $\mathcal{I}^* \leq K_2 e^{(1/2-\mu_1)t/2}\|a^{-1}\|_{L^\infty}\|x^{\mu_1}b\|_{L^2}$, the estimate (3.27) follows from (3.31), (3.4), (3.5), (3.20) and (3.30). This concludes the proof of Lemma 3.2. □

3.3. Energy Estimates

We still assume that the initial data (v_0,w_0) belong to Z^2. Like in Section 2, we study the time decay in Z^2 of the functions f, g defined in (3.2) using several families of energy functionals. We first introduce the functionals

$$\begin{aligned}\mathcal{E}_0(t) &= \frac{1}{2}(\|\mathcal{F}_x(t)\|_{L^2}^2 + \varepsilon e^{-t}\|\mathcal{G}\|_{L^2}^2) \ , \\ \mathcal{E}_1(t) &= \frac{1}{2}\|\mathcal{F}(t)\|_{L^2}^2 + \varepsilon e^{-t}\int_{\mathbf{R}}(\mathcal{F}\mathcal{G})(x,t)\,dx \ , \\ \mathcal{E}_2(t) &= \mathcal{E}_0(t)+\mathcal{E}_1(t) \ ,\end{aligned}$$

where $\mathcal{F}$, $\mathcal{G}$ are given by (3.11).

Remark. In the sequel we shall often use the following obvious estimates

$$\begin{aligned} \max(\|\mathcal{H}_0(t)\|_{L^2},\, \|x^n H_0(t)\|_{L^2},\, \|x^n h_0(t)\|_{L^2}) &\le C_5(|\alpha(t)| + |\dot\alpha(t)|)\;, \\ \max(\|\mathcal{H}_1(t)\|_{L^2},\, \|x^n H_1(t)\|_{L^2},\, \|x^n h_1(t)\|_{L^2}) &\le C_5(|\beta(t)| + |\dot\beta(t)|)\;, \end{aligned} \tag{3.32}$$

for $n = 0, 1, 2, 3, 4$. For later use, we set

$$\mathcal{M}(t) \;=\; \alpha^2(t) + \beta^2(t) + \dot\alpha^2(t) + \dot\beta^2(t)\;.$$

Lemma 3.3. *Assume that* $(v_0, w_0) \in Z^2$. *Then* $\mathcal{E}_2 \in \mathcal{C}^1([t_0, +\infty))$ *and, if* $\varepsilon_0 e^{-t_0} \le 1/8$, *the following inequalities hold,*

$$\frac{1}{4}\left(\|\mathcal{F}\|_{L^2}^2 + 2\|\mathcal{F}_x\|_{L^2}^2 + \varepsilon e^{-t}\|\mathcal{G}\|_{L^2}^2\right) \;\le\; \mathcal{E}_2 \;\le\; \frac{3}{4}\left(\|\mathcal{F}\|_{L^2}^2 + \frac{2}{3}\|\mathcal{F}_x\|_{L^2}^2 + \varepsilon e^{-t}\|\mathcal{G}\|_{L^2}^2\right)\;, \tag{3.33}$$

and

$$\begin{aligned} \dot{\mathcal{E}}_2(t) \;\le\; &-\frac{3}{2}\mathcal{E}_2(t) - \frac{1}{2}\|\mathcal{G}\|_{L^2}^2 - \frac{1}{2}\|F\|_{L^2}^2 + e^{-t/4}\|\mathcal{F}\|_{L^2}^2 \\ &+ C_6\big(e^{-\lambda' t}|\alpha(t)| + e^{-\mu_1 t/2}\|f^*\|_{X^1}\big)\|\mathcal{F}\|_{L^2} \\ &+ C_7\big(e^{-t/4}\|f_x\|_{L^2}^2 + e^{-\mu_1 t}\|f^*\|_{X^1}^2 + e^{-2\lambda' t}\mathcal{M}(t)\big)\;, \end{aligned} \tag{3.34}$$

where $\lambda' = \min(3/4, \mu/2 + 1/4)$, *and* $\mu_1 = \min(\mu, 1)$.

Proof. The inequalities (3.33) are obvious, as soon as $\varepsilon_0 e^{-t_0} \le 1/4$. Since $(\mathcal{F}, \mathcal{G}) \in \mathcal{C}^1([t_0, +\infty), Z^0)$, we have $\mathcal{E}_i(t) \in \mathcal{C}^1([t_0, +\infty))$ for $i = 0, 1, 2$. Clearly,

$$\dot{\mathcal{E}}_0(t) \;=\; \int_{\mathbf{R}} \left(-\mathcal{F}_{xx}\mathcal{F}_t + \varepsilon e^{-t}(\mathcal{G}\mathcal{G}_t - \frac{1}{2}\mathcal{G}^2)\right) dx\;.$$

Using the identity

$$\begin{aligned} \varepsilon e^{-t}(\mathcal{G}\mathcal{G}_t - \frac{1}{2}\mathcal{G}^2) - \mathcal{F}_{xx}\mathcal{F}_t \;=\; &-\mathcal{G}^2 + \frac{1}{2}\mathcal{F}\mathcal{F}_{xx} - \frac{x}{2}\mathcal{F}_x\mathcal{F}_{xx} + \varepsilon e^{-t}\frac{x}{2}\mathcal{G}\mathcal{G}_x \\ &+ \mathcal{G}\int_{-\infty}^{x} (Q_0 B)(y)\,dy - \mathcal{G}\varepsilon(e^{-t}\mathcal{H}_0 + e^{-3t/2}\mathcal{H}_1)\;, \end{aligned}$$

which follows from (3.15), we find

$$\begin{aligned} \dot{\mathcal{E}}_0(t) \;=\; &-\frac{1}{4}\|\mathcal{F}_x\|_{L^2}^2 - (1 + \frac{\varepsilon e^{-t}}{4})\|\mathcal{G}\|_{L^2}^2 + \int_{\mathbf{R}} \mathcal{G}\left(\int_{-\infty}^{x}(Q_0 B)(y)\,dy\right) dx \\ &- \int_{\mathbf{R}} \mathcal{G}\varepsilon\left(e^{-t}\mathcal{H}_0 + e^{-3t/2}\mathcal{H}_1\right) dx\;. \end{aligned} \tag{3.35}$$

Likewise using the identity

$$\begin{aligned}\varepsilon e^{-t}((\mathcal{F}\mathcal{G})_t - \mathcal{F}\mathcal{G}) + \mathcal{F}\mathcal{F}_t \;&=\; \frac{x}{2}\mathcal{F}\mathcal{F}_x + \mathcal{F}\mathcal{F}_{xx} - \frac{1}{2}\mathcal{F}^2 + \mathcal{F}\int_{-\infty}^{x} (Q_0 B)(y)\,dy \\ &\quad + \varepsilon e^{-t}\left(\mathcal{G}^2 - \mathcal{F}\mathcal{G} + \frac{x}{2}(\mathcal{F}\mathcal{G})_x\right) - \mathcal{F}\varepsilon(e^{-t}\mathcal{H}_0 + e^{-3t/2}\mathcal{H}_1)\ ,\end{aligned}$$

we obtain

$$\begin{aligned}\dot{\mathcal{E}}_1(t) \;&=\; -\,\|\mathcal{F}_x\|^2_{L^2} - \frac{3}{4}\|\mathcal{F}\|^2_{L^2} + \varepsilon e^{-t}\int_{\mathbf{R}} (\mathcal{G}^2 - \frac{3}{2}\mathcal{F}\mathcal{G})\,dx \\ &\quad + \int_{\mathbf{R}} \mathcal{F}\left(\int_{-\infty}^{x} (Q_0 B)(y)\,dy\right)dx - \int_{\mathbf{R}} \mathcal{F}\varepsilon(e^{-t}\mathcal{H}_0 + e^{-3t/2}\mathcal{H}_1)\,dx\ .\end{aligned} \tag{3.36}$$

Using the Young inequality, we deduce from (3.35) and (3.36) that, for $\varepsilon e^{-t_0} \le 1/8$,

$$\dot{\mathcal{E}}_2(t) + \frac{3}{2}\mathcal{E}_2(t) \;\le\; -\frac{1}{2}\|\mathcal{F}_x\|^2_{L^2} - \frac{1}{2}\|\mathcal{G}\|^2_{L^2} + \|\mathcal{F}\|_{L^2}\mathcal{J} + \mathcal{J}^2\ , \tag{3.37}$$

where

$$\mathcal{J} = \|\int_{-\infty}^{x} (Q_0 B)(y)\,dy)\|_{L^2} + \varepsilon(e^{-t}\|\mathcal{H}_0\|_{L^2} + e^{-3t/2}\|\mathcal{H}_1\|_{L^2})\ .$$

We recall that, by (2.14),

$$\|\int_{-\infty}^{x} (Q_0 B)(y)\,dy)\|_{L^2} \;\le\; 2(\|xB\|_{L^2} + |\int_{\mathbf{R}} B(y,t)\,dy|\;\|x\varphi_0\|_{L^2})\ . \tag{3.38}$$

Since on the one hand,

$$\|xB\|_{L^2} \;\le\; e^{-\lambda' t}|\alpha(t)|\;\||x|^{\mu_1}b\|_{L^2}\||x|^{1-\mu_1}\varphi_0'\|_{L^\infty} + e^{-\mu_1 t/2}\||x|^{\mu_1}b\|_{L^\infty}\||x|^{1-\mu_1}f_x^*\|_{L^2}\ , \tag{3.39}$$

and on the other hand

$$\begin{aligned}|\int_{\mathbf{R}} B(y,t)\,dy| \;&\le\; e^{-\lambda' t}|\alpha(t)|\;\||x|^{\mu_1}b\|_{L^2}\||x|^{-\mu_1}\varphi_0'\|_{L^2} + e^{-t}|\beta(t)|\|b\|_{L^1}\|\varphi_1'\|_{L^\infty} \\ &\quad + e^{-t/4}\|b\|_{L^2}\|f_x\|_{L^2}\ ,\end{aligned} \tag{3.40}$$

we obtain,

$$\|\int_{-\infty}^{x} (Q_0 B)(y)\,dy)\|_{L^2} \;\le\; C_6(e^{-\lambda' t}|\alpha(t)| + e^{-t}|\beta(t)| + e^{-t/4}\|f_x\|_{L^2} + e^{-\mu_1 t/2}\|f^*\|_{X^1})\ . \tag{3.41}$$

Applying the Young inequality and noting that $C_6 e^{-t/4}\|\mathcal{F}\|_{L^2}\|f_x\|_{L^2} \le e^{-t/4}(\frac{1}{2}\|\mathcal{F}\|^2_{L^2} + \frac{1}{2}C_6^2\|f_x\|^2_{L^2})$, we at once infer (3.34) from (3.37), (3.32) and (3.41). This concludes the proof of Lemma 3.3. □

We next define, for $n = 0, 1$,

$$\begin{aligned}
E_{0,n}(t) &= \frac{1}{2}\int_{\mathbf{R}} x^{2n}\left(F_x^2 + \varepsilon e^{-t}\frac{G^2}{a(xe^{t/2})}\right)\,dx\ ,\\
E_{1,n}(t) &= \int_{\mathbf{R}} \frac{x^{2n}}{a(xe^{t/2})}\left(\frac{1}{2}F^2 + \varepsilon e^{-t}FG\right)\,dx\ ,\\
E_{2,n}(t) &= E_{0,n}(t) + 2E_{1,n}(t)\ ,
\end{aligned}$$

where F, G are given by (2.13).

Lemma 3.4. *Assume that* $(v_0, w_0) \in Z^2$. *Then* $E_{2,n} \in \mathcal{C}^1([t_0, +\infty))$ *and, if* $\varepsilon_0 e^{-t_0} \leq 1/16$, *the following inequalities hold, for* $n = 0,\ 1$,

$$\begin{aligned}
\frac{1}{4\|a\|_{L^\infty}}(2\|x^nF\|_{L^2}^2 + \varepsilon e^{-t}\|x^nG\|_{L^2}^2) + \frac{1}{2}\|x^nF_x\|_{L^2}^2 &\leq E_{2,n}\\
&\leq \frac{3}{4}\|a^{-1}\|_{L^\infty}(2\|x^nF\|_{L^2}^2 + \varepsilon e^{-t}\|x^nG\|_{L^2}^2) + \frac{1}{2}\|x^nF_x\|_{L^2}^2\ ,
\end{aligned}\tag{3.42}$$

and

$$\begin{aligned}
\dot{E}_{2,n}(t) \leq &-\frac{3}{2}E_{2,n}(t) - \|x^nF_x\|_{L^2}^2 + 3(1-n)\int_{\mathbf{R}} x^{2n}a^{-1}(xe^{t/2})F^2\,dx\\
&-\frac{1}{2}\int_{\mathbf{R}} x^{2n}a^{-1}(xe^{t/2})G^2\,dx + C_8e^{-t/2}\|f_x\|_{L^2}^2\\
&+ C_9e^{-2\lambda' t}\mathcal{M}(t) + 4n(1+\|a^{-1}\|_{L^\infty})(\|F\|_{L^2}^2 + \|G\|_{L^2}^2)\ .
\end{aligned}\tag{3.43}$$

Proof. The estimate (3.42) is obvious, as soon as $\varepsilon_0 e^{-t_0} \leq 1/8$. As in the proof of Lemma 2.5, we first note that

$$E_{0,n}(t) = \frac{1}{2}\int_{\mathbf{R}} x^{2n}F_x^2(x,t)\,dx + \frac{\varepsilon}{2}\,e^{-(3/2+n)t}\int_{\mathbf{R}} \xi^{2n}\frac{\tilde{G}^2(\xi,t)}{a(\xi)}\,d\xi\ ,$$

where $\tilde{G}(\xi, t) = G(\xi e^{-t/2}, t)$. Since $F \in \mathcal{C}^1([t_0, +\infty), X^1)$ and $\tilde{G} \in \mathcal{C}^1([t_0, +\infty), Y^1)$, it follows that $E_{0,n} \in \mathcal{C}^1([t_0, +\infty))$, and a direct calculation gives

$$\dot{E}_{0,n}(t) = \int_{\mathbf{R}}\Big(-x^{2n}F_{xx}F_t - 2nx^{2n-1}F_xF_t + \frac{\varepsilon e^{-t}x^{2n}}{a(xe^{t/2})}(GG_t - \frac{x}{2}GG_x - \frac{3+2n}{4}G^2)\Big)\,dx\,.$$

Using the identity

$$\varepsilon e^{-t}a^{-1}(xe^{t/2})(GG_t - \frac{x}{2}GG_x - G^2) - F_{xx}F_t = -\frac{x}{2}F_xF_{xx} + a^{-1}(xe^{t/2})(-G^2 + G\mathcal{R})\ ,$$

where $\mathcal{R} = (A - P_0 B) - \varepsilon(e^{-t} H_0 + e^{-3t/2} H_1)$, we find

$$\begin{aligned} \dot{E}_{0,n} \;&=\; \frac{1-2n}{4}\, \|x^n F_x\|_{L^2}^2 - \Big(1 - \varepsilon e^{-t}(\frac{1-2n}{4})\Big) \int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2}) G^2 \, dx \\ &\quad + \int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2}) G\mathcal{R} \, dx - 2n \int_{\mathbf{R}} x^{2n-1} F_x G \, dx \;. \end{aligned} \tag{3.44}$$

In a similar way, using the identity

$$\varepsilon e^{-t}((FG)_t - \frac{x}{2}(FG)_x - FG) + FF_t - \frac{x}{2} FF_x \;=\; a(xe^{t/2}) FF_{xx} + \varepsilon e^{-t} G^2 + F\mathcal{R} \;,$$

we obtain,

$$\begin{aligned} \dot{E}_{1,n} \;&=\; -\|x^n F_x\|_{L^2}^2 - \frac{1+2n}{4} \int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2}) F^2 \, dx \\ &\quad + \varepsilon e^{-t} \int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2})(G^2 - \frac{1+2n}{2} FG) \, dx \\ &\quad + \int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2}) F\mathcal{R} \, dx + n(2n-1)\|x^{n-1} F\|_{L^2}^2 \;. \end{aligned} \tag{3.45}$$

Combining (3.44) and (3.45) and using the Young inequality, we find, for $n = 0, 1$,

$$\begin{aligned} \dot{E}_{2,n}(t) + \frac{3}{2} E_{2,n}(t) \;&\le\; -\|x^n F_x\|_{L^2}^2 + (1-n)(1 + \varepsilon e^{-t}) \int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2}) F^2 \, dx \\ &\quad - \frac{3}{4} \int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2}) G^2 \, dx + 2n(\|F\|_{L^2}^2 + \|G\|_{L^2}^2) \\ &\quad + \int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2})(G + 2F)\mathcal{R} \, dx \;. \end{aligned} \tag{3.46}$$

Since, for instance,

$$\begin{aligned} \int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2})(G + 2F)\mathcal{R} \, dx \;&\le (1-n) \int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2}) \left(\frac{1}{4} G^2 + F^2 + 2\mathcal{R}^2\right) dx \\ &\quad + n \int_{\mathbf{R}} a^{-1}(xe^{t/2}) \left(2G^2 + 2F^2 + \frac{5}{8} x^{4n} \mathcal{R}^2\right) dx \;, \end{aligned}$$

we deduce from (3.46) that, for $n = 0, 1$,

$$\begin{aligned} \dot{E}_{2,n}(t) + \frac{3}{2} E_{2,n}(t) \;&\le\; -\|x^n F_x\|_{L^2}^2 + 3(1-n) \int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2}) F^2 \, dx \\ &\quad - \frac{1}{2} \int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2}) G^2 \, dx + 4n(1 + \|a^{-1}\|_{L^\infty})(\|F\|_{L^2}^2 + \|G\|_{L^2}^2) + \mathcal{I}_0 \;, \end{aligned} \tag{3.47}$$

where

$$\mathcal{I}_0 \;=\; 8\|a^{-1}\|_{L^\infty}\Big(\|x^{2n}A\|_{L^2}^2 + |\int_{\mathbf{R}} B\,dx|^2\|x^{2n}\varphi_0\|_{L^2}^2 + \varepsilon^2(e^{-2t}\|x^{2n}H_0\|_{L^2}^2 + e^{-3t}\|x^{2n}H_1\|_{L^2}^2)\Big)\;.$$

Arguing as in (3.39), we can write, for $m = 0,\,1,\,2,\,3,\,4,$

$$\|x^m A\|_{L^2} \;\le\; e^{-\lambda' t}|\alpha(t)|\;\||x|^{\mu_1}b\|_{L^2}\||x|^{m-\mu_1}\varphi_0'\|_{L^\infty} + e^{-3t/4}|\beta(t)|\;\|b\|_{L^2}\|x^m\varphi_1'\|_{L^\infty}\;. \tag{3.48}$$

Finally, from (3.32), (3.40) and (3.48), we deduce that

$$\mathcal{I}_0 \;\le\; C_8 e^{-t/2}\|f_x\|_{L^2}^2 + C_9 e^{-2\lambda' t}\mathcal{M}\;, \tag{3.49}$$

where $C_8 \;=\; 24\|b\|_{L^2}^2\|a^{-1}\|_{L^\infty}\|x^{2n}\varphi_0\|_{L^2}^2$. The estimate (3.43) follows directly from (3.47) and (3.49). This concludes the proof of Lemma 3.4. $\square$

Finally, to control the decay of f, f_x and g, we introduce the following functionals, for $n = 0,\,1,\,2$:

$$\begin{aligned}
E_{3,n}(t) \;&= \frac{1}{2}\int_{\mathbf{R}} x^{2n}\left(a(xe^{t/2})f_x^2 + 2A(x,t)f_x + \varepsilon e^{-t}g^2\right)\,dx + e^{-2\lambda' t}(\alpha^2 C_{0n} + \beta^2 C_{1n})\;,\\
E_{4,n}(t) \;&= \frac{1}{2}\|x^n f\|_{L^2}^2 + \varepsilon e^{-t}\int_{\mathbf{R}} x^{2n} f g\,dx\;,\\
E_{5,n}(t) \;&= E_{3,n}(t) + 2E_{4,n}(t)\;,
\end{aligned}$$

where f, g are defined by (3.2), and C_{0n}, C_{1n} are positive constants satisfying

$$C_{0n} \;\ge\; 8\underline{a}^{-1}\||x|^{n-\mu_1}\varphi_0'\|_{L^\infty}^2\||x|^{\mu_1}b\|_{L^2}^2\;,\quad C_{1n} \;\ge\; 8\underline{a}^{-1}\|x^n\varphi_1'\|_{L^\infty}^2\|b\|_{L^2}^2\;. \tag{3.50}$$

Lemma 3.5. *Assume that* $(v_0, w_0) \in Z^2$. *Then,* $E_{5,n} \in \mathcal{C}^1([t_0, +\infty))$ *and, if* $\varepsilon_0 e^{-t_0} \le 1/16$, *the following inequalities hold, for* $n = 0,\,1,\,2,$

$$\begin{aligned}
&\frac{1}{4}(2\|x^n f\|_{L^2}^2 + \varepsilon e^{-t}\|x^n g\|_{L^2}^2 + \underline{a}\|x^n f_x\|_{L^2}^2 + 2e^{-2\lambda' t}(\alpha^2(t)C_{0n} + \beta^2(t)C_{1n}))\\
&\qquad \le\; E_{5,n}(t) \;\le\\
&\frac{3}{4}(2\|x^n f\|_{L^2}^2 + \varepsilon e^{-t}\|x^n g\|_{L^2}^2 + \|a\|_{L^\infty}\|x^n f_x\|_{L^2}^2 + 2e^{-2\lambda' t}(\alpha^2(t)C_{0n} + \beta^2(t)C_{1n}))\;,
\end{aligned} \tag{3.51}$$

and

$$
\begin{aligned}
\dot{E}_{5,n}(t) \ \le\ & -\frac{3}{2}E_{5,n}(t) - \frac{1}{4}\int_{\mathbf{R}} x^{2n} a(xe^{t/2}) f_x^2 \, dx + C_{10}e^{-t/2}\|f_x\|_{L^2}^2 \\
& -\frac{1}{2}\|x^n g\|_{L^2}^2 + 2(2-n)\|x^n f\|_{L^2}^2 + \frac{n}{8}(4\|f\|_{L^2}^2 + \|g\|_{L^2}^2) \\
& + 8n(1+\|a\|_{L^\infty})(4\|x^{n-1}f\|_{L^2}^2 + \|x^{n-1}g\|_{L^2}^2) + C_{11}e^{-2\lambda' t}\mathcal{M}(t) \ .
\end{aligned}
\tag{3.52}
$$

Proof. The estimate (3.51) follows easily from (3.48), (3.50) if $\varepsilon_0 e^{-t_0} \le 1/8$. As in Lemma 2.6, one shows that $E_{3,n}$ and $E_{4,n}$ belong to $\mathcal{C}^1([t_0,+\infty))$. A direct calculation yields

$$
\begin{aligned}
\dot{E}_{3,n}(t) \ =\ & \int_{\mathbf{R}} x^{2n}\Big(a(xe^{t/2})\big(f_x f_{xt} - \frac{x}{2}f_x f_{xx} - \frac{2n+1}{4}f_x^2\big) + \varepsilon e^{-t}\big(gg_t - \frac{1}{2}g^2\big)\Big)\, dx \\
& + \alpha \int_{\mathbf{R}} x^{2n} b(xe^{t/2})\Big(\varphi_0'\big(f_{xt} - \frac{x}{2}f_{xx} - (n+\frac{1}{2})f_x\big) - \frac{x}{2}\psi_0 f_x\Big)\, dx \\
& + e^{-t/2}\beta \int_{\mathbf{R}} x^{2n} b(xe^{t/2})\Big(\varphi_1'\big(f_{xt} - \frac{x}{2}f_{xx} - (n+1)f_x\big) - \frac{x}{2}\psi_1 f_x\Big)\, dx \\
& + e^{-t/2}\dot{\beta}\int_{\mathbf{R}} x^{2n} b(xe^{t/2})\varphi_1' f_x \, dx + \dot{\alpha}\int_{\mathbf{R}} x^{2n} b(xe^{t/2})\varphi_0' f_x\, dx \\
& + e^{-2\lambda' t}[(2\alpha\dot{\alpha} - 2\lambda'\alpha^2)C_{0n} + (2\beta\dot{\beta} - 2\lambda'\beta^2)C_{1n}] \ ,
\end{aligned}
\tag{3.53}
$$

and

$$
\dot{E}_{4,n}(t) \ =\ \int_{\mathbf{R}} x^{2n}\left(ff_t + \varepsilon e^{-t}\big((fg)_t - fg\big)\right)\, dx \ . \tag{3.54}
$$

Using the identities

$$
\begin{aligned}
& \varepsilon e^{-t}\big(gg_t - \frac{1}{2}g^2\big) - (a(xe^{t/2})f_x)_x\big(f_t - \frac{x}{2}f_x - \frac{1}{2}f\big) \ = \\
& = \ -g^2 + g[A_x + (\int_{\mathbf{R}} B\, dx)\varphi_1] + \varepsilon e^{-t}(g^2 + \frac{x}{2}gg_x) - \varepsilon g(e^{-t}h_0 + e^{-3t/2}h_1) \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
\varepsilon e^{-t}\big((fg)_t - fg\big) + ff_t \ =\ & (a(xe^{t/2})f_x)_x f + \frac{x}{2}ff_x + \frac{1}{2}f^2 + f(A_x + (\int_{\mathbf{R}} B\, dx)\varphi_1) \\
& + \varepsilon e^{-t}\big(g^2 + \frac{x}{2}(fg)_x + fg\big) - \varepsilon f(e^{-t}h_0 + e^{-3t/2}h_1) \ ,
\end{aligned}
$$

as well as $f_{xt} - \frac{x}{2}f_{xx} - f_x = g_x$, which follow from (3.9), we deduce from (3.53) and

(3.54) that, for $n = 0, 1, 2$,

$$\begin{aligned}\dot{E}_{5,n}(t) + \frac{3}{2}E_{5,n}(t) \;=\; & -\frac{n+1}{2}\int_{\mathbf{R}} x^{2n} a(xe^{t/2}) f_x^2\, dx - (n-2)\|x^n f\|_{L^2}^2 \\ & - \left(1 + (\frac{n-7}{2})\varepsilon e^{-t}\right)\|x^n g\|_{L^2}^2 - n\int_{\mathbf{R}} x^{2n} A f_x\, dx \\ & + 2\varepsilon e^{-t}(2-n)\int_{\mathbf{R}} x^{2n} f g\, dx + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 \ ,\end{aligned} \tag{3.55}$$

where

$$\begin{aligned}\mathcal{I}_1 \;&= e^{-2\lambda' t}\Big((2\alpha\dot{\alpha} + (\frac{3}{2} - 2\lambda')\alpha^2)C_{0n} + (2\beta\dot{\beta} + (\frac{3}{2} - 2\lambda')\beta^2)C_{1n}\Big) \ , \\ \mathcal{I}_2 \;&= -2n\int_{\mathbf{R}} x^{2n-1}(g + 2f)(A + a(xe^{t/2})f_x)\, dx \ , \\ \mathcal{I}_3 \;&= \int_{\mathbf{R}} x^{2n}(g + 2f)\left((\int_{\mathbf{R}} B\, dx)\varphi_1 - \varepsilon(e^{-t}h_0 + e^{-3t/2}h_1)\right) dx \ , \\ \mathcal{I}_4 \;&= -\frac{1}{2}\alpha(t)\int_{\mathbf{R}} x^{2n+1} b(xe^{t/2}) f_x \psi_0\, dx + \dot{\alpha}(t)\int_{\mathbf{R}} x^{2n} b(xe^{t/2}) f_x \varphi_0'\, dx \\ & \quad - \frac{1}{2}e^{-t/2}\beta(t)\int_{\mathbf{R}} b(xe^{t/2})(x^{2n+1}\psi_1 + x^{2n}\varphi_1') f_x\, dx \\ & \quad + e^{-t/2}\dot{\beta}(t)\int_{\mathbf{R}} x^{2n} b(xe^{t/2}) f_x \varphi_1'\, dx \ .\end{aligned}$$

Using the Young inequality, we can write,

$$\begin{aligned} n\int_{\mathbf{R}} x^{2n} A f_x\, dx \;&\le\; \frac{n}{4}\int_{\mathbf{R}} x^{2n} a(xe^{t/2}) f_x^2\, dx + n\int_{\mathbf{R}} x^{2n} a^{-1}(xe^{t/2}) A^2\, dx \ , \\ 2(2-n)\int_{\mathbf{R}} x^{2n} f g\, dx \;&\le\; (2-n)(8\|x^n f\|_{L^2}^2 + \frac{1}{8}\|x^n g\|_{L^2}^2) \ ,\end{aligned} \tag{3.56}$$

and

$$\begin{aligned}\mathcal{I}_1 \;&\le e^{-2\lambda' t}(\frac{5}{2} - 2\lambda')(C_{0n} + C_{1n})\mathcal{M}(t) \ , \\ \mathcal{I}_2 \;&\le n(\frac{1}{4}\int_{\mathbf{R}} x^{2n} a(xe^{t/2}) f_x^2\, dx + \frac{1}{4}\|x^n A\|_{L^2}^2) \\ & \quad + 8n(1 + \|a\|_{L^\infty})(\|x^{n-1} g\|_{L^2}^2 + 4\|x^{n-1} f\|_{L^2}^2) \ .\end{aligned} \tag{3.57}$$

Likewise, writing $\mathcal{I}_3 = \frac{n}{2}\mathcal{I}_3 + \frac{2-n}{2}\mathcal{I}_3$ and using the Young inequality in two different ways, we obtain

$$\begin{aligned}\mathcal{I}_3 \;\le\; & \frac{n}{8}(4\|f\|_{L^2}^2 + \|g\|_{L^2}^2) + \frac{2-n}{8}(4\|x^n f\|_{L^2}^2 + \|x^n g\|_{L^2}^2) \\ & + 3n\left(\varepsilon^2 e^{-2t}\|x^{2n} h_0\|_{L^2}^2 + \varepsilon^2 e^{-3t}\|x^{2n} h_1\|_{L^2}^2 + (\int_{\mathbf{R}} B\, dx)^2\|x^{2n}\varphi_1\|_{L^2}^2\right) \\ & + 3(2-n)\left(\varepsilon^2 e^{-2t}\|x^n h_0\|_{L^2}^2 + \varepsilon^2 e^{-3t}\|x^n h_1\|_{L^2}^2 + (\int_{\mathbf{R}} B\, dx)^2\|x^n \varphi_1\|_{L^2}^2\right) \ .\end{aligned} \tag{3.58}$$

Finally, the Young inequality and the hypothesis (H2) imply

$$
\begin{aligned}
\mathcal{I}_4 \leq & \frac{1}{4}\int_{\mathbf{R}} x^{2n} a(xe^{t/2}) f_x^2\, dx \\
& + 5\|a^{-1}\|_{L^\infty}\Big(e^{-2\lambda' t}\||x|^{\mu_1} b\|_{L^2}^2 \big(\frac{\alpha^2}{4}\||x|^{n+1-\mu_1}\psi_0\|_{L^\infty}^2 + \dot{\alpha}^2\||x|^{n-\mu_1}\varphi_0'\|_{L^\infty}^2\big) \\
& + e^{-3t/2}\|b\|_{L^2}^2\big(\frac{\beta^2}{4}(\|x^{n+1}\psi_1\|_{L^\infty}^2 + \|x^n\varphi_1'\|_{L^\infty}^2) + \dot{\beta}^2\|x^n\varphi_1'\|_{L^\infty}^2\big)\Big)\ .
\end{aligned}
\tag{3.59}
$$

Now the estimate (3.52) is a direct consequence of (3.55), (3.56), (3.57), (3.58), (3.59), (3.48) and (3.40), with $C_{10} = 9\|b\|_{L^2}^2(n\|x^{2n}\varphi_1\|_{L^2}^2 + (2-n)\|x^n\varphi_1\|_{L^2}^2)$. This concludes the proof of Lemma 3.5. □

We now summarize our results and prove Theorem 1.2. For convenience, we introduce the following notation, for $n = 0, 1, 2$,

$$
\begin{aligned}
\Sigma_{f,n}(t) &= (2-n)\big((1-n)\|\mathcal{F}(t)\|_{L^2}^2 + \|x^n F(t)\|_{L^2}^2\big) + \|x^n f(t)\|_{L^2}^2 + \|x^n f_x(t)\|_{L^2}^2\ , \\
\Sigma_{f,n}^*(t) &= (2-n)\big((1-n)\|x^n F(t)\|_{L^2}^2 + \|x^n f(t)\|_{L^2}^2\big) + \|x^n f_x(t)\|_{L^2}^2\ , \\
\Sigma_{g,n}(t) &= (2-n)\big((1-n)\|\mathcal{G}(t)\|_{L^2}^2 + \|x^n G(t)\|_{L^2}^2\big) + \|x^n g(t)\|_{L^2}^2\ .
\end{aligned}
\tag{3.60}
$$

We first consider the case $n = 0$. Let

$$
E(t) \;=\; E_{5,0}(t) + 5E_{2,0}(t) + C_{12}\mathcal{E}_2(t)\ , \tag{3.61}
$$

where $C_{12} = 2(15\|a^{-1}\|_{L^\infty} + 1)$. We remark that, due to the estimates (3.33), (3.42) and (3.51), there exists a constant $C_{13} > 1$ such that, if $\varepsilon_0 e^{-t_0} \leq 1/16$,

$$
C_{13}^{-1}(\Sigma_{f,0} + \varepsilon e^{-t}\Sigma_{g,0}) \;\leq\; E \;\leq\; C_{13}\Big(\Sigma_{f,0} + \varepsilon e^{-t}\Sigma_{g,0} + e^{-2\lambda' t}(\alpha^2 + \beta^2)\Big)\ . \tag{3.62}
$$

Below, we shall also choose t_0 so that

$$
e^{-t_0/2}(C_{10} + 5C_8) + e^{-t_0/4}C_7 C_{12} \;\leq\; \frac{1}{8}\,\underline{a}\ . \tag{3.63}
$$

Proposition 3.6. *Assume that $(v_0, w_0) \in Z^2$, that $\varepsilon_0 e^{-t_0} \leq 1/16$, and that t_0 satisfies the condition (3.63). Then there exists a positive constant C_{14} (independent of t_0) such that, for $\varepsilon \leq \varepsilon_0$, for $t \geq t_0$,*

$$
\begin{aligned}
\Sigma_{f,0}(t) + \varepsilon e^{-t}\Sigma_{g,0}(t) + \int_{t_0}^{t} e^{-2\lambda'(t-s)}(\Sigma_{f,0}^*(s) + \Sigma_{g,0}(s))\, ds \\
\leq\; C_{14}(1 + (t-t_0)^2)e^{-2\lambda'(t-t_0)}\Phi_2(\varepsilon e^{-t_0}, v_0, w_0)\ ,
\end{aligned}
\tag{3.64}
$$

where $\lambda' = \min(3/4, 1/4 + \mu/2)$.

Proof. The inequalities (3.4), (3.18), (3.33), (3.34), (3.43) and (3.52) at once imply, if t_0 satisfies (3.63),

$$\begin{aligned}\dot{E}(t) + \frac{3}{2}E(t) \leq & -\frac{1}{8}\underline{a}\|f_x\|_{L^2}^2 - \|f\|_{L^2}^2 - \|F\|_{L^2}^2 - \frac{1}{2}\|g\|_{L^2}^2 \\ & - \frac{5}{2}\|a\|_{L^\infty}^{-1}\|G\|_{L^2}^2 - \frac{C_{12}}{2}\|\mathcal{G}\|_{L^2}^2 + 4e^{-t/4}E \\ & + C_{15}(t_0)e^{-\lambda' t}\sqrt{E} + C_{16}e^{-2\lambda' t}(\mathcal{M}(t) + e^{t_0/2}\Phi_2(\varepsilon e^{-t_0}, v_0, w_0)) \ ,\end{aligned} \tag{3.65}$$

where $C_{15}(t_0) = C_{17}e^{t_0/4}(\Phi_2(\varepsilon e^{-t_0}, v_0, w_0))^{1/2}$. If we introduce the new functional $E^*(t) = \exp(2\lambda' t - 16(1 - e^{-t/4}))E(t)$, the differential inequality (3.65) implies, for $t \geq t_0$,

$$\dot{E}^*(t) \leq C_{15}(t_0)\sqrt{E^*} + C_{16}(\mathcal{M}(t) + e^{t_0/2}\Phi_2(\varepsilon e^{-t_0}, v_0, w_0)) \ . \tag{3.66}$$

Remark that, as long as $E^*(t) \geq (1 + \Phi_2(\varepsilon e^{-t_0}, v_0, w_0))e^{t_0}$ for example, the inequality (3.66) can be written as

$$\dot{E}^*(t) \leq \left(C_{15}(t_0) + \frac{C_{16}(\mathcal{M}(t) + e^{t_0/2}\Phi_2(\varepsilon e^{-t_0}, v_0, w_0))}{(1 + \Phi_2(\varepsilon e^{-t_0}, v_0, w_0))^{1/2}e^{t_0/2}}\right)\sqrt{E^*(t)} \ .$$

It follows that, for $t \geq t_0$,

$$\begin{aligned}E^*(t) \leq & K_1\Big(\frac{1}{(1 + \Phi_2(\varepsilon e^{-t_0}, v_0, w_0))e^{t_0}}\big(\int_{t_0}^t \mathcal{M}(s)\,ds\big)^2 \\ & + ((t - t_0)^2 + e^{t_0/2})e^{t_0/2}\Phi_2(\varepsilon e^{-t_0}, v_0, w_0) + E^*(t_0)\Big) \ .\end{aligned} \tag{3.67}$$

But, Lemma 3.1 and Lemma 3.2 imply

$$\big(\int_{t_0}^t \mathcal{M}(s)\,ds\big)^2 \leq C_{18}(e^{2t_0} + 1)(1 + (t - t_0)^2)(\Phi_2(\varepsilon e^{-t_0}, v_0, w_0))^2 \ . \tag{3.68}$$

Thus, we infer from (3.67) and (3.68) that

$$E(t) \leq K_2 e^{-2\lambda'(t - t_0)}\Big(E(t_0) + (1 + (t - t_0)^2)\Phi_2(\varepsilon e^{-t_0}, v_0, w_0)\Big) \ . \tag{3.69}$$

Integrating again the inequality (3.65) and using the estimate (3.69), we obtain, for $t \geq t_0$,

$$\begin{aligned}E(t) + \int_{t_0}^t & e^{-2\lambda'(t-s)}(\Sigma_{f,0}^*(s) + \Sigma_{g,0}(s))\,ds \\ & \leq K_3 e^{-2\lambda'(t - t_0)}\Big(E(t_0) + (1 + (t - t_0)^2)\Phi_2(\varepsilon e^{-t_0}, v_0, w_0)\Big) \ .\end{aligned}$$

Since $E(t_0) \le K_4 \Phi_2(\varepsilon e^{-t_0}, v_0, w_0)$, the estimate (3.64) is a direct consequence of the above inequality and (3.62). This concludes the proof of Proposition 3.6. $\square$

In the next proposition, we prove the decay of the functions f, f_x and g in the weighted norms.

Proposition 3.7. *Under the hypotheses of Proposition 3.6, there exists a positive constant C_{19} (independent of t_0) such that, for $\varepsilon \le \varepsilon_0$, for $t \ge t_0$ and for $n = 1, 2$,*

$$\begin{aligned} \Sigma_{f,n}(t) + \varepsilon e^{-t} \Sigma_{g,n}(t) + \int_{t_0}^t e^{-2\lambda'(t-s)} (\Sigma^*_{f,n}(s) + \Sigma_{g,n}(s))\, ds \\ \le C_{19}(1 + (t - t_0)^2) e^{-2\lambda'(t-t_0)} \Phi_2(\varepsilon e^{-t_0}, v_0, w_0) \ . \end{aligned} \tag{3.70}$$

Proof. If we integrate the inequality (3.43) for $n = 1$ and apply Gronwall's lemma, we obtain, due to the estimates (3.64) and (3.68),

$$\begin{aligned} E_{2,1}(t) + \int_{t_0}^t e^{-2\lambda'(t-s)} (\|xf(s)\|^2_{L^2} + \frac{1}{2\|a\|_{L^\infty}} \|xG(s)\|^2_{L^2})\, ds \\ \le K_1(1 + (t - t_0)^2) e^{-2\lambda'(t-t_0)} \Phi_2(\varepsilon e^{-t_0}, v_0, w_0) \ . \end{aligned} \tag{3.71}$$

Likewise, integrating the inequality (3.52) for $n = 1$ and using the estimates (3.64), (3.68) and (3.71), we can write, for $t \ge t_0$,

$$\begin{aligned} E_{5,1}(t) + \int_{t_0}^t e^{-2\lambda'(t-s)} (\|xf_x(s)\|^2_{L^2} + \|xg(s)\|^2_{L^2})\, ds \\ \le K_2(1 + (t - t_0)^2) e^{-2\lambda'(t-t_0)} \Phi_2(\varepsilon e^{-t_0}, v_0, w_0) \ . \end{aligned} \tag{3.72}$$

Then, (3.70) for $n = 1$ is an immediate consequence of (3.71), (3.72) and the inequalities (3.42), (3.51). In a similar way, the estimate (3.70) for $n = 2$ is proved by integrating the differential inequality (3.52) for $n = 2$ and using the estimates (3.64), (3.68), (3.70) for $n = 1$ as well as (3.51). This concludes the proof of Proposition 3.7. $\square$

Proof of Theorem 1.2 Fix $\varepsilon_0 > 0$ and t_0 satisfying the conditions of Proposition 3.6. We recall that, according to Lemma 3.1 and Lemma 3.2, $\lim_{t\to+\infty} \alpha(t) = \alpha^*$ and $\lim_{t\to+\infty} \beta(t) = \beta^*$ exist and are given by (3.5) and (3.26) respectively. Since, by (3.2),

$$\begin{aligned} \|v(t) - \alpha^*\varphi_0 - e^{-t/2}\beta^*\varphi_1\|_{X^2} &\le |\alpha(t) - \alpha^*| \|\varphi_0\|_{X^2} + e^{-t/2}|\beta - \beta^*| \|\varphi_1\|_{X^2} + \|f(t)\|_{X^2} \\ \|w(t) - \alpha^*\psi_0 - e^{-t/2}\beta^*\psi_1\|_{Y^2} &\le |\dot\alpha(t)| \|\varphi_0\|_{Y^2} + |\alpha(t) - \alpha^*| \|\psi_0\|_{Y^2} + \|g(t)\|_{Y^2} \\ &\quad + e^{-t/2}(|\dot\beta(t)| \|\varphi_1\|_{Y^2} + |\beta(t) - \beta^*| \|\psi_1\|_{Y^2}) \ . \end{aligned}$$

the estimate (1.19) is an easy consequence of (3.64), (3.70) and Lemma 3.1, Lemma 3.2. This concludes the proof of Theorem 1.2. $\square$

Remark. Theorem 1.2 is still true in the general case where the diffusion term $a(\xi)$ satisfies the hypotheses (H1) and (H2) with $\mu > 1/2$ and $a_+ \neq a_-$. The proof follows exactly the same lines, except for an important difference: if $a_+ \neq a_-$, the second eigenfunction φ_1 of the limiting operator $\tilde{\mathcal{L}}$ is no longer smooth, since $\tilde{a}(x)\varphi_1'(x)$ is continuous at the origin, see (1.14). As a consequence, we cannot split the expression $(a(xe^{t/2})v_x)_x$ using the decomposition $v = \alpha\varphi_0 + \beta e^{-t/2}\varphi_1 + f$ as in (3.9), since $a(xe^{t/2})\varphi_1'(x)$ is discontinuous at $x = 0$. The way out of this difficulty is to replace $\varphi_1(x), \psi_1(x)$ in (3.2) with approximate eigenfunctions $\varphi_1^*(x,t), \psi_1^*(x,t)$ satisfying

$$\int_{\mathbf{R}} \varphi_1^*(x,t)\,dx \;=\; 0\;, \quad \int_{\mathbf{R}} \frac{x}{\tilde{a}(x)}\varphi_1^*(x,t)\,dx \;=\; 1\;, \quad \psi_1^* \;=\; -\frac{x}{2}\varphi_{1x}^* - \varphi_1^* + \varphi_{1t}^*\;.$$

Then, defining $\alpha(t) = \int v(x,t)\,dx$, $\beta(t) = e^{t/2}\int \tilde{a}(x)^{-1} x v(x,t)\,dx$, and proceeding as in section 3.1, we obtain for f, g a system of the form (3.9), where A, B, h_0 and h_1 are modified accordingly. A possible choice for φ_1^* is $\varphi_1^*(x,t) = (1+\delta_2(t))^{-1}(\tilde{\varphi}_1(x,t) - \delta_1(t)\varphi_0(x))$, where

$$\begin{aligned} \tilde{\varphi}_1(x,t) \;&=\; \frac{1}{\sqrt{4\pi}}\,\frac{\sqrt{a_+a_-}}{\sqrt{a_+}+\sqrt{a_-}}\,\frac{x}{\hat{a}(xe^{t/2})}\,\exp\left(-\frac{x^2}{4\tilde{a}(x)}\right)\;, \\ \delta_1(t) \;&=\; \int_{\mathbf{R}} \tilde{\varphi}_1(x,t)\,dx\;, \quad 1+\delta_2(t) \;=\; \int_{\mathbf{R}} \frac{x}{\tilde{a}(x)}\tilde{\varphi}_1(x,t)\,dx\;, \end{aligned}$$

and $\hat{a}(\xi)$ is any smooth, positive function such that $\hat{a} - \tilde{a}$ has compact support in $\mathbf{R}$. With this choice, we have

$$\|\varphi_1^* - \varphi_1\|_{Y^2} + \|\psi_1^* - \psi_1\|_{Y^2} \;=\; \mathcal{O}(e^{-3t/4})\;, \quad \|\varphi_{1x}^* - \varphi_1'\|_{Y^2} \;=\; \mathcal{O}(e^{-t/4})\;,$$

as $t \to +\infty$, and this allows to obtain exactly the same estimates as in Theorem 1.2. Since φ_1^*, ψ_1^* are now time-dependent, the calculations are somewhat longer, but the modifications are inessential and can be left to the reader.

Appendix A. Spectral Properties of the Limiting Operator

In this section, we study the spectral properties of the differential operator $\tilde{\mathcal{L}}$ defined by (1.13) in the function space $Y^\ell = \mathrm{L}^2(\mathbf{R}, (1+x^2)^\ell dx)$. Although they are not explicitly used in the proofs of Theorems 1.1 or 1.2, the results here are important since they explain the form of the asymptotic expansions (1.15). We begin with the easy case where the diffusion coefficient a is constant (and equal to one).

Proposition A.1. *Let $\ell \in \mathbf{N}$ and let $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to Y^\ell$ be the differential operator defined by $\mathcal{D}(\mathcal{L}) = \{v \in Y^\ell \,|\, v'' \in Y^\ell\,,\, xv' \in Y^\ell\}$ and $\mathcal{L}v = v'' + \frac{x}{2}v' + \frac{1}{2}v$ for $v \in \mathcal{D}(\mathcal{L})$. Then the spectrum of $\mathcal{L}$ is*

$$\sigma(\mathcal{L}) = \Big\{-\frac{n}{2}\;\Big|\; n \in \mathbf{N}\Big\} \cup \Big\{z \in \mathbf{C}\;\Big|\; \mathrm{Re}\,(z) \le \frac{1}{4} - \frac{\ell}{2}\Big\}\;. \tag{A.1}$$

For all $n \in \mathbf{N}$, $n \le \ell - 1$, the eigenvalue $\lambda_n = -n/2$ is simple, and the corresponding spectral projection is given by $P_n^0(v) = (p_n^0, v)\varphi_n^0$, where φ_n^0 is the Hermite function

$$\varphi_n^0(x) = \frac{1}{\sqrt{4\pi}}\,(-1)^n \frac{d^n}{dx^n}\, e^{-x^2/4}\;, \tag{A.2}$$

and p_n^0 is the Hermite polynomial

$$p_n^0(x) = \frac{2^n}{n!}\, e^{x^2/4}(-1)^n \frac{d^n}{dx^n}\, e^{-x^2/4}\;. \tag{A.3}$$

Proof. Using the Fourier transform $\hat{v}(k) = \int v(x) e^{-ikx}\, dx$, it is equivalent to consider the operator $\hat{\mathcal{L}} : \mathcal{D}(\hat{\mathcal{L}}) \to \mathrm{H}^\ell(\mathbf{R})$ (where H^ℓ is the usual Sobolev space of order ℓ) defined by

$$\mathcal{D}(\hat{\mathcal{L}}) = \big\{w \in \mathrm{H}^\ell \,|\, k^2 w \in \mathrm{H}^\ell\,,\, kw' \in \mathrm{H}^\ell\big\}\;,\quad \hat{\mathcal{L}}w = -k^2 w - \frac{1}{2}kw'\;.$$

Given $\lambda \in \mathbf{C}$, any solution of the eigenvalue equation $\hat{\mathcal{L}}\psi = \lambda\psi$ is of the form $c_+\psi_\lambda^+ + c_-\psi_\lambda^-$, where $\psi_\lambda^+(k) = |k|^{-2\lambda}e^{-k^2}$, $\psi_\lambda^-(k) = \mathrm{sign}(k)|k|^{-2\lambda}e^{-k^2}$. If $\mathrm{Re}\,(\lambda) < 1/4 - \ell/2$, then $\psi_\lambda^\pm \in \mathcal{D}(\hat{\mathcal{L}})$, hence λ is a double eigenvalue of $\hat{\mathcal{L}}$. If $\mathrm{Re}\,(\lambda) \ge 1/4 - \ell/2$, then $\psi_\lambda^+ \in \mathcal{D}(\hat{\mathcal{L}})$ if and only if $\lambda = -n/2$ with n even, and $\psi_\lambda^- \in \mathcal{D}(\hat{\mathcal{L}})$ if and only if $\lambda = -n/2$ with n odd; in both cases, $\lambda_n = -n/2$ is a simple eigenvalue with eigenfunction $\psi_n(k) = (-ik)^n e^{-k^2}$. This shows that the discrete spectrum of $\hat{\mathcal{L}}$ is the set $\{-n/2 \,|\, n \in \mathbf{N}\} \cup \{z \in \mathbf{C} \,|\, \mathrm{Re}\,(z) < 1/4 - \ell/2\}$.

On the other hand, let $\lambda \in \mathbf{C}$ satisfying $\mathrm{Re}\,(\lambda) > 1/4 - \ell/2$, $-2\lambda \notin \mathbf{N}$, and let $g \in \mathrm{H}^\ell$. We decompose g as

$$g(k) \;=\; g_\ell(k) + \sum_{n=0}^{\ell-1} Q_n(g)\psi_n(k)\ ,$$

where $Q_n : \mathrm{H}^\ell \to \mathbf{C}$ is the continuous linear form

$$Q_n(g) \;=\; \frac{i^n}{n!}\,\frac{d^n}{dk^n}\Bigl(e^{k^2}g(k)\Bigr)\Big|_{k=0}\ .$$

Note that the coefficients $Q_n(g)$ are chosen so that $g_\ell(0) = g_\ell'(0) = \ldots = g_\ell^{(\ell-1)}(0) = 0$. Therefore, it is straightforward to verify that the function f defined by

$$f(k) \;=\; \frac{2}{|k|^{2\lambda}}\,e^{-k^2}\int_0^k |p|^{2\lambda}p^{-1}e^{-p^2}g_\ell(p)\,dp + \sum_{n=0}^{\ell-1}\frac{Q_n(g)}{\lambda+n/2}\,\psi_n(k)\ , \quad k \in \mathbf{R}\ ,$$

satisfies the resolvent equation $(\lambda - \hat{\mathcal{L}})f = g$, and that $\|f\|_{\mathrm{H}^\ell} \le C(\lambda)\|g\|_{\mathrm{H}^\ell}$ for some $C(\lambda) > 0$. This proves that the spectrum of $\hat{\mathcal{L}}$ is exactly the set (A.1) and that the linear mapping $w \mapsto Q_n(w)\psi_n$ is the spectral projection corresponding to the eigenvalue $\lambda_n = -n/2$. Returning to the original variables, we note that the inverse Fourier transform of ψ_n is the Hermite function (A.2), and that $Q_n(\hat{v}) = (p_n^0, v)$ for all $v \in Y^\ell$, where p_n^0 is the Hermite polynomial (A.3). This concludes the proof of Proposition A.1. $\square$

We now extend this result to the differential operator $\tilde{\mathcal{L}} : \mathcal{D}(\tilde{\mathcal{L}}) \to Y^\ell$ defined by

$$\mathcal{D}(\tilde{\mathcal{L}}) \;=\; \bigl\{v \in Y^\ell \,|\, (\tilde{a}v')' \in Y^\ell\,,\ xv' \in Y^\ell\bigr\}\ ,\quad \tilde{\mathcal{L}}v \;=\; (\tilde{a}v')' + \frac{x}{2}v' + \frac{1}{2}v\ , \tag{A.4}$$

where $\tilde{a}(x)$ is the piecewise constant function (1.3). Note that $\mathcal{D}(\tilde{\mathcal{L}}) \ne \mathcal{D}(\mathcal{L})$ if $a_+ \ne a_-$, since any $v \in \mathcal{D}(\tilde{\mathcal{L}})$ has to satisfy the matching condition $a_+v'(0+) = a_-v'(0-)$.

Proposition A.2. *Let $\ell \in \mathbf{N}$, and let $\tilde{\mathcal{L}}$ be the differential operator (A.4). Then the spectrum of $\tilde{\mathcal{L}}$ is the set (A.1). For all $n \in \mathbf{N}$, $n \le \ell - 1$, the eigenvalue $\lambda_n = -n/2$ is simple, with eigenfunction*

$$\begin{aligned}
\varphi_n(x) \;&=\; \frac{2}{\sqrt{a_+}+\sqrt{a_-}}\,\varphi_n^0\Bigl(\frac{x}{\sqrt{\tilde{a}(x)}}\Bigr) \quad (n \text{ even})\ ,\\
\varphi_n(x) \;&=\; \frac{2\sqrt{a_+a_-}}{\sqrt{a_+}+\sqrt{a_-}}\,\frac{1}{\sqrt{\tilde{a}(x)}}\,\varphi_n^0\Bigl(\frac{x}{\sqrt{\tilde{a}(x)}}\Bigr) \quad (n \text{ odd})\ .
\end{aligned} \tag{A.5}$$

The corresponding spectral projection is $P_n(v) = (p_n, v)\varphi_n$*, where*

$$p_n(x) \;=\; p_n^0\left(\frac{x}{\sqrt{\tilde a(x)}}\right) \quad (n\ \text{even})\;, \quad p_n(x) \;=\; \frac{1}{\sqrt{\tilde a(x)}}\,p_n^0\left(\frac{x}{\sqrt{\tilde a(x)}}\right) \quad (n\ \text{odd})\;.$$

Proof. Throughout the proof, we write $a(x)$ instead of $\tilde a(x)$ for simplicity. Using the change of variables $w(x) = v(x\sqrt{a(x)})$, we note that $(\tilde{\mathcal{L}}v)(x\sqrt{a(x)}) = (\Lambda w)(x)$, where $\Lambda : \mathcal{D}(\Lambda) \to Y^\ell$ is defined by

$$\mathcal{D}(\Lambda) \;=\; \bigl\{w \in Y^\ell\,|\,(\sqrt{a}w')' \in Y^\ell\,,\, xw' \in Y^\ell\bigr\}\;, \quad \Lambda w \;=\; \frac{1}{\sqrt{a}}(\sqrt{a}w')' + \frac{x}{2}w' + \frac{1}{2}w\;.$$

To study the spectral properties of Λ, we decompose Y^ℓ as $Y^\ell_+ \oplus Y^\ell_-$, where Y^ℓ_+, Y^ℓ_- are respectively the even and odd subspaces of Y^ℓ. If $w \in \mathcal{D}(\Lambda)$, the odd part w_- of w satisfies $w''_- \in Y^\ell$, whereas the matching condition $\sqrt{a_-}w'(0-) = \sqrt{a_+}w'(0+)$ implies that the even part w_+ of w satisfies $w'_+(0-) = -w'_+(0+) = Aw'_-(0)$, where $A = (\sqrt{a_+} - \sqrt{a_-})/(\sqrt{a_+} + \sqrt{a_-})$. Therefore, any $w \in \mathcal{D}(\Lambda)$ can be written in a unique way as $w = w_- + \tilde w_+ + Aw'_-(0)e^{-|x|}$, with $w_- \in Y^\ell_- \cap \mathcal{D}(\mathcal{L})$, $\tilde w_+ \in Y^\ell_+ \cap \mathcal{D}(\mathcal{L})$. Moreover, since Λ coincides with $\mathcal{L}$ outside the origin, the action of Λ on w is given by

$$\Lambda w \;=\; \mathcal{L}w_- + \mathcal{L}\tilde w_+ + Aw'_-(0)e^{-|x|}(3 - |x|)/2\;.$$

As a consequence, if $\lambda \in \mathbf{C}$ and $f = f_- + f_+ \in Y^\ell$, the resolvent equation $(\lambda - \Lambda)w = f$ is equivalent to the system

$$(\lambda - \mathcal{L})w_- \;=\; f_-\;, \quad (\lambda - \mathcal{L})\tilde w_+ \;=\; f_+ + Aw'_-(0)e^{-|x|}(3 - |x| - 2\lambda)/2\;.$$

Obviously, this system has a unique solution $(w_-, \tilde w_+)$ for all $f \in Y^\ell$ if and only if the equation $(\lambda - \mathcal{L})v = g$ has a unique solution v for all $g \in Y^\ell$. This proves that the spectrum of Λ (hence of $\tilde{\mathcal{L}}$) coincides with $\sigma(\mathcal{L})$. Now, since $\mathcal{L}\varphi_n^0 = \lambda_n\varphi_n^0$, it is straightforward to verify that $\tilde{\mathcal{L}}\varphi_n = \lambda_n\varphi_n$ if φ_n is defined by (A.5). Similarly, one verifies that $\tilde{\mathcal{L}}^* p_n = \lambda_n p_n$, where $\tilde{\mathcal{L}}^*$ is the adjoint of $\tilde{\mathcal{L}}$, and that $(p_n, \varphi_n) = 1$. This concludes the proof of Proposition A.2. □

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References

[BV] A.V. Babin and M.I. Vishik: *Attractors of Evolutionary Equations*, North-Holland (1989).

[Ba] J. M. Ball: Strongly Continuous Semigroups, Weak Solutions and the Variation of Constants Formula, Proc. Amer. Math. Soc. **63** (1977), 370–373.

[BK1] J. Bricmont and A. Kupiainen: Universality in Blow-up for Non Linear Heat Equations, Nonlinearity **7** (1994), 539–575.

[BK2] J. Bricmont and A. Kupiainen: Stable Non-Gaussian Diffusive Profiles, Nonlinear Anal., Theory Methods Appl. **26** (1996), 583–593.

[BKL] J. Bricmont, A. Kupiainen and G. Lin: Renormalization Group and Asymptotics of Solutions of Nonlinear Parabolic Equations, Comm. Pure Appl. Math. **47** (1994), 893–922.

[CH] Th. Cazenave and A. Haraux: *Introduction aux Problèmes d'Evolution Semi-linéaires*, Mathématiques et Applications **1**, Ellipses (1990).

[CK] J.F. Crow and M. Kimura: *An Introduction to Population Genetics Theory*, Harper and Row, Publishers, New York (1969).

[DO] S.R. Dunbar and H.G. Othmer: On a Nonlinear Hyperbolic Equation Describing Transmission Lines, Cell Movement, and Branching Random Walks, in *Nonlinear Oscillations in Biology and Chemistry*, H.G. Othmer (Ed.), Lect. Notes in Biomathematics **66**, Springer (1986).

[EKM] M. Escobedo, O. Kavian and H. Matano: Large Time Behavior of Solutions of a Dissipative Semi-linear Heat Equation, Comm. Partial Diff. Equations **20** (1995), 1427–1452.

[EWW] J.-P. Eckmann, C.E. Wayne and P. Wittwer: Geometric Stability Analysis for Periodic Solutions of the Swift-Hohenberg Equation, Comm. Math. Phys. (to appear).

[EZ] M. Escobedo and E. Zuazua: Large-time Behavior for Convection Diffusion Equations in $\mathbf{R}^N$, J. Funct. Anal. **100** (1991), 119–161.

[FG] W. Fulks and B. Guenther: Damped Wave Equations and the Heat Equation, Czechosl. Math. J. **21** (1971), 683–695.

[GKS] V.A. Galaktionov, S. P. Kudyumov and A.A. Samarskii: On Asymptotic Eigenfunctions of the Cauchy Problem for a Nonlinear Parabolic Equation, Math.

USSR Sbornik **54** (1986), 421–455.

[GR1] Th. Gallay and G. Raugel: Stability of Travelling Waves for a Damped Hyperbolic Equation, ZAMP **48** (1997), 451–479.

[GR2] Th. Gallay and G. Raugel: Scaling Variables and Stability of Hyperbolic Fronts, in preparation.

[GK] Y. Giga and R.V. Kohn: Characterizing Blowup Using Similarity Variables, Indiana J. **36** (1987), 1–40.

[Go] S. Goldstein: On Diffusion by Discontinuous Movements and the Telegraph Equation, Quart. J. Mech. Appl. Math. **4** (1951), 129–156.

[Ha] K.P. Hadeler: Reaction Telegraph Equations and Random Walk Systems, to appear in *Dynamical Systems and their Applications in Science*, S. van Strien, S. Verduyn Lunel (Eds.), Academy Colloquium, Royal Academy of the Netherlands (1995).

[HR1] J.K. Hale and G. Raugel: Upper Semicontinuity of the Attractor for a Singularly Perturbed Hyperbolic Equation, J. Differential Equations **73** (1988), 197–214.

[HR2] J.K. Hale and G. Raugel: Lower Semicontinuity of the Attractor for a Singularly Perturbed Hyperbolic Equation, J. Dyn. Diff. Equations **2** (1990), 19–67.

[HR3] J.K. Hale and G. Raugel: A Damped Hyperbolic Equation on Thin Domains, Trans. Am. Math. Soc. **329** (1992), 185–219.

[HLP] G.H. Hardy, J.E. Littlewood and G. Polya: *Inequalities*, Cambridge University Press (1934), reprinted (1964).

[HV1] M.A. Herrero and J.J.L. Velazquez: Blow-up Profiles in One-dimensional, Semilinear Parabolic Equations, Comm. in P.D.E. **17** (1992), 205–219.

[HV2] M.A. Herrero and J.J.L. Velazquez: Blow-up Behavior of One-dimensional Semilinear Parabolic Equations, Ann. Inst. H. Poincaré **10** (1993), 131–189.

[HL1] L. Hsiao and T. Liu: Convergence to Nonlinear Diffusion Waves for Solution of a System of Hyperbolic Conservation Laws with Damping, Comm. Math. Phys. **143** (1992), 599–605.

[HL2] L. Hsiao and T. Liu: Nonlinear Diffusive Phenomena of Nonlinear Hyperbolic Systems, Chin. Ann. of Math. **14B:4** (1993), 465–480.

[Kac] M. Kac: A Stochastic Model Related to the Telegrapher's Equation, Rocky Mountain J. Math. **4** (1974), 497–509.

[Ka] O. Kavian: Remarks on the Large Time Behavior of a Nonlinear Diffusion Equation, Ann. Inst. Henri Poincaré **4** (1987), 423–452.

[LFDFH] S.G. Lachenmann, G. Filatrella, T. Doderer, J.C. Fernandez and R.P. Huebener: Soliton Dynamics in Two-dimensional Josephson Tunnel Junctions, Physical Review B **48** (1993), 16623–16629.

[LDLH] A. Laub, T. Doderer, S.G. Lachenmann and R.P. Huebener: Lorentz Contraction of Flux Quanta Observed in Experiments with Annular Josephson Tunnel Junctions, Physical Review Letters **75** (1995), 1372–1375.

[MZ] F. Merle and H. Zaag: Optimal Estimates for Blow-up Rate and Behavior for Nonlinear Heat Equations, Comm. Pure Appl. Math. (to appear).

[MSM] X. Mora and J. Solà Morales: The Singular Limit of Semilinear Damped Equations, J. Differential Equations **78** (1989), 262–307.

[Mu] J.D. Murray: *Mathematical Biology* 2nd ed., Biomathematics **19**, Springer Verlag (1993).

[Ni] K. Nishihara: Convergence Rates to Nonlinear Diffusion Waves for Solutions of System of Hyperbolic Conservation Laws with Damping, J. Differential Equations **131** (1996), 171–188.

[Ok] A. Okubo: *Diffusion and Ecological Problems: Mathematical Models*, Biomathematics **10**, Springer Verlag (1980).

[Pa] A. Pazy: *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. **44**, Springer, New-York (1983).

[Ra] G. Raugel: Dynamics of Partial Differential Equations on Thin Domains, in *Dynamical Systems*, CIME Course, R. Johnson (Ed.), Lect. Notes in Mathematics **1609**, Springer (1995).

[Sch] G. Schneider: Diffusive Stability of Spatial Periodic Solutions of the Swift-Hohenberg equation, Comm. Math. Phys. **178** (1996), 679–702.

[Ta] G.I. Taylor: Diffusion by Continuous Movements, Proc. London Math. Soc. **20** (1920), 196–212.

[Wa] C. E. Wayne: Invariant Manifolds for Parabolic Partial Differential Equations in Unbounded Domains, Arch. Rat. Mech. Anal. **138** (1997), 279–306.

[Ya] E. Yanagida: Existence of Stable Stationary Solutions of Scalar Reaction Diffusion Equations in Thin Tubular Domains, Applicable Anal. **36** (1990), 171–188.

[Za] E. Zauderer: *Partial Differential Equations of Applied Mathematics*, New York, John Wiley (1983).

[Zl1] M. Zlámal: Sur l'Equation des Télégraphistes avec un Petit Paramètre, Att. Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. **27** (1959), 324–332.

[Zl2] M. Zlámal: The Mixed Problem for Hyperbolic Equations with a Small Parameter, Czechosl. Math. J. **10** (1960), 83–122.

[Zl3] M. Zlámal: The Parabolic Equations as a Limiting Case of Hyperbolic and Elliptic Equations, in *Differential Equations and Their Applications*, Proc. Conf., Prague, Academic Press, New York (1963).

[Zu] E. Zuazua: A Dynamical System Approach to the Self-similar Large Time Behavior in Scalar Convection-diffusion Equations, J. Differential Equations **108** (1994), 1–35.