Fixed point theorems in the Arnol'd model about instability of the action-variables in phase-space

Paolo Perfetti
Dipartimento di matematica II Università di Roma
via della Ricerca Scientifica 00133 Roma, Italy

Abstract We consider the hamiltonian $H = \frac{1}{2}(I_1^2+I_2^2)+\varepsilon(\cos\varphi_1-1)(1+\mu(\sin\varphi_2+\cos t))$ $I \in \mathbb{R}^2$ ("Arnol'd model about diffusion"); by means of fixed point theorems, the existence of the stable and unstable manifolds *(whiskers)* of invariant, "a priori unstable tori", for any vector-frequency $(\omega, 1) \in \mathbb{R}^2$ is proven. Our aim is to provide detailed proofs which are missing in Arnol'd's paper, namely prove the content of the `Assertion B` pag.583 of [A]. Our proofs are based on technical tools suggested by Arnol'd i.e. the contraction mapping method togheter with the "conical metric" (see the footnote ** of pag. 583 of [A]).

§**0.Introduction** By means of contraction techniques we prove the existence of *whiskered tori* (i.e. invariant tori that possess stable and unstable manifolds) for the dynamical system whose hamiltonian is ("Arnol'd model")

$$H(I_1, I_2, \varphi_1, \varphi_2, t) = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\varphi_1 - 1) + \varepsilon\mu(\cos\varphi_1 - 1)(\sin\varphi_2 + \cos t).$$

The hamiltonian H is defined over the "extended phase-space" $\tilde{\mathcal{M}} = \mathcal{M} \times \mathbb{T} = \mathbb{R}^2 \times \mathbb{T}^3$; $(I_1, I_2) \in \mathbb{R}^2$, $(\varphi_1, \varphi_2, t) \in \mathbb{T}^3$ $(\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z})$, ε and μ are real parameters. Physically H represents a pendulum $(\frac{1}{2}I_1^2 + \varepsilon(\cos\varphi_1 - 1))$ coupled to a rotator $(\frac{1}{2}I_2^2)$ through a force periodically depending on φ_1, φ_2 and t. The canonical equations of motion are

$$\begin{cases} \dot\theta = 1, \qquad \dot\varphi_1 = I_1, \qquad \dot\varphi_2 = I_2 \\ \dot I_1 = \varepsilon\sin\varphi_1(1 + \mu(\sin\varphi_2 + \cos\theta)) \qquad \dot I_2 = -\varepsilon\mu\cos\varphi_2(\cos\varphi_1 - 1) \end{cases} \tag{0.0}$$

and we denote by $\phi_t^\mu \colon \tilde M \to \tilde M$ the associated hamiltonian flow.
For $\varepsilon = 0$ the system is integrable. For $0 < \varepsilon \le \varepsilon_o$ (ε_o small enough), as well known by KAM theorem, most (in the sense of Lebesgue measure) of the tori are continued. The measure of the initial data excluded by the theorem is $O(\sqrt{\varepsilon})$ (although they are dense).

Now for $\varepsilon \neq 0$ and $\mu = 0$, following [A], we consider the motions that "develop along the separatrix of the pendulum". We therefore write the equations

$$H^1(I_1,\varphi_1) = \frac{1}{2}I_1^2 + \varepsilon(\cos\varphi_1 - 1) = 0, \qquad H^2(I_2,\varphi_2) = \frac{1}{2}I_2^2 = \frac{1}{2}\omega^2$$

and consider their orbits. A standard calculation gives

$$\begin{aligned} &\theta(t) = \theta^o + t, \qquad I_2(t) \equiv \omega, \qquad I_1(t) \doteq \pm\beta(t) = \pm\frac{2\sqrt{\varepsilon}}{\cosh(\sqrt{\varepsilon}t)} \\ &\varphi_2(t) = \varphi_2^o + \omega t, \qquad \varphi_1(t) \doteq \pm\alpha(t) = \pi \pm 2\arctan\sinh(\sqrt{\varepsilon}t). \end{aligned} \tag{0.1}$$

$\phi_t^0(\pm 2\sqrt{\varepsilon},\pi,\omega,\varphi_2^o,\theta^o) : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ is the flow generated by the hamiltonian $H^0 = H^1 + H^2$ when the initial datum, for what concerns the variables (I_1,φ_1), is $(\pm 2\sqrt{\varepsilon},\pi)$. The invariant set $\mathbb{T}_\omega = \{(I_1,\varphi_1,I_2,\varphi_2,\theta)|(I_1 = 0, \varphi_1 = 0, I_2 = \omega, (\varphi_2,\theta) \in \mathbb{T}^2\}$ is geometrically an unstable torus (from which the definition "a priori unstable" used in [CG]) and it is run linearly: $\theta^o \to \theta^o + t$, $\varphi_2^o \to \varphi_2^o + \omega t$. Adopting the definitions of [A] we call Y_ω^- the set of points such that, $dist(\phi_t^0(\pm 2\sqrt{\varepsilon},\pi,\omega,\varphi_2^o,\theta^o),\mathbb{T}_\omega) \to 0$ as $t \to \infty$ (it is called the *stable manifold* of $\mathbb{T}_\omega$). It is a three dimensional manifold in a five dimensional phase-space. The *unstable manifold* at $\mathbb{T}_\omega$, Y_ω^+, is defined as the set of points such that $dist(\phi_t^0(\pm 2\sqrt{\varepsilon},\pi,\omega,\varphi_2^o,\theta^o),\mathbb{T}_\omega) \to 0$ as $t \to -\infty$; because of the degeneracy of the model we have $Y_\omega^- \equiv Y_\omega^+$ and $Y_\omega^+ \cap Y_{\omega'}^- = \emptyset \ \forall \omega \neq \omega'$. This last relation is connected to the impossibility of the trajectories (0.1) to make a "large variation" in the (I_1, I_2) variables; in fact I_2 is a constant while $|I_1| \leq 2\sqrt{\varepsilon}$.

Now set $\varepsilon \neq 0$ and $\mu \neq 0$. The torus $\mathbb{T}_\omega$ continues to be invariant for any value of μ and in [A] (`Theorem 3` pag.582) the following theorem ("Arnol'd diffusion") is stated

Assume $0 < A < B$. For any $\varepsilon > 0$ there exists a $\mu_o(\varepsilon, A, B)$ such that if $0 < \mu < \mu_o$ then $|I_2(t^o) - I_2(0)| \geq (B - A)$ where $t^o = t^o(\mu)$[1]

The proof is based on two fundamental steps:

1) `Assertion B` of [A]: "$\mathbb{T}_\omega$ *is a whiskered transition torus for μ small enough*"

2) $Y_\omega^+ \cap Y_{\omega'}^- \neq \emptyset$ for ω' close to ω.

The proof of Part 1) is the object of this paper. Part 2) is proven in [A].
In order to prove 1) we shall need:

---

[1] $\mu_o \sim e^{-1/\sqrt{\varepsilon}}$. Moreover it is easy to verify that $\mu_o \to 0$ as $B \to \infty$.

i) the existence (continuation) for $\mu \neq 0$ of the *stable manifold*, Y^-_ω, and *unstable manifold*, Y^+_ω, at $\mathbb{T}_\omega$ for any real ω (proved in §**2**; we have retained for the manifolds the same notation of the case $\mu = 0$). This is what in the step 1) is represented by the word "$\mathbb{T}_\omega$ is a whiskered torus for μ small enough"

ii) "The transition property" of $\mathbb{T}_\omega$ with $\omega \in \mathbb{R}\backslash\mathbb{Q}$ so that "$\mathbb{T}_\omega$ is a whiskered transition torus" (see §**4**).

Point ii) means: given an arbitrary point p of the *stable manifold* at $\mathbb{T}_\omega$ let us consider an arbitrary neighborhood U of p and set $\Omega = \cup_{t\geq 0}\phi_t(U)$ (ϕ_t is the flow of the system (0.0)). Now consider the *unstable manifold*, an arbitrary point q of it and an arbitrary manifold $\mathcal{N}$ that crosses Y^+_ω at q. Moreover the manifold $\mathcal{N}$ is such that $T_q\mathcal{N} \oplus T_qY^+_\omega = T_qM$ (i.e.$T_q\mathcal{N}$ and $T_qY^+_\omega$ span the whole tangent space; it is said that $\mathcal{N}$ *complements* Y^+_ω at q); then Ω intersects $\mathcal{N}$.

We do not prove ii) as it has been done in [Ma] and extended in [Cr] (they consider the hamiltonian functions previously considered in [Gr] and [Tr] respectively). The idea of the proof is based on a suitable generalization of the $\lambda-$lemma by Palis (see [PM] pag.80, [Wi] pag. 324, [Ru] pag.88)

As said above we make use of contraction arguments in order to prove the existence of *stable* and *unstable* manifolds. We will work in the analytic class as the Cauchy theorem makes the estimates very easy. Frequently other less regular function spaces are introduced in connection with fixed-point theorems but they imply more involved estimates adding little to the comprehension of the problem.

In recent years there have been many papers about instability of quasi-integrable dynamical system ([Xia1], [CG], [Xia2], [Ga2], [Xia3], [Ge1], [Ge2], [C1], [Moe], [RW1], [RW2], [C2], [Ni]). We just point out that the above papers use KAM theorem to prove the existence of the invariant tori or their stable and ustable manifolds. This forces to exclude, as usual, a dense set of initial data of small Lebesgue measure. The presence of holes in the space of the actions complicates a lot the construction of the transition chains (see [CG]).

¿From a non-perturbative point of view we cite the work of Bessi [Be]. Bessi uses variational techniques in order to prove the existence of trajectories connecting far away regions of phase-space and finds good estimates about the times of transition.

The plan of the paper is the following: in §**1** we construct a suitable Poincaré map, say M, for the system (0.0); in §**2** we construct the stable and unstable manifold, called $\mathcal{Y}^-$ and $\mathcal{Y}^+$ respectively, for M; in §**3** using the equations of $\mathcal{Y}^-$ and $\mathcal{Y}^+$ we linearize the motion on $\mathcal{Y}^\pm$; finally in §**4** we show how [Ma] applies to our setting in order to prove the transition property (see ii) above).

Acknowledgments I thank Luigi Chierchia for many careful comments, advises and suggestions.

§**1.Setup** In this section we do a suitable Poincaré map on the system (0.0). Therefore the relevant quantities that we shall consider: tori, stable and unstable manifolds are all relative to the map.

With the transformation over the variables $(I_1, \varphi_1, I_2, \varphi_2, \theta) = U(q, p, I_2, \varphi_2, \theta)$ made of two canonical maps plus a rescaling the equations (0.0) become (more details are in Appendix 1)

$$\begin{cases} \dot\theta = 1, \qquad \dot\varphi_2 = I_2, \qquad \dot I_2 = -\varepsilon\mu\cos\varphi_2(\cos\tilde\varphi_1(q,p) - 1) \\ \dot q = \sqrt{\varepsilon}q(1+F_J) - \dfrac{\sqrt{\varepsilon}}{\sqrt{2}}\mu\sin\tilde\varphi_1(q,p)(\sin\varphi_2 + \cos\theta) \\ \dot p = -\sqrt{\varepsilon}p(1+F_J) + \dfrac{\sqrt{\varepsilon}}{\sqrt{2}}\mu\sin\tilde\varphi_1(q,p)(\sin\varphi_2 + \cos\theta) \end{cases} \tag{1.0}$$

$F(J) = O(J^2)$ and $\tilde\varphi_1$ are analytic functions of q and p in the region $|q| + |p| \le O(1)$; $\tilde\varphi_1(0,0) = 0$.

¿From the equations (1.0) one immediately realizes that:

1) $\mathbb{T}_\omega = \{(q, p, I_2, \varphi_2, \theta) | q = 0, p = 0, I_2 = \omega, (\varphi_2, \theta) \in \mathbb{T}^2\}$ is an invariant set for any value of the parameter μ and is linearly i.e. $\varphi_2(t) = \varphi_2^o + \omega t$, $\theta(t) = \theta^o + t$.

2) if $\mu = 0$ the q and p axis are invariant. $F(J)$ is a first integral (actually any function of J is a prime integral) and for $|q|+|p|$ small enough it is $1+F_J > 0$. Then the points of the p axis are exponentially shrunk to 0 while the points of the q axis are exponentially pushed away from the origin. In the new variables $(q, p, I_2, \varphi_2, \theta)$ the *stable manifold*, Y_ω^- is written as $\{(q, p, I_2, \varphi_2, \theta) | q = 0, p \in (-\sigma, \sigma), I_2 = \omega, (\varphi_2, \theta) \in \mathbb{T}^2\}$ while the *unstable manifold* Y_ω^+ as $\{(q, p, I_2, \varphi_2, \theta) | q \in (-\sigma, \sigma), p = 0, I_2 = \omega, (\varphi_2, \theta) \in \mathbb{T}^2\}$ $(\sigma = O(1))$.

Now we do the following Poincaré-map: let $\phi_t^\mu(q^o, p^o, I_2^o, \varphi_2^o, 0) \colon \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ be the flow of the system (1.0) (0 stands for $\theta^o = 0$ and we omit it for brevity). Denoting $r^o = (q^o, p^o, I_2^o, \varphi_2^o)$, for any integer n we define $\phi_{2n\pi}^\mu(r^o, 0)|_{\mathcal{M}} \doteq \hat\phi_{2n\pi}^\mu(r^o) \colon B_\sigma^2 \times \mathbb{R} \times \mathbb{T} \to B_\sigma^2 \times \mathbb{R} \times \mathbb{T}$ (B_σ^2 is a two-dimensional sphere of radius $2\sigma^2$ in the (q, p) space). The map $\hat{\phi}^{\mu}_{2\pi}$ is written symbolically as $(r^o)' \doteq (x', y', I', \varphi') = \hat{\phi}^{\mu}_{2\pi}(x, y, I, \varphi) \doteq \hat{\phi}^{\mu}_{2\pi}(r^o)$

$$\begin{cases} \theta' = \theta = 0 \\ y' = ye^{-2\pi\sqrt{\varepsilon}(1+F_J)} + \mu\sqrt{\varepsilon}f_1(x, y, I, \varphi, \mu) \\ x' = xe^{2\pi\sqrt{\varepsilon}(1+F_J)} - \mu\sqrt{\varepsilon}f_2(x, y, I, \varphi, \mu) \\ \varphi' = \varphi + 2\pi I + \mu\varepsilon f_3(x, y, I, \varphi, \mu) \\ I' = I - \mu\varepsilon f_4(x, y, I, \varphi, \mu) \end{cases} \tag{1.1}$$

where $f_1,\ f_2 = O(|x| + |y|) \qquad f_3,\ f_4 = O((|x| + |y|)^2)$. In fact we have

$$\begin{cases} q(2\pi; r^o) = q^o e^{2\pi\sqrt{\varepsilon}(1+F_J)} - \mu\dfrac{\sqrt{\varepsilon}}{\sqrt{2}}\displaystyle\int_0^{2\pi} ds \sin\tilde{\varphi}_1(q, p)(\sin\varphi_2 + \cos s) \\ \qquad\qquad + \sqrt{\varepsilon}\displaystyle\int_0^{2\pi} ds\Big[\big(q(1 + F_J)\big)(s; r^o, \mu) - \big(q(1 + F_J)\big)(s; r^o, 0)\Big] \\ p(2\pi; r^o) = p^o e^{-2\pi\sqrt{\varepsilon}(1+F_J)} - \mu\dfrac{\sqrt{\varepsilon}}{\sqrt{2}}\displaystyle\int_0^{2\pi} ds \sin\tilde{\varphi}_1(q, p)(\sin\varphi_2 + \cos s) - \\ \qquad\qquad - \sqrt{\varepsilon}\displaystyle\int_0^{2\pi} ds\Big[\big(p(1 + F_J)\big)(s; r^o, \mu) - \big(p(1 + F_J)\big)(s; r^o, 0)\Big] \\ \varphi_2(2\pi; r^o) = \varphi_2^o + 2\pi I_2^o + \mu\varepsilon\displaystyle\int_0^{2\pi} dt\int_0^t ds \cos\varphi_2(\cos\tilde{\varphi}_1(q, p) - 1) \\ I_2(2\pi; r^o) = I_2^o - \mu\varepsilon\displaystyle\int_0^{2\pi} ds \cos\varphi_2(\cos\tilde{\varphi}_1(q, p) - 1) \end{cases} \tag{1.2}$$

In analogy with the differential equations (0.0) we define: $\mathcal{Y}^-_\omega = \{r^o \in B^2_\sigma \times \mathbb{R} \times \mathbb{T} | dist(\hat{\phi}^{\mu}_{2\pi n}(r^o), \mathcal{T}_\omega) \to 0$ as $n \to +\infty$ (the *stable manifold* at $\mathcal{T}_\omega$) and $\mathcal{Y}^+_\omega = \{r^o \in B^2_\sigma \times \mathbb{R} \times \mathbb{T} | dist(\hat{\phi}^{\mu}_{-2\pi n}(r^o), \mathcal{T}_\omega) \to 0$ as $n \to \infty$ (the *unstable manifold* at $\mathcal{T}_\omega$). A different value of θ^o would generate a different Poincaré map. If we call $\mathcal{Y}^-_\omega(\theta^o)$ the stable manifold, $\mathcal{Y}^+_\omega(\theta^o)$ the unstable one and $\mathcal{T}_\omega(\theta^o)$ the invariant torus of frequency ω for the Poincaré map at θ^o i.e. taking as surface map $\{(p, q) \in B^2_\sigma, I \in \mathbb{R}, (\varphi, \theta) \in \mathbb{T}^2 | \theta = \theta^o\}$, then we have $Y^-_\omega = \cup_{0 \le \theta^o \le 2\pi}(\mathcal{Y}^-_\omega)(\theta^o)$, $Y^+_\omega = \cup_{0 \le \theta^o \le 2\pi}(\mathcal{Y}^+_\omega)(\theta^o)$ and $\mathbb{T}_\omega = \cup_{0 \le \theta^o \le 2\pi}\mathcal{T}_\omega(\theta^o)$ (see Appendix 1). We then reduce to the study of the existence of $\mathcal{Y}^-_\omega$ and $\mathcal{Y}^+_\omega$; the existence of $\mathcal{T}_\omega$ is trivial matter due to the peculiarity of the model.

We have to keep in mind that after the integrations with respect to time, there remain functions only of the initial data r^o. For any $I^o \in \mathbb{R}$ and for any μ the set $\mathcal{T}_{I^o} = \{q = 0, p = 0, I = I^o, \varphi \in \mathbb{T}\}$ is invariant. Geometrically it is a circle contained in the hyperplane $\theta = 0$; on it the motion is linear $\varphi_2^{(n)} = \varphi_2^{(0)} + 2\pi I^o n$, $n \in \mathbb{N}$; it is periodic if $I^o \in \mathbb{Q}$ and quasi-periodic if $I^o \in \mathbb{R} \backslash \mathbb{Q}$. Of course we have $\mathbb{T}_{I^o} = \cup_{0 \le \theta^o \le 2\pi}\mathcal{T}_{I^o}(\theta^o)$.

When $\mu = 0$ the following sets are also invariant: $\mathcal{Y}^-_{I^o} = \{q = 0, p \in (-\sigma, +\sigma), I = I^o, \varphi \in \mathbb{T}\}$ (the stable manifold) and $\mathcal{Y}^+_{I^o} = \{q \in (-\sigma, +\sigma), p = 0, I = I^o, \varphi \in \mathbb{T}\}$ (the unstable manifold); they are two dimensional in the four dimensional space $\mathbb{R}^3 \times \mathbb{T}$.

§2.Existence of perturbed stable and unstable manifolds

Now we investigate the point i) of the introduction. ¿From now on we shall consider only the map (1.1).

What follows is closely related to the Hadamard's theorem about the existence of stable and unstable manifolds of fully hyperbolic systems and in general to the theory of invariant sets for flows and maps defined on a manifold (possibly $\mathbb{R}^N$). There is an abundant literature on this argument and we mention only few references [Ke], [Fe1], [Fe2], [Fe3], [La2], [La1], [Ga3], [Ze], [HPS], [Wi], [PM], [Ir], [Ru].
Following Hadamard's approach and using a suggestion by Arnol'd (see footnote ** of pag.583 of [A]) we look for the stable manifold at $\mathcal{T}_\omega$ as $\mathcal{Y}^-_\omega = \{(x, y, I, \varphi) | x = \mu y A^s(y, \varphi; \omega, \mu), y \in (-\sigma, +\sigma), I = \omega + \mu\sqrt{\varepsilon} y \mathcal{I}^s(y, \varphi; \omega, \mu), \varphi \in \mathbb{T}\}$ while $\mathcal{Y}^+_\omega = \{(x, y, I, \varphi) | x \in (-\sigma, +\sigma), y = \mu x A^u(x, \varphi; \omega, \mu), I = \omega + \mu\sqrt{\varepsilon} y \mathcal{I}^u(x, \varphi; \omega, \mu), \varphi \in \mathbb{T}\}$. In order to simplify the estimates (leaving all the essential difficulties) we shall consider the space of the holomorphic functions $\mathcal{A}(\Delta^{\sigma,\xi}_{y,\varphi})$ (for $\mathcal{Y}^-_\omega$) given by the analytic functions over $(-\sigma, \sigma) \times \mathbb{T}$ that admit an holomorphic extension to the complex domain $\Delta^{\sigma,\xi}_{y,\varphi} = \{(y, \varphi) \in \mathbb{C}^2 |\ |y| < \sigma, |Im\varphi| < \xi < 1\}$ and are continuous on $\overline{\Delta}^{\sigma,\xi}_{y,\varphi}$. With the norm $\|g\| = \sup_{\overline{\Delta}^{\sigma,\xi}_{y\varphi}} |g(y, \varphi)|$, $\mathcal{A}(\Delta^{\sigma,\xi}_{y,\varphi})$ is a Banach space. Of course for $\mathcal{Y}^+_\omega$ the definition of the space of functions is exactly the same except that there is x in place of y and u in place of s. Now we prove the following

Theorem 2.0 *For any $\omega \in \mathbb{R}\backslash\{0\}$ and for any $|\mu| \leq \mu_o$ there exist two unique functions $(A^s, \mathcal{I}^s): [-\frac{\sigma}{2}, \frac{\sigma}{2}] \times \mathbb{T} \to \mathbb{R} \times \mathbb{R}$ such that the set $\mathcal{Y}^-_\omega = \{(x, y, I, \varphi) = (\mu y A^s(y, \varphi; \omega, \mu, \varepsilon), y, \omega + \mu\sqrt{\varepsilon} y \mathcal{I}^s(y, \varphi; \omega, \mu, \varepsilon), \varphi) | y \in [-\frac{\sigma}{2}, \frac{\sigma}{2}],\ \varphi \in \mathbb{T}\}$ is invariant respect to the Poincaré map (1.1). A^s and $\mathcal{I}^s$ are analytic in (y, φ), in $\omega \in \mathbb{R}\backslash\{0\}$ and μ.*

Remarks i) in the functions A^s and $\mathcal{I}^s$, φ and y are the variables while the presence of ω, μ and ε is of parametric type ii) clearly the theorem could be formulated choosing less regular function spaces (see e.g. [La1]) iii) $\sigma = O(1)$; μ_o is a function of σ, ε, ξ, and of the Poincaré map; moreover $\mu_o \to 0$ as $\sigma \to 0$, as $\varepsilon \to 0$ and $\varepsilon \to \infty$ iv) in the next section we will prove that $\mathcal{Y}^-_\omega$ is the stable manifold

Proof. Let's consider the closed set $\bar{\mathcal{A}} \subset \mathcal{A}(\Delta^{\sigma,\xi}_{y\varphi})$ whose functions have the norm bounded by a constatnt $\mathcal{A}$ $\|g\|_{\sigma,\xi} = \sup_{\Delta^{\sigma,\xi}_{y,\varphi}} |g(y,\varphi)| \le A_o$. Moreover we will consider the functions from $\Delta^{\sigma,\xi,\rho}_{x,y,I,\varphi}$ into $\mathbb{C}$ where $\Delta^{\sigma,\xi,\rho}_{x,y,I,\varphi} = \{(x,y,I,\varphi) \in \mathbb{C}^4 |\ |x| < \sigma, |y| < \sigma, |I-\omega| < \rho, |Im\varphi| < \xi < 1\}$. Analogously to what happens in $\mathcal{A}(\Delta^{\sigma,\xi}_{y\varphi})$ we define $\|h\|_{\sigma,\xi,\rho} = \sup_{\overline{\Delta}^{\sigma,\xi,\rho}_{x,y,I,\varphi}} |h(x,y,I,\varphi)|$ for functions defined and analytic on $\Delta^{\sigma,\xi,\rho}_{x,y,I,\varphi}$ and continuous on $\overline{\Delta}^{\sigma,\xi,\rho}_{x,y,I,\varphi}$.

The function $(A^s, \mathcal{I}^s)$ will be found as fixed point of a contraction map from $\bar{\mathcal{A}} \times \bar{\mathcal{A}}$ into itself. Then the uniqueness and analiticity immediately follow.

First of all we impose the invariance. A brief computation gives

$$\begin{aligned}
A = &\frac{\sqrt{\varepsilon}}{ay} f_2(\mu y A, y, \omega + \mu\sqrt{\varepsilon} y\mathcal{I}, \varphi) + \frac{1}{a}\Big(\frac{1}{a} + \frac{\mu\sqrt{\varepsilon}}{y} f_1(\mu y A, y, \omega + \mu\sqrt{\varepsilon} y\mathcal{I}, \varphi)\Big) A\Big(\frac{y}{a} + \\
&+ \mu\sqrt{\varepsilon} f_1(\mu y A, y, \omega + \mu\sqrt{\varepsilon} y \mathcal{I}, \varphi), \\
&\qquad , \varphi + 2\pi\omega + 2\pi\mu\sqrt{\varepsilon} y\mathcal{I} + \varepsilon\mu f_3(\mu y A, y, \omega + \mu\sqrt{\varepsilon} y\mathcal{I}, \varphi)\Big) \\
\mathcal{I} = &\frac{\sqrt{\varepsilon}}{y} f_4(\mu y A, y, \omega + \mu\sqrt{\varepsilon} y\mathcal{I}, \varphi) + \Big(\frac{1}{a} + \frac{\mu\sqrt{\varepsilon}}{y} f_1(\mu y A, y, \omega + \mu\sqrt{\varepsilon} y\mathcal{I}, \varphi)\Big)\mathcal{I}\Big(\frac{y}{a} + \\
&+ \mu\sqrt{\varepsilon} f_1(\mu y A, y, \omega + \mu\sqrt{\varepsilon} y\mathcal{I}, \varphi), \\
&\qquad , \varphi + 2\pi\omega + 2\pi\mu\sqrt{\varepsilon} y\mathcal{I} + \mu\varepsilon f_3(\mu y A, y, \omega + \mu\sqrt{\varepsilon} y\mathcal{I}, \varphi)\Big)
\end{aligned} \tag{2.0}$$

and $a = e^{2\pi\sqrt{\varepsilon}(1+F_J)}$.

We indicate (2.0) as $M(A,\mathcal{I})(y,\varphi) = \big(M_1(A,\mathcal{I}), M_2(A,\mathcal{I})\big)(y,\varphi)$ and try to solve it by means of contraction techniques: we look for two functions $(A^s, \mathcal{I}^s)$ such that $M_1(A^s,\mathcal{I}^s) = A^s, M_2(A^s,\mathcal{I}^s) = \mathcal{I}^s$. We show that there exists a real number $0 < q < 1$ such that

$$\|M(A,\mathcal{I}) - M(A',\mathcal{I}')\|_{\sigma',\xi'} \le q\|(A,\mathcal{I}) - (A',\mathcal{I}')\|_{\sigma',\xi'} \qquad \sigma' = \frac{\sigma}{2} \quad \xi' = \frac{\xi}{4}$$

and $\|M(A,\mathcal{I})\| = \|M_1(A,\mathcal{I})\| + \|M_2(A,\mathcal{I})\|$.
More precisely

$$\|M(A,\mathcal{I}) - M(A',\mathcal{I}')\|_{\sigma',\xi'} \le q\big(\|A - A')\|_{\sigma',\xi'} + \|\mathcal{I} - \mathcal{I}'\|_{\sigma',\xi'}\big) \tag{2.1}$$

and $q \le e^{-\pi\sqrt{\varepsilon}} + \mu\frac{P(\varepsilon)A_o^2}{\sigma\xi}$, $(P(\varepsilon) = 1 + 87\sqrt{\varepsilon} + 1521\varepsilon)$. $q < 1$ implies $\mu\frac{A_o^2 P(\varepsilon)}{\sigma\xi} < 1 - e^{-\pi\sqrt{\varepsilon}}$ and a sufficient condition is $\mu \le \mu_o(\sigma,\xi,A_o,\varepsilon) = \frac{\sigma\xi}{2P(\varepsilon)A_o^2}(1 - e^{-\pi\sqrt{\varepsilon}})$;[2] for $\mu = \mu_o$ it is $q = \frac{1}{2}(1 - e^{-\pi\sqrt{\varepsilon}}) < 1$

[2]The estimate of the size of μ looks a little complicated. We can bound $\mu_o < \frac{\sigma\xi}{2A_o^2}\pi\sqrt{\varepsilon}$ which is the asymptotic formula when ε goes to zero.

About the proof we use the Cauchy estimates on the holomorphic functions (see the Appendix 1). We take $\mu \leq \frac{1}{2A_o}\min\left\{\sigma, \frac{\xi}{5\pi\sqrt{\varepsilon}}\right\}$, $\sigma \leq 1, \sqrt{\varepsilon} \leq \rho$, $\sup_{|J|\leq\sigma^2}|F_J| \leq \frac{1}{2}$, $A_o \geq \max\left\{1, \frac{2\tilde{F}}{\pi}\right\}$, $e^{\pi\sqrt{\varepsilon}} \leq a \leq e^{\frac{3}{2}\pi\sqrt{\varepsilon}}$

$$\max_{j=1,2}\max_{s=0,1}\{\|(x^2+y^2)^{-s/2}f_j\|_{\sigma,\xi,\rho} = \sup_{\Delta^{\sigma,\xi,\rho}_{x,y,I,\varphi}}|(x^2+y^2)^{-s/2}f_j(x,y,I,\varphi)|\} = F$$
$$\max_{s=0,1,2}\{\|(x^2+y^2)^{-s/2}f_4\|_{\sigma,\xi,\rho} = \sup_{\Delta^{\sigma,\xi,\rho}_{x,y,I,\varphi}}|(x^2+y^2)^{-s/2}f_4(x,y,I,\varphi)|\} = F_1$$
$$\max_{s=0,1,2}\{\|(x^2+y^2)^{-s/2}\rho f_3\|_{\sigma,\xi,\rho} = \sup_{\Delta^{\sigma,\xi,\rho}_{x,y,I,\varphi}}|(x^2+y^2)^{-s/2}\rho f_3(x,y,I,\varphi)|\} = F_2$$

and $\tilde{F} = \max\{F, F_1, F_2\}$; then obtain the estimates

$$\begin{aligned}\|M_1(A,\mathcal{I}) - M_1(A',\mathcal{I}')\|_{\sigma',\xi'} \leq &\|A-A'\|_{\sigma',\xi'}(e^{-2\pi\sqrt{\varepsilon}} + (1+39\sqrt{\varepsilon})^2\frac{\mu}{3\sigma\xi}A_o^2)+\\ &+\|I-I'\|_{\sigma',\xi'}2(1+\sqrt{\varepsilon})\frac{\mu\sqrt{\varepsilon}}{\xi}A_o^2\end{aligned} \tag{2.2}$$

$$\begin{aligned}\|M_2(A,\mathcal{I}) - M_2(A',\mathcal{I}')\|_{\sigma',\xi'} \leq &\|I-I'\|_{\sigma',\xi'}(e^{-\pi\sqrt{\varepsilon}} + (1+6\pi\sqrt{\varepsilon})^2\frac{\mu}{\sigma\xi}A_o)+\\ &+\|A-A'\|_{\sigma',\xi'}\frac{20}{3}(\frac{4}{3}+\frac{\pi\sqrt{\varepsilon}}{4})\frac{\mu\sqrt{\varepsilon}}{\sigma\xi}A_o^2\end{aligned} \tag{2.3}$$

Reuniting (2.2) and (2.3) we get (2.1) ∎

A theorem analogous to 2.0 could be formulated in order to find the *unstable manifold* $\mathcal{Y}^+_\omega$. We write only its statement

Theorem 2.1 *For any* $\omega \in \mathbb{R}\backslash\{0\}$, $|\mu| \leq \mu_o$ *there exist two unique functions* $(A^u, \mathcal{I}^u)\colon[-\frac{\sigma}{2},\frac{\sigma}{2}]\times\mathbb{T} \to \mathbb{R}\times\mathbb{R}$ *such that the set* $\mathcal{Y}^+_\omega = \{(x,y,I,\varphi) = (x, \mu x A^u(x,\varphi;\omega,\mu,\varepsilon), \omega + \mu\sqrt{\varepsilon}x\mathcal{I}^u(x,\varphi;\omega,\mu,\varepsilon),\varphi) | x \in [-\frac{\sigma}{2},\frac{\sigma}{2}],\ \varphi\in\mathbb{T}\}$ *is invariant respect to the Poincaré map (1.1). The functions* A^u *and* $\mathcal{I}^u$ *are analytic in* (x,φ), *in* $\omega\in\mathbb{R}\backslash\{0\}$ *and in* μ.

§**3.Linearizations** In this section we write the linearizations of the motion. Roughly speaking we find new variables and functions, say p and $\Gamma(p)$, such that if we indicate the Poincaré map with $x' = F(x)$ and with $x = \Gamma(p)$ the change of coordinates, we then have $p' = \Gamma^{-1}\circ F\circ\Gamma(p) = \alpha p$ and α a constant. Unfortunately using the contaction arguments we can linearize all the variables only “on the whiskers” while near them we should use the KAM theorem. In [Ma] and [Cr] the authors prove that this is enough for an extension of the $\lambda-$lemma to this type of problem.

We make the following, regular, change of variables
$(\xi,\eta,\bar I,\bar\varphi)=U(x,y,I,\varphi)\colon \Delta^{\sigma_o,\xi/4}_{x,y,I,\varphi}\to\Delta^{\sigma_1,\xi/4}_{\xi,\eta,\bar I,\bar\varphi}$ where
$U(x,y,I,\varphi)=(x-\mu yA^s(y,\varphi;\omega,\mu),y-\mu xA^u(x,\varphi;\omega,\mu),I,\varphi)$ $(\sigma_o\doteq\frac{\sigma}{2}-\frac{\sigma}{2}\mu_oA_o$ $(\sigma_1=\frac{\sigma}{2}-\frac{3}{2}\sigma\mu_oA_o$ and being μ_oA_o very small the map U is onto and invertible obtaining

$$x=X(\xi,\eta,\bar\varphi;\omega,\mu,\varepsilon)=\xi+O(\mu),\quad y=Y(\xi,\eta,\bar\varphi;\omega,\mu,\varepsilon)=\eta+O(\mu),\quad I=\bar I,\quad \varphi=\bar\varphi.$$

We have of course[3] $X(0,0,\varphi)=Y(0,0,\varphi)\equiv 0$
$X(0,\eta,\varphi)=\mu Y(0,\eta,\varphi)A^s\big(Y(0,\eta,\varphi),\bar\varphi\big)$ $Y(\xi,0,\varphi)=\mu X(\xi,0,\varphi)A^u\big(X(\xi,0,\varphi),\bar\varphi\big)$
The Poincaré map relative to the new variables $(\xi,\eta,\bar I,\bar\varphi)$, is

$$\begin{aligned}
\xi'&=\xi a(XY)+\mu g(X(\xi,\eta,\varphi),Y(\xi,\eta,\varphi),I,\varphi)\\
\eta'&=\frac{\eta}{a(XY)}+\mu h(X(\xi,\eta,\varphi),Y(\xi,\eta,\varphi),I,\varphi)\\
\varphi'&=\varphi+2\pi I+\mu\varepsilon f_3(X(\xi,\eta,\varphi),Y(\xi,\eta,\varphi),I,\varphi)\\
I'&=I-\mu\varepsilon f_4(X(\xi,\eta,\varphi),Y(\xi,\eta,\varphi),I,\varphi)
\end{aligned}\tag{3.0}$$

where

$$\begin{aligned}
g(X(\xi,\eta,\varphi),Y(\xi,\eta,\varphi),I,\varphi)&=a(XY)YA^s\big(Y,\varphi\big)-\sqrt{\varepsilon}f_2(X,Y,I,\varphi)-\frac{Y}{a(XY)}A^s\big(\frac{Y}{a}+\\
&+\sqrt{\varepsilon}\mu f_1(X,Y,I,\varphi),\varphi+2\pi I+\mu\varepsilon f_3(X,Y,I,\varphi)\big)-\sqrt{\varepsilon}\mu f_1(X,Y,I,\varphi)A^s\big(\frac{Y}{a}+\\
&\qquad+\sqrt{\varepsilon}\mu f_1(X,Y,I,\varphi),\varphi+2\pi I+\mu\varepsilon f_3(X,Y,I,\varphi)\big),
\end{aligned}$$

$$\begin{aligned}
h(X(\xi,\eta,\varphi),Y(\xi,\eta,\varphi),I,\varphi)&=\frac{X}{a(XY)}A^u(X,\varphi)+\sqrt{\varepsilon}f_1(X,Y,I,\varphi)-Xa(XY)\cdot\\
&\cdot A^u\big(a(XY)X-\mu\sqrt{\varepsilon}f_2(X,Y,I,\varphi),\varphi+2\pi I+\mu\varepsilon f_3(X,Y,I,\varphi)\big)-\mu\sqrt{\varepsilon}f_2(X,Y,I,\varphi)\cdot\\
&\cdot A^u\big(a(XY)X-\mu\sqrt{\varepsilon}f_2(X,Y,I,\varphi),\varphi+2\pi I+\mu\varepsilon f_3(X,Y,I,\varphi)\big).
\end{aligned}$$

The theorems 2.0 and 2.1 of course imply that

$$\begin{aligned}
&g(X(0,\eta,\varphi),Y(0,\eta,\varphi),\omega+\mu\sqrt{\varepsilon}Y(0,\eta,\varphi)\mathcal{I}^s(Y(0,\eta,\varphi),\varphi)\equiv 0\\
&h(X(\xi,0,\varphi),Y(\xi,0,\varphi),\omega+\mu\sqrt{\varepsilon}X(\xi,0,\varphi)\mathcal{I}^u(X(\xi,0,\varphi),\varphi)\equiv 0.
\end{aligned}$$

[3] In almost all the function we meet or define there is a parametric ω, μ and ε dependences but sometimes we do not write them for brevity.

Now let's define

$$
\begin{aligned}
&a(XY) \equiv b(\xi,\eta,\varphi), \quad b(\xi,\eta,\varphi)|_{\mu=0} = a(\xi\eta) = e^{2\pi\sqrt{\varepsilon}(1+F_J(J))}, \quad b(0,0,\varphi) \equiv e^{2\pi\sqrt{\varepsilon}} \\
&\omega + \mu\sqrt{\varepsilon}Y(0,\eta,\varphi)\mathcal{I}^s(Y(0,\eta,\varphi),\varphi) \doteq \omega + \mu\sqrt{\varepsilon}\mathcal{I}^s_\omega(\eta,\varphi) \\
&\omega + \mu\sqrt{\varepsilon}X(\xi,0,\varphi)\mathcal{I}^u(X(\xi,0,\varphi),\varphi) \doteq \omega + \mu\sqrt{\varepsilon}\mathcal{I}^u_\omega(\xi,\varphi)
\end{aligned}
\tag{3.1}
$$

and write the Poincaré map restricted respectively on the *stable manifold* and *unstable manifold* (which are invariant sets of course)

$$
\begin{cases}
\eta' = \dfrac{\eta}{b(\eta,\varphi)} + \mu h(\eta,\varphi) & \xi = 0, \qquad I = \omega + \mu\sqrt{\varepsilon}\mathcal{I}^s_\omega(\eta,\varphi) \\
\varphi' = \varphi + 2\pi\omega + 2\pi\mu\sqrt{\varepsilon}\mathcal{I}^s_\omega(\eta,\varphi) + \mu\varepsilon F_3(\eta,\varphi) &
\end{cases}
\tag{3.2}
$$

$$
\begin{cases}
\xi' = \xi b(\xi,\varphi) + \mu g(\xi,\varphi) & \eta = 0, \qquad I = \omega + \mu\sqrt{\varepsilon}\mathcal{I}^u_\omega(\xi,\varphi) \\
\varphi' = \varphi + 2\pi\omega + 2\pi\mu\sqrt{\varepsilon}\mathcal{I}^u_\omega(\xi,\varphi) + \mu\varepsilon F_3'(\xi,\varphi) &
\end{cases}
\tag{3.3}
$$

where

$$
\begin{aligned}
&h(\mu Y(0,\eta,\varphi)A^s(Y(0,\eta,\varphi),\varphi), Y(0,\eta,\varphi), \omega + \mu\sqrt{\varepsilon}\mathcal{I}^s_\omega(\eta,\varphi),\varphi) \doteq h(\eta,\varphi) \\
&g(X(\xi,0,\varphi), \mu X(\xi,0,\varphi)A^u(X(\xi,0,\varphi),\varphi), \omega + \mu\sqrt{\varepsilon}\mathcal{I}^u_\omega(\xi,\varphi),\varphi) \doteq g(\xi,\varphi) \\
&f_3(\mu Y(0,\eta,\varphi)A^s(Y(0,\eta,\varphi),\varphi), Y(0,\eta,\varphi), \omega + \mu\sqrt{\varepsilon}\mathcal{I}^s_\omega(\eta,\varphi),\varphi,\mu) \doteq F_3(\eta,\varphi) \\
&f_4(X(\xi,0,\varphi), \mu X(\xi,0,\varphi)A^u(X(\xi,0,\varphi),\varphi), \omega + \mu\sqrt{\varepsilon}\mathcal{I}^u_\omega(\xi,\varphi),\varphi,\mu) \doteq F_3'(\xi,\varphi) \\
&a(\mu Y^2(0,\eta,\varphi)A^s(Y(0,\eta,\varphi),\varphi)) = b(\eta,\varphi), \qquad \mathcal{I}^s_\omega(0,\varphi) \equiv 0 \\
&a(\mu X^2(\xi,0,\varphi)A^u(X(\xi,0,\varphi),\varphi)) = b(\xi,\varphi), \qquad \mathcal{I}^u_\omega(0,\varphi) \equiv 0
\end{aligned}
\tag{3.4}
$$

Actually we have $\lim_{\eta\to 0}\eta^{-2}\mathcal{I}^s_\omega(\eta,\varphi) \equiv 0$ and $\lim_{\xi\to 0}\xi^{-2}\mathcal{I}^u_\omega(\xi,\varphi) \equiv 0$. In fact we observe that $Y(0,\eta,\varphi) \to 0$ as $\eta \to 0$ and $X(\xi,0,\varphi) \to 0$ as $\xi \to 0$ and being X, Y analytic they make finite the following limit $\lim_{\eta\to 0}\eta^{-1}Y(0,\eta,\varphi)$ as well as $\lim_{\xi\to 0}\xi^{-1}X(\xi,0,\varphi)$. So we are left with the relations $\lim_{\eta\to 0}\eta^{-1}\mathcal{I}^s(Y(0,\eta,\varphi)\varphi) \equiv 0$ and $\lim_{\xi\to 0}\xi^{-1}\mathcal{I}^u(X(\xi,0,\varphi)\varphi) \equiv 0$. Now we set $y = 0$ in the second of (2.0) and obtain $\mathcal{I}(\varphi) = (\frac{1}{a} + \mu\sqrt{\varepsilon}g(\omega,\varphi))\mathcal{I}(\varphi + 2\pi\omega)$ (where $g(\omega,\varphi) = \lim_{y\to 0}\frac{1}{y}f_1(\mu y A, y, \omega + \mu y\mathcal{I},\varphi)$ while $\lim_{y\to 0}\frac{1}{y}f_4(\mu y A, y, \omega + \mu y\mathcal{I},\varphi) \equiv 0$. Using again the contraction argument we find the solution of the last functional equation. Being $\mathcal{I} \equiv 0$ a solution, by uniqueness , we can say that the only solution is the identically zero one.

Now (3.2)-(3.3) and the Theorems 2.0-2.1 respectively give the next two theorems about the linearization of the motion when we are set "on" the whiskers.

Theorem 3.0 *For any $\omega \in \mathbb{R}\backslash\{0\}$ and for any $|\mu| \leq \mu_1$ there exist two analytic functions (R^s, Φ^s): $[-\sigma_2, \sigma_2] \times \mathbb{T} \to \mathbb{R} \times \mathbb{T}$ such that on the stable manifold (see system (3.2)), in the variables $(r, \theta) \in \mathbb{R} \times \mathbb{T}$ given by $\eta = r + \mu R^s(r, \theta)$, $\varphi = \theta + \mu r^2 \Phi^s(r, \theta)$, the motion is linear i.e. $r' = re^{-2\pi\sqrt{\varepsilon}}$, $\theta' = \theta + 2\pi\omega$.*

Remarks i) The proof implies $R^s(0, \theta) \equiv 0$ ii) $\sigma_2 = \frac{\sigma_1}{4}$,
$\mu_1 = \frac{1}{2}(1 - e^{-2\pi\sqrt{\varepsilon}})e^{-3\pi\sqrt{\varepsilon}}\mu_o \frac{\sigma^2 \xi}{130\pi A_o}$

The proof is obtained by means of the contraction method again and more details are given in Appendix 2

Of course we have the unstable counterpart of Theorem 3.0. By the implicit function theorem invert the system (3.3)

$$\xi = \xi' e^{-2\pi\sqrt{\varepsilon}} + \mu\xi' A(\xi', \varphi'), \qquad \varphi = \varphi' - 2\pi\omega + \mu B(\xi', \varphi')$$

with suitable functions A and B. Then by the following change of variables $\xi' = q + \mu R^u(q, \theta)$, $\varphi' = \theta + \mu q^2 \Phi^u(q, \theta)$ we can linearize the inverse of system (3.3). Of course we have $q \to qe^{-2\pi\sqrt{\varepsilon}}$ and $\theta \to \theta - 2\pi\omega$. About the domains we loose something because of the inversion of the system and the Cauchy estimates of the derivatives needed in the contraction theorem;

§4. About the obstruction mechanism

Here we want just to show how the proof of the obstruction mechanism of [Ma] and [Cr] can be adapted to our construction. Before we write a very simple argument which implies that the we can deal with the obstruction mechanism for the Poincaré map $(\mathcal{T}_\omega)$ instead of the flow $(\mathbb{T}_\omega)$.

Suppose to have proved the obstruction for $\mathcal{T}_\omega(\theta^o)$ for any θ^o. Then the same is true for $\mathbb{T}_\omega$.

It is a simple corollary of the result of the Lemma 1 in Appendix 1. Suppose in fact that the obstruction mechanism is not verified for $\mathbb{T}_\omega$. This means that there exists a point $p \in Y_\omega^+$, a manifold $\mathcal{N}$ that complements Y_ω^+ in p, a neighbourhhood $U \ni p$, a point $q \in Y_\omega^-$, a neighbourood $V \ni q$, such that $\phi_t(\bar{q}) \notin U$ for any $\bar{q} \in V$ and for any time t. If we take a section at $\theta = \theta^o$ mod.2π of the previous quantities we are considering the projection of the evolution on the surface-map at θ^o i.e. a Poincaré map. But a fortiori the projected torus $\mathcal{T}_\omega(\theta^o)$ will not verify the obstruction mechanism falling in contradiction.∎

Then we change the variables $\xi \to \xi$, $\eta \to \eta$, $\varphi \to \varphi$, $I \to z = I - \omega - \mu\sqrt{\varepsilon}Y(0,\eta,\varphi)\mathcal{I}^s(Y(0,\eta,\varphi),\varphi) - \mu\sqrt{\varepsilon}X(\xi,0,\varphi)\mathcal{I}^u(X(\xi,0,\varphi),\varphi) = I - \omega - Z$ which permits to write (3.0) as

$$\begin{aligned}
\xi' &= \xi e^{2\pi\sqrt{\varepsilon}} + \xi e^{2\pi\sqrt{\varepsilon}}(e^{2\pi\sqrt{\varepsilon}F_J(\xi\eta)} - 1) + \mu(\xi g_1 + g) \\
\eta' &= \eta e^{-2\pi\sqrt{\varepsilon}} + \eta e^{-2\pi\sqrt{\varepsilon}}(e^{-2\pi\sqrt{\varepsilon}F_J(\xi\eta)} - 1) + \mu(\eta h_1 + h) \\
\varphi' &= \varphi + 2\pi z + 2\pi\omega + \mu\varepsilon f_3(X(\xi,\eta,\varphi), Y(\xi,\eta,\varphi), z + \omega + Z(\xi,\eta,\varphi),\varphi) \\
z' &= z - \mu\varepsilon\tilde{f}_4(X(\xi,\eta,\varphi), Y(\xi,\eta,\varphi), z + \omega + Z(\xi,\eta,\varphi),\varphi)
\end{aligned} \tag{4.0}$$

where $g = g(X(\xi,\eta,\varphi), Y(\xi,\eta,\varphi), z+\omega+Z(\xi,\eta,\varphi),\varphi) = O(|\xi| + |\eta|)$,
$h = h(X(\xi,\eta,\varphi), Y(\xi,\eta,\varphi), z+\omega+Z(\xi,\eta,\varphi),\varphi) = O(|\xi| + |\eta|)$,
$h_1 = \frac{1}{\mu}\int_0^1 d\lambda \frac{d}{d\lambda}\frac{1}{a(XY;\lambda\mu)}$, $g_1 = \frac{1}{\mu}\int_0^1 d\lambda\frac{d}{d\lambda}a(XY;\lambda\mu)$, see (3.1)

$$\begin{aligned}
\tilde{f}_4 = \big\{&f_4(X,Y,z+\omega+Z,\varphi) - f_4(X,Y,z+\omega+Z,\varphi)|_{\xi=0} - f_4(X,Y,z+\omega+ \\
&+ Z,\varphi)|_{\eta=0}\big\} + \big\{\mathcal{I}^s_\omega(\eta',\varphi') - \mathcal{I}^s_\omega(\eta',\varphi')|_{\xi=0}\big\} + \big\{\mathcal{I}^u_\omega(\xi',\varphi') - \mathcal{I}^u_\omega(\xi',\varphi')|_{\eta=0}\big\}.
\end{aligned}$$

Of course $\tilde{f}_4(X(0,\eta,\varphi), Y(0,\eta,\varphi), \omega + Z,\varphi) \equiv \tilde{f}_4(X(\xi,0,\varphi), Y(\xi,0,\varphi),\omega+Z,\varphi) \equiv 0$ as now $\mathcal{Y}^+_\omega = \{(\xi,\eta,z,\varphi)|\eta = 0, z = 0\}$ while $\mathcal{Y}^-_\omega = \{(\xi,\eta,z,\varphi)|\xi = 0, z = 0\}$. Now all the conditions of the Lemme of pag.11 of [Ma] are verified[4] and we can apply to (4.0) the Condition (OS) of pag.7 which is the obstruction mechanism when $\omega \in \mathbb{R}\backslash\mathbb{Q}$

Condition (OS) *For any submanifold Δ that intersects transversaly $\mathcal{Y}^-_\omega$, for any point $b \in \mathcal{Y}^+_\omega$, for any neighborhood B of b there exists an integer k such that $\hat{\phi}_{2\pi k}(\Delta) \cap B \neq \emptyset$*

Appendix 1 The transformation U consists of a composition of three transformations (the last of them is actually the composition of infinite canonical transformations). The first one (canonical) is defined in the set $\{(I_1,\varphi_1)|\ |\varphi_1| < \pi,\ |I_1 - |\beta(t)|| < \frac{1}{2}\sqrt{\varepsilon}\}$, (see (0.1)) and it is given by

$$x = \frac{1}{\sqrt{2}}(\varepsilon^{-\frac{1}{4}}I_1 - \varepsilon^{\frac{1}{4}}\varphi_1) \qquad y = \frac{1}{\sqrt{2}}(\varepsilon^{-\frac{1}{4}}I_1 + \varepsilon^{\frac{1}{4}}\varphi_1)$$

so (x,y) are defined in a neighborhood of radius $O(\varepsilon^{1/4})$ of the origin. In the second

[4] The reader could note that $\xi g_1 + g$ and $\eta h_1 + h$ are of order one in ξ, η, z and f_3 and $\tilde{f}_4$ are of order 2 while all the analogous terms in the Lemme of pag.11 of [Ma] have order 2. We think that this causes no problem in the proof because the order two is necessary to write the map as a sum of two addends, the second one much smaller that the first. In our case, for what concerns the first two equations the presence of μ helps us.

one we rescale by a constant factor $x' = x\varepsilon^{-\frac{1}{4}}$, $y' = y\varepsilon^{-\frac{1}{4}}$ The equations (0.0) become

$$\dot\theta = 1, \qquad \dot\varphi_2 = I_2, \qquad \dot I_2 = -\varepsilon\mu\cos\varphi_2(\cos\varphi_1(x',y')-1)$$
$$\dot x' = \sqrt{\varepsilon}x' + \frac{\sqrt{\varepsilon}}{\sqrt{2}}\frac{\partial f}{\partial\varphi_1}(\varphi_1(x',y')) - \frac{\sqrt{\varepsilon}}{\sqrt{2}}\mu\sin\varphi_1(x',y')(\sin\varphi_2+\cos\theta)$$
$$\dot y' = -\sqrt{\varepsilon}y' + \frac{\sqrt{\varepsilon}}{\sqrt{2}}\frac{\partial f}{\partial\varphi_1}(\varphi_1(x',y')) - \frac{\sqrt{\varepsilon}}{\sqrt{2}}\mu\sin\varphi_1(x',y')(\sin\varphi_2+\cos\theta)$$

where $f(x',y') = \cos\varphi_1(x',y') - 1 + \frac{1}{2}\varphi_1^2(x',y')$. The previous system is non-canonical for two reasons: the first one (less important) is the presence of $\dot\theta$ while the second and more important one is due to the rescaling only on the (x,y) variables. Nevertheless the following equations are canonical with hamiltonian $\tilde H = \sqrt{\varepsilon}x'y' + \sqrt{\varepsilon}f(\varphi_1(x',y'))$

$$\dot x' = \sqrt{\varepsilon}x' + \frac{\sqrt{\varepsilon}}{\sqrt{2}}\frac{\partial f}{\partial\varphi_1}(\varphi_1(x',y')) \qquad \dot y' = -\sqrt{\varepsilon}y' + \frac{\sqrt{\varepsilon}}{\sqrt{2}}\frac{\partial f}{\partial\varphi_1}(\varphi_1(x',y'))$$

At this point we do a third transformation only on the (x',y') introducing the normal coordinates (q,p) in such a way that $\tilde H(x'(q,p),y'(q,p)) = K(qp) = qp + F(qp)$ and $F(qp) = O((qp)^2)$ (the dependence in only through the product qp). The variables (q,p) are analytic functions of (x',y') and the series (that we do not write) begin with a linear term. This suitable linear dependence and the linearity of each other transformation we have already introduced imply that (q,p) are defined in a neighborhood of radius $O(1)$ around $(0,0)\in\mathbb{R}^2$. The reference [CG] (lemma 0 of §5) is a good one where to read (and possibly learn) the construction of this normal form.
$\tilde\varphi_1(q,p) = \varphi_1(x'(q,p),y'(q,p))$, $\quad \varphi_1(x',y') = \frac{y'-x'}{\sqrt{2}} = \varepsilon^{-1\backslash 4}\frac{y-x}{\sqrt{2}}$, $J = qp$, $F_J = \frac{\partial F}{\partial J}$

Lemma 1 Let us write the system of differential equations (0.0) as

$$\begin{cases} \dot x = f(x,\theta), & \dot\theta = 1 \\ x(0) = p, \quad \theta(0) = \theta^o & \qquad\qquad f(x,\theta+2\pi) = f(x,\theta) \end{cases}$$

where $x\in\mathcal{M}$, $\theta\in\mathbb{T}$ and let be $\phi_t\colon\tilde{\mathcal{M}}\to\tilde{\mathcal{M}}$ $((p,\theta^o)\to\phi_t(p,\theta^o))$ the flow of the system. Let $\hat\phi_{2k\pi}(p)\colon\mathcal{M}\to\mathcal{M}$, $k\in\mathbb{Z}$ be the Poincaré map at time θ^o and suppose that $M^s_{\theta^o}$ is its stable manifold. Of course the Poincaré map is defined taking the projection on $\mathcal{M}$ of the following equations[5]

$$\begin{cases} x(2k\pi;p,\theta^o) = p + \displaystyle\int_0^{2k\pi} d\tau f\big(x(\tau;p,\theta^o),\theta^o+\tau\big) \\ \theta(2k\pi;\theta^o) - \theta(0;\theta^o) = 2\pi = 0\ (mod.2\pi). \end{cases} \tag{4.1}$$

[5]The Poincaré map written down is equal to (1.2) with $s+\theta^o$ in place of s inside the integrals

We show that $M^s \doteq \cup_{0\le\theta^o\le 2\pi} M^s_{\theta^o}$ is the stable manifold of the flow ϕ_t.

$\Rightarrow$: $(p,\theta^o) \in M^s$ and then $\lim_{t\to\infty}\phi_t(p,\theta^o) \in \mathbb{T}_\omega$ is the hypotheses. Being trivially $\mathbb{T}_\omega = \cup_{0\le\theta^o\le 2\pi}\mathcal{T}_\omega(\theta^o)$ we get $\lim_{t\to\infty}\phi_t(p,\theta^o)|_{\mathcal{M}} \in \mathcal{T}_\omega(\theta^o)$ so that $p \in M^s_{\theta^o}$.

$\Leftarrow$: Now suppose that $p \in M^s_{\theta^o}$ and consider

$$\begin{aligned}
&\lim_{k\to\infty} p + \int_0^{\theta^o+2\pi k} d\tau f(x(\tau;p,0),\tau) = \\
&= \lim_{k\to\infty} p + \int_0^{2\pi k} d\tau f(x(\tau;p,0),\tau) + \int_0^{\theta^o} d\tau f(x(\tau+2\pi k;p,0),\tau) = \\
&= \hat{\phi}_{\theta^o} \circ \lim_{k\to\infty} \phi_{2\pi k}(p,0).
\end{aligned}$$

Being $\lim_{k\to\infty}\hat{\phi}_{2\pi k}(p,0) \in \mathcal{T}_\omega(0)$ by hypotheses, we have $\lim_{k\to\infty}\hat{\phi}_{\theta^o}\circ\phi_{2\pi k}(p,0) \in \mathcal{T}_\omega(\theta^o) \subset \mathbb{T}_\omega$ and then $(p,\theta^o) \in M^s$.■

Here we write the basic (elementary) tool (a "Cauchy estimate") which is fundamental if we would write down explicitely the calculations leading to (2.1) and the other contraction estimates.

Lemma 2 *Let g be an analytic function from $\mathcal{D} \subset \mathbb{C}^N \to \mathbb{C}$ ($\partial\mathcal{D}$ smooth). Then for any subdomain $\mathcal{D}' \subset \mathcal{D}$ with $\delta \equiv dist(\mathcal{D}',\partial\mathcal{D}) > 0$ and any $n \in \mathbb{N}^N$, one has:*

$$\|\partial_z^n g\|_{\mathcal{D}'} \equiv \sup_{\mathcal{D}'} \left|\frac{\partial^{|n|} g}{\partial_{z_1}^{n_1}\cdots\partial_{z_N}^{n_N}}\right| \le |n|!\ \delta^{-|n|}\ \|g\|_{\mathcal{D}}$$

The proof is a straightforward exercise based upon Cauchy's integral formula using the contour $|z'_h - z_h| = \delta$, $h = 1,...,N$ ($z \in \mathcal{D}'$ fixed and $z' \in \mathcal{D}$ variable of integration).

Appendix 2 Sketch of the proof of theorem 3.0. We rewrite the system (3.2) as

$$\begin{cases} \eta' = \eta e^{-2\pi\sqrt{\varepsilon}} + \mu\eta\big(h_1+h_2\big)(\eta,\varphi) \\ \varphi' = \varphi + 2\pi\omega + 2\pi\mu\sqrt{\varepsilon}\mathcal{I}^s_\omega(\eta,\varphi) + \mu\varepsilon F_3(\eta,\varphi) \end{cases} \tag{4.2}$$

where

$$h_1 = \frac{1}{\eta\mu}\int_0^1 d\lambda \frac{d}{d\lambda}\frac{1}{b(\eta,\varphi;\lambda\mu)} = \frac{1}{\eta}\int_0^1 -d\lambda\frac{1}{b^2}\frac{d}{d\zeta}a[\zeta Y^2(\eta,\varphi;\zeta)A^s\Big(Y(\eta,\varphi;\zeta),\varphi\Big)]|_{\zeta=\lambda\mu}$$

$|\eta| \le \sigma_1$, $|Im\varphi| \le \frac{\xi}{4}$; for the definition of $a[\cdot]$ see (3.4). Now

$$\frac{d}{d\zeta}a[\zeta Y^2(\eta,\varphi;\zeta)A^s\big(Y(\eta,\varphi;\zeta),\varphi\big)] = \Big(Y^2A^s + 2\zeta Y A^s\frac{\partial Y}{\partial\zeta} + \zeta Y^2\frac{\partial Y}{\partial\zeta}\frac{\partial A^s}{\partial Y}\Big)\frac{da}{dJ}$$

and in the domain $|\eta| \le \sigma_1$, $|\mu| \le \frac{1}{2}\mu_o$, $|Im\varphi| \le \frac{\xi}{4}$ we have $\|\frac{1}{\eta}\frac{d}{d\zeta}a[\cdot]\| \le \frac{A_o e^{3\pi\sqrt{\varepsilon}}}{\sigma\mu_o}$ and then $|-\frac{1}{\eta\mu}\int_0^1 d\lambda b^{-2}(\eta,\varphi;\lambda\mu)\frac{d}{d\lambda}b(\eta,\varphi;\lambda\mu)| = |h_1(\varphi;\lambda\mu)| \le \frac{4A_o}{\mu_o}e^{\pi\sqrt{\varepsilon}}$.

For what concerns h_2 we observe that $Y(\eta,\varphi)\to 0$ as $\eta\to 0$ and then $h(\eta,\varphi)\to 0$ as $\eta\to 0$ (see (3.4)). Using the maximum principle for holomorphic functions in the form of Schwarz's lemma we bound $\|\frac{h}{\eta}\|_{\sigma_1,\frac{\xi}{4}} \le \frac{\pi A_o}{\sigma_1}\sqrt{\varepsilon}e^{3\pi\sqrt{\varepsilon}}$.

Now substitute $\eta = r+\mu R(r,\theta)$ and $\varphi = \theta + \mu r^2\Phi(r,\theta)$ into the system (4.2) and write a fixed point equation

$$R(re^{-2\pi\sqrt{\varepsilon}},\theta+2\pi\omega) = e^{-2\pi\sqrt{\varepsilon}}R(r,\theta) + \big(r+\mu R(r,\theta)\big)\cdot$$
$$\cdot\big(h_1+h_2\big)(r+\mu R(r,\theta),\theta+\mu r^2\Phi(r,\theta)) \doteq V_1(R,\Phi)$$

$$\Phi(r,\theta) = e^{-4\pi\sqrt{\varepsilon}}\Phi(re^{-2\pi\sqrt{\varepsilon}},\theta+2\pi\omega) - 2\pi\frac{\sqrt{\varepsilon}}{r^2}\mathcal{I}^s_\omega(\eta(r,\theta),\varphi(r\theta)) -$$
$$-\frac{\varepsilon}{r^2}F_3(\eta(r,\theta),\varphi(r\theta)) \doteq V_2(R,\Phi).$$

As in the previous cases the Cauchy estimates enable us to say that in the domain $\Delta^{\sigma_2,\frac{\xi}{16}}_{\eta,\theta}$ and for $|\mu| \le \mu_1$ there exists a unique solution (fixed point) of the previous equations that we call (R^s,Φ^s). About their norm we have taken $\|R^s\|_{\sigma_2,\frac{\xi}{16}} \ge \frac{20A_o\sigma_1}{\mu_o\sigma_o}e^{3\pi\sqrt{\varepsilon}}(1-e^{-2\pi\sqrt{\varepsilon}})^{-1}$, $\|\Phi^s\|_{\sigma_2,\frac{\xi}{16}} \ge \frac{7A_o\sigma_o\sqrt{\varepsilon}}{\sigma_2^2}(1-e^{-4\pi\sqrt{\varepsilon}})^{-1}$.
It is important to note that $R^s(0,\varphi)\equiv 0$ and being R^s an analytic function it can be written as $r\hat{R}^s$. Moreover for writing $\frac{\mathcal{I}^s_\omega}{r^2}$ meaningful we need to keep in mind the content of what written immediately before the Theorem 3.0.

References

[A] V.I.ARNOL'D *Instabilities of dynamical systems with several degrees of freedom* Sov.math.Dokl. **6** (1964) p.581-585
[Be] U.BESSI *An Approach to Arnold Diffusion through the Calculus of Variations*, Nonlinear Analysis, T.M.A., vol.26 num.6 (1996)
[C1] L.CHIERCHIA *Arnol'd instability for nearly-integrable analytic hamiltonian systems*, in "Local and Variational Methods in the Study of Hamiltonian Systems" Ambrosetti-Dell'Antonio eds.World Scientific Pub. (1995)
[C2] L.CHIERCHIA *Non degenerate Arnol'd diffusion*, in print in Ergodic theory and Dynamical Systems
[CG] L.CHIERCHIA-G.GALLAVOTTI *Drift and Diffusion in Phase Space*, Ann. Inst. H.Poincaré (Physique Teorique), **60** No.1 (1994) p.1-144
and

Carr Reports in Mathematical Physics No.15/92 (July 1992)
[Cr] J.CRESSON *A λ-lemma for partially tori and the obstruction property*, Institut de Mathématique de Jussieu, Prépublication n.114 Avril 1997
[Fe1] N.FENICHEL *Persistance and smoothness of invariant manifolds for flows*, Ind. Univ. Math. J.**21**, No.3 (1971), 193-225
[Fe2] N.FENICHEL *Asymptotic stability with rate conditions*, Ind. Univ. Math. J.**23**, No.12 (1974), 1109-1137
[Fe3] N.FENICHEL *Asymptotic stability with rate conditions, II*, Ind. Univ. Math. J.**26**, No.1 (1977), 81-93
[Ga1] G.GALLAVOTTI *Quasi-integrable Mechanical Systems*, K.Osterwalder and R.Stora, eds. Les Houches, Session XLIII, 1984 *Critical phenomena, random systems, gauge theories* Elsevier Science Publishers B.V.(1986) p.539-623
[Ga2] G.GALLAVOTTI *Twistless Kam Tori, Quasi Flat Homoclinic Intersections, and Other Cancellations in the Perturbation Series of Certain Completely Integrable Hamiltonian Systems. A Review.*, Reviews on Mathematical Physics, **6**, (1994) p.343-411
[Ga3] G.GALLAVOTTI "The Elements of Mechanics", Springer-Verlag ed., New-York (1983) p.407-441
[Ge1] G.GENTILE *A Proof of Whiskered tori with a quasi flat homoclinic intersections in a class of almost integrable hamiltonian systems*, Forum Mathematicum **7** 709-753
[Ge2] G.GENTILE *Whiskered tori with prefixed frequencies and Lyapunov spectrum*, Dynam. Stability of Systems **10** 1995, 269-308
[Gr] S.GRAFF *On the Conservation of Hyperbolic Invariant Tori for Hamiltonian Systems*, Jour. of Differential Equations **15** (1974), p.1-69
[GH] J.GUCKENHEIMER, P.HOLMES "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields", Applied Mathematical Sciences 42, Third Printing, Revised and Corrected, Springer Verlag ed. New York, (1990) chapter 4.
[HPS] M.W.HIRSCH, C.C.PUGH, M.SHUB "Invariant Manifolds", Lecture Notes in Mathematics N.583, Springer Verlag 1977
[Ir] M.C.IRWIN "Smooth Dynamical Systems", Academic Press 1980
[Ke] A.KELLEY *The stable, center stable, center, center unstable and unstable manifolds*, Jour.Differential.Equations., **3** (1967), p.546-570
[La1] O.E.LANFORD *Lectures given at the University of Rome*, September (1981) (Notes of Prof. E.Presutti)
[La2] O.E.LANFORD *Bifurcation of periodic solutions into invariant tori. The work of Ruelle and Takens*, Lect.Notes in Math. No.322 (1973) p. 159-192

[Me] V.K.MELNIKOV *On the stability of the center for time periodic perturbations*, Trans. Moscow Math.Soc., 12, (1963), 1-57

[Ma] J.P.MARCO *Transition le long des chaines des tores invariantes pour le systèmes hamiltoniens analytique*, Annales de l'I.H.P., Physique Theorique, Vol. **64**, n.2, (1996), p.205-252 (our citations are taken from: Prébublication 34, Juin 1995, Institut de Mathématiques de Jussieu, Unité Mixte de Recherque 9994, Université Paris VI et Paris VII/CNRS)

[Moe] R.MOECKEL *Transition Tori in the Five-Body Problems*, Jour. Differential Equations **129** (1996) p. 290-314

[Ni] L.NIEDERMAN *Dynamic Around a Chain of Simple Resonant Tori in Nearly Integrable Hamiltonian Systems*, Prepublication 97-19 Université de Paris-Sud Mathématiques Bâtiment 425 91405 Orsay France

[PM] J.PALIS, Jr., W.de MELO "Geometric Theory of Dynamical Systems An Introduction", Springer Verlag 1980

[Po] H.POINCARÉ "Les methodes nouvelles de la mecanique celeste", Tome I-II-III, Gauthier-Villars, 1892, 1893, 1899 (Reprinted by "Librairie Albert Blanchard" rue de Médicis 75006 Paris) 1987

[Ru] D.RUELLE, "Elements of Differentiable Dynamics and Bifurcation Theory", ACADEMIC PRESS; INC. (1989)

[RW1] M.RUDNEV, S.WIGGINS *Existence of Exponentially Small Splittings and Homoclinic and Heteroclinic Connections Between Whiskered Tori in Weakly Hyperbolic Near-integrable Hamiltonian Systems*, `mp_arc@math.utexas.edu`(for informations send an empty mail to the archive)

[RW2] M.RUDNEV, S.WIGGINS *KAM theory multiplicity one resonant surfaces in perturbation of a-priori stable Hamiltonian systems*, `mp_arc@math.utexas.edu`

[Tr] D.V.TRESHCEV *The Mechanism of Destruction of Resonant Tori of Hamitonian Systems*, Math. USSR Sbornik vol.68 (1991) No.1

[Wi] S.WIGGINS "Gobal Bifurcation and Chaos", Applied Mathematical Sciences, 1988 Springer Verlag

[Xia1] Z.XIA *Melnikov Method and Transversal Homoclinic Orbits in the Restricted Three-Body Problem*, Jour. Differential Equations **96** (1992) p. 170-184

[Xia2] Z.XIA *Arnol'd Diffusion in the Elliptic Restricted Three-Body Problem*, J. Dynamics and Diff. Equations **5** (1993) p. 219-240

[Xia3] Z.XIA *Arnol'd Diffusion and Oscillatory Solutions in the Planar Three-Body Problem*, Jour. Differential Equations **110** (1994) p. 289-321

[Ze] E.ZEHNDER *Stability and Instability in Celestial Mechanics*, Troisieme Cycle de la Suisse Romande

E-mail: perfetti@mat.utovrm.it