Mathematical Derivation of Chiral Anomaly in Lattice Gauge Theory with Wilson's Action

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Abstract

Chiral $U(1)$ anomaly is derived with mathematical rigor for a Euclidean fermion coupled to a smooth external $U(1)$ gauge field on an even dimensional torus as a continuum limit of lattice regularized fermion field theory with the Wilson term in the action. The present work rigorously proves for the first time that the Wilson term correctly reproduces the chiral anomaly.

Running title: Mathematical Derivation of Chiral Anomaly

1 Introduction

It is widely believed that continuum limit of a lattice regularized theory of quantized fermion gives the correct chiral anomaly [1, 3, 4] if we have Wilson terms in the lattice fermion action [13]. Since Karsten and Smit [6] observed this fact by a perturbative argument, there appeared mathematically more careful analyses [7, 9], in which it was claimed that the Wilson fermion has the expected continuum limit and gives the correct chiral anomaly under some mathematical ansatz on perturbative expansions. However, a further investigation on the validity of the ansatz has not been published so far, and from a mathematical view point, a proof that Wilson's formalism for lattice fermion is a correct scheme giving the expected anomaly, has not been completed.

On the other hand, in modern physics, the Wilson fermion provides not only a mathematical basis for analytic investigations but also a practical scheme for numerical studies on Euclidean quantum field theories including fermions. In view of this, we cannot help asking for a mathematically rigorous proof that the Wilson term correctly reproduces the chiral anomaly in the continuum limit.

In the present paper, we study Wilson's formalism for a Euclidean lattice fermion coupled to a smooth external $U(1)$ gauge field defined on an even dimensional torus and derive the expected chiral anomaly in the continuum limit with mathematical rigor. The present work rigorously proves for the first time that the Wilson term correctly reproduces the chiral anomaly.

We summarize below three essential mathematical problems in the present study and the strategies to solve the problems.

1. Chiral oscillation. Let $\mathbb{T}_a^d = (a\mathbb{Z}/L\mathbb{Z})^d$ be a discrete torus with period L and lattice spacing a, where L/a is a positive integer. We denote by $\mathbb{V}_a^d$ the vector space consisting of all functions $u : \mathbb{T}_a^d \longrightarrow \mathbb{C}^{2^{d/2}}$ defined on $\mathbb{T}_a^d$ with values in the 'fibre' $\mathbb{C}^{2^{d/2}}$. The vacuum expectation of divergence of chiral current, which is our main quantity of interest, is written in the form of $\mathrm{Tr}\,(K\gamma_{d+1})$, where K is an operator acting on $\mathbb{V}_a^d$, and γ_{d+1} is the chirality in d dimensions (see (3.1)). [Note that γ_{d+1} is regarded as an operator acting on $\mathbb{V}_a^d$ instead of the fibre $\mathbb{C}^{2^{d/2}}$ and therefore the multiplicities of the eigenvalues ± 1 are not uniformly bounded in the lattice spacing a.]

To take the trace, we choose, as in [7, 9], the 'planewave basis', i.e. the set of eigenfunctions of the (free) translations on the lattice (see (4.4)). Though this choice may be standard, it should be underlined because the Wilson term is too weak to make the *trace norm* of $K\gamma_{d+1}$ uniformly bounded in the lattice spacing a. In other words, it is essential to 'cancel' (see e.g. (6.9)) the highly degenerate positive and negative eigenvalues of γ_{d+1} (the 'chiral oscillation') before taking the continuum limit. The planewave basis is convenient for this purpose.

Strategy 1: *Take the trace with respect to the planewave basis in order to explicitly cancel the chiral oscillation.*

It may be illustrative to compare the situation with a previous related work by one of the authors [10, 11, 12]. There, the chiral oscillation is controlled by introducing an additional heat-kernel regularization, with which the operator in question lies in the trace class, and the continuum limit can be taken without explicit cancellation of chiral oscillation. As a result, the index of the continuum Dirac operator appears, which is equal to the Chern class by the index theorem [2], hence the proof is completed.

2. Perturbative expansion. After performing the fermion integration, we have a formula with inverse of operators (see (3.9)). If we try to expand the inverse operators perturbatively, the convergence of the expansion becomes a problem, as is usual with formal perturbation series. For example, the operator C defined in (3.8) is decomposed as

$$C = C_0 + C_1 \tag{1.1}$$

into a *free part* C_0 (for vanishing gauge potentials) and an *interaction part* C_1, but the Neumann series

$$C^{-1} = \sum_{\ell=0}^{\infty}(-C_0^{-1}C_1)^{\ell}C_0^{-1} \tag{1.2}$$

is not absolutely convergent, unless the gauge field is sufficiently small. The following idea was suggeted in [9] without details:

Strategy 2: *Terminate a perturbative expansion of inverse operator up to a finite order, and use 'positivity' to control the remainder terms.*

Namely, we use an identity

$$C^{-1} \;=\; C_0^{-1} + \sum_{j=1}^{m-1} (-C_0^{-1}C_1)^j C_0^{-1} + (-C_0^{-1}C_1)^m C^{-1}\ , \quad m = 1, 2, \dots\ , \tag{1.3}$$

for a sufficiently large m. What we need is the power-counting property of the operator $(-C_0^{-1}C_1)^j$, and an *a priori bound* arising from positivity of C, to control the remainder terms. See Section 4 and Section 5 for details. It may be suggestive to note that remainder estimates based on *positivity* is used in the convergence proof of cluster expansions in constructive field theory.

3. Power counting. The strategy 2 should be accompanied by a rigorous power-counting argument. Furthermore, we need asymptotic estimates of operators, when we take the continuum limit. Our last point is to employ the power-counting arguments with mathematical rigor in order to extract the asymptotic behaviors.

Strategy 3: *Show and use the fact that the operators in question are 'quasi diagonal' with respect to the planewave basis.*

The class of *quasi diagonal operators* defined in Section 4 is closed with respect to the summation and the product, and, under a simple assumption, it is also closed with respect to the inverse operation. This calculus, which is a systematic arrangement of the asymptotic analysis of lattice operators in the spirit of power-counting, fits our analysis in this paper well. The smoothness assumption on the gauge fields is of relevance here. See Section 4 for details.

According to the above strategies, we take the continuum limit of $\mathrm{Tr}\,(K\gamma_{d+1})$ and confirm that the Wilson fermion gives the correct chiral anomaly.

This paper is organized as follows. We describe the model and state the result (Theorem 2.1) in Section 2. Theorem 2.1 is proved in the subsequent sections. In Section 3, we derive the expression of chiral anomaly in the form of chiral oscillation. Section 4 is devoted to the calculus of quasi diagonal operators. In Section 5, we apply the framework of Section 4 to several operators and determine their orders. The proof of Theorem 2.1 is completed in Section 6.

The propositions in Section 3 and Section 6 may guide the readers who wish to have an overview of our argument.

2 Problem and Result

In this section, we formulate the problem and then state the result in Theorem 2.1.

2.1 Notation

Let $\mathbb{T}_a^d$ be a discrete torus with lattice spacing a and period L. Throught this paper, we assume that $0 < a < 1/2, L/a \in \mathbb{Z}$. Let $\mathbb{V}_a^d$ be the set of all mappings

$$u = (u_s)_{s=1,2,\dots,2^{d/2}} : \;\; \mathbb{T}_a^d \longrightarrow \mathbb{C}^{2^{d/2}}. \tag{2.1}$$

We impose the anti-periodic boundary condition for definiteness, though boundary conditions play no essential roles in this paper.[1]

Let $(\ ,\ )$ be the inner product on $\mathbb{V}_a^d$ defined by

$$(v,u) \;=\; a^d \sum_{x\in\mathbb{T}_a^d} (v(x),u(x))_x\,, \tag{2.2}$$

where $(\ ,\ )_x$ is the (standard) inner product on $\mathbb{C}^{2^{d/2}}$:

$$(v(x),u(x))_x \;=\; \sum_{s=1}^{2^{d/2}} \bar{v}_s(x)u_s(x)\,.$$

Let $A_\mu(x), \mu=1,2,\ldots,d$, be a fixed smooth $U(1)$ gauge field on the continuum torus $\mathbb{T}^d=(\mathbb{R}/L\mathbb{Z})^d$. Using A_μ, we assign the gauge group element

$$U_\mu(x) \;=\; \exp\left(-iQ\int_0^a A_\mu(x+se_\mu)\,ds\right) \tag{2.3}$$

to a directed link $(x,x+e_\mu), x\in\mathbb{T}_a^d, \mu=1,2,\ldots,d$, where $Q\in\mathbb{R}$ is a charge and e_μ is a unit vector along the μ-th axis for $\mu=1,2,\ldots,d$. Denote the free and the covariant translations by $T_{\mu 0}$ and T_μ, respectively:

$$T_{\mu 0}u(x) \;=\; u(x+ae_\mu),\ \mu=1,2,\ldots,d,\ x\in\mathbb{T}_a^d,\ u\in\mathbb{V}_a^d, \tag{2.4}$$

$$T_\mu \;=\; U_\mu T_{\mu 0}\,,\ \mu=1,2,\ldots,d, \tag{2.5}$$

where U_μ denotes the multiplication operator defined by

$$U_\mu u(x) \;=\; U_\mu(x)u(x),\ \mu=1,2,\ldots,d,\ x\in\mathbb{T}_a^d,\ u\in\mathbb{V}_a^d. \tag{2.6}$$

Note that $T_\mu^*=T_\mu^{-1}$ holds, where T_μ^* is the adjoint with respect to the inner product $(\ ,\ )$. Put

$$D_\mu \;=\; \frac{1}{2a}(T_\mu-T_\mu^*)\,,\ \mu=1,2,\ldots,d, \tag{2.7}$$

and let $\gamma_\mu, \mu=1,2,\ldots,d$, be the d dimensional anti-hermitian Dirac matrices. Namely,

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu \;=\; -2\delta_{\mu\nu}I_{2^{d/2}}\ , \quad \mu,\nu=1,2,\ldots,d, \tag{2.8}$$

where I_ℓ denotes the unit matrix of order ℓ. In what follows, we abbreviate $\gamma_\mu\otimes\dot{I}$ to γ_μ, where $\dot{I}$ denotes the identity operator on the space $\dot{\mathbb{V}}_a^d=\{\varphi\ :\ \mathbb{T}_a^d\to\mathbb{C}\}$. Then, the lattice Dirac operator on the discrete torus $\mathbb{T}_a^d$ is by definition

$$\not{D} \;=\; \sum_{\mu=1}^d \gamma_\mu D_\mu\,. \tag{2.9}$$

[1]The anti-periodic boundary condition is required in the proof of the reflection positivity [8].

2.2 Lattice fermion

As is well-known, the operator $\not{D}$ has eigenfunctions called *doublers* which do not approximate any eigenfunctions of the continuum Dirac operator. In order to suppress the contributions of doublers to vacuum expectation values, we introduce the Wilson term:

$$W \;=\; -\sum_{\mu=1}^{d}(2I - T_\mu - T_\mu^*)\,, \tag{2.10}$$

where I denotes the identity operator on $\mathbb{V}_a^d$. Then our lattice fermion action is written as

$$S(\bar{\psi},\psi) \;=\; (\bar{\psi},(i\not{D} + MI - \frac{r}{2a}W)\psi), \tag{2.11}$$

where $M>0$ is the fermion mass and r is a positive constant.[2] Using the action S, we define the vacuum expectation by

$$\langle \Phi(\bar{\psi},\psi)\rangle \;=\; \frac{1}{Z(A)}\int \prod_{x\in\mathbb{T}_a^d} d\bar{\psi}(x)d\psi(x)\exp(-S(\bar{\psi},\psi))\Phi(\bar{\psi},\psi), \tag{2.12}$$

where the integrations with respect to $\bar{\psi}$ and ψ are the Grassmann integrations [3, 8] and the partition function $Z(A)$ is defined by

$$\begin{aligned} Z(A) &= \int \prod_{x\in\mathbb{T}_a^d} d\bar{\psi}(x)d\psi(x)\exp(-S(\bar{\psi},\psi)) \\ &= \det(i\not{D} + MI - \frac{r}{2a}W). \end{aligned} \tag{2.13}$$

Note that the semi-positivity of $-W$

$$(u,-Wu) \;=\; \sum_{\mu=1}^{d} \|\,(I-T_\mu)u\,\|^2 \geq\; 0 \tag{2.14}$$

implies

$$(u,(i\not{D} + MI - \frac{r}{2a}W)u) \;\neq\; 0\;,\; u\neq 0, \tag{2.15}$$

hence $Z(A)\neq 0$.

2.3 Lattice Chiral Current

We define the lattice chiral current by

$$\begin{aligned} J_\mu(x) \;=\; & \frac{1}{2}(\bar{\psi}(x),\gamma_{d+1}\gamma_\mu(T_\mu\psi)(x))_x + \frac{1}{2}((T_\mu\bar{\psi})(x),\gamma_{d+1}\gamma_\mu\psi(x))_x\,, \\ & \mu = 1,2,\ldots,d, \end{aligned} \tag{2.16}$$

[2] Our analysis allows any $r>0$, but the proof of reflection positivity [8] requires $0<r\leq 1$.

where the chirality γ_{d+1} is by definition

$$\gamma_{d+1} \quad = \quad i^{d/2}\gamma_1\gamma_2\cdots\gamma_d\,. \tag{2.17}$$

Put

$$Y(x) \quad = \quad \frac{1}{a}\sum_{\mu=1}^{d}(J_\mu(x)-J_\mu(x-ae_\mu)) - 2Mi(\bar{\psi}(x),\gamma_{d+1}\psi(x))_x,\ \ x\in\mathbb{T}_a^d, \tag{2.18}$$

and smear it as

$$Y(\xi) \quad = \quad a^d\sum_{x\in\mathbb{T}_a^d}\xi(x)Y(x) \tag{2.19}$$

by an arbitrary real-valued smooth function ξ defined on the continuum torus. The functional $Y(x)$ is the 'divergence' of lattice chiral current with a mass correction. Our problem is to calculate $\lim_{a\to 0}\langle Y(\xi)\rangle$.

2.4 Result

Let us state our result.

Theorem 2.1. *For an arbitrary smooth gauge field $A=(A_\mu)$ and for an arbitrary smooth function ξ both defined on the continuum torus $\mathbb{T}^d$, it holds that*

$$\lim_{a\to 0}\langle Y(\xi)\rangle \quad = \quad \frac{-2iQ^{d/2}}{(4\pi)^{d/2}(d/2)!}\int_{\mathbb{T}^d}dx\ \xi(x)\sum_{\mu_1,\mu_2,\ldots,\mu_d=1}^{d}\epsilon_{\mu_1\mu_2\cdots\mu_d} F_{\mu_1\mu_2}(x)F_{\mu_3\mu_4}(x)\cdots F_{\mu_{d-1}\mu_d}(x), \tag{2.20}$$

where $\epsilon_{\mu_1\mu_2\cdots\mu_d}$ is the totally antisymmetric tensor and

$$F_{\mu\nu}(x) \quad = \quad \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{2.21}$$

Namely, $\langle Y(x)\rangle$ weakly converges to the Chern class in the continuum limit. In the sebsequent sections, we prove this fact with mathematical rigor.

3 Chiral Oscillation

In this section, we carry out the Grassmann integrations, and write the chiral anomaly as a trace of an operator on $\mathbb{V}_a^d$.

Proposition 3.1 ([7, 9]). *It holds that*

$$\langle Y(\xi)\rangle \quad = \quad -2i\mathrm{Tr}_{\mathbb{V}_a^d}\left[\Lambda\xi\gamma_{d+1}\right], \tag{3.1}$$

where

$$\Lambda \quad = \quad \frac{1}{2}(i\not{D}+MI-\frac{r}{2a}W)^{-1}(i\not{D}+MI)+\frac{1}{2}(i\not{D}+MI)(i\not{D}+MI-\frac{r}{2a}W)^{-1}. \tag{3.2}$$

Proof. It is easy to see that

$$\begin{aligned} Y(\xi) &= a^d \sum_{x\in\mathbb{T}_a^d} \xi(x)Y(x) && (3.3)\\ &= (\bar\psi, \xi\gamma_{d+1}\not{D}\psi) + (\not{D}\bar\psi, \xi\gamma_{d+1}\psi) - 2Mi(\bar\psi, \xi\gamma_{d+1}\psi), && (3.4)\end{aligned}$$

where in (3.4) we used the same symbol ξ for the multiplication operator given by $(\xi u)(x) = \xi(x)u(x), x \in \mathbb{T}_a^d$. Using (3.4) and manipulating the Grassmann integrations, we obtain (3.1). Here, (2.15) ensures the existence of the inverse of $i\not{D} + MI - \frac{r}{2a}W$. □

Taking into account the behavior of the Wilson term $-\frac{r}{2a}W$ for doublers, one may intuitively see that the Λ works as a 'projection' to decouple doublers, and according to this picture, one can write the scenario that when the doublers go out of the continuum world, they take away a part of lattice chiral current and produces the chiral unbalance in the continuum world. One may say that this is the origin of chiral anomaly and nonzero index of the Dirac operator.

Put

$$\begin{aligned} L &= MI - \frac{r}{2a}W, && (3.5)\\ X_{\mu\nu} &= D_\mu L^{-1} D_\nu - D_\nu L^{-1} D_\mu \ , \ \mu,\nu = 1,2,\ldots,d, && (3.6)\\ B &= \frac{1}{2}\sum_{\mu,\nu=1}^{d} \gamma_\mu\gamma_\nu X_{\mu\nu}, && (3.7)\\ C &= L - \sum_{\mu=1}^{d} D_\mu L^{-1} D_\mu. && (3.8)\end{aligned}$$

Proposition 3.2 ([7, 9]). *It holds that*

$$\langle Y(\xi)\rangle = -i\mathrm{Tr}_{\mathbb{V}_a^d}\left[\frac{r}{2a}(\xi W + W\xi)(C+B)^{-1}\gamma_{d+1}\right]. \qquad (3.9)$$

Proof. Put

$$\begin{aligned} H_0 &= i\not{D} + MI, && (3.10)\\ H_1 &= i\not{D} + L, && (3.11)\end{aligned}$$

and rewrite (3.1) as

$$\langle Y(\xi)\rangle = -i\mathrm{Tr}_{\mathbb{V}_a^d}\left[\gamma_{d+1}(H_0H_1^{-1} + H_1^{-1}H_0)\xi\right]. \qquad (3.12)$$

Since $H_0 = H_1 + \frac{r}{2a}W$ and $\mathrm{Tr}_{\mathbb{V}_a^d}[\xi\gamma_{d+1}] = 0$, it holds that

$$\langle Y(\xi)\rangle = -i\mathrm{Tr}_{\mathbb{V}_a^d}\left[\gamma_{d+1}\frac{r}{2a}(WH_1^{-1} + H_1^{-1}W)\xi\right]. \qquad (3.13)$$

Note that

$$\begin{aligned} H_1L^{-1}H_1^* &= L + \not{D}L^{-1}\not{D}\\ &= C + B && (3.14)\end{aligned}$$

implies

$$H_1^{-1} \;=\; L^{-1}H_1^*(C+B)^{-1}. \tag{3.15}$$

Furthermore,

$$H_1\gamma_{d+1} \;=\; \gamma_{d+1}H_1^* \tag{3.16}$$

holds, while L, $\xi W+W\xi$, and $(C+B)^{-1}$ commute with γ_{d+1}. Then we have

$$\begin{aligned}
\langle Y(\xi)\rangle &= -i\mathrm{Tr}_{\mathbb{V}_a^d}\left[\frac{r}{2a}(\xi W+W\xi)L^{-1}H_1^*(C+B)^{-1}\gamma_{d+1}\right]\\
&= -i\mathrm{Tr}_{\mathbb{V}_a^d}\left[\frac{r}{2a}(\xi W+W\xi)L^{-1}\frac{H_1+H_1^*}{2}(C+B)^{-1}\gamma_{d+1}\right]\\
&= -i\mathrm{Tr}_{\mathbb{V}_a^d}\left[\frac{r}{2a}(\xi W+W\xi)(C+B)^{-1}\gamma_{d+1}\right].
\end{aligned} \tag{3.17}$$

□

4 Quasi Diagonal Operators

All the lattice operators appearing in this paper are 'quasi diagonal' with respect to the planewave basis. In this section, we define the class of quasi diagonal operators, give a few examples (Lemma 4.1, Lemma 4.2), and prove basic properties (Lemma 4.3, Lemma 4.4).

4.1 Definition of quasi diagonal operators

We first fix an orthonormal basis of $\mathbb{V}_a^d$. Put

$$\mathbb{T}_a^{d*} \;=\; \{p=(p_\mu)\in\mathbb{R}^d \mid p_\mu=\frac{2k_\mu+1}{L}\pi, k_\mu\in\mathbb{Z}, |p_\mu|<\frac{2}{a}\pi\}. \tag{4.1}$$

We regard $\mathbb{T}_a^{d*}$ as a discrete torus with period $\frac{2}{a}\pi$ and spacing $\frac{2\pi}{L}$ and denote by $\langle p\rangle$ the periodic distance on $\mathbb{T}_a^{d*}$:

$$\begin{aligned}
\langle p\rangle &= \max_{\mu=1,2,\ldots,d}\langle p_\mu\rangle\\
&= \max_{\mu=1,2,\ldots,d}\min_{n\in\mathbb{Z}}|p_\mu-\frac{2n}{a}\pi|.
\end{aligned} \tag{4.2}$$

For $p\in\mathbb{T}_a^{d*}$, define the anti-periodic planewave u_p on $\mathbb{T}_a^d$ with a momentum p by

$$u_p(x) \;=\; L^{-d/2}\exp(ipx)\,, \quad x\in\mathbb{T}_a^d. \tag{4.3}$$

Furthermore, let $\chi_\alpha, \alpha=1,2,\ldots,2^{d/2}$, be the canonical orthonormal basis of $\mathbb{C}^{2^{d/2}}$ with respect to the standard inner product. Then,

$$u_{\alpha p} \;=\; \chi_\alpha\otimes u_p\in\mathbb{V}_a^d\,, \quad \alpha=1,2,\ldots,2^{d/2}, p\in\mathbb{T}_a^{d*}, \tag{4.4}$$

constitute an orthonormal basis of $\mathbb{V}_a^d$:

$$(u_{\alpha p}, u_{\beta q}) \;=\; \delta_{\alpha\beta}\delta_{pq}\ ,\ \ \alpha,\beta = 1,2,\ldots,2^{d/2},\ \ p,q \in \mathbb{T}_a^{d*}, \tag{4.5}$$

where the inner product (,) is the one defined by (2.2).

Let us formulate the class of quasi diagonal operators. Operators such as T_μ or C acting on $\mathbb{V}_a^d$ have dependences on the lattice spacing a, so that when we define such operators, we in fact define families of operators $T_\mu = T_\mu^{(a)}, a > 0$, or $C = C^{(a)}, a > 0$.

Definition. Let $k = k(a,p), p \in \mathbb{T}_a^{d*}$, be a non-negative function satisfying

$$k(a,p) \;\le\; c(1 + \langle p-q \rangle)^\tau k(a,q)\ ,\ \ p,q \in \mathbb{T}_a^{d*}, \tag{4.6}$$

for some constants c and τ independent of a, p, and q. Then, for a family of linear operators $K^{(a)}$ on $\mathbb{V}_a^d$, we say that $K^{(a)}$ is *a quasi diagonal operator of the order* k and write $K^{(a)} = \mathcal{O}(k)$, if

$$|(u_{\alpha p}, K^{(a)} u_{\beta q})| \;\le\; c_\sigma \frac{k(a,p)}{(1+\langle p-q \rangle)^\sigma}\ , \tag{4.7}$$

holds for all $\sigma \ge 0$, $p,q \in \mathbb{T}_a^{d*}$, and $\alpha,\beta = 1,2,\ldots,2^{d/2}$. Here the constant c_σ may depend on σ but not on a,p,q,α,β.

Note that (4.6) and (4.7) imply

$$|(u_{\alpha p}, K^{(a)} u_{\beta q})| \;\le\; c'_\sigma \frac{k(a,q)}{(1+\langle p-q \rangle)^\sigma} \tag{4.8}$$

for another constant c'_σ. In other words,

$$K^{(a)} = \mathcal{O}(k) \;\Longrightarrow\; K^{(a)*} = \mathcal{O}(k). \tag{4.9}$$

In the following, we often suppress writing a-dependences of operators explicitly and write K instead of $K^{(a)}$. Then we will simply say that K is quasi diagonal and write $K = \mathcal{O}(k)$.

4.2 Free parts and multiplication operators

Define the 'free parts' of D_μ, W, L, and C by

$$D_{\mu 0} \;=\; \frac{1}{2a}(T_{\mu 0} - T_{\mu 0}^*), \tag{4.10}$$

$$W_0 \;=\; -\sum_{\mu=1}^{d}(2 - T_{\mu 0} - T_{\mu 0}^*), \tag{4.11}$$

$$L_0 \;=\; MI - \frac{r}{2a}W_0, \tag{4.12}$$

$$C_0 \;=\; L_0 - \sum_{\mu=1}^{d} L_0^{-1} D_{\mu 0}^2, \tag{4.13}$$

respectively. A free part of an operator is equal to the operator when the gauge potentials $A_\mu, \mu = 1, 2, \ldots, d$, vanish. The free part of B vanishes. Since $-W_0$ is semi-positive (see (2.14)), L_0 and C_0 are positive definite and therefore invertible.

These operators are *exactly diagonal* with respect to the planewave basis and are the first examples of quasi diagonal operators.

Lemma 4.1. *We have the following.*

$$\begin{aligned}
T_{\mu 0} &= \mathcal{O}(1), && (4.14)\\
T_{\mu 0} - I &= \mathcal{O}(a(1 + \langle p \rangle)), && (4.15)\\
D_{\mu 0} &= \mathcal{O}(1 + \langle p \rangle), && (4.16)\\
W_0 &= \mathcal{O}(a^2(1 + \langle p \rangle^2)), && (4.17)\\
L_0 &= \mathcal{O}(1 + a\langle p \rangle^2), && (4.18)\\
L_0^{-1} &= \mathcal{O}(\frac{1}{1 + a\langle p \rangle^2}), && (4.19)\\
C_0 &= \mathcal{O}(\frac{1 + \langle p \rangle^2}{1 + a\langle p \rangle^2}), && (4.20)\\
C_0^{-1} &= \mathcal{O}(\frac{1 + a\langle p \rangle^2}{1 + \langle p \rangle^2}). && (4.21)
\end{aligned}$$

Proof. (4.14)-(4.17) follow from

$$\begin{aligned}
T_{\mu 0} u_{\alpha p} &= e^{iap_\mu} u_{\alpha p}, && (4.22)\\
D_{\mu 0} u_{\alpha p} &= \frac{i}{a} \sin(ap_\mu) u_{\alpha p}, && (4.23)\\
W_{\mu 0} u_{\alpha p} &= -2 \sum_{\mu=1}^{d} (1 - \cos(ap_\mu)) u_{\alpha p}. && (4.24)
\end{aligned}$$

For example, if $K = T_{\mu 0} - I$, then (4.7) is satisfied by putting $k(a, p) = a\langle p \rangle$. We add extra '1' in (4.15) in order to ensure (4.6): the function $k(a, p) = a(1 + \langle p \rangle)$ satisfies (4.6), because

$$\begin{aligned}
1 + \langle q + r \rangle &\leq 1 + \langle q \rangle + \langle r \rangle\\
&\leq (1 + \langle r \rangle)(1 + \langle q \rangle), \;\; q, r \in \mathbb{T}_a^{d*}. && (4.25)
\end{aligned}$$

For the case $K = W_0$, we need

$$\begin{aligned}
1 + \langle q + r \rangle^2 &\leq 1 + 2(\langle q \rangle^2 + \langle r \rangle^2)\\
&\leq 2(1 + \langle r \rangle^2)(1 + \langle q \rangle^2), \;\; q, r \in \mathbb{T}_a^{d*}. && (4.26)
\end{aligned}$$

The other bounds are shown by using

$$L_0 u_{\alpha p} = \left(M + \frac{r}{2a} \sum_{\mu=1}^{d} (1 - \cos ap_\mu) \right) u_{\alpha p}, \qquad (4.27)$$

$$C_0 u_{\alpha p} = \left(M + \frac{r}{2a} \sum_{\mu=1}^{d} (1 - \cos a p_\mu) + \frac{\displaystyle\sum_{\mu=1}^{d} \frac{1}{a^2} \sin^2 a p_\mu}{M + \displaystyle\frac{r}{2a} \sum_{\mu=1}^{d} (1 - \cos a p_\mu)} \right) u_{\alpha p}. \qquad (4.28)$$

□

A multiplication operator determined by a smooth function on $\mathbb{T}^d$ is another example of quasi diagonal operators.

Lemma 4.2. *Let ϕ be a multiplication operator defined by*

$$\phi u(x) = \varphi(x) u(x) \ , \ x \in \mathbb{T}_a^d, u \in \mathbb{V}_a^d, \qquad (4.29)$$

for a complex valued smooth (periodic) function $\varphi(x)$ on the continuum torus $\mathbb{T}^d$. Then, we have $\phi = \mathcal{O}(1)$.

Furthermore, $\phi = \mathcal{O}(1)$ holds, if ϕ is defined by

$$\phi u(x) = \varphi_a(x) u(x) \ , \ x \in \mathbb{T}_a^d, u \in \mathbb{V}_a^d, \qquad (4.30)$$

for a family of functions $\varphi_a(x)$ on $\mathbb{T}^d$ that are 'uniformly smooth', i.e.

$$\sup_{x \in \mathbb{T}^d} |\partial^\alpha \varphi_a(x)| < c^{(\alpha)} \ , \ \alpha = (\alpha_1, \ldots, \alpha_d) \in \{0, 1, 2, \ldots\}^d, \qquad (4.31)$$

where $\{c^{(\alpha)} \mid \alpha \in \{0, 1, \cdots\}^d\}$ is a family of constants independent of a, and ∂^α stands for $\prod_{k=1}^d \partial_k^{\alpha_k}$.

Proof. We show the first part of the lemma. Since $\varphi(x)$ is smooth and satisfies

$$(u_{\alpha p}, \phi u_{\beta q}) = \delta_{\alpha\beta} a^d \sum_{x \in \mathbb{T}_a^d} e^{i(q-p)x} \varphi(x), \ \alpha, \beta = 1, 2, \ldots, 2^{d/2}, p, q \in \mathbb{T}_a^{d*}, \qquad (4.32)$$

there exists c_σ such that

$$|(u_{\alpha p}, \phi u_{\beta q})| \leq \frac{c_\sigma}{(1 + \langle p - q \rangle)^\sigma} \ , \sigma \geq 0 \qquad (4.33)$$

for $p, q \in \mathbb{T}_a^{d*}$ and for $\alpha, \beta = 1, 2, \ldots, 2^{d/2}$. Then, $K_a = \phi$ and $k(a, p) = 1$ satisfy (4.6) and (4.7). The other part of the lemma can be proved in a similar way. □

4.3 Basic properties

The class of quasi diagonal operators is closed with respect to summation, product, and the inverse operation in the following sense.

Lemma 4.3. *Let* $K_j = \mathcal{O}(k_j), j = 1, 2$. *Then, we have* $K_1 + K_2 = \mathcal{O}(k_1 + k_2)$ *and* $K_1 K_2 = \mathcal{O}(k_1 k_2)$.

The proof is given in Section 4.4. Note that, when we say $K = \mathcal{O}(k)$, it is implied that k has the property (4.6) with some constants c, τ.

Lemma 4.4. *Suppose that* $K = K_0 + K_1$ *satisfies the following, where 'const.' stands for some positive constant independent of* a *and* p*:*

(1) There exists k_0 *satisfying*

$$k_0(p) \quad \le \quad const.(1 + \langle p \rangle)^{\nu} \ , \ p \in \mathbb{T}_a^{d*}, \tag{4.34}$$

for some constant ν *independent of* a *and* p*, such that* K_0 *is exactly of the order* k_0 *in the sense that* $K_0 = \mathcal{O}(k_0)$ *and* $K_0^{-1} = \mathcal{O}(1/k_0)$ *hold.*

(2) K_1 *is of a lower order than* K_0 *in the sense that* $K_1 = \mathcal{O}(k_1)$ *and*

$$\frac{k_1(p)}{k_0(p)} \quad \le \quad const.(1 + \langle p \rangle)^{-\delta} \ , \ p \in \mathbb{T}_a^{d*}, \tag{4.35}$$

for some constant $\delta > 0$ *independent of* a *and* p.

(3) K^{-1} *exists and is uniformly bounded, i.e.,*

$$\| K^{-1} \| \quad = \quad \sup_{\substack{u \in \mathbb{V}_a^d \\ u \neq 0}} \frac{\| K^{-1} u \|}{\| u \|} \le c \tag{4.36}$$

for some constant c *independent of* a.

Then we have $K^{-1} = \mathcal{O}(1/k_0)$.

The proof is given in Section 4.6.

By means of Lemma 4.3 and Lemma 4.4, we can estimate traces on $\mathbb{V}_a^d$ using the order of operators. As may be seen explicitly from Lemma 4.1, such estimates provide a mathematical justification of the power counting argument.

4.4 Proof of Lemma 4.3

Let us first show $K_1 + K_2 = \mathcal{O}(k_1 + k_2)$. Assume

$$\begin{aligned} k_1(a,p) &\le c_1(1+\langle p-q\rangle)^{\tau_1} k_1(a,q)\ , & (4.37)\\ k_2(a,p) &\le c_2(1+\langle p-q\rangle)^{\tau_2} k_2(a,q)\ , & (4.38)\\ |(u_{\alpha p}, K_1 u_{\beta q})| &\le c_{1\sigma}\frac{k_1(a,p)}{(1+\langle p-q\rangle)^{\sigma}}\ , & (4.39)\\ |(u_{\alpha p}, K_2 u_{\beta q})| &\le c_{2\sigma}\frac{k_2(a,p)}{(1+\langle p-q\rangle)^{\sigma}}\ , & (4.40) \end{aligned}$$

for $\sigma \ge 0$, $p,q \in \mathbb{T}_a^{d*}$, and for $\alpha,\beta = 1,2,\ldots,2^{d/2}$. Then, putting $K = K_1 + K_2, k = k_1 + k_2, \tau = \max(\tau_1,\tau_2), c = \max(c_1,c_2)$, and $c_\sigma = \max(c_{1\sigma}, c_{2\sigma})$, we have (4.6) and (4.7), and consequently, $K_1 + K_2 = \mathcal{O}(k_1 + k_2)$.

Let us show $K = K_1K_2 = \mathcal{O}(k_1k_2)$. Put $k = k_1k_2$. Then, for $c = c_1c_2$ and for $\tau = \tau_1 + \tau_2$, (4.6) holds. In order to show (4.7), we need the following lemma.

Lemma 4.5. *If $0 \le \rho \le \min\{\sigma,\tau\}$ and $\sigma + \tau - \rho \ge d+1$ hold, we have*

$$\sum_{r\in\mathbb{T}_a^{d*}} (1+\langle p-r\rangle)^{-\sigma}(1+\langle r-q\rangle)^{-\tau} \le \mathit{const.}\,(1+\langle p-q\rangle)^{-\rho}, \tag{4.41}$$

where 'const.' is independent of p, q, and a.

Proof. Decompose the summation in the left hand side of (4.41) as

$$\begin{aligned} &\sum_{r\in\mathbb{T}_a^{d*}} (1+\langle p-r\rangle)^{-\sigma}(1+\langle r-q\rangle)^{-\tau}\\ &= \left(\sum_{\substack{r\in\mathbb{T}_a^{d*}\\ \langle p-r\rangle\ge\langle r-q\rangle}} + \sum_{\substack{r\in\mathbb{T}_a^{d*}\\ \langle p-r\rangle<\langle r-q\rangle}}\right)(1+\langle p-r\rangle)^{-\sigma}(1+\langle r-q\rangle)^{-\tau}. \end{aligned} \tag{4.42}$$

Let us estimate the first sum. If $\langle p-r\rangle \ge \langle r-q\rangle$, then it holds that

$$2\langle p-r\rangle \ge \langle p-r\rangle + \langle r-q\rangle \ge \langle p-q\rangle, \tag{4.43}$$

hence

$$\begin{aligned} &\sum_{\substack{r\in\mathbb{T}_a^{d*}\\ \langle p-r\rangle\ge\langle r-q\rangle}} (1+\langle p-r\rangle)^{-\sigma}(1+\langle r-q\rangle)^{-\tau}\\ \le\ &\sum_{\substack{r\in\mathbb{T}_a^{d*}\\ \langle p-r\rangle\ge\langle r-q\rangle}} (1+\langle p-r\rangle)^{-\rho}(1+\langle p-r\rangle)^{-\sigma+\rho}(1+\langle r-q\rangle)^{-\tau}\\ \le\ &\sum_{r\in\mathbb{T}_a^{d*}} (1+\frac{1}{2}\langle p-q\rangle)^{-\rho}(1+\langle r-q\rangle)^{-\tau-\sigma+\rho}\\ \le\ &\mathit{const.}\,(1+\frac{1}{2}\langle p-q\rangle)^{-\rho} \end{aligned} \tag{4.44}$$

where we also used the assumptions $0 \le \rho \le \sigma$ and $\tau + \sigma - \rho \ge d + 1$. The second sum in the right hand side of (4.42) can be estimated in a similar way. □

Using (4.39),(4.40),(4.38) and (4.41), we have

$$
\begin{aligned}
|(u_{\alpha p}, K_1 K_2 u_{\beta q})| &\le \sum_{\gamma=1}^{2^{d/2}} \sum_{r \in \mathbb{T}_a^{d*}} |(u_{\alpha p}, K_1 u_{\gamma r})||(u_{\gamma r}, K_2 u_{\beta q})| \\
&\le \sum_{\gamma=1}^{2^{d/2}} \sum_{r \in \mathbb{T}_a^{d*}} \frac{c_{1\sigma} c_{2\tau} k_1(a,p) k_2(a,r)}{(1 + \langle p - r \rangle)^{\sigma} (1 + \langle r - q \rangle)^{\tau}} \\
&\le \sum_{\gamma=1}^{2^{d/2}} \sum_{r \in \mathbb{T}_a^{d*}} \frac{c_{1\sigma} c_{2\tau} c_2 k_1(a,p) k_2(a,p)}{(1 + \langle p - r \rangle)^{\sigma - \tau_2} (1 + \langle r - q \rangle)^{\tau}} \\
&\le const. \frac{c_{1\sigma} c_{2\tau} c_2 k_1(a,p) k_2(a,p)}{(1 + \langle p - q \rangle)^{\rho}}
\end{aligned} \tag{4.45}
$$

by choosing sufficiently large σ and τ for each ρ. This implies (4.7) for $K = K_1 + K_2$, hence a proof of Lemma 4.3 is completed.

4.5 Preparation for a Proof of Lemma 4.4

In this section, we prepare Lemma 4.6, Lemma 4.7, and Lemma 4.8 for the proof of Lemma 4.4.

The idea of the proof of Lemma 4.4 is to estimate the right hand side of the identity

$$
K^{-1} = \sum_{j=0}^{n-1} J^j K_0^{-1} + J^n K^{-1}, \ n = 1, 2, \ldots, \tag{4.46}
$$

where

$$
J = -K_0^{-1} K_1. \tag{4.47}
$$

Lemma 4.3 implies that the first term in the right hand side of (4.46) is quasi diagonal. For the second term, as we shall see later, $J^n K^{-1}$ for a sufficiently large n has a good power counting property in spite of the poor information (4.36) on K^{-1}. The key point is that the order of J is strictly lower than 1:

Lemma 4.6. *It holds that*

$$
J = \mathcal{O}((1 + \langle p \rangle)^{-\delta}). \tag{4.48}
$$

Furthermore, for $n \ge \nu/\delta$,

$$
J^n = \mathcal{O}(1/k_0) \tag{4.49}
$$

holds.

Proof. The assumtion (4.35) and Lemma 4.3 imply (4.48). Then we have

$$J^n \;=\; \mathcal{O}((1+\langle p\rangle)^{-n\delta}) \;,\; n=1,2,\dots . \tag{4.50}$$

In view of (4.34), we can bound the order of J^n as follows:

$$(1+\langle p\rangle)^{-n\delta} \;\leq\; const.\frac{(1+\langle p\rangle)^{\nu-n\delta}}{k_0(p)}. \tag{4.51}$$

Choosing $n \geq \nu/\delta$, we obtain (4.49). □

The above lemma implies that multiplying by J improves the power counting properties of operators.

In order to determine the order of the second term in the right hand side of (4.46), we have to bound

$$\langle p_\mu - q_\mu\rangle^\rho |(u_{\alpha p}, J^n K^{-1} u_{\beta q})| \tag{4.52}$$

for $\rho \geq 0, p, q \in \mathbb{T}_a^{d*}$, and for $\alpha, \beta = 1, 2, \dots, 2^{d/2}$. To this end, we use commutators with $\frac{1}{a}T_{\mu 0}$. Let us write

$$\mathrm{ad}_\mu(R) \;=\; \left[\frac{1}{a}T_{\mu 0}, R\right] \tag{4.53}$$

for an operator R on $\mathbb{V}_a^d$.

Lemma 4.7. *For an operator R on $\mathbb{V}_a^d$, it holds that*

$$|(u_{\alpha p}, \mathrm{ad}_\mu(R) u_{\beta q})| \quad \begin{cases} \leq \langle p_\mu - q_\mu\rangle |(u_{\alpha p}, R u_{\beta q})|, \\ \geq \frac{1}{\pi}\langle p_\mu - q_\mu\rangle |(u_{\alpha p}, R u_{\beta q})|, \end{cases} \tag{4.54}$$
$$p, q \in \mathbb{T}_a^{d*},\ \alpha, \beta = 1, 2, \dots, 2^{d/2},\ \mu = 1, 2, \dots, d.$$

Furthermore, if $R = \mathcal{O}(r)$, then $\mathrm{ad}_\mu(R) = \mathcal{O}(r)$.

Proof. The bounds (4.54) follows from

$$\begin{aligned} |(u_{\alpha p}, \mathrm{ad}_\mu(R) u_{\beta q})| &= \frac{1}{a}|(\exp(ip_\mu a) - \exp(iq_\mu a))||(u_{\alpha p}, R u_{\beta q})| \\ &= \frac{2}{a}|\sin(\frac{1}{2}(p_\mu - q_\mu)a)||(u_{\alpha p}, R u_{\beta q})|. \end{aligned} \tag{4.55}$$

The estimate $\mathrm{ad}_\mu(R) = \mathcal{O}(r)$ follows from (4.54) and (4.7). □

Lemma 4.8. *(1) For $R = \mathcal{O}(r)$, we have*

$$|(u_{\alpha p}, RK^{-1}u_{\beta q})| \;\leq\; const.r(p)\ ,\ p,q \in \mathbb{T}_a^{d*},\ \alpha,\beta = 1,2,\ldots,2^{d/2}, \quad (4.56)$$

where 'const.' is a constant independent of a, p, and q.

(2) Assume that $R = \mathcal{O}(r)$ is at most of the same order as K, i.e.,

$$r(p) \;\leq\; k_0(p)\ ,\ p \in \mathbb{T}_a^{d*}. \quad (4.57)$$

Then, we have

$$\sum_{s' \in \mathbb{T}_a^{d*}} |(u_{\alpha s'}, RK^{-1}u_{\beta s})| \;\leq\; const.\ ,\ s \in \mathbb{T}_a^{d*},\ \alpha,\beta = 1,2,\ldots,2^{d/2}, \quad (4.58)$$

where 'const.' is a constant independent of a and s.

Proof. (1) (4.7) and (4.36) imply

$$|(u_{\alpha p}, Ru_{\gamma s})| \;\leq\; \frac{const.r(p)}{(1+\langle p-s\rangle)^{d+1}}\ , \quad (4.59)$$
$$|(u_{\gamma s}, K^{-1}u_{\beta p})| \;\leq\; c, \quad (4.60)$$

for $p, s \in \mathbb{T}_a^{d*}$ and for $\alpha, \beta, \gamma = 1, 2, \ldots, 2^{d/2}$. Then, we have

$$\begin{aligned}|(u_{\alpha p}, RK^{-1}u_{\beta q})| &\leq \sum_{s \in \mathbb{T}_a^{d*}} |(u_{\alpha p}, Ru_{\gamma s})||(u_{\gamma s}, K^{-1}u_{\beta q})| \\ &\leq \sum_{s \in \mathbb{T}_a^{d*}} \frac{const.r(p)}{(1+\langle p-s\rangle)^{d+1}} \\ &\leq const.r(p), \end{aligned} \quad (4.61)$$

where we used

$$\sum_{s \in \mathbb{T}_a^{d*}} (1+\langle s\rangle)^{-d-1} \;\leq\; const. \quad (4.62)$$

(2) Using (4.46), we have

$$\begin{aligned}&\sum_{s' \in \mathbb{T}_a^{d*}} |(u_{\alpha s'}, RK^{-1}u_{\beta s})| \\ \leq\ &\sum_{s' \in \mathbb{T}_a^{d*}} \sum_{j=0}^{n-1} |(u_{\alpha s'}, RJ^j K_0^{-1}u_{\beta s})| + \sum_{s' \in \mathbb{T}_a^{d*}} |(u_{\alpha s'}, RJ^n K^{-1}u_{\beta s})|. \end{aligned} \quad (4.63)$$

Let us bound the right hand side of the above inequality. For the first term, (4.48) implies

$$J = \mathcal{O}(1), \quad (4.64)$$

hence

$$RJ^j K_0^{-1} \;=\; \mathcal{O}(\frac{r}{k_0}) \quad (4.65)$$

holds. Then, from (4.65), (4.8), and (4.57), we obtain

$$\begin{aligned}\sum_{s'\in\mathbb{T}_a^{d*}}\sum_{j=0}^{n-1}|(u_{\alpha s'},RJ^jK_0^{-1}u_{\beta s})| &\le const.\frac{r(s)}{k_0(s)}\sum_{s'\in\mathbb{T}_a^{d*}}(1+\langle s'-s\rangle)^{-d-1}\\ &\le const.\,,\quad s\in\mathbb{T}_a^{d*},\end{aligned}\tag{4.66}$$

for a '*const.*' independent of a and s. For the second term, note that (4.34) and $r(p)\le k_0(p)$ imply that by choosing a sufficiently large n, we can make the order $r(p)(1+\langle p\rangle)^{-n\delta}$ of RJ^n lower than $(1+\langle p\rangle)^{-d-1}$. Then, using (4.36) for K^{-1} and (4.6) for RJ^n with $\sigma=d+1$, we have

$$\begin{aligned}\sum_{s'\in\mathbb{T}_a^{d*}}|(u_{\alpha s'},RJ^nK^{-1}u_{\beta s})| &\le \sum_{s',s''\in\mathbb{T}_a^{d*}}\sum_{\gamma=1}^{2^{d/2}}|(u_{\alpha s'},RJ^nu_{\gamma s''})||(u_{\alpha s''},K^{-1}u_{\beta s})|\\ &\le \sum_{s',s''\in\mathbb{T}_a^{d*}}\sum_{\gamma=1}^{2^{d/2}}const.(1+\langle s'\rangle)^{-d-1}(1+\langle s'-s''\rangle)^{-d-1}\\ &\le const..\end{aligned}\tag{4.67}$$

□

4.6 Proof of Lemma 4.4

In order to show Lemma 4.4, it suffices to estimate the right hand side of (4.46). For the first term, using (4.64), we have the following bound:

$$\sum_{j=0}^{n-1}J^jK_0^{-1} = \mathcal{O}(1/k_0).\tag{4.68}$$

For the second term, we have to estimate

$$\langle p_\mu-q_\mu\rangle^\rho|(u_{\alpha p},J^nK^{-1}u_{\beta q})|\tag{4.69}$$

for $\rho\ge 0, p,q\in\mathbb{T}_a^{d*}$ and for $\alpha,\beta=1,2,\dots,2^{d/2}$.

If $\rho=0$, (4.49) and (4.56) yield

$$|(u_{\alpha p},J^nK^{-1}u_{\beta p})| \le \frac{const.}{k_0(p)}\,,\ p,q\in\mathbb{T}_a^{d*},\ \alpha,\beta=1,2,\dots,2^{d/2}.\tag{4.70}$$

Now assumue $\rho>0$. Then Lemma 4.7 implies

$$\begin{aligned}\langle p_\mu-q_\mu\rangle^\rho|(u_{\alpha p},J^nK^{-1}u_{\beta q})| &\le \pi^\rho|(u_{\alpha p},\mathrm{ad}_\mu^\rho(J^nK^{-1})u_{\beta q})|\,,\\ &\qquad p,q\in\mathbb{T}_a^{d*},\ \alpha,\beta=1,2,\dots,2^{d/2}.\end{aligned}\tag{4.71}$$

Here we note the equalities

$$\mathrm{ad}_\mu^\rho(J^nK^{-1}) = \sum_{m=0}^{\rho}\begin{pmatrix}\rho\\ m\end{pmatrix}\mathrm{ad}_\mu^{\rho-m}(J^n)\mathrm{ad}_\mu^m(K^{-1})\tag{4.72}$$

and

$$\begin{aligned}
&\mathrm{ad}_\mu^m(K^{-1}) \\
&= \sum_{\ell=1}^{m} \sum_{(m_1,m_2,\ldots,m_\ell)} \frac{(-1)^\ell m!}{m_1!m_2!\cdots m_\ell!} K^{-1}\mathrm{ad}_\mu^{m_1}(K)K^{-1}\mathrm{ad}_\mu^{m_2}(K)K^{-1}\cdots\mathrm{ad}_\mu^{m_\ell}(K)K^{-1}, \\
&\qquad m = 1,2,\ldots, \qquad (4.73)
\end{aligned}$$

where the summation $\sum_{(m_1,m_2,\ldots,m_\ell)}$ is taken over all $(m_1,m_2,\ldots,m_\ell)$'s such that

$$m_1 + m_2 + \cdots + m_\ell = m, \qquad (4.74)$$

$$m_1, m_2, \ldots, m_\ell \geq 1. \qquad (4.75)$$

Combining them, we obtain

$$\begin{aligned}
\mathrm{ad}_\mu^\rho(J^n K^{-1}) &= \mathrm{ad}_\mu^\rho(J^n)K^{-1} \\
&+ \sum_{m=1}^{\rho} \begin{pmatrix} \rho \\ m \end{pmatrix} \sum_{\ell=1}^{m} \sum_{(m_1,m_2,\ldots,m_\ell)} \frac{(-1)^\ell m!}{m_1!m_2!\cdots m_\ell!} \\
&\quad \mathrm{ad}_\mu^{\rho-m}(J^n)K^{-1}\mathrm{ad}_\mu^{m_1}(K)K^{-1}\mathrm{ad}_\mu^{m_2}(K)K^{-1}\cdots\mathrm{ad}_\mu^{m_\ell}(K)K^{-1}.
\end{aligned} \qquad (4.76)$$

Then, it suffices to bound $\mathrm{ad}_\mu^{m'}(J^n)K^{-1}$ and $\mathrm{ad}_\mu^m(K)K^{-1}$.

Note that (4.49) and the latter part of Lemma 4.7 imply $\mathrm{ad}_\mu^{m'}(J^n) = \mathcal{O}(1/k_0)$ for sufficiently large n. Then, using (4.56), we have

$$|(u_{\alpha p}, \mathrm{ad}_\mu^{m'}(J^n)K^{-1}u_{\beta s})| \leq \frac{const.}{k_0(p)}, \; p,s \in \mathbb{T}_a^{d*}, \; \alpha,\beta = 1,2,\ldots,2^{d/2}. \qquad (4.77)$$

Furthermore, the assumptions on the orders of K_0 and K_1 imply $K = \mathcal{O}(k_0)$. This fact and Lemma 4.7 and (4.58) imply

$$\sum_{s\in\mathbb{T}_a^{d*}} \sum_{\beta=1}^{2^{d/2}} |(u_{\beta s}, \mathrm{ad}_\mu^m(K)K^{-1}u_{\gamma s'})| \leq const.\,, \; s' \in \mathbb{T}_a^{d*}, \; \gamma = 1,2,\ldots,2^{d/2}. \qquad (4.78)$$

Then, (4.71), (4.76), (4.77), and (4.78) imply

$$\begin{aligned}
\langle p_\mu - q_\mu\rangle^\rho |(u_{\alpha p}, J^n K^{-1} u_{\beta q})| &\leq \frac{const.}{k_0(p)}, \\
\rho > 0, \; p,q &\in \mathbb{T}_a^{d*}, \; \alpha,\beta = 1,2,\ldots,2^{d/2}.
\end{aligned} \qquad (4.79)$$

This together with (4.70) yields $J^n K^{-1} = \mathcal{O}(1/k_0)$, which concludes the analysis on the second term in the right hand side of (4.46), hence the proof of Lemma 4.4.

5 Orders of Operators

Using the framework of quasi diagonal operators, we determine the orders of lattice operators.

5.1 Orders of Interaction parts

Interaction parts of the operators T_μ, D_μ, W, L, and C are by definition

$$\begin{aligned} T_{\mu 1} &= T_\mu - T_{\mu 0}, &(5.1)\\ D_{\mu 1} &= D_\mu - D_{\mu 0}, &(5.2)\\ W_1 &= W - W_0, &(5.3)\\ L_1 &= L - L_0, &(5.4)\\ C_1 &= C - C_0. &(5.5)\end{aligned}$$

Namely,

$$\begin{aligned} T_{\mu 1} &= (U_\mu - I)T_{\mu 0}, &(5.6)\\ D_{\mu 1} &= \frac{1}{2a}(T_{\mu 1} - T_{\mu 1}^*), &(5.7)\\ W_1 &= \sum_{\mu=1}^{d}\Big((U_\mu - I)(T_{\mu 0} - I) + (T_{\mu 0}^* - I)(U_\mu^* - I) + (U_\mu + U_\mu^* - 2I)\Big), &(5.8)\\ L_1 &= -\frac{r}{2a}W_1, &(5.9)\\ C_1 &= L_1 - \sum_{\mu=1}^{d}\Big(D_\mu(L^{-1} - L_0^{-1})D_\mu + D_{\mu 1}L_0^{-1}D_\mu + D_{\mu 0}L_0^{-1}D_{\mu 1}\Big). &(5.10)\end{aligned}$$

Lemma 5.1. *We have the following order estimates.*

(1) The orders of the interaction parts:

$$\begin{aligned} T_{\mu 1} &= \mathcal{O}(a), &(5.11)\\ D_{\mu 1} &= \mathcal{O}(1), &(5.12)\\ W_1 &= \mathcal{O}(a^2(1 + \langle p\rangle)), &(5.13)\\ L_1 &= \mathcal{O}(a(1 + \langle p\rangle)). &(5.14)\end{aligned}$$

(2) The orders of the full operators:

$$\begin{aligned} T_\mu &= \mathcal{O}(1), &(5.15)\\ D_\mu &= \mathcal{O}(1 + \langle p\rangle), &(5.16)\\ W &= \mathcal{O}(a^2(1 + \langle p\rangle^2)), &(5.17)\\ L &= \mathcal{O}(1 + a\langle p\rangle^2). &(5.18)\end{aligned}$$

Proof. (1) follows from (5.6)–(5.9) and Lemma 4.3 together with Lemma 4.1 and

$$\begin{aligned} U_\mu - I, U_\mu^* - I &= \mathcal{O}(a), &(5.19)\\ U_\mu + U_\mu^* - 2I &= \mathcal{O}(a^2). &(5.20)\end{aligned}$$

(2) follows from (1) and Lemma 4.1.

□

5.2 Orders of L^{-1} and C^{-1}

Lemma 5.2. *The operator L is positive definite and satisfies*

$$
\begin{aligned}
(u, Lu) &\geq M \parallel u \parallel^2, && (5.21)\\
\parallel L^{-1} \parallel &\leq M^{-1}. && (5.22)
\end{aligned}
$$

Furthermore, it holds that

$$
\begin{aligned}
L^{-1} &= \mathcal{O}(\frac{1}{1+a\langle p\rangle^2}), && (5.23)\\
L^{-1} - L_0^{-1} &= \mathcal{O}(\frac{a(1+\langle p\rangle)}{(1+a\langle p\rangle^2)^2}). && (5.24)
\end{aligned}
$$

Proof. The semi-positivity of $-W$ (see (2.14)) yields (5.21), from which (5.22) follows. Furthermore, we have (5.23) from (4.18), (4.19), (5.14), and Lemma 4.4, because

$$
\frac{a(1+\langle p\rangle)}{1+a\langle p\rangle^2} \leq \frac{2}{1+\langle p\rangle}. \tag{5.25}
$$

Finally, estimating the right hand side of

$$
L^{-1} - L_0^{-1} = -L^{-1}L_1L_0^{-1}, \tag{5.26}
$$

we obtain (5.24). □

Lemma 5.3. *The operator C is positive definite and satisfies*

$$
\begin{aligned}
(u, Cu) &\geq M \parallel u \parallel^2, && (5.27)\\
\parallel C^{-1} \parallel &\leq M^{-1}. && (5.28)
\end{aligned}
$$

Furthermore, it holds that

$$
\begin{aligned}
C_1 &= \mathcal{O}(\frac{1+\langle p\rangle}{1+a\langle p\rangle^2}), && (5.29)\\
C &= \mathcal{O}(\frac{1+\langle p\rangle^2}{1+a\langle p\rangle^2}), && (5.30)\\
C^{-1} &= \mathcal{O}(\frac{1+a\langle p\rangle^2}{1+\langle p\rangle^2}). && (5.31)
\end{aligned}
$$

Proof. In the right hand side of (3.8), L is positive definite (Lemma 5.2) and

$$
D_\mu^* = -D_\mu \ , \ \mu = 1, 2, \ldots, d. \tag{5.32}
$$

Then, (5.27) and (5.28) hold. Applying Lemma 4.1, Lemma 5.1, and Lemma 5.2 to operators in the right hand side of (5.10), we have (5.29), which, with (4.20), implies (5.30). Finally, we obtain (5.31) from (4.20), (4.21), and (5.28). □

5.3 Order of $X_{\mu\nu}$

We begin with the following.

Lemma 5.4. *If ϕ is a multiplication operator which satisfies (4.30) and (4.31), we have, for $\mu = 1, 2, \dots, d$,*

$$
\begin{aligned}
[T_{\mu 0}, \phi] &= \mathcal{O}(a), && (5.33)\\
\left[T^*_{\mu 0}, \phi\right] &= \mathcal{O}(a), && (5.34)\\
\left[T_{\mu 0} + T^*_{\mu 0}, \phi\right] &= \mathcal{O}(a^2(1 + \langle p \rangle)). && (5.35)
\end{aligned}
$$

Proof. We bound the right hand side of

$$[T_{\mu 0}, \phi] = (T_{\mu 0}\phi T^*_{\mu 0} - \phi)T_{\mu 0}. \tag{5.36}$$

Since $T_{\mu 0}\phi T^*_{\mu 0}$ is a multiplication operator determined by the translation of φ, it holds that

$$T_{\mu 0}\phi T^*_{\mu 0} - \phi = \mathcal{O}(a). \tag{5.37}$$

This proves (5.33). The estimate (5.34) is obtained in a similar way. The estimate (5.35) follows from

$$
\begin{aligned}
\left[T_{\mu 0} + T^*_{\mu 0}, \phi\right] T_{\mu 0} &= 2a(T_{\mu 0}\phi T^*_{\mu 0} - \phi)D_{\mu 0}\\
&\quad +(T_{\mu 0}\phi T^*_{\mu 0} + T^*_{\mu 0}\phi T_{\mu 0} - 2\phi)T^*_{\mu 0} && (5.38)
\end{aligned}
$$

with a help of

$$T_{\mu 0}\phi T^*_{\mu 0} + T^*_{\mu 0}\phi T_{\mu 0} - 2\phi = \mathcal{O}(a^2). \tag{5.39}$$

□

The refined bound (5.35) due to the cancellation between $T_{\mu 0}$ and $T^*_{\mu 0}$ is essential in the proof of Proposition 6.1.

Lemma 5.5. *We have the following order estimates:*

$$
\begin{aligned}
[D_{\mu 1}, D_{\nu 0}] &= \mathcal{O}(1), && (5.40)\\
[D_{\mu 1}, D_{\nu 1}] &= \mathcal{O}(a), && (5.41)\\
[D_{\mu}, D_{\nu}] &= \mathcal{O}(1), && (5.42)\\
[L_0, D_{\nu 1}] &= \mathcal{O}(a(1 + \langle p \rangle)), && (5.43)\\
[L_1, D_{\nu 0}] &= \mathcal{O}(a(1 + \langle p \rangle)), && (5.44)\\
[L_1, D_{\nu 1}] &= \mathcal{O}(a), && (5.45)\\
[L, D_{\nu}] &= \mathcal{O}(a(1 + \langle p \rangle)), && (5.46)\\
X_{\mu\nu} &= \mathcal{O}(\frac{a(1 + \langle p \rangle^2)}{(1 + a\langle p \rangle^2)^2}), && (5.47)
\end{aligned}
$$

where $\mu, \nu = 1, 2, \dots, d$.

Proof. Put $\eta_\mu = U_\mu - I$. Then, $\eta_\mu = \mathcal{O}(a)$ and

$$[T_{\mu 1}, T_{\nu 0}] \;=\; [\eta_\mu, T_{\nu 0}]T_{\mu 0} \tag{5.48}$$

hold. Using (5.33) for $\phi = \frac{1}{a}\eta_\mu$ (the functions $U_\mu(x)$ and $\eta_\mu(x)$ can be extended on the whole $\mathbb{T}^d$), we obtain

$$[T_{\mu 1}, T_{\nu 0}] \;=\; \mathcal{O}(a^2). \tag{5.49}$$

Similarly, we have

$$\left[T^*_{\mu 1}, T_{\nu 0}\right] \;,\; [T_{\mu 1}, T^*_{\nu 0}] \;,\; \left[T^*_{\mu 1}, T^*_{\nu 0}\right] \;=\; \mathcal{O}(a^2). \tag{5.50}$$

Then, (5.40) holds. The estimates

$$[T_{\mu 1}, T_{\nu 1}] \;=\; -\eta_\mu[\eta_\nu, T_{\mu 0}]T_{\nu 0} + \eta_\nu[\eta_\mu, T_{\nu 0}]T_{\mu 0} \;=\; \mathcal{O}(a^3), \tag{5.51}$$

$$\left[T^*_{\mu 1}, T_{\nu 1}\right],\; [T_{\mu 1}, T^*_{\nu 1}],\; \left[T^*_{\mu 1}, T^*_{\nu 1}\right] \;=\; \mathcal{O}(a^3), \tag{5.52}$$

are obtained similarly, from which we have (5.41). The estimate (5.42) follows from (5.40) and (5.41), because $D_{\mu 0}$ and $D_{\nu 0}$ commutes.

Next, using

$$\left[T_{\mu 0} + T^*_{\mu 0}, T_{\nu 1}\right] \;=\; \left[T_{\mu 0} + T^*_{\mu 0}, \eta_\nu\right] T_{\nu 0}\,, \tag{5.53}$$

$$[T_{\mu 0}, T_{\nu 1} + T^*_{\nu 1}] \;=\; T_{\nu 0}\,[T_{\mu 0}, \eta_\nu + \eta^*_\nu] - 2aD_{\nu 0}\,[T_{\mu 0}, \eta^*_\nu]\,, \tag{5.54}$$

and Lemma 5.4, and the fact that $\eta_\nu + \eta^*_\nu = \mathcal{O}(a^2)$, we have

$$\left[T_{\mu 0} + T^*_{\mu 0}, T_{\nu 1}\right] \;=\; \mathcal{O}(a^3(1 + \langle p\rangle)), \tag{5.55}$$

$$[T_{\mu 0}, T_{\nu 1} + T^*_{\nu 1}] \;=\; \mathcal{O}(a^3(1 + \langle p\rangle)). \tag{5.56}$$

Similarly,

$$\left[T_{\mu 0} + T^*_{\mu 0}, T^*_{\nu 1}\right] \;=\; \mathcal{O}(a^3(1 + \langle p\rangle)), \tag{5.57}$$

$$\left[T^*_{\mu 0}, T_{\nu 1} + T^*_{\nu 1}\right] \;=\; \mathcal{O}(a^3(1 + \langle p\rangle)) \tag{5.58}$$

hold and we obtain (5.43) and (5.44). Furthermore, (5.45) is shown by estimating the right hand sides of

$$\begin{aligned}\left[T_{\mu 1} + T^*_{\mu 1}, T_{\nu 1}\right] \;=\;& \eta_\mu[T_{\mu 0}, \eta_\nu]T_{\nu 0} + \eta_\nu[\eta_\mu, T_{\nu 0}]T_{\mu 0} \\ &+[T^*_{\mu 0}, \eta_\nu]\eta^*_\mu T_{\nu 0} + \eta_\nu T^*_{\mu 0}[\eta^*_\mu, T_{\nu 0}]\end{aligned} \tag{5.59}$$

and its adjoint. The estimate (5.46) follows from (5.43)–(5.45). Finally, note that

$$X_{\mu\nu} \;=\; D_\mu L^{-1}[D_\nu, L]L^{-1} - D_\nu L^{-1}[D_\mu, L]L^{-1} + [D_\mu, D_\nu]L^{-1}\,. \tag{5.60}$$

Estimating the right hand side of (5.60) by means of (5.16), (5.23), (5.42), and (5.46), we obtain (5.47). □

Lemma 5.6. *The operator $C+B$ is positive definite and satisfies*

$$(u,(C+B)u) \geq M \parallel u \parallel^2, \tag{5.61}$$

$$\parallel (C+B)^{-1} \parallel \leq M^{-1}. \tag{5.62}$$

Furthermore, it holds that

$$B = \mathcal{O}(\frac{a(1+\langle p\rangle^2)}{(1+a\langle p\rangle^2)^2}), \tag{5.63}$$

$$BC^{-1} = \mathcal{O}(\frac{a}{1+a\langle p\rangle^2}), \tag{5.64}$$

$$(C+B)^{-1} = \mathcal{O}(\frac{1+a\langle p\rangle^2}{1+\langle p\rangle^2}). \tag{5.65}$$

Proof. (5.61) and (5.62) are obtained from

$$(u,(C+B)u) = (u,Lu)+\sum_{\mu=1}^{d}(D_\mu u, L^{-1}D_\mu u) \tag{5.66}$$

and (5.21). (5.63) is a consequence of (3.7) and (5.47). (5.64) follows from (5.63) and (5.31). Lemma 4.4 then yields (5.65). □

5.4 Leading Term of $X_{\mu\nu}$

We extract the leading term of $X_{\mu\nu}$ and bound the remainder. Put

$$E_{\mu 0} = \frac{1}{2}(T_{\mu 0}+T^*_{\mu 0}) \ , \ \mu = 1,2,\ldots,d. \tag{5.67}$$

Lemma 5.7. *For $\mu,\nu = 1,2,\ldots,d$, we have*

$$[D_\mu, D_\nu] + iQF_{\mu\nu}E_{\mu 0}E_{\nu 0} = \mathcal{O}(a), \tag{5.68}$$

$$[L, D_\mu] + iarQ\sum_{\rho=1}^{d} F_{\mu\rho}D_{\rho 0}E_{\mu 0} = \mathcal{O}(a), \tag{5.69}$$

where Q is the charge and $F_{\mu\nu}$ is the field strength defined by (2.21).

Proof. Put $\eta_\mu = U_\mu - I$. Recalling (2.3), we have

$$\begin{aligned}\eta_\mu - T_{\nu 0}\eta_\mu T^*_{\nu 0} &= -a\partial_\nu\eta_\mu + \mathcal{O}(a^3)\\ &= -ia^2Q\partial_\nu A_\mu + \mathcal{O}(a^3).\end{aligned} \tag{5.70}$$

Then, (5.48) and (5.36) yield

$$[T_{\mu 1}, T_{\nu 0}] = ia^2Q(\partial_\nu A_\mu)T_{\mu 0}T_{\nu 0} + \mathcal{O}(a^3). \tag{5.71}$$

Similarly, we have

$$\begin{aligned}
[T_{\mu 1}, T^*_{\nu 0}] &= -ia^2 Q(\partial_\nu A_\mu) T_{\mu 0} T^*_{\nu 0} + \mathcal{O}(a^3), &(5.72)\\
\left[T^*_{\mu 1}, T_{\nu 0}\right] &= -ia^2 Q(\partial_\nu A_\mu) T^*_{\mu 0} T_{\nu 0} + \mathcal{O}(a^3), &(5.73)\\
\left[T^*_{\mu 1}, T^*_{\nu 0}\right] &= ia^2 Q(\partial_\nu A_\mu) T^*_{\mu 0} T^*_{\nu 0} + \mathcal{O}(a^3), &(5.74)
\end{aligned}$$

hence

$$[D_{\mu 1}, D_{\nu 0}] = iQ(\partial_\nu A_\mu) E_{\mu 0} E_{\nu 0} + \mathcal{O}(a). \tag{5.75}$$

This together with (5.41) yields (5.68). Let us show (5.69). (5.38) and (5.53) yield

$$\left[T_{\rho 0} + T^*_{\rho 0}, T_{\mu 1}\right] = -2ia^3 Q(\partial_\rho A_\mu) D_{\rho 0} T_{\mu 0} + \mathcal{O}(a^3). \tag{5.76}$$

Similarly, we have

$$\left[T_{\rho 0} + T^*_{\rho 0}, T^*_{\mu 1}\right] = 2ia^3 Q(\partial_\rho A_\mu) D_{\rho 0} T^*_{\mu 0} + \mathcal{O}(a^3). \tag{5.77}$$

Then,

$$[L_0, D_{\mu 1}] = iarQ \sum_{\rho=1}^{d} (\partial_\rho A_\mu) D_{\rho 0} E_{\mu 0} + \mathcal{O}(a) \tag{5.78}$$

follows. Furthermore, (5.54) implies

$$[L_1, D_{\mu 0}] = -iarQ \sum_{\rho=1}^{d} (\partial_\mu A_\rho) D_{\rho 0} E_{\mu 0} + \mathcal{O}(a). \tag{5.79}$$

Then, (5.69) follows from (5.78), (5.79), and (5.45). □

Lemma 5.8. *It holds that*

$$\begin{aligned}
X_{\mu\nu} &+ iQ \left(F_{\mu\nu} E_{\mu 0} E_{\nu 0} L_0^{-1} + ar \sum_{\rho=1}^{d} F_{\rho\nu} L_0^{-2} D_{\rho 0} D_{\mu 0} E_{\nu 0} - ar \sum_{\rho=1}^{d} F_{\rho\mu} L_0^{-2} D_{\rho 0} D_{\nu 0} E_{\mu 0} \right) \\
&= \mathcal{O}\left(\frac{a(1+\langle p\rangle)}{(1+a\langle p\rangle^2)^2}\right)
\end{aligned} \tag{5.80}$$

for $\mu, \nu = 1, 2, \ldots, d$.

Proof. We extract the leading term from the right hand side of (5.60). Firstly, (5.68) and (5.24) imply

$$[D_\mu, D_\nu]\, L^{-1} = -iQ F_{\mu\nu} E_{\mu 0} E_{\nu 0} L_0^{-1} + \mathcal{O}\left(\frac{a(1+\langle p\rangle)}{(1+a\langle p\rangle^2)^2}\right). \tag{5.81}$$

Similarly, from (5.69), we have

$$D_\mu L^{-1} [D_\nu, L]\, L^{-1} = -iarQ \sum_\rho D_{\mu 0} L_0^{-1} F_{\rho\nu} D_{\rho 0} E_{\nu 0} L_0^{-1} + \mathcal{O}\left(\frac{a(1+\langle p\rangle)}{(1+a\langle p\rangle^2)^2}\right). \tag{5.82}$$

In the right hand side of the above equality, we may move $F_{\rho\nu}$ to the front of the term by using the following lemma. (The difference is absorbed in the second term of the right hand side of (5.82).) Thus we have the desired result. □

Lemma 5.9. *If ϕ is a multiplication operator which satisfies (4.30) and (4.31), we have, for $\mu = 1, 2, \dots, d$,*

$$\begin{aligned}
[D_{\mu 0}, \phi] &= \mathcal{O}(1), && (5.83)\\
[E_{\mu 0}, \phi] &= \mathcal{O}(a), && (5.84)\\
[W_0, \phi] &= \mathcal{O}(a^2(1+\langle p\rangle)), && (5.85)\\
\left[L_0^{-1}, \phi\right] &= \mathcal{O}(\frac{1}{(1+a\langle p\rangle^2)(1+\langle p\rangle)}), && (5.86)\\
\left[C_0^{-1}, \phi\right] &= \mathcal{O}(\frac{1+a\langle p\rangle^2}{(1+\langle p\rangle)^3}). && (5.87)
\end{aligned}$$

Proof. The estimates (5.83) – (5.85) follow from (5.33) – (5.35). The commutators with L_0^{-1} and with C_0^{-1} are estimated using commutators with L_0 and with C_0, respectively. □

6 Proof of Theorem 2.1

6.1 Irrelevant Terms

If we formally expand $(C+B)^{-1}$ in the right hand side of (3.9), we have

$$\langle Y(\xi)\rangle \overset{\text{formally}}{=} \sum_{j=0}^{\infty} z^{(j)}, \tag{6.1}$$

where

$$z^{(j)} = -i\mathrm{Tr}_{\mathbb{V}_a^d}\left[\frac{r}{2a}(\xi W + W\xi)C^{-1}(-BC^{-1})^j\gamma_{d+1}\right],\ j = 0, 1, 2, \dots. \tag{6.2}$$

The following proposition is a consquence of the order estimates in Section 5.

Proposition 6.1. *For $j > d/2$, $z^{(j)}$ is irrelevant, i.e.*

$$\lim_{a\to 0}(\langle Y(\xi)\rangle - \sum_{j=0}^{d/2} z^{(j)}) = 0\,. \tag{6.3}$$

Proof. It suffices to bound the right hand side of

$$\langle Y(\xi)\rangle - \sum_{j=0}^{d/2} z^{(j)} = -i\mathrm{Tr}_{\mathbb{V}_a^d}\left[\frac{r}{2a}(\xi W + W\xi)(C+B)^{-1}(-BC^{-1})^{d/2+1}\gamma_{d+1}\right]. \tag{6.4}$$

The order estimates (5.17), (5.65), and (5.64) imply

$$\frac{r}{2a}(\xi W + W\xi)(C+B)^{-1}(-BC^{-1})^{d/2+1}\gamma_{d+1} = \mathcal{O}(\frac{a^{d/2+2}}{(1+a\langle p\rangle^2)^{d/2}}), \tag{6.5}$$

with the help of Lemma 4.2 and Lemma 4.3. Let us take the trace on $\mathbb{V}_a^d$ by means of the planewave basis (4.4). Then, we have from (4.7) with $q = p$

$$\begin{aligned} |\langle Y(\xi)\rangle - \sum_{j=0}^{d/2} z^{(j)}| &\leq const. \sum_{p\in\mathbb{T}_a^{d\,*}} \frac{a^{d/2+2}}{(1+a\langle p\rangle^2)^{d/2}} \\ &\leq const. a^2 \log(1/a) \\ &\to 0 \,,\, a \to 0, \end{aligned} \tag{6.6}$$

where we used

$$\sum_{p\in\mathbb{T}_a^{d\,*}} \frac{a^\ell}{(1+a\langle p\rangle^2)^\ell} \leq const. \begin{cases} a^{\ell-d/2}, & \ell > d/2, \\ \log(1/a)\,, & \ell = d/2. \end{cases} \tag{6.7}$$

□

6.2 Spin Properties

Next we take into account the 'spin properties' of operators. We classify operators on $\mathbb{V}_a^d$ with respect to the homogeneous degrees in γ matrices. For example, C and B have 0 and 2 homogeneous degrees, respectively. An operator with homogeneous degree 0 is regarded as an operator on $\dot{\mathbb{V}}_a^d = \{\varphi \ :\ \mathbb{T}_a^d \to \mathbb{C}\}$. Since $\mathbb{V}_a^d = \mathbb{C}^{d/2} \otimes \dot{\mathbb{V}}_a^d$, the trace is factorized as:

$$\mathrm{Tr}_{\mathbb{V}_a^d} = \mathrm{Tr}_{\dot{\mathbb{V}}_a^d} \mathrm{Tr}_{\mathbb{C}^{d/2}}\,. \tag{6.8}$$

Proposition 6.2. *It holds that*

$$z^{(j)} = 0 \,,\ j = 0, 1, \ldots, d/2 - 1, \tag{6.9}$$

$$\begin{aligned} z^{(d/2)} = -i^{d/2+1} \sum_{\mu_1,\mu_2,\ldots,\mu_d=1}^{d} \epsilon_{\mu_1\mu_2\ldots\mu_d} \mathrm{Tr}_{\dot{\mathbb{V}}_a^d}[\frac{r}{2a}(\xi W + W\xi) \\ C^{-1} X_{\mu_1\mu_2} C^{-1} X_{\mu_3\mu_4} C^{-1} \cdots X_{\mu_{d-1}\mu_d} C^{-1}], \end{aligned} \tag{6.10}$$

where $\epsilon_{\mu_1\mu_2\ldots\mu_d}$ denotes the totally antisymmetric tensor, and all the operators, having homogeneous degree 0, are regarded as defined on the space $\dot{\mathbb{V}}_a^d$.

Proof. The operator $(\xi W + W\xi)C^{-1}(-BC^{-1})^j$ has homogeneous degree $2j$ in γ's. Then, (6.9) is obvious from

$$\mathrm{Tr}_{\mathbb{C}^{2^{d/2}}} (\gamma_{\mu_1}\gamma_{\mu_2} \cdots \gamma_{\mu_\ell}\gamma_{d+1}) = 0,\ 0 \leq \ell < d,\ \mu_1, \mu_2, \ldots, \mu_\ell = 1, 2, \ldots, d. \tag{6.11}$$

On the other hand, (6.10) follows from (3.7) and

$$\begin{aligned} \mathrm{Tr}_{\mathbb{C}^{2^{d/2}}} (\gamma_{\mu_1}\gamma_{\mu_2} \cdots \gamma_{\mu_d}\gamma_{d+1}) &= (-2i)^{d/2} \epsilon_{\mu_1\mu_2\ldots\mu_d}, \\ &\mu_1, \mu_2, \ldots, \mu_d = 1, 2, \ldots, d. \end{aligned} \tag{6.12}$$

□

6.3 Leading Terms

We extract the leading terms from (6.10). Let us introduce the 'normal product' : : that moves multiplication operators to front, e.g.

$$:\xi W_0 + W_0\xi: \; = \; 2\xi W_0, \tag{6.13}$$

$$:F_{\mu\nu}E_{\mu 0}E_{\nu 0}L_0^{-1}F_{\mu'\nu'}E_{\mu' 0}E_{\nu' 0}L_0^{-1}: \; = \; F_{\mu\nu}F_{\mu'\nu'}E_{\mu 0}E_{\nu 0}L_0^{-1}E_{\mu' 0}E_{\nu' 0}L_0^{-1}. \tag{6.14}$$

Proposition 6.3. *Put, for $\mu, \nu = 1, 2, \ldots, d$,*

$$\bar{X}_{\mu\nu} \; = \; -iQ\left(F_{\mu\nu}E_{\mu 0}E_{\nu 0}L_0^{-1} + 2ar\sum_{\rho=1}^{d} F_{\rho\nu}L_0^{-2}D_{\rho 0}D_{\mu 0}E_{\nu 0}\right). \tag{6.15}$$

and

$$z_0 \; = \; -i^{d/2+1}\sum_{\mu_1,\mu_2,\ldots,\mu_{d/2}=1}^{d}\;\sum_{\nu_1,\nu_2,\ldots,\nu_{d/2}=1}^{d}\epsilon_{\mu_1\nu_1\ldots\mu_{d/2}\nu_{d/2}}$$

$$\mathrm{Tr}_{\mathbb{V}_a^d}[:\frac{r}{a}\xi W_0 C_0^{-1}\bar{X}_{\mu_1\nu_1}C_0^{-1}\bar{X}_{\mu_2\nu_2}C_0^{-1}\cdots\bar{X}_{\mu_{d/2}\nu_{d/2}}C_0^{-1}:], \tag{6.16}$$

Then, we have

$$\lim_{a\to 0}(z^{(d/2)} - z_0) \; = \; 0. \tag{6.17}$$

Proof. Insert

$$C^{-1} \; = \; C_0^{-1} - C_0^{-1}C_1C^{-1} \tag{6.18}$$

into the right hand side of (6.10) and expand it with respect to C_1. Lemma 4.1 and Lemma 5.3 imply that C_0^{-1} has the same order as C^{-1} and

$$C_0^{-1}C_1C^{-1} \; = \; \mathcal{O}\left(\frac{1}{1+\langle p\rangle}\frac{1+a\langle p\rangle^2}{1+\langle p\rangle^2}\right) \tag{6.19}$$

is of lower order than C^{-1} by the factor $1/(1+\langle p\rangle)$. Then, terms containing C_1 vanish in the continuum limit in a similar way as in the proof of Proposition 6.1. By the same reason, we can replace W by W_0 (see (5.13), (5.17), and (4.17)). Furthermore, we can replace $X_{\mu\nu}$ by

$$-iQ\left(F_{\mu\nu}E_{\mu 0}E_{\nu 0}L_0^{-1} + ar\sum_{\rho=1}^{d}F_{\rho\nu}L_0^{-2}D_{\rho 0}D_{\mu 0}E_{\nu 0} - ar\sum_{\rho=1}^{d}F_{\rho\mu}L_0^{-2}D_{\rho 0}D_{\nu 0}E_{\mu 0}\right) \tag{6.20}$$

(see (5.80)). Here, the second and third terms of (6.20) give the same contributions to $z^{(d/2)}$ when (5.80) is inserted into the right hand side of (6.10). Then, we can replace $X_{\mu\nu}$ by $\bar{X}_{\mu\nu}$. Finally in order to replace $\xi W_0 + W_0\xi$ by $2\xi W_0$ and to take the normal product, it suffices to bound commutators $[W_0, \xi], [C_0^{-1}, F_{\mu\nu}], [E_{\mu' 0}E_{\nu' 0}L_0^{-1}, F_{\mu\nu}]$ etc. Those commutators are estimated using Lemma 5.9: By taking a commutator with a multiplication operator, the order of the free part of an operator (with the suffix 0 like $D_{\mu 0}$) is reduced by the facter $1/(1+\langle p\rangle)$, hence the trace containing at least one commutators vanishes in the continuum limit. □

6.4 Symmetry arguments

In view of (6.15) and (6.16), we see that z_0 is a polynomial in r of order $d/2+1$ if we ignore the r-dependences of L_0^{-1} and C_0^{-1}. The following proposition claims that the cubic and higher order terms vanish.

Proposition 6.4. *Put*

$$\begin{aligned} z_1 &= -iQ^{d/2} \sum_{\mu_1,\mu_2,\ldots,\mu_{d/2}=1}^{d} \sum_{\nu_1,\nu_2,\ldots,\nu_{d/2}=1}^{d} \epsilon_{\mu_1\nu_1\ldots\mu_{d/2}\nu_{d/2}} \\ &\quad \frac{r}{a}\mathrm{Tr}_{\dot{\mathbb{V}}_a^d}[: \xi W_0 C_0^{-(d/2+1)} L_0^{-d/2} \prod_{j=1}^{d/2}\left(F_{\mu_j\nu_j}E_{\mu_j 0}E_{\nu_j 0}\right) :], \end{aligned} \tag{6.21}$$

$$\begin{aligned} z_2 &= -iQ^{d/2} \sum_{\mu_1,\mu_2,\ldots,\mu_{d/2}=1}^{d} \sum_{\nu_1,\nu_2,\ldots,\nu_{d/2}=1}^{d} \epsilon_{\mu_1\nu_1\ldots\mu_{d/2}\nu_{d/2}} \\ &\quad 2r^2\mathrm{Tr}_{\dot{\mathbb{V}}_a^d}[: \xi W_0 C_0^{-(d/2+1)} L_0^{-(d/2+1)} \sum_{\ell=1}^{d/2}\sum_{\rho=1}^{d} F_{\rho\nu_\ell}D_{\rho 0}D_{\mu_\ell 0}E_{\nu_\ell 0} \\ &\quad \prod_{\substack{j=1\\ j\neq\ell}}^{d/2}\left(F_{\mu_j\nu_j}E_{\mu_j 0}E_{\nu_j 0}\right) :]. \end{aligned} \tag{6.22}$$

Then, we have

$$z_0 = z_1 + z_2. \tag{6.23}$$

Moreover, in the right hand side of (6.22), we can fix the value of ρ to μ_ℓ, i.e.

$$\begin{aligned} z_2 &= -iQ^{d/2} \sum_{\mu_1,\mu_2,\ldots,\mu_{d/2}=1}^{d} \sum_{\nu_1,\nu_2,\ldots,\nu_{d/2}=1}^{d} \epsilon_{\mu_1\nu_1\ldots\mu_{d/2}\nu_{d/2}} \\ &\quad 2r^2\mathrm{Tr}_{\dot{\mathbb{V}}_a^d}[: \xi W_0 C_0^{-(d/2+1)} L_0^{-(d/2+1)} \sum_{\ell=1}^{d/2} F_{\mu_\ell\nu_\ell}D_{\mu_\ell 0}^2 E_{\nu_\ell 0} \\ &\quad \prod_{\substack{j=1\\ j\neq\ell}}^{d/2}\left(F_{\mu_j\nu_j}E_{\mu_j 0}E_{\nu_j 0}\right) :]. \end{aligned} \tag{6.24}$$

Proof. Insert (6.15) into the summand in the right hand side of (6.16) and expand it. Let us look at a term generated by the expansion that contains at least two factors of the form $2ar\sum_{\rho=1}^{d} F_{\rho\nu}L_0^{-2}D_{\rho 0}D_{\mu 0}E_{\nu 0}$ (the second term of the right hand side of (6.15)). Such a term has the form

$$: \cdots 2ar\sum_{\rho_\ell=1}^{d} F_{\rho_\ell\nu_\ell}L_0^{-2}D_{\rho_\ell 0}D_{\mu_\ell 0}E_{\nu_\ell 0}\cdots 2ar\sum_{\rho_{\ell'}=1}^{d} F_{\rho_{\ell'}\nu_{\ell'}}L_0^{-2}D_{\rho_{\ell'} 0}D_{\mu_{\ell'} 0}E_{\nu_{\ell'} 0}\cdots :. \tag{6.25}$$

Then, it is symmetric with respect to μ_ℓ and $\mu_{\ell'}$, hence it vanishes when it is summed up with respect to μ_ℓ and $\mu_{\ell'}$ under the presence of the totally antisymmetric tensor. This gives (6.23).

Next, note that $\mu_1, \nu_1, \ldots, \mu_{d/2}, \nu_{d/2}$ are mutually distinct because of the totally anti-symmetric tensor. Then, the sum (6.22) is decomposed into partial sums in which ρ is fixed to μ_j or ν_j for $j = 1, 2, \ldots, d/2$. If $\rho = \nu_\ell$, $F_{\rho\nu_\ell}$ vanishes. It therefore suffices to consider the case $\rho = \mu_j, \nu_j$ for some $j \neq \ell$. Assume $\rho = \mu_j$. In this case, the summand has the form

$$: \cdots F_{\mu_j \nu_\ell} D_{\mu_j 0} D_{\mu_\ell 0} E_{\nu_\ell 0} \cdots F_{\mu_j \nu_j} E_{\mu_j 0} E_{\nu_j 0} \cdots : \tag{6.26}$$

Then, it is symmetric with respect to ν_ℓ and ν_j, hence it vanishes when it is summed up with respect to ν_ℓ and ν_j. The term for $\rho = \nu_j$ similarly vanishes. Thus we have (6.24). □

Combining Proposition 6.1, Proposition 6.2, Proposition 6.3, and Proposition 6.4, we obtain:

Proposition 6.5. *Put*

$$G = \frac{r}{a} W_0 C_0^{-(d/2+1)} L_0^{-(d/2+1)} \left(L_0 + ar \sum_{\ell=1}^{d} \frac{D_{\ell 0}^2}{E_{\ell 0}} \right) \prod_{j=1}^{d} E_{j0}, \tag{6.27}$$

where we write (with a slight abuse of notaition)

$$\frac{D_{\ell 0}^2}{E_{\ell 0}} \prod_{j=1}^{d} E_{j0} = D_{\ell 0}^2 \prod_{\substack{j=1 \\ j \neq \ell}}^{d} E_{j0}. \tag{6.28}$$

Then we have

$$\lim_{a \to 0} \left(\langle Y(\xi) \rangle + i Q^{d/2} \mathrm{Tr}_{\,\dot{\mathbb{V}}_a^d} (\xi \epsilon F^{d/2} G) \right) = 0, \tag{6.29}$$

where

$$\epsilon F^{d/2} = \sum_{\mu_1, \mu_2, \ldots, \mu_{d/2} = 1}^{d} \sum_{\nu_1, \nu_2, \ldots, \nu_{d/2} = 1}^{d} \epsilon_{\mu_1 \nu_1 \ldots \mu_{d/2} \nu_{d/2}} \prod_{j=1}^{d/2} F_{\mu_j \nu_j}. \tag{6.30}$$

Proof. Rewrite the right hand sides of (6.21) and (6.24) as:

$$\begin{aligned} z_1 &= -i Q^{d/2} \mathrm{Tr}_{\,\dot{\mathbb{V}}_a^d} \left(\xi \epsilon F^{d/2} \frac{r}{a} W_0 C_0^{-(d/2+1)} L_0^{-d/2} \prod_{j=1}^{d} E_{j0} \right), \\ z_2 &= -i Q^{d/2} \mathrm{Tr}_{\,\dot{\mathbb{V}}_a^d} \Bigg(\xi \sum_{\mu_1, \mu_2, \ldots, \mu_{d/2} = 1}^{d} \sum_{\nu_1, \nu_2, \ldots, \nu_{d/2} = 1}^{d} \epsilon_{\mu_1 \nu_1 \ldots \mu_{d/2} \nu_{d/2}} \prod_{j=1}^{d/2} F_{\mu_j \nu_j} \\ &\qquad 2 r^2 W_0 C_0^{-(d/2+1)} L_0^{-(d/2+1)} \sum_{\ell=1}^{d/2} \frac{D_{\mu_\ell 0}^2}{E_{\mu_\ell 0}} \prod_{j=1}^{d} E_{j0} \Bigg) \end{aligned} \tag{6.31}$$

$$\begin{aligned} &= -iQ^{d/2}\mathrm{Tr}_{\dot{\mathbb{V}}_a^d}\Bigg(\xi \sum_{\mu_1,\mu_2,\ldots,\mu_{d/2}=1}^{d} \sum_{\nu_1,\nu_2,\ldots,\nu_{d/2}=1}^{d} \epsilon_{\mu_1\nu_1\ldots\mu_{d/2}\nu_{d/2}} \prod_{j=1}^{d/2} F_{\mu_j\nu_j} \\ &\qquad 2r^2W_0C_0^{-(d/2+1)}L_0^{-(d/2+1)}\frac{1}{2}\sum_{\ell=1}^{d/2}\Big(\frac{D^2_{\mu_\ell 0}}{E_{\mu_\ell 0}}+\frac{D^2_{\nu_\ell 0}}{E_{\nu_\ell 0}}\Big)\prod_{j=1}^{d} E_{j0}\Bigg) \\ &= -iQ^{d/2}\mathrm{Tr}_{\dot{\mathbb{V}}_a^d}\Bigg(\xi\epsilon F^{d/2}2r^2W_0C_0^{-(d/2+1)}L_0^{-(d/2+1)}\frac{1}{2}\sum_{\ell=1}^{d/2}\frac{D^2_{\ell 0}}{E_{\ell 0}}\prod_{j=1}^{d}E_{j0}\Bigg). \qquad (6.32) \end{aligned}$$

Then, the proposition follows from (6.3), (6.9), (6.17), (6.23), (6.31), and (6.32). □

6.5 Coefficients

We take the trace of the left hand side of (6.29) with respect to the (spinless) planewave basis $u_p, p \in \mathbb{T}_a^{d*}$, defined by (4.3). In the following proposition, $G(p)$ is the eigenvalue of G, i.e. $Gu_p = G(p)u_p$ (see the proof of Lemma 4.1).

Proposition 6.6. *Put*

$$\begin{aligned} G(p) &= -2r^2a^d\sum_{\mu=1}^{d}(1-\cos ap_\mu)\prod_{\lambda=1}^{d}\cos ap_\lambda \\ &\qquad \times\frac{\dfrac{Ma}{r}+\sum_{\nu=1}^{d}(1-\cos ap_\nu)-\sum_{\nu=1}^{d}\dfrac{\sin^2 ap_\nu}{\cos ap_\nu}}{\left[(Ma+r\sum_{\nu=1}^{d}(1-\cos ap_\nu))^2+\sum_{\nu=1}^{d}\sin^2 ap_\nu\right]^{d/2+1}}, \qquad (6.33) \\ \kappa_a &= \frac{1}{L^d}\sum_{p\in\mathbb{T}_a^{d*}} G(p). \qquad (6.34) \end{aligned}$$

Then, we have

$$\lim_{a\to 0}\Bigg(\langle Y(\xi)\rangle + i\kappa_a Q^{d/2}a^d\sum_{x\in\mathbb{T}_a^d}\xi(x)\epsilon F^{d/2}(x)\Bigg) = 0, \qquad (6.35)$$

where

$$\epsilon F^{d/2}(x) = \sum_{\mu_1,\mu_2,\ldots,\mu_{d/2}=1}^{d}\sum_{\nu_1,\nu_2,\ldots,\nu_{d/2}=1}^{d}\epsilon_{\mu_1\nu_1\ldots\mu_{d/2}\nu_{d/2}}\prod_{j=1}^{d/2}F_{\mu_j\nu_j}(x)\ ,\ \ x\in\mathbb{T}_a^d. \qquad (6.36)$$

Proof. Let ϕ be a multiplication operator determined by a function φ which satisfies (4.30) and (4.31), and let K_0 be a free operator on $\dot{\mathbb{V}}_a^d$, i.e. an operator with $u_p, p \in \mathbb{T}_a^{d*}$ as the set of eigenfunctions. We denote the eigenvalue by $K_0(p)$, i.e. $K_0u_p = K_0(p)u_p$. Then we see that

$$\mathrm{Tr}_{\dot{\mathbb{V}}_a^d}(\phi K_0) = \sum_{p\in\mathbb{T}_a^{d*}}(u_p, \phi K_0 u_p)$$

$$= \sum_{p \in \mathbb{T}_a^{d\,*}} K_0(p)(u_p, \phi u_p)$$

$$= \sum_{p \in \mathbb{T}_a^{d\,*}} K_0(p) \frac{a^d}{L^d} \sum_{x \in \mathbb{T}_a^d} \varphi(x). \tag{6.37}$$

The proposition is a consequence of (6.29) and (6.37). □

Now, the proof of Theorem 2.1 is completed if we confirm the following.

Proposition 6.7 ([9]). *It holds that*

$$\lim_{a \to 0} \kappa_a = \frac{2}{(4\pi)^{d/2}(d/2)!}. \tag{6.38}$$

Proof. Put

$$\begin{aligned} g(a,q) &= a^{-d} G(q/a) \\ &= -2r^2 \sum_{\mu=1}^{d} (1 - \cos q_\mu) \prod_{\lambda=1}^{d} \cos q_\lambda \\ &\quad \times \frac{\dfrac{Ma}{r} + \sum_{\nu=1}^{d}(1-\cos q_\nu) - \sum_{\nu=1}^{d} \dfrac{\sin^2 q_\nu}{\cos q_\nu}}{\left[(Ma + r \sum_{\nu=1}^{d} (1-\cos q_\nu))^2 + \sum_{\nu=1}^{d} \sin^2 q_\nu \right]^{d/2+1}} \end{aligned} \tag{6.39}$$

for $a > 0$, and

$$g(0,q) = \lim_{a \downarrow 0} g(a,q). \tag{6.40}$$

Using the funcion g, we write

$$\kappa_a = \left(\frac{a}{L}\right)^d \sum_{q \in a\mathbb{T}_a^{d\,*}} g(a,q). \tag{6.41}$$

For $q \in \mathbb{T}^{d*} = [-\pi,\pi]^d$, let $[q]$ denote the point in $a\mathbb{T}_a^{d*}$ nearest to q. $[q]$ is defined almost everywhere in $\mathbb{T}^{d*}$. Then, it holds that

$$\kappa_a = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^{d\,*}} dq\, g(a,[q]). \tag{6.42}$$

Note that, for an arbitrarily small $\epsilon > 0$, we can choose $\delta > 0$ such that

$$\frac{1}{(2\pi)^d} \int_{B_\delta} dq\, |g(a,[q])| < \epsilon, \tag{6.43}$$

hold for any $a \geq 0$, where $B_\delta = \{q \in \mathbb{R}^d \mid |q| < \delta\}$. Removing the small ball B_δ from the hypercube $\mathbb{T}^{d^*}$, we can take the continuum limit:

$$\lim_{a\to 0} \frac{1}{(2\pi)^d} \int_{\mathbb{T}^{d^*}\setminus B_\delta} dq\, g(a,[q]) \;=\; \frac{1}{(2\pi)^d} \int_{\mathbb{T}^{d^*}\setminus B_\delta} dq\, g(0,q). \tag{6.44}$$

Then, it holds that

$$\lim_{a\to 0} \kappa_a \;=\; \frac{1}{(2\pi)^d} \int_{\mathbb{T}^{d^*}} dq\, g(0,q). \tag{6.45}$$

It therefore suffices to show

$$\int_{\mathbb{T}^{d^*}} dq\, g(0,q) \;=\; \frac{2}{d}\Omega_d, \tag{6.46}$$

where Ω_d stands for the area of the d-dimensional unit sphere, i.e. $\Omega_d = (2\pi)^{d/2}/(d-2)!!$. This fact is proved for $d = 4$ in [6] and for all d and all r in [9].[3] □

Appendix

We list up the operators that are (globally) used in this paper.

$$T_{\mu 0} u(x) = u(x + a e_\mu) \tag{2.4}$$

$$T_\mu = U_\mu T_{\mu 0} \tag{2.5}$$

$$D_\mu = \frac{1}{2a}(T_\mu - T_\mu^*) \tag{2.7}$$

$$\not{D} = \sum_{\mu=1}^{d} \gamma_\mu D_\mu \tag{2.9}$$

$$W = -\sum_{\mu=1}^{d} (2I - T_\mu - T_\mu^*) \tag{2.10}$$

$$L = MI - \frac{r}{2a} W \tag{3.5}$$

$$X_{\mu\nu} = D_\mu L^{-1} D_\nu - D_\nu L^{-1} D_\mu \tag{3.6}$$

$$B = \frac{1}{2} \sum_{\mu,\nu=1}^{d} \gamma_\mu \gamma_\nu X_{\mu\nu} \tag{3.7}$$

$$C = L - \sum_{\mu=1}^{d} D_\mu L^{-1} D_\mu \tag{3.8}$$

$$E_{\mu 0} = \frac{1}{2}(T_{\mu 0} + T_{\mu 0}^*) \tag{5.67}$$

[3] To be strict, our conclusion is consistent with that of [9], if we include $\prod_\mu \epsilon_\mu$ and replace $s^d - 1$ by s^{d-1} in the integrand of (A20) in [9].

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