α-Hölder linearization

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Abstract

A well known theorem of Hartman-Grobman says that a C^2 diffeomorphism $f : \mathbf{R^n} \to \mathbf{R^n}$ with a hyperbolic fixed point at 0 can be topologically conjugated to the linear diffeomorphism $L = df(0)$ (in a neighborhood of 0). On the other hand, if a non-planar map has resonance, then linearization may not be C^1. A counter-example is due to P. Hartman (see [H2]). In this paper we will show that for any $\alpha \in (0, 1)$ there exists an α-Hölder linearization in a neighborhood of 0 for the counter-example of Hartman. No resonance condition will be required. A linearization of a more general map will be discussed.

1 Introduction

Let f be a C^3 diffeomorphism with a hyperbolic fixed point at 0 and linear part L. We will say that f *can be linearized* if there exist a neighborhood U of 0 and homeomorphism $h : U \to h(U)$ with $h(0) = 0$ such that $h \circ f = L \circ h$ on U.

A well known theorem of Hartman-Grobman says that such a diffeomorphism can be topologically linearized (locally). See [H1]. A simpler proof can be found in [P].

However, if linearization is only C^0, it is inconvenient for various applications. Of course, if we will make additional assumptions on resonance, we will obtain smooth linearization. (See [S]). But this technical assumption is unnatural for applications.

P. Hartman showed ([**?**]) that in the planar case, a C^2 diffeomorphism with a hyperbolic fixed point can be C^1 linearized.

G. Belitskii and S. van Strien ([B], [ABZ], [v-S]) have considered this problem in higher dimensions and proved that for some α less then 1 there exists a local linearization in the α-Hölder class. In their works this α depend on the resonance.

Hartman's counter-example ([H2]) shows that a non-planar map may not be C^1 linearizable, and bounds the search for linearization.

In this paper, first, we will show that for any $\alpha \in (0, 1)$ there exists an α-Hölder linearization of the Hartman map in a neighborhood of 0. No resonance condition will be required.

Then, we will deduce the same conclusion for a more general map, i.e., we are going to prove the following:

Let f be a C^3 diffeomorphism of $\mathbf{R^3}$ into itself with a hyperbolic fixed point at 0,

$$f(x) = Lx + \mu(x),\ \ x = (x_1, x_2, x_3),$$

such that L is diagonal and f leaves invariant x and y axis, and (x, y) and (y, z) planes.

Then, for any α arbitrarily close to but less than 1, there exists a neighborhood $U \subset \mathbf{R^3}$ of the origin and a homeomorphism $R_\alpha : U \to R_\alpha(U)$ such that $R_\alpha \in D^\alpha$ (see Definition 1 below) and

$$R_\alpha \circ f = L \circ R_\alpha.$$

We conclude the paper with the following conjecture:

Let f be a C^3 diffeomorphism of $\mathbf{R^n}$ into itself with a hyperbolic fixed point at 0,

$$f(x) = Lx + \lambda(x),$$

such that all first-order partial derivatives of λ at 0 are zero.

Then for any α arbitrarily close to but less than 1 there exists a neighborhood $U \subset \mathbf{R^n}$ of the origin and a homeomorphism $R_\alpha : U \to R_\alpha(U)$ such that $R_\alpha \in D^\alpha$ and

$$R_\alpha \circ f = L \circ R_\alpha,$$

(i.e., our result holds even without the hypotheses on μ.)

This research was motivated by the study of degenerate homoclinic crossings of stable and unstable manifolds, see [R].

2 Linearization Theorem

In 1960, P. Hartman presented an example ([H2]) of a map (flow) with resonance (see the map f in Theorem 1) that does not have a C^1 linearization. First, we are going to show that this map can be linearized in the class D^α (see the definition below) with α less than but arbitrarily close to 1.
Using this fact, we will show how to obtain the same result for a more general map with a hyperbolic fixed point (with no resonance assumption). See Theorem **??**.

Definition 1 *Let Φ be a homeomorphism from $\mathbf{R^n}$ to $\mathbf{R^n}$, $\Phi(0) = 0$. We say that $\Phi \in D^\alpha$, $(0 < \alpha < 1)$, if there exists a positive constant A such that*

$$\frac{|\Phi(x) - \Phi(y)|}{|x-y|^\alpha} \leq A,$$

and

$$\frac{|\Phi^{-1}(x) - \Phi^{-1}(y)|}{|x-y|^\alpha} \leq A,$$

for all sufficiently small $|x| \neq |y|$, x, y different from 0.

The theorem presented below for Hartman's map will be generalized in the next section.

Theorem 1 *Let f be the mapping of R^3 into itself*

$$f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} ax \\ ac(y + xz) \\ cz \end{pmatrix}$$

where $c < 1 < ac < a$ are constants.
Then for any α arbitrarily close to 1 ($\alpha < 1$) there exists a neighborhood $U \in \mathbf{R^3}$ and a homeomorphism $R_\alpha : U \to R_\alpha(U)$ such that $R_\alpha \in D^\alpha$ and

$$R_\alpha \circ f = L \cdot R_\alpha$$

where L is the linear part of f at 0,

$$L = \begin{bmatrix} a & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Proof:
In this proof we will use the following notations:

$$\lambda(x, y, z) = \begin{pmatrix} 0 \\ acxz \\ 0 \end{pmatrix},$$

$\mathbf{R}_+ = (\mathbf{x}, \mathbf{y})-$plane and $\mathbf{R}_- = \mathbf{z}-$axis.
Then $\mathbf{R^3} = \mathbf{R}_+ \oplus \mathbf{R}_-$.
Introduce the following pseudonorm:

$$\| g \| = \| g \|_\delta = \sup \left\{ \frac{| g(q_1) - g(q_2) |}{| x_1 z_1 - x_2 z_2 |^{1-\delta}} : \ x_1 z_1 \neq x_2 z_2, \quad q_1, q_2 \in \mathbf{R^3} \right\}$$

where $q_i = (x_i, y_i, z_i), \ i = 1, 2$, and where $g : \mathbf{R^3} \to \mathbf{R^3}$.
Let E_δ be the following space:
$E_\delta = \{g : \mathbf{R^3} \to \mathbf{R^3}$, g is continuous, g is uniformly bounded, $g(0) = 0$, and $\| g\|_\delta \leq 1\}$.
Its easy to show that the pseudonorm introduced above is actually a norm on the space
$\{g : \mathbf{R^3} \to \mathbf{R^3}$, g is continuous, g is uniformly bounded, $g(0) = 0$, and $\| g\|_\delta$ is bounded $\}$.
For $0 < \epsilon \leq 1$ define

$$\lambda_\epsilon(q) = \beta \left(\frac{|x \cdot z|}{\epsilon} \right) \cdot \lambda(q), \ q = (x, y, z),$$

where $\beta : \mathbf{R} \to [\mathbf{0}, \mathbf{1}]$ is C^∞, $Lip(\beta)$ is bounded, and $0 \leq \beta(t) \leq 1$ for all t, $\beta(t) = 0$ for $|t| \geq 1$ and $\beta(t) = 1$ for $|t| \leq \frac{1}{2}$.
Now we can define

$$f_\epsilon(q) = Lq + \lambda_\epsilon(q).$$

Existence of a homeomorphism $R_\alpha = id + g$ is equivalent (see Lemma 4) to the existence of two fixed points g and g' of the operators

$$P, P' : E_\delta \mapsto E_\delta,$$

such that

$$\begin{cases} (Pg)_+ &= L_+^{-1}[g_+ \circ f_\epsilon + \lambda_{\epsilon+}] \\ (Pg)_- &= L_- g_- \circ f_\epsilon^{-1} \end{cases}$$

and

$$\begin{cases} (P'g')_+ &= L_+^{-1}[g'_+ \circ L - \lambda_{\epsilon+} \circ (id + g')] \\ (P'g')_- &= L_- g'_- \circ L^{-1} \end{cases}$$

Here $L_+, L_-, g_+, g_-, g'_+, g'_-, \lambda_+, \lambda_-$ are the projections on the corresponding $\mathbf{R}_+$ and $\mathbf{R}_-$ spaces ;
$f_\epsilon^{\pm 1}$ is defined by the equations

$$f_\epsilon(q) = \left(\begin{array}{c} ax \\ (ac)\left(y + \gamma^+\left(\frac{|xz|}{\epsilon}\right) xz\right) \\ cz \end{array} \right),$$

$$f_\epsilon^{-1}(q) = \left(\begin{array}{c} a^{-1}x \\ (ac)^{-1}\left(y + \gamma^-\left(\frac{|xz|}{\epsilon}\right) xz\right) \\ c^{-1}z \end{array} \right),$$

where $\gamma^+(t) = \beta(t)$, $\gamma^-(t) = \beta\left(\frac{t}{ac}\right)$

Lemma 4 will show that the map R_α is a homeomorphism and it belongs to D^α, if $g, g' \in E_\delta$ (here α is $1 - \delta$).
For simplicity, we will denote E_δ by E.
To prove the existence of the fixed points g, g', it is suffices to prove that P and P' are contracting operators, from the metric space E into itself, and E is complete with respect to the norm $\| \, \|_\delta$, defined above. (The proofs of these lemmas will be presented in the next section.) Then, by the Fixed Point Theorem there exist fixed points $g \in E_\delta$ and $g' \in E_\delta$ such that $Pg = g$ and $Pg' = g'$.
Now we can show that these g and g' are the maps we sought.

$$g = Pg = L^{-1}(g \circ f_\epsilon + \lambda_\epsilon),$$

or

$$L(id+g) = f_\epsilon \circ (id+g).$$

Similarly,

$$(id+g)\circ L = (id+g)\circ f_\epsilon.$$

In the $\frac{\epsilon}{2}$ neighborhood of the fixed point, $f_\epsilon = f$. By Lemma 4 $id+g$ and $id+g'$ are inverse of each other and of the class D^α.
Therefore, locally f can be conjugated to its linear part in the class D^α.
This completes the proof of Theorem 1.
QED

3 Proofs of Lemmas

Lemma 1

$$P: \begin{array}{ccc} E & \hookrightarrow & E \\ g & \mapsto & ((Pg)_+, (Pg)_-) \end{array}$$

$$P': \begin{array}{ccc} E & \hookrightarrow & E \\ g' & \mapsto & ((P'g')_+, (P'g')_-) \end{array}$$

Proof:
We first treat the P_- component. It is easier.

$$\frac{|(Pg)_-(q_1) - (Pg)_-(q_2)|}{|x_1z_1 - x_2z_2|^{1-\delta}} \leq$$

$$\|L_-\| \cdot \frac{|g_- \circ f_\epsilon^{-1}(q_1) - g_- \circ f_\epsilon^{-1}(q_2)|}{|(f_\epsilon^{-1}(q_1))_1(f_\epsilon^{-1}(q_1))_3 - (f_\epsilon^{-1}(q_2))_1(f_\epsilon^{-1}(q_2))_3|^{1-\delta}} \cdot$$

$$\left(\frac{|(f_\epsilon^{-1}(q_1))_1(f_\epsilon^{-1}(q_1))_3 - (f_\epsilon^{-1}(q_2))_1(f_\epsilon^{-1}(q_2))_3|}{|x_1z_1 - x_2z_2|}\right)^{1-\delta} \leq$$

$$c \cdot 1 \cdot \left((ac)^{-1}\right)^{1-\delta} = c^{\delta} \cdot a^{\delta-1} < 1.$$

Now for P_+ component,

$$\frac{|(Pg)_+(q_1) - (Pg)_+(q_2)|}{|x_1 z_1 - x_2 z_2|^{1-\delta}} \leq$$

$$\|L_+^{-1}\| \cdot \frac{|g_+ \circ f_\epsilon(q_1) - g_+ \circ f_\epsilon(q_2)|}{|(f_\epsilon(q_1))_1 (f_\epsilon(q_1))_3 - (f_\epsilon(q_2))_1 (f_\epsilon(q_2))_3|^{1-\delta}} \cdot$$

$$\left(\frac{|(f_\epsilon(q_1))_1 (f_\epsilon(q_1))_3 - (f_\epsilon(q_2))_1 (f_\epsilon(q_2))_3|}{|x_1 z_1 - x_2 z_2|}\right)^{1-\delta} +$$

$$\|L_+^{-1}\| \cdot \frac{|\lambda_{\epsilon+}(q_1) - \lambda_{\epsilon+}(q_2)|}{|x_1 z_1 - x_2 z_2|^{1-\delta}} \leq$$

$$\max\{\frac{1}{a}, \frac{1}{ac}\} \cdot 1 \cdot (ac)^{1-\delta} + \left(\epsilon^{\delta} - small\ term\right)$$

$$\leq \left(\frac{1}{ac}\right)^{\delta} + \left(\epsilon^{\delta} - small\ term\right) < 1.$$

By $\epsilon^{\delta} - small$ we mean functions of the class $o(\epsilon^{\delta})$, as $\epsilon \to 0$ (here δ is a fixed positive number).
The proof, that $\frac{|\lambda_{\epsilon+}(q_1) - \lambda_{\epsilon+}(q_2)|}{|x_1 z_1 - x_2 z_2|^{1-\delta}}$ is $\epsilon^{\delta}-$ small, involves several special cases and is given below.

1. Suppose that $|x_1 z_1| < \epsilon$ and $|x_2 z_2| < \epsilon$.
Then

$$\frac{|\lambda_\epsilon(q_1) - \lambda_\epsilon(q_2)|}{|x_1 z_1 - x_2 z_2|^{1-\delta}} =$$

$$\frac{|x_1 z_1 \beta\left(\frac{|x_1 z_1|}{\epsilon}\right) - x_2 z_2 \beta\left(\frac{|x_2 z_2|}{\epsilon}\right)|}{|x_1 z_1 - x_2 z_2|^{1-\delta}} \leq$$

$$\frac{|x_1z_1 - x_2z_2| \cdot \beta\left(\frac{|x_1z_1|}{\epsilon}\right) + |x_2z_2| \cdot |\beta\left(\frac{|x_1z_1|}{\epsilon}\right) - \beta\left(\frac{|x_2z_2|}{\epsilon}\right)|}{|x_1z_1 - x_2z_2|^{1-\delta}} \leq$$

$$|x_1z_1 - x_2z_2|^{\delta} \cdot 1 + |x_2z_2| \cdot \frac{Lip(\beta)\frac{1}{\epsilon}||x_1z_1| - |x_2z_2||}{|x_1z_1 - x_2z_2|^{1-\delta}} \leq$$

$$|x_1z_1 - x_2z_2|^{\delta} + Lip(\beta) \cdot |x_1z_1 - x_2z_2|^{\delta} \leq \kappa\epsilon^{\delta}$$

2. Suppose that $|x_1z_1| < \epsilon$ and $|x_2z_2| \geq \epsilon$.
(i). $|x_1z_1 - x_2z_2| \geq \epsilon$.
Then

$$\frac{|\lambda_\epsilon(q_1) - \lambda_\epsilon(q_2)|}{|x_1z_1 - x_2z_2|^{1-\delta}} =$$

$$\frac{|x_1z_1|\beta\left(\frac{|x_1z_1|}{\epsilon}\right)}{|x_1z_1 - x_2z_2|^{1-\delta}} \leq$$

$$\frac{\epsilon \cdot 1}{\epsilon^{1-\delta}} = \epsilon^{\delta}$$

(ii). $|x_1z_1 - x_2z_2| < \epsilon$.
Then

$$\frac{|\lambda_\epsilon(q_1) - \lambda_\epsilon(q_2)|}{|x_1z_1 - x_2z_2|^{1-\delta}} =$$

$$\frac{|x_1z_1|\beta\left(\frac{|x_1z_1|}{\epsilon}\right)}{|x_1z_1 - x_2z_2|^{1-\delta}} =$$

$$\frac{|x_1z_1|\left(\beta\left(\frac{|x_1z_1|}{\epsilon}\right) - \beta\left(\frac{|x_2z_2|}{\epsilon}\right)\right)}{|x_1z_1 - x_2z_2|^{1-\delta}} \leq$$

$$\frac{|x_1 z_1| \cdot Lip(\beta) \frac{\big||x_1 z_1| - |x_2 z_2|\big|}{\epsilon}}{|x_1 z_1 - x_2 z_2|^{1-\delta}} \leq$$

$$Lip(\beta) \cdot \frac{|x_1 z_1| \cdot |x_1 z_1| - |x_2 z_2|}{\epsilon |x_1 z_1 - x_2 z_2|^{\delta}} \leq$$

$$Lip(\beta) \cdot |x_1 z_1 - x_2 z_2|^{\delta} \leq$$

$$Lip(\beta) \cdot \epsilon^{\delta}.$$

Thus, we have proved, that

$$\|Pg\|_{\delta} = \sup \left\{ \frac{|\ (Pg)(q_1) - (Pg)(q_2)\ |}{|\ x_1 z_1 - x_2 z_2|^{1-\delta}} :\ x_1 z_1 \neq x_2 z_2, \quad q_1, q_2 \in \mathbf{R^3} \right\} < 1$$

if $\|g\| < 1$.
Obviously, Pg is continuous, uniformly bounded and $Pg(0) = 0$.
Therefore $Pg : E \to E$.

The similar proof for the operator P' is the following:

$$\frac{|(P'g')_+(q_1) - (P'g')_+(q_2)|}{|x_1 z_1 - x_2 z_2|^{1-\delta}} \leq$$

$$\|L_+^{-1}\| \cdot \frac{|g'_+(Lq_1) - g'_+(Lq_2)|}{|ax_1 \cdot cz_1 - ax_2 \cdot cz_2|^{1-\delta}} \cdot$$

$$\left(\frac{|ax_1 \cdot cz_1 - ax_2 \cdot cz_2|}{|x_1 z_1 - x_2 z_2|} \right)^{1-\delta} +$$

$$\|L_+^{-1}\| \cdot \frac{|\lambda_{\epsilon+}(q_1 + g(q_1)) - \lambda_{\epsilon+}(q_2 + g(q_2))|}{|x_1 z_1 - x_2 z_2|^{1-\delta}} \leq$$

$$\max\{\frac{1}{a}, \frac{1}{ac}\} \cdot 1 \cdot (ac)^{1-\delta} + \Big(\epsilon^\delta - small\ term\Big)$$

$$\leq \left(\frac{1}{ac}\right)^\delta + \Big(\epsilon^\delta - small\ term\Big) < 1.$$

(The proof that $\frac{|\lambda_{\epsilon+}(q_1+g(q_1))-\lambda_{\epsilon+}(q_2+g(q_2))|}{|x_1z_1-x_2z_2|^{1-\delta}}$ $\epsilon^\delta-$ small is very similar to the proof, that $\frac{|\lambda_{\epsilon+}(q_1)-\lambda_{\epsilon+}(q_2)|}{|x_1z_1-x_2z_2|^{1-\delta}}$ $\epsilon^\delta-$ small.)
Obviously,

$$\frac{|(P'g')_+(q_1) - (P'g')_+(q_2)|}{|x_1z_1 - x_2z_2|^{1-\delta}} \leq$$

$$c \cdot 1 \cdot \Big((ac)^{-1}\Big)^{1-\delta} = c^\delta \cdot a^{\delta-1} < 1.$$

We have now completed the proof that P and P' map E into itself.
QED

Lemma 2 *Let P and P' be the operators defined above.*
Then, P and P' contract.

Proof:
$\| (Pg)_- - (Ph)_- \|_\delta$ is the supremum of

$$\frac{|L_- \cdot (g_-(q_1) - g_-(q_2) - h_-(q_1) - h_-(q_2))|}{|x_1z_1 - x_2z_2|^{1-\delta}} \leq$$

$$\|L_-\| \cdot \frac{|g_- \circ f_\epsilon^{-1}(q_1) - g_- \circ f_\epsilon^{-1}(q_2) - h_- \circ f_\epsilon^{-1}(q_1) - h_- \circ f_\epsilon^{-1}(q_2)|}{|(f_\epsilon^{-1}(q_1))_1(f_\epsilon^{-1}(q_1))_3 - (f_\epsilon^{-1}(q_2))_1(f_\epsilon^{-1}(q_2))_3|^{1-\delta}} \cdot$$

$$\left(\frac{|(f_\epsilon^{-1}(q_1))_1(f_\epsilon^{-1}(q_1))_3 - (f_\epsilon^{-1}(q_2))_1(f_\epsilon^{-1}(q_2))_3|}{|x_1z_1 - x_2z_2|}\right)^{1-\delta} \leq$$

$$\|L_-\| \cdot \left(\frac{|(f_\epsilon^{-1}(q_1))_1(f_\epsilon^{-1}(q_1))_3 - (f_\epsilon^{-1}(q_2))_1(f_\epsilon^{-1}(q_2))_3|}{|x_1z_1 - x_2z_2|}\right)^{1-\delta} \cdot \|g - h\|_\delta$$

as q_1 and q_2 vary in $\mathbf{R^3}$, $x_1 z_1 \neq x_2 z_2$.
Let α^+ be the supremum of

$$\|L_-\| \cdot \left(\frac{|(f_\epsilon^{-1}(q_1))_1 (f_\epsilon^{-1}(q_1))_3 - (f_\epsilon^{-1}(q_2))_1 (f_\epsilon^{-1}(q_2))_3|}{|x_1 z_1 - x_2 z_2|} \right)^{1-\delta}$$

as q_1 and q_2 vary in $\mathbf{R^3}$, $x_1 z_1 \neq x_2 z_2$.
Then

$$\alpha^+ \leq c \left(\frac{1}{ac} \right)^{1-\delta} < 1.$$

This gives the estimate

$$\| (Pg)_- - (Ph)_- \|_\delta \leq \alpha^+ \|g - h\|_\delta.$$

Similarly one can show that

$$\| (Pg)_+ - (Ph)_+ \|_\delta \leq \alpha^- \|g - h\|_\delta,$$

with

$$\alpha^- \leq \frac{1}{ac} (ac)^{1-\delta} = \left(\frac{1}{ac} \right)^{\delta} < 1,$$

for any fixed $\delta > 0$.
Let

$$\alpha = \max \left(c \left(\frac{1}{ac} \right)^{1-\delta}, \left(\frac{1}{ac} \right)^{\delta} \right).$$

Then

$$\|Pg - Ph\|_\delta \leq \alpha \|g - h\|_\delta,$$

with $\alpha < 1$.
Therefore P contracts.
The proof for the operator P' is similar to the one above.
QED

Lemma 3 *E is complete with respect to $\|.\|_\delta$.*

Proof:

Take a sequence $\{g_n\} \subset E$, $n = 1, 2, ...$, which is Cauchy with respect to $||.||_\delta$.
Let C_0 be the set of uniformly bounded continuous functions.
Then

$$||g_n - g_m||_{C_0} =$$

$$\sup_{q_1 \in \mathbf{R^3}} |g_n(q_1) - g_m(q_1)| \leq$$

$$\sup_{q_1 \in \mathbf{R^3}} \frac{|g_n(q_1) - g_m(q_1)|}{|x_1 z_1|^{1-\delta}} =$$

$$\sup_{q_1 \in \mathbf{R^3}} \frac{|g_n(q_1) - g_m(q_1) - g_n(0) + g_m(0)|}{|x_1 z_1 - 0 \cdot 0|^{1-\delta}} \leq$$

$$\sup_{q_1, q_2 \in \mathbf{R^3}} \frac{|g_n(q_1) - g_m(q_1) - g_n(q_2) + g_m(q_2)|}{|x_1 z_1 - x_2 z_2|^{1-\delta}} < \epsilon,$$

for all $x_1 z_1 \neq x_2 z_2$.
Therefore $\{g_n\}$ is Cauchy in C_0 with respect to the supremum norm and converges in C_0 to a limit g.

Fix an arbitrarily small $\epsilon > 0$. Then there exists N such that for all $n, m > N$

$$\sup_{\substack{x_1 z_1 \neq x_2 z_2 \\ q_1, q_2 \in \mathbf{R^3}}} \frac{|g_n(q_1) - g_m(q_1) - g_n(q_2) + g_m(q_2)|}{|x_1 z_1 - x_2 z_2|^{1-\delta}} < \epsilon.$$

Taking the limit as m approaches ∞, we will obtain

$$\frac{|g_n(q_1) - g(q_1) - g_n(q_2) + g(q_2)|}{|x_1 z_1 - x_2 z_2|^{1-\delta}} \leq \epsilon$$

for all $q_1, q_2 \in \mathbf{R^3}$, $x_1 z_1 \neq x_2 z_2$.
This implies that $||g_n - g||_\delta \leq \epsilon$.
One can prove directly (without using property of the norm), that

$$\|g\|_\delta \leq \|g_n - g\|_\delta + \|g_n\|_\delta.$$

Then $\|g\|_\delta \leq \epsilon + 1$ for arbitrarily small ϵ.
Thus, $\|g\|_\delta \leq 1$.

Clearly, g is continuous, bounded and $g(0) = 0$. Therefore E is complete with respect to $\|.\|_\delta$.
QED

Thus, we have contracting operators P and P' that act on the complete space E_δ and map this space into itself. (See Lemma 1, Lemma 2, Lemma 3). By the Fixed Point Theorem there exist fixed points $g \in E_\delta$ and $g' \in E_\delta$ such that $Pg = g$ and $Pg' = g'$.

We now propose that $id + g$ and $id + g'$ are the locally conjugating homeomorphisms we sought.

Lemma 4 *Assume*

$$(id + g) \circ f_\epsilon = L \cdot (id + g), \quad g \in E_\delta$$

and

$$(id + g') \circ L = f_\epsilon \circ (id + g'), \quad g' \in E_\delta$$

Then the $R_\alpha = id + g$ *(*$1 - \delta = \alpha$*) is a homeomorphism,* $R_\alpha \in D^\alpha$*, and* $R_\alpha^{-1} = id + g'$*.*

Proof:

For simplicity we will denote R_α by R.
Then $R = id + g$, g is in E_δ.
Let $h' = id + g'$, g' is in E_δ.
First we need to show that $R^{-1} = id + g' = h'$.
To show this, first we will prove that $h \circ h' = id$.
We will show that $L \circ (h \circ h') = (h \circ h') \circ L$.

$$L \circ (h \circ h') = h \circ f \circ f^{-1} \circ h' \circ L = (h \circ h') \circ L.$$

By uniqueness of the globally conjugating map, $h \circ h' = id$.
Similarly, $h' \circ h = id$.
This implies that $R^{-1} = id + g' = h'$.

To demonstrate, that $R_\alpha \in D^\alpha$ we will prove that $\frac{|g(q_1)-g(q_2)|}{|q_1-q_2|^{1-\delta}}$ and $\frac{|g'(q_1)-g'(q_2)|}{|q_1-q_2|^{1-\delta}}$ are bounded above for sufficiently small $|q_1| \neq |q_2|$.
Suppose that $|q_1| \leq 1$, $|q_1| \leq 1$.

$$|x_1z_1 - x_2z_2| \leq |x_1| \cdot |z_1 - z_2| + |z_2| \cdot |x_1 - x_2| \leq$$

$$2|q_1 - q_2|$$

Then

$$\frac{|g(q_1) - g(q_2)|}{|q_1 - q_2|^{1-\delta}} \leq$$

$$(2)^{1-\delta} \cdot \frac{|g(q_1) - g(q_2)|}{|x_1z_1 - x_2z_2|^{1-\delta}} \leq (2)^{1-\delta}.$$

Similarly one can show the same property for g'.
Therefore $R_\alpha \in D^\alpha$.
QED

4 Discussion of Further Results

Let f be the mapping of R^3 into itself

$$f : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} (a+o(1))x \\ (ac+o(1))(y+xz), \quad \text{as} |x|,\ |y|,\ |z| \longrightarrow 0 \\ (c+o(1))z \end{pmatrix}$$

where $c < 1 < ac < a$ are constants.

Since $\frac{1}{constant+o(1)} = frac1constant + o(1)$, we have the following expression for the inverse map:

$$f^{-1} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} (\frac{1}{a}+o(1))x \\ (\frac{1}{ac}+o(1))(y-xz), \quad \text{as} |x|,\ |y|,\ |z| \longrightarrow 0 \\ (\frac{1}{c}+o(1))z \end{pmatrix}$$

Introduce the following notations: $f = L + \mu^+$ and $f^1 = L^{-1} + \mu^-$, where L

and L^{-1} represent the linear parts of f and f^{-1} correspondantly.
Then

$$\mu(x,y,z) = \begin{pmatrix} o(1)x \\ acxz + o(1)(y+xz) \\ o(1)z \end{pmatrix}$$

and

$$\mu^{-1}(x,y,z) = \begin{pmatrix} o(1)x \\ acxz + o(1)(y-xz) \\ o(1)z \end{pmatrix}.$$

It is clear that all estimates from Lemmas 1-4 remains true for this more general function. Thus we have the following theorem:

Theorem 2 *Let f be a C^3 diffeomorphism of $\mathbf{R}^3$ into itself with a hyperbolic fixed point at* 0*,*

$$f(x) = Lx + \mu(x), \; x = (x_1, x_2, x_3),$$

such that L is diagonal and f leaves invariant x and y axis, and (x,y) and (y,z) planes.

Then, for any α arbitrarily close to but less than 1*, there exists a neighborhood $U \subset \mathbf{R}^3$ of the origin and a homeomorphism $R_\alpha : U \to R_\alpha(U)$ such that $R_\alpha \in D^\alpha$ (see Definition 1 below) and*

$$R_\alpha \circ f = L \circ R_\alpha.$$

Conjecture 1 *Let f be a C^3 diffeomorphism of $\mathbf{R}^\mathbf{n}$ into itself with a hyperbolic fixed point at 0,*

$$f = Lx + \lambda(x),$$

such that all first order partial derivatives of λ at 0 *are zero.*

Then for any α arbitrarily close to but less than 1 *there exists a neighborhood $U \subset \mathbf{R}^\mathbf{n}$ of the origin and a homeomorphism $R_\alpha : U \to R_\alpha(U)$ such that $R_\alpha \in D^\alpha$ and*

$$R_\alpha \circ f = L \circ R_\alpha.$$

I believe that this conjecture can be proved. More over, this result is closely related to the theorem stated in the paper by van Strien (see [v-S]). He was trying to prove the following:

Let $f : \mathbf{R^n} \to \mathbf{R^n}$ be a C^2 diffeomorphism with a hyperbolic fixed point at 0. Let $L = Df(0)$. Then there exists a neighborhood U of 0 and a homeomorphism $h : U \to h(U)$ such that

$$h \circ f = L \circ h \quad \text{on} \quad U;$$

and h, h^{-1} are differentiable at 0 (i.e. $(Dh)_0$ and $(Dh^{-1})_0$ exist).
There are some errors in his proof of this theorem and, in fact, the theorem remains an open question. But some ideas of his proof and some correct and interesting lemmas could be employed in the subject of the linearization.
The technique of our proof is similar to van Strien's.

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