Geometric Forces on Point Fluxes in Quantum Hall Fluids

J. E. Avron and P. G. Zograf

Department of Physics, Technion, 32000 Haifa, Israel

Steklov Mathematical Institute, 191011 St.-Petersburg, Russia

September 17, 1997

Abstract

We study the forces that act on a point flux carrying an integral number of flux units in quantum Hall fluids. Forces due to external fields, Lorentz and Magnus type forces, and the forces due to mutual interaction of point fluxes are considered. The forces are related to the adiabatic curvature associated with families of Landau Hamiltonians. The problem displays distinct features of the quantum Hall fluids with point fluxes on the plane and on the torus, which, however, agree at the "thermodynamic" limit.

PACS numbers : 73.40.Hm, 74.60.Ge

A *point flux* in two dimensions is the magnetic analog of a point charge. It is created by an (infenitesimally thin) Aharonov-Bohm flux tube that orthogonally pierces the two dimensional surface in question. For reasons that shall become clear later, we restrict ourselves to cases where the flux Φ is an integer multiple of the quantum flux unit $\Phi_0 = \frac{hc}{e}$.

A point flux in vacuum does not interact directly with electric or magnetic fields, nor with other point fluxes (by the linearity of Maxwell equations). However, point fluxes associated with, say, a two dimensional quantum Hall fluid can interact through the electrons that fill the Landau level.

There are several forces that we consider: the force due to an external electric field, the force due to the point flux motion and the mutual forces that a collection of point fluxes apply one on another. It turns out that the forces depend, among other things, on the topology of the two dimensional manifold in question: the forces that act in the plane and on the torus are different. In both cases the forces are geometric in character and are related to the adiabatic curvature [1, 2].

The problem we study is related to, but distinct from, the study of the forces that act on vortices in superfluids and superconductors [3, 4, 5]. The gauge field of a point flux is formally identical to the velocity field of a vortex (the flux is an analog of the vorticity), and in both cases the forces that arise have geometric interpretation [5]. However, the physics is different and so are the results.

There are also relations to the fractional quantum Hall effect [6, 7, 8] and to anyon physics [9, 10] where weakly interacting composite objects made up of a charged particle attached to a point flux, play a central role.

Consider a family $H(A)$ of Landau Hamiltonians in two dimensions associated with gauge fields $A(\phi, p_1, \ldots, p_N)$ depending on $N+1$ complex parameters . The parameter $p_j \in \mathbb{C}\,(j = 1, \ldots, N)$ gives the position of the j-th point flux on the two dimensional surface. The parameter $\phi \in \mathbb{C}$ is associated with a (constant in space) gauge field that creates an external electric field, $E = E_1 + iE_2$, in two dimensions. More precisely, the Landau Hamiltonian $H(A)$ is given formally by

$$H(A) = \frac{2\hbar^2}{m} D_A^* D_A, \qquad D_A = \frac{\partial}{\partial \bar{z}} - iA_{\bar{z}}, \tag{1}$$

where $A = A_z dz + A_{\bar{z}} d\bar{z}$. The gauge field A splits into three parts:

$$A(\phi, p_1, \ldots, p_N) = A^0 + A^\phi + \sum_{j=1}^{N} \Phi_j\, A^{p_j}; \tag{2}$$

Φ_j is an integer multiple of the unit of quantum flux Φ_0 (in our units $\Phi_0 = 2\pi$). The gauge field

$$A^0 = -\frac{1}{2} By(dz + d\bar{z}) \tag{3}$$

gives rise to the constant magnetic field B. When we consider such field on a torus of area L^2, we assume that B satisfies Dirac's quantization condition, that is, $BL^2/2\pi$ must be an integer. There is no constraint on B for the Euclidean plane. The gauge field

$$A^\phi = \frac{1}{2}(\bar{\phi}\, dz + \phi\, d\bar{z}), \tag{4}$$

with ϕ a function of time, is associated with a constant (in space) electric field

$$E = -\frac{1}{2c}(\dot{\bar{\phi}}\, dz + \dot{\phi}\, d\bar{z}), \tag{5}$$

where $\dot{\phi}$ is the time derivative of ϕ.

The gauge field A^p is associated to the unit point flux located at p. It is a solution of $dA^p = 2\pi\, \delta(z - p)\, dx \wedge dy$, where $z = x + iy$. In the Euclidean plane a canonical solution is the rotational invariant point flux:

$$A^p = -\frac{i}{2} \left(\frac{dz}{z - p} - \frac{d\bar{z}}{\bar{z} - \bar{p}} \right). \tag{6}$$

This solution is characterized by A_z^p having a simple pole at p with residue $-i/2$. In the case of the (unit) torus, the canonical gauge field associated to a point flux turns out to be

$$A_z^p = -\frac{i}{2}\frac{\vartheta'}{\vartheta}\Big(z-p-\frac{1}{2}-\frac{i}{2}\Big) - \pi\Big(\mathrm{Im}\, p + \frac{1}{2}\Big), \tag{7}$$

where the theta function is defined, as usual, by

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2} e^{2i\pi n z}. \tag{8}$$

When ϕ and p_j depend adiabatically on time, i.e. we consider $\phi(t/\tau)$ and $p_j(t/\tau)$ in the limit $\tau \to \infty$, the electric field $E = O(1/\tau)$ is weak and the velocities $\dot{p}_j = O(1/\tau)$ are small. We are interested in the forces on the point flux to the same order in τ. In particular, we do not calculate any forces that are of order $1/\tau^2$ (e.g. forces that are proportional to, say, $E\dot{p}$).

Our main result is as follows. On the torus represented by the square of size $L \times L$ with opposite sides identified, the quantum expectation value for the force acting on the j-th point flux in a full Landau level is

$$\langle F_j \rangle = \Phi_j \,\frac{e^2}{\hbar c} \left(E - \frac{i}{cL^2} \sum_{k=j}^{N} \Phi_k\, \dot{p}_k \right) + O(1/\tau^2). \tag{9}$$

The expression in brackets in the right hand side of Eq. (9) has physical significance: the motion of the point fluxes creates an emf. With N moving point fluxes the total emf, $V = V_1 + iV_2$, about the parallel and meridian of the torus, as we shall explain, is

$$V = L \left(E - \frac{i}{cL^2} \sum_{k=1}^{N} \Phi_k\, \dot{p}_k \right). \tag{10}$$

Eq. (9) can therefore be re-written as

$$\langle F_j \rangle = \Phi_j \,\frac{e^2}{\hbar c}\,\frac{V}{L} + O(1/\tau^2), \tag{11}$$

where $\frac{e^2}{\hbar c}$ is the fine structure constant. Since $\frac{e^2}{\hbar c}\,\Phi = e\,\frac{\Phi\,e}{\hbar c} = e\,\frac{\Phi}{\Phi_0}$, Eq. (9) may be interpreted as saying that a point flux interacts with an *effective electric field* V/L *as if* it were *electrically charged* with a charge that is a multiple of the electron charge by the number of flux quanta.

In the thermodynamic limit $L \to \infty$, the effective electric field V/L becomes the external electric field E, by Eq. (10). In this limit, or, in other words, in the Euclidean plane, Eq. (9) reduces to

$$\langle F_j \rangle = \frac{e^2}{\hbar c}\,\Phi_j E + O(1/\tau^2). \tag{12}$$

There is no Lorentz or Magnus type force on a point flux in the plane, and point fluxes do not mutually interact. This is, of course, quite unlike vortices in superfluids and superconductors which do interact with each other [3] and experience a Magnus force [4, 5]. We see that the forces on point fluxes in the plane, $L = \infty$, and the torus, $L < \infty$, have distinct character. The term with $k = j$ in the sum in Eq. (9) is a Lorentz (or Magnus) type force which is self-interacting being proportional to Φ_j^2. The sum over $k \neq j$ describes a velocity dependent mutual interaction of point fluxes that violates Newton's third law.

We shall give an outline of a formal derivation of the above results. Let us start with Eq. (7) which gives the gauge field of a point flux on torus. In analogy with the gauge field of a point flux in the plane, it is characterized by a holomorphic function of z, with a simple pole at p of residue $-i/2$ which is doubly periodic in p. For the sake of simplicity let us assume that the torus T is given by the factor of $\mathbb{C}$ modulo the unit square lattice (the case of a square lattice of arbitrary size may be reduced to that one by scaling). The holomorphic function $\vartheta(z)$, Eq. (8), has exactly one simple zero per lattice cell located at the points $(1/2 + m,\ i/2 + in)$ with integer

m, n, and has the following transformation properties with respect to translations by 1 and i:

$$\vartheta(z+1) = \vartheta(z), \quad \vartheta(z+i) = e^{\pi - 2i\pi z}\, \vartheta(z). \tag{13}$$

Put $\zeta = p + 1/2 + i/2$ and consider the function

$$f(z;\zeta) = e^{-2\pi i z\, \mathrm{Im}\,\zeta}\ \vartheta(z-\zeta). \tag{14}$$

It is holomorphic in z and real analytic in ζ. It follows that

$$A_z^p = -\frac{i}{2}\,\frac{f'}{f}(z;\zeta), \tag{15}$$

is meromorphic in $z \in \mathbb{C}$ with simple poles at $p+m+in$, each of residue $-i/2$. It obeys

$$A_z^p(z+1) = A_z^p(z), \quad A_z^p(z+i) = A_z^p(z) + i\pi, \tag{16}$$

and is doubly periodic in p. It means that the gauge potential A^p is independent of the choice of $p \in \mathbb{C}$ modulo the lattice translations. Clearly, $dA^p = 2\pi\delta(z-p)dx\wedge dy$ on the torus T, so Eq. (7) indeed gives a canonical gauge potential of a unit point flux.

The periods of 1-form $\dot{A}$ give the total emf, V, about the parallel and meridian of the torus T. One finds from Eq. (15) that $V = E - 2\pi i\dot{p}$. For N point fluxes with fluxes Φ_k (each a multiple of $\Phi_0 = 2\pi$) on the torus with both parallel and meridian of length L we get Eq. (10) by scaling and putting back the velocity of light c.

Let us consider now the force equations, Eqs. (9, 12). By the principle of virtual work, the *quantum observable* associated with the force on the j-th point flux located at p_j is

$$F_j = F_{j1} + iF_{j2} = -2\,\frac{\partial H}{\partial \bar{p}_j}. \tag{17}$$

The Landau Hamiltonians we consider are such that for any value of the parameters p_j, $j = 0, \ldots, N$ (we write $p_0 = \phi$), the operator is unitarily equivalent to the Landau Hamiltonian $H_0 = H(A^0)$ *without point fluxes and in the absence of external electric field*, that is, $H(A) = U(A) H_0 U^*(A)$ with an appropriate unitary $U(A)$. (It is here that we use the fact that the point flux carries an integer flux). Let P denote the spectral projection on a Landau level of $H(A)$, i.e., $P = \sum_\ell |\phi_\ell\rangle\langle\phi_\ell|$, with $|\phi_\ell\rangle$ being normalized eigenstates that span the Landau level. Clearly, $P = U(A) P_0 U^*(A)$ with P_0 the (fixed) projection on the Landau level of H_0. For such a family of unitarily related Hamiltonians we get from the basic equation of adiabatic transport [11, 12, 13] that

$$\langle F_j \rangle = -i \sum_{k=0}^{N} \Omega_{\bar{p}_j p_k} \dot{p}_k + O(1/\tau^2), \tag{18}$$

where

$$\Omega_{\bar{p}_j p_k} = Tr \left(P \left[\frac{\partial P}{\partial \bar{p}_j}, \frac{\partial P}{\partial p_k} \right] P \right)$$

are the components of the trace of the adiabatic curvature [1, 2]:

$$\Omega(P) = P \, dP \wedge dP \, P. \tag{19}$$

Using the unitarity of U one finds, formally, that

$$Tr \left(\Omega(P) \right) = Tr \left(P_0 [\mathcal{A}, P_0] \wedge [\mathcal{A}, P_0] P_0 \right), \tag{20}$$

where $\mathcal{A} = U^* dU$. In the case of the torus we use the following formally equivalent version of the above equation

$$Tr \left(\Omega(P) \right) = Tr \left(P_0 \mathcal{A} \wedge \mathcal{A} \right). \tag{21}$$

(The two are strictly equivalent when $P_0 \mathcal{A} P_0$ is Hilbert-Schmidt.)

To compute the adiabatic curvatures we need to explicitly construct the unitary families $U(A)$. We shall now outline the construction of U for the case of the torus.

Consider the family of Landau Hamiltonians $H(A)$ for a *unit* point flux given by Eq. (1), with

$$D_A = \frac{\partial}{\partial \bar{z}} + \frac{iBy}{2} - \frac{i\phi}{2} + \frac{1}{2}\frac{\bar{f}'}{\bar{f}}.$$

The total magnetic flux of A through the unit torus T is $\int_T dA = B + 2\pi$. By Dirac's quantization condition, $B/2\pi$ is an integer, and we assume that it is positive. The unitary operators describing magnetic translations which commute with D_A are

$$T_1\psi(z) = \psi(z+1), \quad T_2\psi(z) = e^{i(B+2\pi)x}\,\psi(z+i), \tag{22}$$

that give the usual magnetic translation boundary conditions [14]:

$$T_1\psi = \psi, \quad T_2\psi = \psi. \tag{23}$$

For H_0 we have, instead, the boundary conditions

$$T_1^0\psi = \psi, \quad T_2^0\psi = \psi \tag{24}$$

with $T_1^0\psi(z) = \psi(z+1)$, $T_2^0\psi(z) = e^{iBx}\,\psi(z+i)$.

We represent U as a composition of two auxiliary unitary operators V and W. The first one,

$$V(t)\psi(z) = e^{-iy\,\mathrm{Im}\,t}\psi(z - t/B), \tag{25}$$

commutes with T_1^0, T_2^0 and has the property that

$$V(i\phi)D_{A^0} = D_{A^0+A^\phi}V(i\phi). \tag{26}$$

The second one is a multiplication operator (gauge transformation)

$$W(p)\psi(z) = e^{\pi(\zeta\bar{z}-\bar{\zeta}z)}\,\frac{f}{|f|}(z;\zeta)\,\psi(z), \quad \zeta = p + \frac{1}{2} + \frac{i}{2},$$

that intertwines the boundary conditions in Eqs. (23, 24):

$$W(p)T_1^0 = T_1W(p), \quad W(p)T_2^0 = T_2W(p).$$

Now put $U(A) = W(p)V(t)$ with $t = i\phi - 2\pi\zeta$. We see that

$$U(A)D_{A^0} = D_A U(A), \qquad A = A^0 + A^\phi + A^p. \tag{27}$$

Since the unitary $W(p)$ is a multiplication operator, the eigenstate bundles for the family $H(A) = U(A)H_0U^*(A)$ are topologically equivalent to those of $V(t)H_0V(t)^*$ with $t = i\phi - 2\pi\zeta$. If $W(p)$ were a smooth family, the eigenstate adiabatic curvatures would also coincide, but $W(p)$ is not. This gap can be patched by an appropriate limiting argument. Substituting the formulas

$$\begin{aligned} V(t)^*dV(t) &= -\left(\frac{1}{B}\frac{\partial}{\partial z} + \frac{iy}{2}\right) dt - \left(\frac{1}{B}\frac{\partial}{\partial \bar z} + \frac{iy}{2}\right) d\bar t, \\ V(t)^*dV(t) \wedge V(t)^*dV(t) &= \frac{1}{2B} dt \wedge d\bar t, \end{aligned}$$

into Eq. (21) for the family $P = V(t)P_0V(t)^*$ with $t = i\phi - 2\pi\zeta$, we get

$$Tr\,\Omega\Big(V(it)P_0V^*(it)\Big) = -\frac{1}{4\pi} dt \wedge d\bar t, \tag{28}$$

where $dt = i\,d\phi - 2\pi\,dp$. For N point fluxes p_k with fluxes Φ_k on the torus associated with the $L \times L$ square, by scaling we again get Eq. (28) with

$$dt = iLd\phi - \frac{1}{L}\sum_{k=1}^{N} \Phi_k\, dp_k. \tag{29}$$

Putting back the constants c and $\frac{e}{\hbar}$ and using Eq. (18) we obtain Eq. (9)

One can make an independent formal calculation of the forces on a point flux in the Euclidean plane starting with Eq. (20) and making explicit computations with integral kernels following the methods in [15, 16]. The integral kernel of P_0 is Gaussian, and the unitary U in the plane is an abelian gauge transformation with

$$\mathcal{A} = \frac{1}{2}\left(\sum -\frac{dp_j}{z - p_j} + \frac{d\bar p_j}{\bar z - \bar p_j}\right) + iz d\bar\phi + i\bar z d\phi.$$

The resulting multidimensional (singular) integrals can be evaluated explicitly and give the same result that one obtains from the torus calculation by setting $L \to \infty$. Note that one can not use Eq. (21) (which gives zero since $\mathcal{A}$ is abelian) because P_0 is infinite dimensional, and $P_0 \mathcal{A} P_0$ is not Hilbert-Schmidt.

The effective electric charging of point flux is a way of describing the force that a point flux experiences in an external electric field. This raises the question: is this charging real? For the sake of simplicity and concreteness let us focus on the Euclidean plane. The charge associated with a point flux is a subtle issue in the sense that one gets different answers depending on the precise question one asks. It is a basic fact about integer quantum Hall systems that adding a point flux sends an integral number of charges to infinity [6]. In this sense there is a real charge associated with the point flux. However, an *a priori* different notion of charge associated with a point flux is to look at the charge density in the neighborhood of the flux. In Chern-Simons theory of the quantum Hall effect [7] one has, indeed, an excess charge density near a point flux. On the other hand, for an *ideal point flux* which carries an integral number of flux quanta, it is a consequence of gauge invariance that the charge distribution of a full Landau level in the plane (both for Pauli and Schrödinger Hamiltonian) is blind to presence of the point flux and in this sense no excess charge is attached to a point flux.

The forces that act on point fluxes display neither symmetry nor duality between point charges and point fluxes and appear to conflict with Galilei invariance. This is most dramatic in the case of the Euclidean plane where there is a non-vanishing electric force but no Lorentz force. In the case of the torus one finds both an electric and Lorentz type force, but the two are not related in the way one would expect from Lorentz and Galilei invariance.

In the case of the torus one has a simple reason to doubt arguments based on Lorentz invariance because Lorentz invariance is broken by the identifications that make a torus out of a square. In the case of the plane it is less obvious what has gone awry with Lorentz invariance. The answer to this is that taking the adiabatic limit breaks Lorentz and Galilei invariance. The adiabatic setting stipulates that in the distant past the Hamiltonian is a *time independent* Landau Hamiltonian and the initial state of the system is associated with a spectral projection and a full Landau level. This distinguishes a frame and breaks Galilei invariance.

Acknowledgments

This work was partially supported by a grant from the Israel Academy of Sciences, the Deutsche Forschungsgemeinschaft, and by the Fund for Promotion of Research at the Technion. In addition to that, the work of PZ was supported in part by a grant from the Russian Fund for Fundamental Research and by the Rosenbaum Foundation, and he also acknowledges the hospitality of ITP at the Technion where the first stage of this work was done.

References

[1] M.V. Berry, Proc. Roy. Soc. A **392**, 45 (1984); *The quantum phase: Five years after*, in *Geometric phases in physics*, A. Shapere, F. Wilczek, Eds., World Scientific (1989).

[2] B. Simon, Phys. Rev. Lett. **51**, 2167 (1983).

[3] M. Tinkham, *Introduction to Superconductivity*, McGraw Hill (1975); E.B. Sonin, Rev. Mod. Phys. **59**, 87 (1987), and cond-mat/9606099.

[4] N.B. Kopnin, G.E. Volovik, Ü. Parts, Europys. Lett. **32**, 651 (1995).

[5] D.J. Thouless, P. Ao, Q. Niu, Phys. Rev. Lett. **76**, 3758 (1996).

[6] R. Laughlin, in *"The Quantum Hall Effect"*, R.E. Prange and S.M. Girvin, Eds., Springer (1987).

[7] M. Stone, *The Quantum Hall Effect*, World Scientific, Singapore (1992).

[8] B.I. Halperin, P.A. Lee, N. Reed, Phys. Rev. B **47**, 7312 (1993); R.L. Willet, R.R. Ruel, K.W. West, L.N. Pfieffer, Phys. Rev. Lett. **71**, 3846 (1993).

[9] D.H. Lee, M.P.A. Fisher, Int. J. Mod. Phys. **5**, 2675 (1991).

[10] F. Wilczek, *Fractional statistics and Anyon superconductivity*, World Scientific (1990).

[11] D.J. Thouless, J. Math. Phys. **35**, 1 (1994).

[12] D. J. Thouless, M. Kohmoto, P. Nightingale, M. den Nijs, Phys. Rev. Lett. **49**, 40 (1982).

[13] R. Seiler, in *Recent developments in Quantum Mechanics*, A. Boutet de Monvel et. al. Eds., Kluwer, Netherland (1991).

[14] J. Zak, Phys. Rev. A**134** 1602, ibid. 1607 (1964).

[15] J. E. Avron, R. Seiler, B. Simon, Comm. Math. Phys. **159**, 399 (1994).

[16] J. Bellissard, A. van Elst, H. Schultz-Baldes, J. Math. Phys. **35**, 5373 (1994)