Contact Topology and Hydrodynamics

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Abstract

We draw connections between the field of contact topology and the study of Beltrami fields in hydrodynamics on Riemannian manifolds in dimension three. We demonstrate an equivalence between Reeb fields (vector fields which preserve a transverse nowhere-integrable plane field) up to scaling and rotational Beltrami fields on three-manifolds. Thus, we characterise Beltrami fields in a metric-independant manner.

This correspondence yields a hydrodynamical reformulation of the Weinstein Conjecture, whose recent solution by Hofer (in several cases) implies the existence of closed orbits for all C^∞ rotational Beltrami flows on S^3. This is the key step for a positive solution to the hydrodynamical Seifert Conjecture: all C^ω steady state flows of a perfect incompressible fluid on S^3 possess closed flowlines. In the case of Euler flows on T^3, we give general conditions for closed flowlines derived from the homotopy data of the normal bundle to the flow.

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1 Introduction

The goal of this paper is to establish close relationships between the fields of hydrodynamics on Riemannian manifolds and contact topology in odd dimensions. For convenience and the sake of applications, we restrict to dimension three, although the basic relationships remain true in more general settings.

In hydrodynamics, we consider the class of vector fields whose behavior is most fascinating and whose analysis has been most incomplete: these are the *Beltrami fields*, or fields which are parallel to their own curl. All such fields are solutions to Euler's equations of motion for a perfect incompressible fluid. Flows generated by such fields have several noteworthy properties, such as extremization of an energy functional (Equation 23), as well as the potential to display the phenomenon of *Lagrangian turbulence*, in which a volume-preserving flow has flowlines which fill up regions of space ergodically [Ar2].

We relate the study of such flows with the rapidly developing field of contact topology. On a three-manifold, a contact structure is a maximally nonintegrable plane field. Though long a fixture in the literature on geometric classical mechanics (dating back at least to the work of Lie), it is only recently that topologists have made great progress in classifying such objects. The present state of affairs, brought about via the work of Eliashberg, Gromov, Hofer, and others [El1, El2, El3, Gr, Ho], encompasses several very strong results in this area. In particular, the method of analysing contact structures via *Reeb fields*, or transverse vector fields whose flow preserve the contact form, has recently proven useful [Ho, HWZ].

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In §1.1 and §1.2, we provide the necessary background information from each field, since we hope to promote interaction between both areas of mathematics. Then, in §2, we describe the correspondence between Beltrami fields and Reeb fields:

Theorem *The class of (nonsingular) vector fields on a three-manifold parallel to its (nonsingular) curl is identical to the class of Reeb fields under rescaling.*

This yields a reformulation of the Weinstein Conjecture from symplectic topology into a hydrodynamical context: namely, that Beltrami flows on closed Riemannian three-manifolds must have closed flowlines. We spell out this reformulation in §3.

The question of whether vector fields on three-manifolds are forced to exhibit periodic solutions has a rich history (see, *e.g.*, [Ku, Ho]). The following conjecture of Seifert has been a focus of inquiry:

The Seifert Conjecture *Every C^k vector field on S^3 possesses a closed orbit.*

This conjecture was shown to be false for $k = 1$ by Schweitzer [Sc], for $k = 2$ by Harrison [Ha], and for $k = \omega$ (*i.e.*, real-analytic) by K. Kuperberg [Ku]. A recent theorem of Hofer [Ho] resolves the Weinstein Conjecture on S^3. Via the correspondence between Beltrami and Reeb fields, we derive the startling corollary that all C^∞ rotational Beltrami flows on the three-sphere (and certain other manifolds) possess closed flowlines, no matter what the Riemannian metric implicated. Using this corollary, we provide a positive solution to the Seifert Conjecture for perfect incompressible fluids:

Theorem *Every C^ω steady-state flow of a perfect incompressible fluid on S^3 possesses closed flowlines.*

In other words, the C^ω Kuperberg plug cannot be parallel to its curl under *any* metric. Athough for Beltrami flows, we can relax the smoothness condition to C^∞, it appears very difficult to improve the above theorem to C^∞. For the non-Beltrami flows, we employ techniqes from singularity theory which are dependant upon analyticity.

Finally, in §4, we apply contact-topological methods to a fundamental class of examples in hydrodynamics: spatially periodic flows on $\mathbb{R}^3$ (*i.e.*, flows on T^3). In the case of Euclidean geometry, these include the *ABC flows.* By using a classification theorem of Giroux [Gi], we give conditions which force the existence of closed contractible orbits on T^3 in terms of the algebraic topology of the vector field:

Theorem *Any C^∞ steady-state rotational Beltrami field on T^3 which is homotopically nontrivial must have a contractible closed flowline. More generally, any C^ω steady-state Euler flow on T^3 which is homotopically nontrivial must have a closed flowline.*

It is conjectured that the above theorem holds for *all* C^∞ rotational Beltrami fields on T^3. It certainly does not hold for all Eulerian fluid flows, as irrational linear flow on T^3 is Eulerian (though homotopicaly trivial).

Throughout this paper, we use the language of differential forms, as it provides a convenient common basis for dealing with both hydrodynamics and contact geometry. Specifically, we use $\mathcal{L}_X$ to denote the Lie derivative along the vector field X, and we use ι_X to denote contraction by X.

1.1 Topological hydrodynamics

The principal business of hydrodynamics is to understand the dynamical properties of fluid flows, the simplest class of which are incompressible, inviscid flows described by the Euler equation. The following treatment of hydrodynamics on Riemannian manifolds is based on the approach of Arnold and Khesin [AK], in which the reader will find several excellent references. In order to apply our results to the widest possible class, we adopt the following convention (*cf.* [AK]).

Definition 1.1 The *curl* of a vector field X on a Riemannian 3-manifold M with metric g and a distinguished volume form μ is the unique vector field $\nabla \times X$ given by

$$\iota_{(\nabla\times X)}\mu = d\iota_X g. \tag{1}$$

In the fluids literature, the curl of a velocity field is the *vorticity field.* Note that in the case where μ is the volume form for the metric g, the definition of $\nabla \times X$ assumes the more familiar form [AMR]

$$\nabla \times X = (*d\iota_X g)^{\#}, \tag{2}$$

where $*$ denotes the Hodge star and $^{\#}$ denotes the isomorphism from 1-forms to vector fields derived from g. The uniqueness of $\nabla \times X$ in the above definition comes from the fact that for a fixed volume form μ, the map $\iota.\mu$ is an isomorphism from vector fields to 2-forms.

Definition 1.2 Let $X(t)$ denote a (time-dependant) vector field on a Riemannian 3-manifold M with metric g and distinguished volume form μ. Then $X(t)$ satisfies *Euler's equation for a perfect incompressible fluid* if

$$\begin{array}{rcl} \frac{\partial X}{\partial t} - X \times (\nabla \times X) & = & \nabla p \\ L_X \mu & = & 0, \end{array} \tag{3}$$

for some function $p(t)$ (the reduced pressure). The quantity $\mathcal{L}_X\mu$ vanishes if and only if the flow associated to X is volume-preserving with respect to μ. We call vector fields (or corresponding flows) satisfying (3) *Euler fields* (or *Euler flows* respectively).

The field of hydrodynamics has been most successful in understanding steady Euler flows, or flows without time dependance. *For the remainder of this work, all vector fields will be assumed to be steady-state.* The topology of a steady state Euler flow is almost always very simple:

Theorem 1.3 (Arnold [Ar1]) *Let X be a C^ω nonsingular Euler field on a closed Riemannian three-manifold M. Then, if X is not everywhere parallel to its curl, there exists a compact analytic subset $\Sigma \subset M$ of codimension at least one which splits M into a finite collection of cells $T^2 \times \mathbb{R}$. Each $T^2 \times \{x\}$ is an invariant set for X having flow conjugate to linear flow.*

The proof is straightforward. The reduced pressure term p is an integral for the flow since $X \cdot \nabla p = -X \cdot (X \times (\nabla \times X)) = 0$ (this is Bernoulli's Theorem). This integral is nonzero if and only if the vorticity and velocity fields are independent. In the nondegenerate case, for regular values of p, the preimage is an invariant compact two-manifold possessing a nonsingular flow: $\coprod T^2$. The preimage of the (finite number of) critical values of p forms the singular set Σ. By analyticity, Σ may not contain open sets; hence, the topology of the flowlines are almost everywhere highly constrained *except* in the case where the curl of X is colinear with X.

The class of fields for which the above integral degenerates is extremely important.

Definition 1.4 A vector field X is said to be a *Beltrami field* if it is parallel to its curl: *i.e.*, $\nabla \times X = fX$ for some function f on M. A *rotational* Beltrami field is one for which $f \neq 0$; that is, it has nonzero curl.

Beltrami fields have generated significant interest in the fluids community [Ar2, D, MP, Mo, FGV]. As shown by Arnold [Ar2], Beltrami fields extremize an energy functional within the class of flows conjugate by μ-preserving diffeomorphisms. Furthermore, the flowlines of a Beltrami flow on a three-manifold are not always constrained to lie on a 2-torus, as is the typical case for non-Beltrami steady state Euler flows. Hence, the manifestation of "Lagrangian turbulence" can only appear within the class of Beltrami fields.

The classic examples of Beltrami fields which appear to exhibit Lagrangian turbulence are the *ABC flows*, generated by a certain family of vector fields on T^3 which are eigenfields of the curl operator (see Equation 18 in §4). Although the ABC flows have been repeatedly analysed [Ar2, D, MP, FGV], few results are known, other than for a handful of near-integrable examples. Beltrami fields in general are even less well understood. Typically, one wants to restrict to, say, Beltrami fields on Euclidean $\mathbb{R}^3$ or T^3, under the fixed standard metric and volume form. Under these restrictions, it becomes nearly impossible to do any sort of analysis on the class of Beltrami fields. A small perturbation, even within the class of μ-preserving fields, almost always destroys the Beltrami property. We note in particular the difficulty of answering global questions about Beltrami flows, such as the existence of closed orbits, the presence of hydrodynamic instability, and the minimization of the energy functional. In this work, we provide some topological tools which may prove robust enough to overcome these difficulties.

1.2 Contact topology

For the sake of concreteness and applicatibility, we will restrict all definitions and discussions to the case of contact structures on three-manifolds, noting that several features hold on arbitrary odd-dimensional manifolds. For introductory treatments, see [MS, Ae].

A *contact structure* on a three-manifold M is a maximally nonintegrable plane field. That is, to each point $p \in M$, we assign a hyperplane in T_pM, varying smoothly with p in such a manner that the Frobenius condition fails everywhere. In particular, a contact structure is locally twisted at every point and may be thought of as an "anti-foliation" — no disc may be embedded whose tangent planes agree with the plane field.

Definition 1.5 A *contact form* on M^3 is a one-form α on M such that $\alpha \wedge d\alpha \neq 0$. That is, $\alpha \wedge d\alpha$ defines a volume form on M. A *contact structure* is a plane field which is the kernel of a (locally defined) contact form:

$$\xi = \ker(\alpha) = \{v \in T_pM : \alpha(v) = 0, p \in M\}. \tag{4}$$

It is usually sufficient to consider contact structures which are the kernel of a globally defined contact 1-form: these are called *cooriented* contact structures. Of course, a double-cover suffices to make any plane field coorientable.

Contact structures have often been described as "symplectic structures in odd dimensions" [Ar3], due to the fact that the restriction of $d\alpha$ to the plane field ξ is a closed nondegenerate 2-form. However, given the definition in terms of nonintegrable plane fields, it is easy to see several elementary properties of contact structures. For example, unlike foliations, contact structures are structurally stable, in the sense that perturbing α cannot force ξ to be integrable. On the other hand, the manifestation of the Darboux Theorem in this context implies that every contact structure locally looks like (is contactomorphic to, or diffeomorphic via a map which carries the contact structure to) the kernel of $dz + xdy$ on $\mathbb{R}^3$ (see [MS]). Note the similarity with codimension-one foliations, which are locally equivalent to the kernel of dz on $\mathbb{R}^3$.

Again, as in foliation theory, the global features of a contact structure are closely related to those of the manifold in which it sits. The classification of contact structures follows along lines similar to the Reeb-component versus taut perspective in (codimension-one) foliation theory [ET].

Definition 1.6 Given a three-manifold M with contact structure ξ, let $F \subset M$ be an embedded surface. Then the *characteristic foliation* on F, F_ξ, is the foliation on F generated by the (singular) line field

$$\mathcal{F} = \{T_pF \cap \xi_p : p \in F\}.$$

A contact structure ξ is *overtwisted* if there exists an embedded disc $D \subset M$ such that the characteristic foliation D_ξ has a limit cycle. A contact structure which is not overtwisted is called *tight*.

The classification of overtwisted structures up to contactomorphism coincides with the classification of plane fields up to homotopy [El1] and hence reduces to a problem in algebraic topology. The classification of tight structures, on the other hand, is far from complete. Like taut foliations, tight contact structures exhibit several "rigid" features which make them relatively rare. The classification of tight contact structures is completed only on S^3 [El3], on certain lens spaces [Et2], and on the three-torus T^3 [Gi]. It is unknown whether every closed orientable 3-manifold has a tight contact structure on it.

Given a contact form α generating the contact structure ξ, we may associate to it a vector field which is transverse to ξ and preserves α under the induced flow. Such vector fields were first considered by Reeb [Re].

Definition 1.7 Given a contact form α on M, the *Reeb field* associated to α is the unique vector field X such that

$$\iota_X d\alpha = 0 \qquad \text{and} \qquad \iota_X \alpha = 1. \tag{5}$$

The condition that $\iota_X\alpha = 1$ is a normalisation condition which corresponds to a time reparametrisation of the flow. As we are primarily concerned with the topology of the flowlines, which does not depend on the parametrisation, we will also consider the class of *Reeb-like* fields, for which $\iota_X d\alpha = 0$ and $\iota_X\alpha > 0$.

The relationship between the dynamics of a Reeb field and the topology of the transverse contact structure have been analysed most notably by Hofer *et al.* [Ho, HWZ]. We will consider these results in detail in §3.

Example 1.8 The standard contact structure on the unit $S^3 \subset \mathbb{R}^4$ is given by the kernel of the 1-form

$$\alpha = \frac{1}{2}\left(x_1dx_2 - x_2dx_1 + x_3dx_4 - x_4dx_3\right). \tag{6}$$

The Reeb field associated to α is the unit tangent field to the standard Hopf fibration of S^3; hence, the contact structure ξ can be visualised as the plane field orthogonal to the Hopf fibration (orthogonal with respect to the metric on the unit 3-sphere induced by the standard metric on $\mathbb{R}^4$). It is a foundational result of Bennequin [Be] that this structure is tight; furthermore, by Eliashberg [El3], this is the unique tight contact structure on S^3 up to contactomorphism.

2 Geometric properties of Beltrami flows

It is not coincidental that the Beltrami condition on a vector field — that the flow must continually twist about itself — is reminiscent of the notion of a nonintegrable, everywhere twisting plane field. We obtain a general equivalence between Beltrami and Reeb fields for arbitrary three-manifolds by working with moving frames and metrics adapted to the flow.

Theorem 2.1 *Let M be a Riemannian three-manifold. Any rotational Beltrami field on M is a Reeb-like field for some contact form on M. Conversely, given a contact form α on M with Reeb field X, any nonzero rescaling of X is a rotational Beltrami field for some Riemannian metric on M.*

Proof: Assume that X is a Beltrami field where $\nabla \times X = fX$ for some $f > 0$. Let g denote the metric and μ the volume form on M. On charts for M, choose an orthonormal frame $\{e_i\}_1^3$ such that $e_1 = X/\|X\|$. Note that we must work on charts only in the case where the Euler class of the vector field ($e(X) \in H^2(M;\mathbb{Z})$) is nonzero. Denoting by $\{e^i\}_1^3$ the dual 1-form basis, we have that $\iota_X g = \|X\|e^1$. Let α denote the one-form $\iota_X g = \|X\|e^1$, which is globally defined since X is nonsingular.

The condition $\nabla \times X = fX$ translates to $d\iota_X g = f\iota_X\mu$, or, $d\alpha = f\iota_X\mu$. Since μ is a volume form, its representation on each chart is of the form $he^1 \wedge e^2 \wedge e^3$, with h a nonzero function. Hence, α is a contact form since

$$\alpha \wedge d\alpha = \iota_X g \wedge f\iota_X\mu = fh\|X\|^2 e^1 \wedge e^2 \wedge e^3 \neq 0. \tag{7}$$

Finally, X is Reeb-like with respect to α since

$$\iota_X d\alpha = f\iota_X\iota_X\mu = 0. \tag{8}$$

Conversely, assume further that α is a contact form for M having Reeb field X. Assume that $Y = hX$ for some $h > 0$. Then, on charts for M choose a parallelization $\{e_i\}$ of M such that $e_1 = X$ and e_2 and e_3 are in ξ, the kernel of α, and form a symplectic basis for this plane field (this is always possible since $d\alpha|_\xi$ is a symplectic structure); hence, $d\alpha(e_2, e_3) = 1$. Again let $\{e^i\}$ denote the dual 1-forms to the $\{e_i\}$. On charts, choose the following metric adapted to the flow in the $\{e_i\}$-coordinates:

$$g = \begin{bmatrix} h^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

The transition maps respect this locally defined metric since e_1 is globally defined and the symplectic basis $\{e_2, e_3\}$ differs from chart to chart by an element of $U(1)$, (the plane field is globally defined). We remark that in this metric, $\|X\| = 1/\sqrt{h}$ and $\|Y\| = \sqrt{h}$.

We claim that, under the metric g, Y is volume preserving and parallel to its curl. First, $\iota_Y g = e^1$ since for $i = 1, 2, 3$,

$$\iota_Y g(e_i) = g(Y, e_i) = g(he_1, e_i) = \delta_{i1}, \tag{10}$$

where δ_{ij} is the Kronecker delta. Next, note that $e^1 = \alpha$, since they act on the basis $\{e_i\}$ identically. Specifically, $\iota_{e_1}\alpha = \iota_X\alpha = 1$ and $\iota_{e_2}\alpha = \iota_{e_3}\alpha = 0$ since e_2 and e_3 were chosen to lie in ξ. We now have that $de^1 = e^2 \wedge e^3$ since for $i < j$,

$$de^1(e_i, e_j) = d\alpha(e_i, e_j) = \delta_{2i}\delta_{3j} - \delta_{2j}\delta_{3i}, \tag{11}$$

since we also chose the pair (e_2, e_3) to form a symplectic basis for ξ.

Let μ denote the volume form $h^{-1}e^1 \wedge e^2 \wedge e^3$. Then,

$$d\iota_Y \mu = d(\iota_{he_1} h^{-1} e^1 \wedge e^2 \wedge e^3) = d(e^2 \wedge e^3) = d^2(e^1) = 0; \tag{12}$$

hence, Y is volume preserving with respect to μ. To show that Y is Beltrami with respect to g and μ, it suffices to note that

$$d\iota_Y g = de^1 = e^2 \wedge e^3, \tag{13}$$

as well as

$$\iota_Y \mu = \iota_{he_1} h^{-1} e^1 \wedge e^2 \wedge e^3 = e^2 \wedge e^3. \tag{14}$$

Hence, $\nabla \times Y = Y$. $\Box$

Note that Y is divergence-free under the particular g-induced volume form if and only if the scaling function h is an integral for the flow: *i.e.*, $\mathcal{L}_Y h = 0$.

Remark 2.2 If X is a Beltrami field with singularities, then we may excise the singular points from the manifold and apply Theorem 2.1 to the nonsingular portion of the flow. Then X is still a Reeb flow for a contact form on the punctured manifold.

Corollary 2.3 *Every Reeb-like field generates a steady-state solution to the Euler equations for a perfect incompressible fluid with respect to some metric.*

Corollary 2.4 *Every closed three-manifold has a rotational Beltrami flow on it for some metric. In fact, there are an infinite number, distinct up to homotopy through nonsingular vector fields.*

Proof: The work of Martinet [Ma] and Lutz [Lu] shows that every closed three-manifold has a contact structure. To show that there are a countable collection of homotopically distinct Beltrami fields on any three-manifold, we appeal to the classification of plane fields on three-manifolds, elucidated recently by Gompf [Go]. In the case where the rank of $H^2(M;\mathbb{R}) \neq 0$ (*i.e.*, the cohomology is not pure torsional), there are at least a $\mathbb{Z}$'s worth of distinct Euler classes for plane fields on M; hence for (overtwisted) contact structures and the associated Beltrami fields. In the case where $H^2(M;\mathbb{Z})$ has only torsion elements, the *three-dimensional invariant* of Gompf [Go] implies the existence of an infinite number of homotopy classes of plane fields: again, hence, of (overtwisted) contact structures and thus Beltrami fields. $\Box$

Note that it is *a priori* unclear how one would construct a (nonzero) Beltrami flow on a nontrivial three-manifold, such as $(S^1 \times S^2)\#(S^1 \times S^2)$. Proving global results about such flows would appear to be even more unlikely.

3 Hofer's Theorem and the Hydrodynamical Seifert Conjecture

A fundamental conjecture in global symplectic and contact topology is the Weinstein Conjecture [We], which concerns certain embeddings of codimension-one submanifolds in symplectic manifolds. We briefly recount this conjecture, following [MS]. For excellent treatments, including recent progress, see the books [HZ, MS].

Let M^{2n-1} be a hypersurface in a symplectic manifold (W^{2n}, ω). Then M is said to be of *contact type* if there exists a transverse expanding vector field X on a neighborhood of M. That is, X is transverse to M and $\mathcal{L}_X\omega = \omega$.

Conjecture 3.1 (Weinstein Conjecture, symplectic version [We]) *Any hypersurface M of contact type[3] has closed characteristics. That is, the foliation generated by the line field*

$$\mathcal{F} = \{v \in T_zM : \omega(v, T_zM) = 0\},$$

has a closed leaf.

Following [MS], it follows that any surface M of contact type has a natural 1-form α associated to it via $\alpha = \iota_X\omega$, where X is the expanding vector field. Since $d\alpha = \omega|_M$, it follows that α is a contact form with Reeb field generating the foliation $\mathcal{F}$ above. Similarly, given a Reeb field on M, one can embed M in a symplectic manifold as a hypersurface of contact type [MS]. Hence, the Weinstein Conjecture translates to whether Reeb fields on M have closed orbits. ¿From Theorem 2.1, we thus have:

Conjecture 3.2 (Weinstein Conjecture, hydrodynamics version) *Every smooth[4] rotational Beltrami flow on a Riemannian three-manifold has a closed orbit.*

The simplest example of an Euler flow without closed flowlines is linear irrational flow on the Euclidean three-torus T^3. However, this flow has everywhere vanishing curl. A simple application of Stokes' Theorem implies that this flow (or any flow transverse to a closed surface) cannot be rotational Beltrami under any metric.

Great progress on the Weinstein Conjecture has been made in the past decade (see [HZ] for a historical account). For example, Viterbo [Vi] showed that the Weinstein Conjecture is true on any three-manifold of contact type in $\mathbb{R}^4$ equipped with the standard symplectic form. More important for our applications, though, is the following seminal result of Hofer:

Theorem 3.3 (Hofer [Ho]) *If $\xi = \ker\alpha$ is an overtwisted contact structure on a compact orientable three-manifold, then the associated Reeb field has a closed orbit of finite order in $\pi_1(M)$.*

The proof of Theorem 3.3 is highly nontrivial, relying on the techniques of pseudo-holomorphic curves pioneered by Gromov [Gr]. Such techniques yield, among other things, the following partial resolution to Conjecture 3.1.

Theorem 3.4 (Hofer [Ho]) *The Weinstein Conjecture is true for a three-manifold M in the following cases: (1) $M = S^3$ (or is covered by S^3); (2) $\pi_2(M) \neq 0$.*

In the case of S^3, the Reeb fields associated to overtwisted structures are covered by Theorem 3.3. According to Eliashberg's classification of contact structures on S^3 [El3], any tight contact structure is contactomorphic to that of Example 1.8. The existence of a closed orbit then follows from techniques of Rabinowitz [Ra].

Theorem 3.4 is the key step in the hydrodynamical Seifert Conjecture:

[3] Originally, the conjecture was stated under the additional hypothesis that $H^1(M;\mathbb{Q}) = 0$. All known progress on this conjecture seems to indicate that the cohomological condition is extraneous.

[4] By "smooth" we mean C^∞. However, the question is certainly interesting in other degrees of regularity.

Theorem 3.5 *Any C^ω steady-state Euler flow on S^3 has a closed flowline.*

Proof: Given X a nonsingular Euler field on S^3 under some metric g, Theorem 1.3 presents two possible cases.

Case I: If X is not everywhere colinear with $\nabla \times X$, then the reduced pressure function p is an integral for the flow. Hence, S^3 is decomposed by a compact subset Σ into a finite collection of cells diffeomorphic to $T^2 \times \mathbb{R}$, where the each $T^2 \times \{x\}$ is invariant under the flow. The slope of the vector field on each cell may be constant and irrational; hence, we must consider the singular set, Σ, which is the inverse image of the (finite number of) critical values of the function $p : S^3 \to \mathbb{R}$ from Equation 3.

The key tool in this case comes from singularity theory: the inverse image of a critical value of p is a (Whitney) stratified set, since it is a real-analytic variety [GM]. That is, although the subset is not necessarily a manifold, it can be decomposed into manifolds of varying dimension (≤ 2 in our case) glued together in a sufficiently regular manner: see [GM] for proper definitions and theory.

To determine the structure of Σ, note that it is locally a product in the flow direction. More specifically, let $x \in \Sigma$. Since X is nonsingular, the Flowbox Theorem (see *e.g.*, [Ro]) states that there is a neighborhood $U \cong D^2 \times \mathbb{R}$ of x in S^3 such that $X|_U$ points entirely in the $\mathbb{R}$-component. Denote by D_0 the disc $D^2 \times \{0\}$ containing $\{x\}$. Since $S^3 \setminus \Sigma$ is filled with invariant tori, Σ is invariant and the Flowbox Theorem implies that D_0 is transverse to Σ and $U \cap \Sigma \cong (D_0 \cap \Sigma) \times \mathbb{R}$. Thus, the transverse structure of Σ is invariant along flowlines of X.

By analyticity, Σ is at most two-dimensional. This, along with the Whitney condition (a) for stratified sets [GM], implies that the intersection $D_0 \cap \Sigma$ is homeomorphic to a radial k-pronged tree centred at x. If the number k is zero, then Σ is a 1-manifold, which by the compactness of Σ and the flow-invariance of the transverse structure, implies that X has a compact invariant 1-manifold. If $k = 1$, then Σ has an invariant 1-dimensional boundary component, which again by compactness and flow-invariance implies a closed orbit. If $k > 2$, then, by the Uniqueness Theorem for ODEs, x lies within an invariant 1-stratum, which as before must continue to a compact invariant 1-manifold via flow-invariance and finiteness of the stratification.

Thus, if X has no closed orbits on Σ, then for every $x \in \Sigma$, the transverse prong number k is precisely two, and Σ is locally homeomorphic to $\mathbb{R}^2$. Hence it is an invariant closed 2-manifold with nonsingular vector field: $\Sigma = \coprod T^2$. Thus, S^3 is decomposed into a finite collection of $T^2 \times \mathbb{R}$ cells glued together (pairwise) along sets diffeomorphic to T^2, and we have expressed the three-sphere as a T^2-bundle over S^1, a contradiction.

Case II: If X is everywhere colinear with $\nabla \times X$, then $\nabla \times X = fX$ and X is a Beltrami field. In the case where f is nonzero, X is rotational and the analysis of Case I is completely useless: unlike all other cases, we have a priori no information about the flow. However, by Theorem 2.1, this vector field is, up to a rescaling, a Reeb field. Hence, the associated flows are related by a time reparametrization, which neither creates nor destroys the closed orbit whose existence follows from Theorem 3.4 of Hofer.

If f has zeros, then either f varies, or it is constantly zero. If f is not constant, then f is a nontrivial integral for X as follows. First, $\nabla \times X$ is μ-preserving since

$$d\iota_{\nabla \times X}\mu = d(d\iota_X g) = 0. \tag{15}$$

However, the left hand term simplifies to

$$d\iota_{\nabla \times X}\mu = d(f\iota_X\mu) = df \wedge \iota_X\mu, \tag{16}$$

since X is also μ-preserving. Finally, since we are on a 3-manifold, the 4-form $df \wedge \mu$ vanishes, implying

$$0 = \iota_X(df \wedge \mu) = (\iota_X df)\mu - df \wedge \iota_X \mu = (\iota_X df)\mu. \tag{17}$$

Thus, $\iota_X df = 0$ and f is an integral. The existence of a closed orbit then follows from the singularity-theory arguments of Case I.

Finally, if $\nabla \times X = 0$ everywhere, then the 1-form $\alpha = \iota_X g$ is closed and nondegenerate since $d\iota_X g = f\iota_X \mu = 0$. By the Frobenius condition, $\xi = \ker\alpha$ defines a smooth codimension-one foliation of S^3, to which X is transverse. However, by the Novikov Theorem [No], ξ has a Reeb component: in particular, there is a closed leaf homeomorphic to T^2. This torus separates in S^3 and hence cannot be transverse to the volume-preserving field X, contradicting the assumption that X is nonsingular. □

Remark 3.6 For a typical closed orientable three-manifold M, the Weinstein Conjecture is still open. However, we may conclude that "most" Beltrami fields (in the homotopic sense) possess closed flowlines: by Corollary 2.4, there are an infinite number of homotopically distinct rotational Beltrami fields on M. By a theorem of Kronheimer and Mrowka [KM], there are on M only a finite number of homotopy classes of plane fields which contain a tight contact representative. Hence, by Theorems 2.1 and 3.3, all but finitely many of the rotational Beltrami fields on M (up to homotopy) have closed flowlines.

The important feature of Hofer's Theorem is the relationship between the "twistedness" of a contact structure and the dynamics transverse to it. In order to apply contact-topological methods to manifolds which are of greater interest to fluid dynamicists (T^3 or $\mathbb{R}^3$) we consider algebraic obstructions to tightness.

4 Beltrami flows on T^3

The most important three-manifolds from the point of view of hydrodynamics are the solid torus $D^2 \times S^1$ and the three-torus $T^3 = S^1 \times S^1 \times S^1$. Some authors have suggested using Beltrami "tubes" to model certain domains of turbulent regions in flows, following obervations that suggest the vorticity tends to align with velocity in certain domains [Mo]. We do not discuss the applications of contact geometry to this case, since little is known of the classification of tight contact structures on $D^2 \times S^1$.

Contact structures on the three-torus, however, are more completely understood. In addition, the most important examples of interesting Beltrami fields live on T^3. A fundamental observation of Arnold's is the existence of Beltrami fields on the three-torus T^3 which are nonintegrable: these so-called *ABC flows* (and their generalizations) have been the source of a great deal of inquiry:

$$\begin{array}{l} \dot{x} = A\sin z + C\cos y \\ \dot{y} = B\sin x + A\cos z \\ \dot{z} = C\sin y + B\cos x \end{array}, \tag{18}$$

for some $A, B, C \geq 0$. By symmetry in the variables and parameters, we may assume without loss of generality that $1 = A \geq B \geq C \geq 0$. Under this convention, the vector field is nonsingular if and only if [D]

$$B^2 + C^2 \leq 1. \tag{19}$$

Though the list of publications concerning ABC flows is extensive, there is very little known about the global features of these flows, apart from cases where one of the constants (say C) is zero or a perturbation thereof. Melnikov methods have been used repeatedly to show complex behavior in these near-integrable cases. Of the generalizations of ABC flows to other eigenfields of the curl operator, even less is known. There are seemingly no global results on general Beltrami fields. However, thanks to the following classification theorem of Giroux, we may apply our contact-topological methods to the most general case of Beltrami flows on T^3:

Theorem 4.1 (Giroux [Gi]) *Any tight contact structure on T^3 is (up to a choice of fundamental class in $H_2(T^3)$) isotopic through contact structures to the kernel of α_n for some integer $n > 0$, where*

$$\alpha_n = \sin(nz)dx + \cos(nz)dy. \tag{20}$$

Definition 4.2 Let X be a nonsingular vector field on M. Since all three-manifolds are parallelizable, X gives a map from M to $\mathbb{R}^3$. Under the normalization $\mathbb{R}^3 \setminus \{0\} \to S^2$, X induces a map $M \to S^2$. The *homotopy class* of X is defined to be the homotopy class of the map $M \to S^2$.

In other words, a nonsingular vector field is homotopically trivial if it can be deformed through nonsingular vector fields in such a way that all the vectors "point in the same direction."

Theorem 4.3 *Every homotopically nontrivial rotational Beltrami field on T^3 has contractible closed orbits.*

Proof: Let X denote a homotopically nontrivial field on a Riemannian T^3 satifying $\nabla \times X = fX$ with $f > 0$. Then by Theorem 2.1, there is a natural contact structure ξ transverse to X and uniquely defined up to homotopy. If X is homotopically nontrivial, then so is ξ as a plane field, since the homotopy class of an oriented plane field is defined as the homotopy class of the vector field transverse to it.

By Theorem 4.1, any tight contact structure on T^3 is isotopic to the kernel of the 1-form $\sin(nz)dx + \cos(nz)dy$. The Reeb field X_n associated to this contact form α_n is

$$X_n = \sin(nz)\frac{\partial}{\partial x} + \cos(nz)\frac{\partial}{\partial y}. \tag{21}$$

As the image of the induced map $T^3 \to S^2 \subset \mathbb{R}^3$ lies on the equator $z = 0$, it follows that every tight contact structure on T^3 is homotopically trivial. Hence, X is a Reeb-like field for an overtwisted contact structure, which, by Theorem 3.3, must have a closed orbit which is torsional in $\pi(T^3) \cong \mathbb{Z}^3$ — *i.e.*, it is contractible. $\square$

Remark 4.4 The existence of closed orbits which are *contractible* is of particular importance. One typically works on T^3 in order to model spatially periodic flows on $\mathbb{R}^3$. The existence of a contractible closed orbit on T^3 implies that when lifted to the universal cover $\mathbb{R}^3$, the orbit remains closed. We note that upon numerically integrating examples of Beltrami fields on T^3 (*e.g.*, the ABC equations), the integrable regions are certainly homotopically nontrivial in T^3, whereas the closed orbits in the nonintegrable regions are completely obscured.

Remark 4.5 The determination of the homotopy type of a nonsingular vector field on a three-manifold is a problem of algebraic topology, thanks to recent results of Gompf [Go]. Gompf assigns to a vector field a pair of integer-valued invariants: a two-dimensional refinement of the Euler class, and a more subtle three-dimensional invariant, which is derived from what type of four-manifold M bounds. Together, these invariants, which can be computed in many cases, completely classifiy the homotopy type.

Remark 4.6 It is by no means the case that Theorem 4.3 holds for vector fields in general. As mentioned earlier, the results of Kuperberg [Ku] allow one to insert "plugs" to break isolated closed orbits. Since the Kuperberg plugs do not change the homotopy type of the vector field, there are smooth nonsingular vector fields on T^3 in *every* homotopy class which have no closed orbits.

We may extend Theorem 4.3 to the analogue of the Seifert Conjecture for Euler flows on T^3.

Theorem 4.7 *Any homotopically nontrivial C^ω Euler flow on T^3 has a closed orbit.*

Proof: By Theorems 4.3, 1.3, and the proof of Theorem 3.5, the only remaining cases are when one has an integrable Eulerian field X on T^3, or when $\nabla \times X = 0$.

Consider first the integrable case. As in the proof of Theorem 3.5, we know that either there exists a closed invariant 1-manifold, or we have constructed T^3 as a union of cells $T^2 \times \mathbb{R}$ glued together along invariant 2-tori. In the latter case, if there are no closed orbits, then the rotation number for the flow on each T^2 is irrational and thus constant. Hence, we have a homotopically trivial flow, since each $T^2 \times \{x\}$ is invariant and has linear flow on it (up to conjugacy), so that the image of the Gauss map is contractible in S^2.

In the remaining case where $\nabla \times X = 0$, we again know that the 1-form $\alpha = \iota_X g$ defines a smooth codimension one foliation which is taut since X is a transverse volume-preserving vector field. However, smooth taut foliations can be perturbed to a tight contact structure [ET]. Thus, by Theorem 4.1, the foliation (and the corresponding normal vector field) is homotopically trivial. □

Remark 4.8 One may also utilize the correspondence of Theorem 2.1 to construct interesting examples of Reeb fields. For example, the analysis of Dombre *et al.* [D] indicates the presence of integrable "tubes" within the ABC flows. One may cut out these tubes and replace them in a topologically different manner in such a way as to produce a new contact structure on a different three-manifold. This form of Dehn surgery which is adapted to Reeb fields can be done so that the new Reeb field agrees with the ABC field outside of the integrable tubes [EG]: hence the flow is nonintegrable and exhibits Lagrangian turbulence. Since the integrable tubes present in the ABC flows form a complete set of generators for the first homology group $H_1(T^3;\mathbb{Z})$, we can break all the first homology and obtain an ABC-type flow on the three-sphere S^3.

We close with the specific case of the ABC flows. Unfortunately, we cannot apply Theorem 4.3 to these equations.

Proposition 4.9 *Every nonsingular ABC field is transverse to a tight contact structure.*

Proof: By normailizing the coefficient A to 1 and using Equation 19, we have that the parameter space $\{(B,C) : 0 \leq B^2+C^2 \leq 1\}$ is path-connected; hence, if we show that some ABC field satisfies the proposition, then every other ABC field is homotopic to this through nonsingular Beltrami fields, and so the transverse contact structures are isotopic through contact structures. In the particular case where $B = C = 0$, we have the equations

$$\begin{array}{l} \dot{x} = A\sin z \\ \dot{y} = A\cos z \end{array} , \tag{22}$$

which is the Reeb field for the (tight) contact form $\alpha = A\sin z dx + A\cos z dy$ (*cf.* Equation 21). □

Remark 4.10 An important feature of any Beltrami field X is the fact that it extremizes the energy functional

$$E(\tilde{X}) = \frac{1}{2}\int_M \|\tilde{X}\|^2 d\mu \tag{23}$$

among the class of all vector fields $\tilde{X}$ obtained from X by μ-preserving diffeomorphisms of M [Ar2]. It would be interesting to investigate the relationship between smooth fields which *minimize* the energy functional and the associated transverse contact structures. It follows from remarks in Arnold [Ar2] that the Reeb field associated to the unique tight contact structure on S^3, as well as the ABC flows each minimize energy. In fact, every known example of a smooth energy-minimizing field is the Reeb field for a tight contact structure. It would be interesting to see whether one could always reduce the energy of a Reeb field associated to an overtwisted contact structure: if so, this may be a significant application of the tight / overtwisted dichotomy in contact geometry to a problem in hydrodynamics.

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References

[AMR] R. Abraham, J. Marsden, and T. Ratiu (1988), *Manifolds, Tensor Analysis, and Applications.* Second edition. Springer-Verlag, New York.

[Ae] B. Aebisher, et al. (1994), *Symplectic Geometry.* Progress in Math. **124**, Birkhäuser, Berlin.

[Ar1] V. Arnold (1966), Sur la géométrie differentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier* **16**, 316–361.

[Ar2] V. Arnold (1974), The asymptotic Hopf invariant and its applications. *Proc. of Summer School on Diff. Eqns., 1973, Erevan*, Armenian SSR Acad. Sci. Translation in (1986), *Sel. Math. Sov.* **5**(4), 327–345.

[Ar3] V. Arnold (1989), *Mathematical Methods of Classical Mechanics.* Springer-Verlag, Berlin.

[AK] V. Arnold and B. Khesin (1997), *Topological methods in hydrodynamics.* Springer-Verlag, New York.

[Be] D. Bennequin (1983), Entrelacements et équations de Pfaff. *Astérisque* **107-8**, 83–161.

[D] T. Dombre *et al.* (1986), Chaotic streamlines in the ABC flows, *J. Fluid Mech.* **167**, 353–391.

[El1] Y. Eliashberg (1989), Classification of overtwisted contact structures on 3-manifolds. *Invent. Math.*, **98**, 623–637.

[El2] Y. Eliashberg (1990), Topological characterization of Stein manifolds of dimension > 2. *Int. J. of Math.* **1**, 29–46.

[El3] Y. Eliashberg (1992), Contact 3-manifolds twenty years since J. Martinet's work. *Ann. Inst. Fourier* **42**, 165–192.

[ET] Y. Eliashberg and W. Thurston (1996). Confoliations. Preprint.

[Et1] J. Etnyre (1996), *Symplectic constructions on 4-manifolds.* Dissertation, University of Texas at Austin.

[Et2] J. Etnyre (1997), Contact structures on lens spaces. In preparation.

[EG] J. Etnyre and R. Ghrist (1997), Constructions of tight 3-manifolds and dynamics. In preparation.

[FGV] S. Friedlander, A. Gilbert, and M. Vishik (1993), Hydrodynamic Instability for certain ABC flows. *Geophys. Astrophys. Fluid Dynam.* **73**, 97–107.

[Gi] E. Giroux (1994), Une structure de contact, même tendue est plus ou moins tordue. *Ann. Scient. Ec. Norm. Sup.* **27**, 697–705.

[Go] R. Gompf (1996), Handlebody construction of Stein surfaces. Preprint (1996).

[GM] M. Goresky and R. MacPherson (1988), *Stratified Morse Thoery.* Springer-Verlag, Berlin, Heidelberg.

[Gr] M. Gromov (1985), Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.* **82**, 307–347.

[Ha] J. Harrison (1988) C^2 counterexamples to the Seifert conjecture. *Topology* **27**(3), 249-278.

[Ho] H. Hofer (1993), Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three. *Invent. Math.* **114**, 515–563.

[HWZ] H. Hofer, K. Wysocki and E. Zender (1996), A characterisation of the thight three-sphere. *Duke Math. J.* **81**(1), 159–226.

[HZ] H. Hofer and E. Zehnder (1994), *Hamiltonian Dynamics and Symplectic Invariants.* Birkhäuser, Berlin.

[KM] P. Kronheimer and T. Mrowka (1997), Monopoles and contact structures. Preprint.

[Ku] K. Kuperberg (1994), A smooth counterexample to the Seifert Conjecture in dimension three. *Ann. of Math.* **140**, 723–732.

[Lu] R. Lutz (1977), Structures de contact sur les fibrés principaux en cercles de dimension three. *Ann. Inst. Fourier* **3**, 1–15.

[Ma] J. Martinet (1971), Forms de contact sur les variétés de dimension 3. *Springer Lecture Notes in Math.* **209**, 142–163.

[MS] D. McDuff and D. Salamon (1995), *Introduction to Symplectic Topology.* Oxford University Press, New York, 1995.

[MP] D. McLaughlin and O. Pironneau (1991), Some notes on periodic Beltrami fields in Cartesian geometry. *J. Math. Phys.* **32** (3), 797–804.

[Mo] H. Moffatt (1985), Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology I: fundamentals. *J. Fluid Mech.* **159**, 359–378.

[No] S. Novikov (1967), Topology of foliations. *Trans. Moscow Math. Soc.* **14**, 268–304.

[Ra] P. Rabinowitz (1979), Periodic solutions of Hamiltonian systems on a prescribed energy surface. *J. Diff. Equ.* **33**, 336–352.

[Re] G. Reeb (1952), Sur certaines propriétés topologiques des trajectoires des systèmes dynamicques. *Acad. Roy. de Belg.* **1635**, 1–64.

[Ro] C. Robinson (1995), *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos.* CRC Press, Boca Raton.

[Sc] P. Schweitzer (1974), Counterexamples to the Seifert Conjecture and opening closed leaves of foliations. *Ann. of Math.* **100**, 386–400.

[Vi] C. Viterbo (1987), A proof of Weinstein's conjecture in $\mathbb{R}^{2n}$. *Ann. Inst. Henri Poincaré, Anal. Nonlinéaire* **4**, 337–356.

[We] A. Weinstein (1979), On the hypothesis of Rabinowitz's periodic orbit theorem. *J. Diff. Equ.* **33**, 353–358.