On the Dominant Fourier Modes in the Separatrix Splitting Distance Function for an A-Priori Stable, Three Degree-of-Freedom Hamiltonian System

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Abstract

This note is devoted to the analysis of the Fourier series that one obtains for the splitting distance function in an a-priori stable Hamiltonian system with three degrees-of-freedom, lately studied in connection with the so-called "Arnold diffusion". We give a summary of the theory, developed in Rudnev and Wiggins [1997], and compare it with a number of numerical experiments. These experiments not only illustrate and confirm the former theoretical concepts, but suggest that beyond the aforementioned theory, the same direction of reasoning shall prove useful. In particular, it suggests that for the vast majority of frequencies, and most of the values of the perturbation parameter ϵ, the Fourier series in question has two components, which dominate the rest of it, whose index increases as $\varepsilon \to 0$. These components come from a certain sub-series of the Fourier series under consideration, which corresponds to the sequence of the best approximations to the frequency vector. We believe that the latter situation is generic, no matter what the approximation properties of the frequency vector. Therefore, in the case of three degrees of freedom, the arithmetic issues, which seemed so far one of the major obstructions to dealing quantitatively with exponentially small quantities, can be handled using the more refined approach developed in Rudnev and Wiggins [1997].

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1 Introduction

In 1964 Arnold published an example of a three degree of freedom Hamiltonian system containing a mechanism for a global instability of the action variables which has come to be known as *"Arnold diffusion"*. The purpose of this note is to highlight one of the main difficulties in proving that this mechanism occurs in some generic sense. The difficulties are best motivated by considering Arnold's original example.

Arnold [1964] considered a Hamiltonian of the form

$$H = \frac{I_1^2}{2} + I_2 + \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right) + \mu\varepsilon\left(\cos x - 1\right)\left(\sin\varphi_1 + \cos\varphi_2\right), \tag{1}$$

where $x \in T^1$, $y \in \mathbb{R}$, $(I_1, I_2) \in \mathbb{R}^2$, $(\varphi_1, \varphi_2) \in T^2$, and ε and μ are parameters. The Hamiltonian obtained by setting $\mu = 0$ is referred to as the *unperturbed Hamiltonian*, which one recognizes as the Hamiltonian for three uncoupled systems: a pendulum and two *rotors* described by two action-angle pairs of variables. Hamilton's canonical equations for this system are given by

$$\begin{aligned}
\dot{x} &= \tfrac{\partial H}{\partial y} = y, \\
\dot{y} &= -\tfrac{\partial H}{\partial x} = \varepsilon \sin x + \mu\varepsilon \sin x \left(\sin\varphi_1 + \cos\varphi_2\right), \\
\dot{\varphi}_1 &= \tfrac{\partial H}{\partial I_1} = I_1, \\
\dot{I}_1 &= -\tfrac{\partial H}{\partial \varphi_1} = -\mu\varepsilon \cos\varphi_1 \left(\cos x - 1\right), \\
\dot{\varphi}_2 &= \tfrac{\partial H}{\partial I_2} = 1, \\
\dot{I}_2 &= -\tfrac{\partial H}{\partial \varphi_2} = \mu\varepsilon \sin\varphi_2 \left(\cos x - 1\right).
\end{aligned} \tag{2}$$

We want to describe the geometry of the phase space for $\mu = 0$. In this case the system has a four dimensional normally hyperbolic invariant manifold given by

$$\mathcal{M}_0 = \left\{(x, y, \varphi_1, I_1, \varphi_2, I_2) \,|\, x = y = 0\right\}.$$

whose five dimensional stable and unstable manifolds coincide along

$$W^s\left(\mathcal{M}_0\right) = W^u\left(\mathcal{M}_0\right) = \left\{(x, y, \varphi_1, I_1, \varphi_2, I_2) \,|\, \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right) = 0\right\}.$$

$\mathcal{M}_0$ is foliated by a family of two-tori, and the trajectories on the two-tori are given by

$$\varphi_1(t) = I_1 t + \varphi_{10},$$

$$\varphi_2(t) = t + \varphi_{20}.$$

We denote these two-tori by $\mathcal{T}(I_1)$.

Recall that the dynamics is restricted to lie in the five dimensional energy surface given by

$$h = \frac{I_1^2}{2} + I_2 + \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right),$$

and the four dimensional $\mathcal{M}_0$ intersects the five dimensional energy surface in a three dimensional set. This three dimensional set, denoted by $\mathcal{M}_0^h$, is filled out by a family of two tori, and each two-torus has three-dimensional stable and unstable manifolds that coincide and are given by

$$W^s\left(\mathcal{T}(I_1)\right) = W^u\left(\mathcal{T}(I_1)\right) = \left\{(x, y, \varphi_1, I_1, \varphi_2, I_2) \mid \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right) = 0,\ I_1 \text{ fixed}\right\}.$$

The stable and unstable manifolds of a given torus are often referred to as the *whiskers* of the torus. Moreover, $W^s\left(\mathcal{T}(I_1)\right)$ intersects $W^u\left(\mathcal{T}(I_1)\right)$ non-transversely, which is a situation that we do not expect to persist for μ small, but nonzero, and it is to this situation that we now turn.

The particular form of the perturbation in (1) is such that it does not affect $\mathcal{M}_0$ or any of the tori on it. Nevertheless, for any given torus the perturbation may cause the stable and unstable manifolds to intersect transversely. Moreover, because the tori are arbitrarily close on $\mathcal{M}_0^h$, *and are not isolated by the energy integral for systems with three or more degrees-of-freedom*, the stable and unstable manifolds of nearby tori may intersect and, consequently, give rise to a mechanism for a global drift in the trajectories. Following Lochak [1995], we refer to this as the *Arnold mechanism* and we now describe it in a bit more detail. The tori can be denoted by the parameter I_1. Arnold showed that one could find a sequence of tori, $\{I_1^i\}$, $i = 1, \ldots, N$, with neighboring tori sufficiently close, i.e. $|I_1^i - I_1^{i+1}| = O(|\mu|^2)$, such that the unstable manifold of $\mathcal{T}_\mu(I_1^i)$ transversely intersects the stable manifold of $\mathcal{T}_\mu(I_1^{i+1})$, $i = 1, \ldots, N-1$. This sequence of whiskered tori is said to form a *transition chain*, and Arnold uses it to prove the following theorem.

Theorem 1.1 (Arnold) *For every $\varepsilon, r > 0$ there exists a $\mu_0 > 0$ such that for all $0 < \mu \leq \mu_0$ there are invariant tori $\mathcal{T}_\mu(I_1)$ and $\mathcal{T}_\mu(I_1')$, with $|I_1 - I_1'| > r$, which are connected by a transition chain.*

He then shows that the transition chain is shadowed by a true orbit of the system and in this way one obtains an $O(1)$ drift in the I_1 variable over some finite time interval.

In carrying out this program in general there are two main difficulties that must be overcome. One is concerned with the problem of the splitting of separatrices, and the other is concerned with the construction of transition chains and shadowing orbits. It is the former problem with which this note is concerned.

The purpose of the following discussion is to illustrate the general theory developed in Rudnev and Wiggins [1997] for three degree-of-freedom systems that gives the conditions under which one or two Fourier harmonics surely "dominate" the splitting distance function (not just the first order term given by by the Melnikov integrals) and provides an algorithm to find these harmonics. These Fourier harmonics depend upon the frequency of the torus, ε, and a parameter, σ_0, that quantifies the size of the analyticity domain in the angle variables. A general theorem is proven in Rudnev and Wiggins [1997], but in this note we give a number of numerical simulations for different parameter values, both under conditions where the theory is valid, and under conditions where it is not valid, namely near the resonances. The leitmotif of the whole story will be that in both cases, when the frequency is a two-vector with rationally independent components the behavior of the Fourier series we study is dominated by a sequence, having a relatively small number of terms, which correspond to convergents to an irrational number, which is equal to the ratio of the components of the frequency vector, the latter sequence having generically one or at most two local minima, one of which is the global one, which will account for the asymptotic behavior of the series studied with an exponentially small error.

Before we describe the near integrable three degree-of-freedom Hamiltonian system of interest, it is, perhaps, more useful and instructive to motivate the main problems, issues, and setting in a general framework. From this point of view the central object of interest will be a vector-function $\vec{\mathcal{M}}$ on a 2-torus, given by a Fourier series

$$\vec{\mathcal{M}}(\vec{\alpha}) = \sum_{\vec{k} \in Z^2} \vec{A}_{\vec{k}} \mathcal{E}_{\vec{k}} \sin(\vec{k} \cdot \vec{\alpha}) \tag{3}$$

where $\vec{\alpha} \in T^2$, and the terms $\mathcal{E}_{\vec{k}}$ are represented as follows:

$$\mathcal{E}_{\vec{k}} = \exp\left(-|\vec{k}|\sigma_0 - \frac{\pi|\vec{k}\cdot\vec{\omega}|}{2\sqrt{\varepsilon}}\right) \equiv \exp(-E_{\vec{k}}). \tag{4}$$

In fact, we will call $\mathcal{M}(\vec{\alpha})$ "the splitting distance function", although given the application, the latter is just a particular case of $\mathcal{M}(\vec{\alpha})$.

We will assume that the coefficients $\vec{A}_{\vec{k}} = -A_{-\vec{k}}$ are all strictly nonzero, with their $\vec{k}$ dependence being such that they never grow or vanish faster than some finite power of $|\vec{k}|$ as $|\vec{k}| \to \infty$, where as usual, $|\vec{k}| = |k_1| + |k_2|$. For the purposes of our applications a particular example of such a series might be the Melnikov function, if the perturbation has an infinite number of Fourier components, or, generally, the splitting distance function computed to all orders of perturbation theory.

Correspondingly, we will be interested in zeroes of the function $\mathcal{M}(\vec{\alpha})$, and the determinant $\frac{\partial}{\partial\vec{\alpha}}\mathcal{M}(\alpha)$ evaluated at these zeroes. Instead of upper and lower estimates, with which we had to content ourselves in a general case of n degrees of freedom, when $n = 3$ we were able to give a compact answer for both of these quantities.

In Rudnev and Wiggins [1997] it is proved that for all small enough positive values of ε $(0 < \varepsilon \leq \varepsilon_0 \ll 1)$, except for a set of asymptotically as $\varepsilon \to 0$ zero measure consisting of some small neighborhoods of certain "critical" values of this parameter (the latter forming a sequence which is nowhere dense and accumulating at zero), the exponents $E_{\vec{k}}$ in the Fourier coefficients have an absolute minimum for some $\vec{k}^*(\vec{\omega}, \varepsilon, \sigma_0)$, (and, of course, $-\vec{k}^*$ by symmetry). The location of this minimum depends on the particular frequency vector $\vec{\omega}$, namely $|\vec{k}^*| \sim \mathcal{A}(\vec{\omega}, \sigma_0)\varepsilon^{-\frac{1}{4}}$, for some function $\mathcal{A}$ to be described, which can be computed by following the developed procedure. It is then shown that the absolute value of the corresponding term in the Fourier series will differ from the sum of the absolute values of the rest of the terms by a quantity large enough for it to yield the leading-order asymptotics for the components of the splitting distance function with exponentially small error. The proof will hold true for those values of ε which are away from the aforementioned critical values: otherwise the minimum in consideration turns out to be not sharp enough for us to show that a single term in (3) that dominates.

Transversality of the splitting is quantified by the expression $\partial_{\vec{\alpha}}\vec{\mathcal{M}}$ being nonzero at the intersection point. Unfortunately, a single dominant Fourier harmonic is not sufficient for us to conclude that this expression is nonzero since its sole contribution to $\partial_{\vec{\alpha}}\vec{\mathcal{M}}$ is obviously going to be zero (simply because the zeroes of $\sin(\vec{k}^* \cdot \vec{\alpha})$ are not isolated). Thus, we will need to locate the second smallest exponent, denoted by an index $\vec{k}^{**}$, and to prove that these two terms together will give the leading-order answer for the transversality, or "splitting size". Theorem 4.1 states the main results.

We will outline the procedure which prescribes how $\vec{k}^*$ and $\vec{k}^{**}$ can be found, and then implement the procedure numerically for different values of ω and ε. Our principal tool will be a simple number theoretical lemma, called the Transition lemma, proved in Rudnev and Wiggins [1997], which is the centerpiece of the algorithm of finding $\vec{k}^*$ and $\vec{k}^{**}$.

We remark that Theorem 4.1 strictly holds only for an asymptotically full measure set of "almost" non-resonant frequencies in the sense that they do not come to the resonances closer than ε^{δ_2}, the latter being such that $0 < \delta_2 \ll \frac{1}{2}$ (where the small positive constant δ_2 will be described in more detail later). Moreover, near the "critical values" of ε, some "switching of the modes" occurs, namely when ε is being decreased and it passes the critical value, $\vec{k}^*$ quickly changes to a vector with a bigger norm. So for ε near the critical value there are at least two modes whose contributions are comparable, and we will have two or more terms vying to dominate the whole sum. This is much harder to describe rigorously. Our numerical experiments will also explore these phenomena.

Furthermore, we perform the same experiments for different values of ε with several fairly well approximable frequencies, which correspond to near-resonant cases if ε is not too small; these cases are not covered by the theory. In a way, one still ends up with a similar picture. Several times we also encountered the situation where ε is close to a critical value. In this case the two dominating exponents turn out to be close to each other, but they still differ significantly from all the rest. Hence two terms still suffice to describe the asymptotics. This gives us hope that in the case of three degrees of freedom, apropos the question of the separatrix splitting and the construction of the transition chains, arithmetics can be treated fairly well.

This paper is organized as follows. In the following section we establish the notation and briefly review the results proven in Rudnev and Wiggins [1997], applied to the specific application that we call the Generalized Arnold Model, set up in Section 3, it leads to Theorem 4.1, which is stated in Section 4. After that, we deliver the main conclusions of this note, which are confirmed by the results of several numerical experiments, out of many ran using *Mathematica*, which illustrate a variety of phenomena. The detailed reports for the experiments are given in the Appendix, where they are also discussed in the light of the theory and beyond in the light of what we believe can be treated within a similar ideological framework.

2 Theoretical Aspects

2.1 Minimum Background and Notation from Continued Fractions Theory

Our analysis of the formal series $\mathcal{M}(\vec{\alpha})$ will NOT require that the frequencies $\vec{\omega}$ satisfy the classical diophantine condition, formulated in terms of two parameters γ, τ, $0 < \gamma \leq 1,\ \tau \geq 1$, the latter will be considered fixed. In other words, we do not require that $\vec{\omega} \in \Omega_\gamma$ for

$$\Omega_\gamma \equiv \{\vec{\omega} : |\vec{\omega} \cdot \vec{k}| \geq \gamma |\vec{k}|^{-\tau},\ \forall \vec{k} \neq \vec{0}\}. \tag{5}$$

Instead, we'll only assume so far that the components of $\vec{\omega}$ are rationally independent.

Nevertheless, the diophantine condition is certainly indispensable for the general existence and uniqueness result apropos the whiskered tori and their stable and unstable manifolds (not described in this note), necessary to justify the series $\mathcal{M}(\vec{\alpha})$ as the splitting distance function. Unfortunately, the diophantine condition is not very useful for number theoretic approximation techniques. For this, we will use the theory of Continued fractions (which is why our results are limited to three degree-of-freedom systems).

In order to introduce the main notation and ideas from Continued fraction theory that we will use we first write the dot product $\vec{k} \cdot \vec{\omega} = k_1\omega_1 + k_2\omega_2$ in a form that is more compatible with the standard notation from the theory of approximation of irrational numbers by rational numbers.

Without loss of generality we shall assume that $|\omega_1| > |\omega_2|$ (they can't be equal or zero by the diophantine condition (5)), and denote

$$\omega_+ = |\omega_1|,$$

which we assume to be ε-independent. Moreover, for simplicity let's suppose that both ω_1 and ω_2 are positive (the modifications one will have to make to remove this assumption will be simple bookkeeping).

For the scalar product $\vec{k} \cdot \vec{\omega}$ then we get:

$$\vec{k} \cdot \vec{\omega} = k_1\omega_+ + k_2\omega_2 = \omega_+ \left(k_1 + k_2 \frac{\omega_2}{\omega_+}\right) \equiv \omega_+(-p + q\beta), \tag{6}$$

where we've denoted $\beta = \frac{\omega_2}{\omega_+}$, $p = -k_1$, $q = k_2$ following the standard notation of the approximation theory. So for $\vec{k} \in Z^2$ the question of the asymptotic behavior of the series (3) can be explored in terms of elementary Continued fractions, at least for β with well predictable approximation properties. Moreover, it's enough to consider only $q \geq 0$ by symmetry.

Below we shall give several definitions and facts from the theory of Continued fractions, as a reference we would suggest Schmidt [1980].

For each irrational number β there exists a unique expansion into a *Continued fraction*

$$\beta = [a_0, a_1, a_2, ..],$$

where $[a_0, a_1, a_2, ..]$ is an infinite sequence of integers, all of which are positive, except for possibly a_0, which equals the integer part of β, and if zero, can be omitted in the Continued fraction. Besides, a finite sequence $[a_0, a_1, a_2, .., a_n]$ gives a unique Continued fraction expansion of a rational if we deem the finite Continued fractions $[a_0, a_1, .., a_n]$ with $a_n > 1$ and $[a_0, a_1, .., a_n - 1, 1]$ equivalent.

The following relation explains the term "Continued fraction," whether the sequence under consideration is finite or infinite:

$$\begin{aligned}\beta &= [a_0, a_1, a_2, ..] &&= a_0 + \tfrac{1}{[a_1, a_2, ..]} \\ &= a_0 + \tfrac{1}{a_1 + \frac{1}{[a_2, a_3, ..]}} &&= [a_0, a_1, .., a_{n-1}, [a_n, a_{n+1}, ..]].\end{aligned} \tag{7}$$

We call a fraction $\frac{p}{q}$, where p, q are integers with $q > 0$ a *best approximation*, or a *convergent* to β if for any integer $0 < q' \leq q$ and all p', not equal to p if $q' = q$, one has:

$$\min_{p'} |q'\beta - p'| > |q\beta - p|.$$

Then for every β, rational or irrational, there is a sequence $\{q_n\}$ of positive integers with $q_0 = 1$ and a sequence $\{p_n\}$ of integers such that $\frac{p_n}{q_n}$ are the best approximations to β, these sequences being finite for a rational β and infinite if it is irrational. With some abuse of terminology we will most of the time apply the term "convergents" solely to the members of the sequence $\{q_n\}$, whereas the members of the sequence $\{\frac{p_n}{q_n}\}$ will be referred to as "best approximations".

In the standard notation

$$||q\beta|| = \min_p |q\beta - p|.$$

For any real β the Continued fraction expansion above uniquely encodes the sequences $\{p_n\}$ and $\{q_n\}$ by the recursion rule:

$$\begin{aligned} p_{-2} = 0, &\quad p_{-1} = 1, \\ q_{-2} = 1, &\quad q_{-1} = 0, \end{aligned} \tag{8}$$

and for $n \geq 0$

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2}, \\ q_n &= a_n q_{n-1} + q_{n-2}. \end{aligned} \tag{9}$$

By the Dirichlet theorem (see Schmidt [1980]), every best approximation $\frac{p_n}{q_n}$ to β satisfies

$$||q_n\beta|| = |q_n\beta - p_n| \leq \frac{1}{q_{n+1}}. \tag{10}$$

A number β is called *badly approximable* if there exists a constant $0 < c(\beta) < 1$, such that for every $q > 0$ one has

$$||q\beta|| > \frac{c(\beta)}{q}. \tag{11}$$

In addition given

$$\bar{a} = \max_{i>0} a_i \geq 1, \tag{12}$$

all the 2-vectors $\vec{\omega} = \omega_+(1, \beta)$ will be diophantine with $\tau = 1$ and $\gamma = \frac{\omega_+}{4+2\bar{a}}$ in the standard notation of (5), which implies that for a fixed $\bar{a}$ they form a zero measure set.

Although in our analysis we will be mostly dealing with badly approximable numbers, the ultraviolet cutoff that we are going to apply to the series (3) will eventually enable us to extend our result to a set of frequencies having asymptotically (as $\varepsilon \to 0$) full measure by letting $\bar{a}$ in (12) go to infinity as $\varepsilon \to 0$ and using instead of (12) the following less stringent definition:

$$\bar{a}(\varepsilon) = \max_{0<i\leq n_\varepsilon} a_i, \tag{13}$$

for some large enough n_ε which tends to infinity as $\varepsilon \to 0$.

Our main assumption on the Continued fraction expansion for β is the following.

Assumption 2.1 *Given the ultraviolet cutoff parameter* $q_\varepsilon = [\varepsilon^{-\frac{1}{2}+\delta_0}]$*, where* $[\cdot]$ *stands for an integer part, and* $0 < \delta_0 \ll \frac{1}{2}$*, suppose that for* $i = 1, .., n_\varepsilon$*, the latter being defined as*

$$n_\varepsilon = \max\{i \geq 1 : \; q_i \leq q_\varepsilon\},$$

where q_i *is the denominator in the* i*th best approximation to* β *(an* i*th convergent), the entries* a_i *in the Continued fraction for* β *are bounded as follows:*

$$a_i \leq \bar{a} = \bar{a}_0 \varepsilon^{-\delta_2}$$

for some positive constant $\bar{a}_0$ *and a rather small non-negative* δ_2 *to be specified.*

2.2 Preliminary Simplification of $\vec{\mathcal{M}}(\vec{\alpha})$

In this section we present a decomposition of $\vec{\mathcal{M}}(\vec{\alpha})$ and argue that the terms in the Fourier series corresponding to the sequence of the denominators $\{q_n\}$ of the best approximations (convergents) to β dominate the entire Fourier series. We will further often use the term "convergents" to refer to the members of the sequence $\{q_n\}$ themselves.

If we rewrite (3) in the notation of (6) as two separate sums

$$\begin{aligned} \mathcal{M}(\vec{\alpha}) &= \Sigma_1 + \Sigma_2 \\ &\equiv \textstyle\sum_{p,q\in Z,\ q>0,\ |q\beta-p|<\frac{1}{2}} \exp\left(-(p+q)\sigma_0 - \frac{\pi\omega_+|q\beta-p|}{2\sqrt{\varepsilon}}\right) \vec{A}_{\vec{k}} \sin(\vec{k}\cdot\vec{\alpha}) \\ &+ \textstyle\sum_{p,q\in Z,\ q\geq 0,\ |q\beta-p|\geq\frac{1}{2}} \exp\left(-(p+q)\sigma_0 - \frac{\pi\omega_+|q\beta-p|}{2\sqrt{\varepsilon}}\right) \vec{A}_{\vec{k}} \sin(\vec{k}\cdot\vec{\alpha}), \end{aligned} \tag{14}$$

where $\vec{k} = (-p, q)$, then for any positive δ_0 and ε small enough, the absolute value of the second sum will be bounded as:

$$|\Sigma_2| \leq \exp\left(-\frac{\pi\omega_+}{4\sqrt{\varepsilon}}\right) \sum_{\vec{k}\in Z^2} \left|\vec{A}_{\vec{k}}\right| e^{-|\vec{k}|\sigma_0} \leq \exp\left(-\frac{\pi\omega_+}{8\sqrt{\varepsilon}}\right), \tag{15}$$

where we've used the assumption on the coefficients $\vec{A}_{\vec{k}}$ and the well known equality $\sum_{\vec{k}\in Z^l} e^{-|\vec{k}|\sigma_0} = \coth^l\left(\frac{\sigma_0}{2}\right)$.

This estimate will turn out to be exponentially small compared to the asymptotics of the first sum Σ_1, to which we'll further restrict our attention.

The first sum Σ_1 in (14) has in fact only one index, since for each integer $q > 0$ there is only one p such that $|q\beta - p| < \frac{1}{2}$. To this sum we can apply the ultraviolet cutoff, when q becomes as large as $q_\varepsilon = [\varepsilon^{-\frac{1}{2}+\delta_0}]$, where $[\cdot]$ means an integer part, and $0 < \delta_0 \ll \frac{1}{2}$ to be specified. We write

$$\begin{aligned} \Sigma_1 &= \Sigma_1^< + \Sigma_1^> \\ &\equiv \left(\textstyle\sum_{p,q\in Z,\ 1\leq q\leq q_\varepsilon,\ |q\beta-p|<\frac{1}{2}} + \sum_{p,q\in Z,\ q>q_\varepsilon,\ |q\beta-p|<\frac{1}{2}}\right) \\ &\quad \times \exp\left(-(p+q)\sigma_0 - \omega_+\frac{\pi|q\beta-p|}{2\sqrt{\varepsilon}}\right) \vec{A}_{\vec{k}} \sin(\vec{k}\cdot\vec{\alpha}), \end{aligned} \tag{16}$$

where again and henceforward $\vec{k} = (-p, q)$.

It's easy to see that the ultraviolet part of the sum can be bounded as:

$$|\Sigma_1^>| \leq \exp\left(-\varepsilon^{-\frac{1}{2}+\delta_0+\delta_3}\right), \tag{17}$$

for any small $\delta_3 > 0$ if ε is small enough.

Moreover, one can write

$$\begin{aligned}\Sigma_1^< &= \bar{\Sigma}_1^< + \tilde{\Sigma}_1^< \\ &\equiv \textstyle\sum_{n\geq 0,\ q_n \leq q_\varepsilon} e^{-(p_n+q_n)\sigma_0 - \frac{\pi\omega_+ \|q_n\beta\|}{2\sqrt{\varepsilon}}} A_{\vec{k}^n} \sin(\vec{k}^n\cdot\vec{\alpha}) \\ &+ \textstyle\sum_{n\geq 0,\ q_{n-1}<q<\min(q_n, q_\varepsilon)} e^{-(p+q)\sigma_0 - \frac{\pi\omega_+ |q\beta - p|}{2\sqrt{\varepsilon}}} \vec{A}_{\vec{k}} \sin(\vec{k}\cdot\vec{\alpha}),\end{aligned} \tag{18}$$

with $\vec{k}^n = (-p_n, q_n)$ in the first sum and $\vec{k} = (-p, q)$ in the second one, the first one including the *convergents* q_n to β only (once again, with a slight abuse of terminology we use the term “convergents” to refer to the denominators of the best approximations), and the second one all the q’s that sit in between (recall that for every $q > 0$ there is one only p such that $|\beta q - p| < \frac{1}{2}$).

The ultraviolet cutoff that we have previously made, enables us to define a ”cutoff index” n_ε in (13) as:

$$n_\varepsilon \equiv \max\{j > 0 :\ q_j \leq q_\varepsilon\}. \tag{19}$$

Suppose q_{n-1}, q_n stand for the two consecutive convergents to β. Then in Rudnev and Wiggins [1997] it is shown that for $\tilde{\Sigma}_1^<$ we can estimate the absolute value of the sum of all the terms corresponding to $q_{n-1} < q < q_n$ as follows:

$$\begin{aligned}&\left|\textstyle\sum_{q_{n-1}<q<\min(q_n, q_\varepsilon)} e^{-(p+q)\sigma_0 - \frac{\pi\omega_+ |q\beta - p|}{2\sqrt{\varepsilon}}} \vec{A}_{\vec{k}} \sin(\vec{k}\cdot\vec{\alpha})\right| \\ &\leq q_\varepsilon (\max_{q\leq q_\varepsilon} |\vec{A}_{\vec{k}}|) \exp[-(|p_{n-1}| + q_{n-1})\sigma_0] \exp\left[-\left(1 + \tfrac{1}{\bar{a}+2}\right) \tfrac{\pi\omega_+ \|q_{n-1}\beta\|}{2\sqrt{\varepsilon}}\right].\end{aligned} \tag{20}$$

Then the ratio of the absolute value of the sum, estimated in (20) to a single coefficient, corresponding to the $(n-1)$th convergent in the first sum in (18) will be at most

$$q_\varepsilon \frac{\max_{q\leq q_\varepsilon} |\vec{A}_{\vec{k}}|}{\min_{q\leq q_\varepsilon} |\vec{A}_{\vec{k}}|} \exp\left(-\frac{1}{\bar{a}+2} \frac{\pi\omega_+ \|q_{n-1}\beta\|}{2\sqrt{\varepsilon}}\right) \leq \exp\left(-\frac{\pi\omega_+}{4(\bar{a}+2)^2 \varepsilon^{\delta_0}}\right),$$

where the estimate for $\|q_{n-1}\beta\|$ is obtained in Rudnev and Wiggins [1997] using simple properties of the convergents.

This in turn implies that in the Fourier norm ”with zero width” (which is the sum of the absolute values of the Fourier coefficients)

$$\left|\tilde{\Sigma}_1^<\right|_0 \leq \exp\left(-\frac{\omega_+}{4(\bar{a}+2)^2 \varepsilon^{\delta_0}}\right) \cdot \left|\bar{\Sigma}_1^<\right|_0. \tag{21}$$

The coefficient in the right-hand part of (21) will also be exponentially small for any $\delta_0 > 0$ provided that ε is small enough. Moreover, one can see that we can let $\bar{a}$ in (13) ”slightly” depend on ε, as done in Assumption 2.1, if this still guarantees that the factor in (21) is exponentially small and the estimates (15), (17) are exponentially small in comparison with the asymptotics that we are going to find. In our further estimate we will always keep $\bar{a}$, since we finally expect to let it grow as $\varepsilon \to 0$.

Besides, we notice that out of the three estimates (15), (17), (21) only the last one is non-trivial, and it imposes in fact pretty tight conditions upon the parameters δ_0 and δ_2, requiring in fact that the frequency vector is still far enough from resonances of low order. But we shall point out that this estimate is an artifact of an attempt to give uniform consideration to such frequency vectors, which have asymptotically full measure; the case of proximity to a certain low order resonance can be treated individually and won’t be too difficult, as the numerical results, given further suggest.

So, we are basically left with $\bar{\Sigma}_1^<$ only to evaluate. In other words,

$$\mathcal{M}(\vec{\alpha}) = \sum_{0\leq n\leq n_\varepsilon} \exp\left(-k_n\sigma_0 - \frac{\pi\omega_+ \|q_n\beta\|}{2\sqrt{\varepsilon}}\right) A_{\vec{k}^n} \sin(-p_n\alpha_1 + q_n\alpha_2) + \text{the error term}, \tag{22}$$

the latter being exponentially small. The summation above is taken over all the convergents under the cutoff. Here $\vec{k}^n = (-p_n, q_n)$, $k_n = p_n + q_n$. Moreover, for the exponents in the above formula, risking certain tautology, we shall still keep the notation E_n, as in (4), just keeping in mind for the future that the index n always refers to the Continued fraction, or to the convergents.

Our main tool, called in Rudnev and Wiggins [1997] the Transition lemma and described in the succeeding section, will guarantee that (22) can be represented by its one single term plus an exponentially small error for the vast majority of the values of ε small enough. However to establish non-degeneracy in $\vec{\alpha}$, it will require least two dominant terms.

2.3 The Transition Lemma

The upshot of the case of three degrees of freedom is that first one can easily establish that in the series for the splitting distance function, under the ultraviolet cutoff, those terms that correspond to the best approximations to β significantly exceed all the rest. This was argued in the previous section. Thus, the series can be indexed by one integer rather than two, and moreover, the number of terms in this series, provided the assumption 2.1 holds, is rather small, being roughly $\sim \log \varepsilon^{-1}$. Moreover, in the case when β is well approximable, which is not covered by the theory in Rudnev and Wiggins [1997], the main problem will consist in the uniformity of the estimates. For given a Continued fraction, the result of the judgment whether it is close to a resonance will depend on the value of ε as well as the finite part of the Continued fraction, which is central to this theory. Nevertheless, for near-resonant cases, given ε, it seems that the situation is similar, or even simpler than with the Assumption 2.1. This is illustrated by the numerical experiments 6.3 and 6.4 in the Appendix.

Analyzing the exponents of the terms of the series (22) consisting only of the convergents to β, we establish the following:

Lemma 2.1 (The Transition lemma. Qualitative version) *The sequence of the exponents in the sum (22) under the ultraviolet cutoff, provided that Assumption 2.1 holds for a small* δ_2*, (depending on the cutoff parameter* δ_0*), has either one single local, alias global minimum (V-shape) for some* $\vec{k}^* = (-p_{n_*}, q_{n_*})$*, where* p_{n_*} *and* q_{n_*} *are the numerator and the denominator of the* n_**th best approximation to* β*, or two neighboring local minima (W-shape) for some for* $\vec{k}^* = (-p_{n_*}, q_{n_*})$*, and* $\vec{k}^{**} = (-p_{n_*+2}, q_{n_*+2})$*. The latter case may occur only if the Continued fraction for* β *intermittently contains ones and larger integers.*

In the former (V-shape) case the second minimum exponent will be indexed by either $n_* - 1$ *or* $n_* + 1$*, in the latter case (W-shape)* n_* *and* $n_* + 2$ *yield the two minimum exponents.*

An example of the W-shape, described further in the Appendix (Experiment 6.2) corresponds to $\beta =$ [0222122212221...]. To describe the creation of the W-shape in more detail would require more elaborate necessary and sufficient conditions, which can be found in Rudnev and Wiggins [1997]. Generally, the techniques developed in Rudnev and Wiggins [1997], given ε and β, in principle allow one to find the minimum and the second minimum exponents.

All the other experiments in the Appendix display a sturdy V-shape in the sense of the sequence $\{E_n\}$ having a unique local minimum, even in situations that are beyond the applicability of the theory.

Thus, we are able to find the two maximum terms in the Fourier series for the splitting distance, which will account for its leading-order behavior as well as the measure of transversality (the maximum term alone does not suffice, for then the transversality measure can be written as a monomial $\frac{\partial^2}{\partial \vec{\alpha}^2} \cos(\vec{k} \cdot \vec{\alpha})$, which is zero for any $\vec{k}$ and $\vec{\alpha}$).

Besides, since the exponents explicitly depend on ε, we will have to avoid certain values of ε (referred to as *critical values*) to insure that the two neighboring exponents are never equal to one another, as well as certain intervals near the critical values (referred to as *critical intervals*) to insure that the above minima of the exponents are sharp enough. But as $\varepsilon \to 0$ the relative measure of the critical intervals goes to zero, plus their union is a nowhere dense set, accumulating at zero if ε is small enough.

Finally, a minimum for a badly approximable case, covered by Assumption 2.1 will always be such that the corresponding convergent will be $q_{\min} \sim \varepsilon^{-\frac{1}{4}}$. Moreover, for a well-approximable case such a minimum will be considerably larger and correspond to $q_{\min} << \varepsilon^{-\frac{1}{4}}$.