HIGHER ORDER ESTIMATES IN THE CONJUGATE OPERATOR THEORY

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Abstract. Let H be a self-adjoint operator which admits a locally conjugate operator A (i.e. the set $\mu^A(H)$ defined at the beginning of the introduction is not empty). Set $R(z) = (H-z)^{-1}$, let $\Pi_\pm$ be the spectral projection of A associated to the interval $\pm[0,\infty)$ and let $\mathscr{H}_{s,p}$ ($s \in \mathbb{R}, 1 \le p \le \infty$) be the Besov scale associated to the operator A. We study the regularity properties of the maps $\lambda \mapsto R(\lambda \pm i0)$, $\lambda \mapsto \Pi_\mp R(\lambda \pm i0)$ and $\lambda \mapsto \Pi_\mp R(\lambda \pm i0)\Pi_\pm$ when considered with values in a space of the form $B(\mathscr{H}_{s,p}; \mathscr{H}_{t,q})$. Our results imply optimal local decay and propagation properties of H with respect to A, in particular estimates of the form $\|\langle A\rangle^t \Pi_\mp \exp(\mp i\tau H)\langle A\rangle^{-s}\| \le c\tau^{-\alpha}$ for $\tau \ge 1$.

0. Introduction

Let H, A be two densely defined self-adjoint operators in a Hilbert space $\mathscr{H}$ such that:

(i) the intersection of their domains $D(A) \cap D(H)$ is dense in $D(H)$ (which is equipped with the graph topology) and

(ii) the sesquilinear form defined on $D(A) \cap D(H)$ by $\langle Hg, Af\rangle - \langle Ag, Hf\rangle$ extends to a continuous sesquilinear form, denoted $[H, A]$ on $D(H)$.

Let E be the spectral measure of H and let us set $E(\lambda;\varepsilon) = E((\lambda-\varepsilon, \lambda+\varepsilon))$ for $\lambda \in \mathbb{R}$ and $\varepsilon > 0$. Then $E(\lambda;\varepsilon)[H, iA]E(\lambda;\varepsilon)$ is a bounded self-adjoint operator and one may consider the set $\mu^A(H)$ of real numbers λ such that $E(\lambda;\varepsilon)[H, iA]E(\lambda;\varepsilon) \ge aE(\lambda;\varepsilon)$ for some real $a > 0$, $\varepsilon > 0$. The relevance of this type of inequality in the spectral theory of the operator H has been understood by E. Mourre [M1, M2] and for this reason one calls it a *strict Mourre estimate.* So $\mu^A(H)$ is the (open) set of real points at which a strict Mourre estimate (for H relatively to A) holds.

Mourre has shown that if H is sufficiently regular (cf. (a),(b),(c) below) with respect to A then H has no singularly continuous spectrum in $\mu^A(H)$ and, moreover, for each $s > 1/2$ and each $f \in \mathscr{H}_s := D(|A|^s)$ the limits $\lim_{\mu\to\pm 0}\langle f, R(\lambda + i\mu)f\rangle \equiv \langle f, R(\lambda \pm i0)f\rangle$ exist locally uniformly in $\lambda \in \mu^A(H)$. We have set $R(z) = (H-z)^{-1}$. The regularity conditions required by Mourre were rather restrictive:

(a) the form $[H, A]$ had to be associated to a bounded operator $D(H) \to \mathscr{H}$;

(b) $D(H)$ had to be invariant under the unitary group $W_\tau = \exp(i\tau A)$ generated by A;

(c) the map $\tau \to \langle W_\tau f, HW_\tau f\rangle$ had to be of class C^2, for each $f \in D(H)$.

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The condition (a) has been slightly relaxed in [PSS] (with, however significant consequences in the case of N-body hamiltonians). In [ABG1] and [BGM] it has been shown that the class C^2 in (c) can be replaced by $C^{1+\varepsilon}$ for some $\varepsilon > 0$ (and even $\varepsilon = +0$ in a certain sense); here $C^{1+\varepsilon}$ is the Besov class $B_\infty^{1+\varepsilon,\infty}(\mathbb{R})$. In [BG5] the conditions (a) and (b) were completely eliminated and C^2 in (c) has been replaced by the Besov class $B_\infty^{1,1}(\mathbb{R})$; moreover, it was shown that the regularity class $B_\infty^{1,1}$ is unimprovable on the Besov scale $B_\infty^{s,p}$ (if one wants to get the limiting absorption principle in the form $|\langle f, R(\lambda + i\mu)f\rangle| \le c(\lambda) < \infty$ for $f \in \cap_{k\in\mathbb{N}} D(A^k)$ and $\lambda \in \mu^A(H)$). In [M1, M2] and [PSS] the operator H was assumed bounded from below. In [CFKS] and [ABG1] it has been remarked that such a condition is not needed under the hypotheses (b) and the version of (a) required in [PSS]. However, the elimination of (a) and (b) in [BG5] was possible only under the assumption that H has a spectral gap. In [JP] it has been observed that an estimate proved by Mourre in [M2] allows one to replace the assumption $f \in \mathscr{H}_s$ with $s > 1/2$ by $f \in \mathscr{H}_{1/2,1}$ (which is a certain Besov type space associated to A). A complete presentation of the theory with several improvements can be found in Chapter 7 of [ABG2].

A natural question is the degree of regularity of the map $\lambda \mapsto \langle f, R(\lambda \pm i0)f\rangle$ assuming that f and H have certain regularity properties with respect to A. Results in this direction have been obtained in [PSS], [W], [ABG1], [BGM], [JMP]. For example in [PSS], under the hypotheses (a),(b),(c), it has been shown that $\lambda \mapsto \langle f, R(\lambda + i0)f\rangle$ is locally Hőlder continuous of order $\theta = 2/3(s - 1/2)$ if $f \in \mathscr{H}_s$ for some $s \in (1/2, 1]$, and in [W] it has been proved that one may take $\theta = (s + 1/2)^{-1}(s - 1/2)$. The case of less regular hamiltonians H is treated in [ABG1] and [BGM] with similar results. However, the optimal answer is $\theta = s - 1/2$ and was obtained in [BG2, BG3] under optimal regularity conditions on H. The regularity of $\lambda \mapsto \langle f, R(\lambda + i0)f\rangle$ improves if $f \in \mathscr{H}_s$ with $s > 1$ and if C^2 in condition (c) is replaced by C^k for some integer $k \ge 3$, as it has been shown in [JMP], where results of the same nature (so not optimal) as those in [PSS] and [W] are obtained. Again, the optimal result is obtained in [BG2, BG3].

Let us identify $\mathscr{H}^* = \mathscr{H}$ with the help of the Riesz isomorphism ($\mathscr{H}^*$ is the space of antilinear continuous functionals). Then we shall have $\mathscr{H}_s \subset \mathscr{H} \subset \mathscr{H}_s^* = \mathscr{H}_{-s}$ and the results described above concern the operators $R(\lambda \pm i0) : \mathscr{H}_s \to \mathscr{H}_{-s}$ for $s > 1/2$. Let Π_+, Π_- be the spectral projections of A associated to the intervals $[0, \infty)$ and $(-\infty, 0]$. It was observed in [M2] that the operators $\Pi_\mp R(\lambda \pm i0)$ have much better continuity properties than $R(\lambda \pm i0)$ (and there are further improvements for $\Pi_\mp R(\lambda \pm i0)\Pi_\pm$). These operators were studied and their importance in scattering theory was pointed out in [M2], [JMP], [J]. In [J] it was shown, for example, that $\Pi_\mp R(\lambda \pm i0)$ send $\mathscr{H}_s$ into $\mathscr{H}_{s-1}$ if $s > 1/2$ and if H is sufficiently regular. Some regularity properties of the maps $\lambda \mapsto \Pi_\mp R(\lambda \pm i0) \in B(\mathscr{H}_s; \mathscr{H}_t)$ are described in [JMP]. However, their results are far from optimal, as it will be clear from a comparison with our results. Similar problems concerning the maps $\lambda \mapsto \Pi_\mp R(\lambda \pm i0)\Pi_\pm$ are treated in [JMP] and [J].

We follow Mourre and we say that A is *locally (strictly) conjugated to H on an interval J* if $J \subset \mu^A(H)$. Loosely speaking, we say that A is locally conjugated to H if $\mu^A(H) \neq \varnothing$. The simplest example of a pair (H, A) such that A is locally conjugated to H is obtained as follows: $\mathscr{H} = L^2(\mathbb{R})$, H is the operator of multiplication by a function $h : \mathbb{R} \to \mathbb{R}$ of class C^1 with $h'(x) > 0$ $(\forall x)$, and $A = i(d/dx)$; later on we refer to this situation as the *classical case*. The developments of this paper suggest that the following proposition is, in some sense, true: when properly formulated, each assertion which holds in the classical case should hold in general. Of course, the "proper formulation" should involve only terms which make sense in an abstract Hilbert space setting. We hope that Section 1, which contains a quite detailed description of our results, will clarify this point of view.

Our purpose in this paper is to make a systematic study of the behaviour as $\mu \to \pm 0$ of the resolvent $R(\lambda + i\mu)$ in the framework of the conjugate operator theory $(\lambda \in \mu^A(H))$. We shall obtain the best possible (on a scale that will be defined in Section 1) regularity properties of the maps $\lambda \mapsto R(\lambda \pm i0)$, $\lambda \mapsto \Pi_\mp R(\lambda \pm i0)$ and $\lambda \mapsto \Pi_\mp R(\lambda \pm i0)\Pi_\pm$ when considered with values in $B(\mathscr{H}_{s,p}; \mathscr{H}_{t,q})$, where $\{\mathscr{H}_{s,p}\}$ is the Besov scale associated to A in $\mathscr{H}$. This will be done under optimal A-regularity hypotheses on H, in a sense that will be made clear in Section 1. In our main theorems we require H to have a spectral gap. The advantage is that this allows us to avoid any condition concerning the domain of H. Moreover, we may then treat as easily the case when H "lives" in a Hilbert space which is smaller than that in which "lives" A. In other terms, H could be a non-densely defined operator in $\mathscr{H}$, and this is useful in several applications (see [BGS] for example). Of course, the spectral gap hypothesis is annoying in some applications, e.g. to Stark effect hamiltonians or simply characteristic operators. For operators H of A-regularity class $\mathscr{C}^\alpha(A)$ with $1 < \alpha < 3/2$ this condition has been eliminated in [S1, S2].

The paper is organized as follows.

In Section 1 the main definitions and results are stated and discussed in detail. Moreover, we try to motivate the statements and we prove their optimality by studying the "classical case".

In Section 2 we describe the Besov scale $\{\mathscr{H}_{s,p} \mid s \in \mathbb{R}, 1 \leq p \leq \infty\}$ associated to A in $\mathscr{H}$. The main result of the section is the theorem proved in §2.3, which will play an important role in Sections 5 and 6. We are also going to take advantage of this theorem in §2.5 and we shall prove several interesting facts concerning the spaces $\mathscr{H}_{s,p}$ (we could have deduced these results from more general ones obtained in Chapter 3 of [ABG2]). In §2.6 we show that the Banach space $\mathscr{H}_{s,1}$ is weakly sequentially complete (we need this in the statement of Theorems D and E).

In Section 3 we make a similar study of a Besov scale $\{\mathscr{C}^{s,p}(A) \mid s \in (0, \infty), p \in [1, \infty]\}$ associated to the operator $\mathrm{ad}_A : S \mapsto [A, S]$ acting in the Banach space $B(\mathscr{H})$. The regularity with respect to A of the hamiltonian H will be expressed by conditions of the form $(H - z)^{-1} \in \mathscr{C}^{s,p}(A)$. The most important statement of the section is the theorem from §3.7 (the proof is given in [BG3]) which gives a Littlewood-Paley type description of the space $\mathscr{C}^{s,p}(A)$: this is the main ingredient of the proof of our main results from Sections 5 and 6. Several other propositions from Section 3 could have been obtained as corollaries of the results from Ch. 5 in [ABG2], but we have made an effort in order to make the section as self-contained as possible (all the unproven facts are straightforward consequences of the theorem from §3.7).

Section 4 is devoted to the study of the resolvent (or pseudo-resolvent) families, i.e. resolvents of non-densely defined self-adjoint operators, in the framework of the conjugate operator theory. A similar, but less systematic, study may be found in [BG5] and [BGS]. In §4.2 we give a new and very simple proof of a version of the Helffer-Sjöstrand formula (although we shall not use it in the rest of the paper; cf. [D1, D2]). In §4.5 we define the Mourre sets $\mu^A(H)$ and $\tilde{\mu}^A(H)$ and we prove our first main theorem (Theorem A from Section 1). In §4.6 we explain how one may reduce the proof of the other theorems from the Section 1 to the case when H is a bounded everywhere defined operator. It is at this point that we require the hamiltonian to have a spectral gap.

Sections 5 and 6 are the technical core of the paper. Section 5 contains estimates on the "distorted" hamiltonian and in Section 6 are proven all the theorems stated in Section 1 (for everywhere defined H). Note that we do not use the method of differential ineqialities in the study of the regularity properties of the map $\lambda \mapsto R(\lambda + i0)$, neither the method of [J] in order to prove (for example) that $\Pi_- R(\lambda + i0)\mathscr{H}_{s,p} \subset \mathscr{H}_{s-1,p}$. It seems to us that these methods would require significantly stronger A-regularity properties for H. Our techniques (based on Lemmas 6.1, 6.2) are natural extensions of those from [BG2, BG3].

The idea of using a "twisted" (or "distorted") hamiltonian H_ε in the study of the behaviour of $(H - \lambda - i\mu)^{-1}$ as $\mu \to +0$ goes back to [AC], [BC] (the case of A-analytic H) and to [M1, M2], [JMP]. The connection with the dilation analyticity (or complex scaling) techniques is particularly striking in our approach. Indeed, the twisted hamiltonian H_ε is constructed as follows. First, in order to compensate the lack of A-regularity of H, we "regularize" H with respect to A (this is the analogue of regularizing a function h by using a convolution). This gives a self-adjoint operator $H(\varepsilon)$ which is A-analytic and such that $\|H(\varepsilon) - H\| \leq C\varepsilon^s$ if (and only if) $H \in \mathscr{C}^{s,\infty}(A)$. Then we take $H_\varepsilon = e^{-\varepsilon A} H(\varepsilon) e^{\varepsilon A}$ for $\varepsilon > 0$. If H itself is A-analytic, the first step is not necessary and the proofs become extremely natural and transparent. In fact we strongly recommend to anyone who really wants to understand the conjugate operator method to go through the proofs in the particular case of A-analytic hamiltonians with the distorsion $H_\varepsilon = e^{-\varepsilon A} H(\varepsilon) e^{\varepsilon A}$ (see §4.12 in [BG3] and also [S1]).

N.B. This text first appeared as a preprint of the Institut de Mathématiques de Jussieu in January 1996 (some minor mistakes of the preprint are corrected here). The paper "Boundary Values of Regular Resolvent Families", submitted for publication to Journal of Functional Analysis in May 1996, contains in a rather condensed form (the next 62 pages are reduced to 33) the main results of the preprint, with one significant improvement (Proposition 3.1, which clarifies the considerations of §4.5 below). This paper is available as preprint number 96-506 from the Mathematical Physics Preprint Archive of the University of Texas (http://www.ma.utexas.edu/mp_arc/). We decided to make available to a larger audience the present detailed version because we think that the developments of the following four sections are of some independent interest.

1. Description of the Main Results

1.1. Generally, a family $\{R(z) \mid z \in \mathbb{C} \setminus \mathbb{R}\}$ of bounded operators in the Hilbert space $\mathscr{H}$ is called a *self-adjoint pseudo-resolvent family* if the following relations are satisfied: $R(z)^* = R(\bar{z})$ and $R(z_1) - R(z_2) = (z_1 - z_2)R(z_1)R(z_2)$; in this paper we are going to abbreviate the name and just say *resolvent family.* We say that such a family is *regular with respect to a densely defined self-adjoint operator A in $\mathscr{H}$*, or simply *A-regular*, if for some z the following condition is fulfilled ($W_\tau = \exp(i\tau A)$):

$$\int_0^1 \|W_{2\tau}^* R(z) W_{2\tau} - 2W_\tau^* R(z) W_\tau + R(z)\| \tau^{-2} d\tau < \infty. \tag{1.1}$$

Intuitively this means that the map $\tau \mapsto W_\tau^* R(z) W_\tau \in B(\mathscr{H})$ is slightly more than of class C^1 (in norm). Conditions of this type will be discussed in some detail later on.

The most important class of resolvent families is that associated to everywhere defined (hence bounded) self-adjoint operators H through the formula $R(z) = (H - z)^{-1}$. In this case (1.1) is equivalent to

$$\int_0^1 \|W_{2\tau}^* H W_{2\tau} - 2W_\tau^* H W_\tau + H\| \tau^{-2} d\tau < \infty. \tag{1.2}$$

A more general situation is that when $R(z) = (H - z)^{-1}$ with H a densely defined self-adjoint operator in $\mathscr{H}$. If the domain of H is invariant under all the operators W_τ ($\tau \in \mathbb{R}$) then the condition (1.1) is equivalent to

$$\int_0^1 \|W_{2\tau}^* H W_{2\tau} - 2W_\tau^* H W_\tau + H\|_{D(H) \to D(H)^*} \tau^{-2} d\tau < \infty.$$

Here $D(H)$ is the domain of H equipped with the graph topology, $D(H)^*$ is the adjoint space, and we identify $D(H) \subset \mathscr{H} \equiv \mathscr{H}^* \subset D(\mathscr{H})^*$ with the help of the Riesz's lemma. In particular $B(\mathscr{H}) \subset B(D(H); \mathscr{H}) \subset B(D(H); D(H)^*)$.

If $\{R(z)\}$ is an arbitrary resolvent family, then the map $z \mapsto R(z) \in B(\mathscr{H})$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and one has $\|R(z)\| \leq |\operatorname{Im} z|^{-1}$. Moreover, if we fix a real number λ, then there are two possibilities: either there is a constant $C = C(\lambda) < \infty$ such that $\|R(\lambda + i\mu)\| \leq C$ for all $\mu \neq 0$, and this happens if and only if the map $z \mapsto R(z) \in B(\mathscr{H})$ has a holomorphic extension to a neighbourhood of λ in $\mathbb{C}$, or $\|R(\lambda + i\mu)\| = |\mu|^{-1}$ for all real $\mu \neq 0$. The points λ of the second type form the *spectrum of the resolvent family* and it is clear that $\lim_{n\to\infty} R(\lambda + i\mu_n)$ cannot exist weakly in $B(\mathscr{H})$ if λ belongs to the spectrum and $\{\mu_n\}$ is a sequence of real numbers which tends to zero. However, the two limits $\lim_{\mu\to\pm 0} R(\lambda + i\mu) \equiv R(\lambda \pm i0)$ could exist in a space larger than $B(\mathscr{H})$ and this fact has important consequenes in spectral and scattering theory.

The purpose of our work is to study the behaviour of $R(\lambda + i\mu)$ as $\mu \to \pm 0$ and the properties of the limit operators $R(\lambda \pm i0)$ in the framework of the so-called *conjugate operator method.* The class of "spaces larger than $B(\mathscr{H})$" usually considered in applications consists of spaces of the form $B(\mathscr{E}; \mathscr{F})$, where $\mathscr{E}$, $\mathscr{F}$ are Banach spaces such that $\mathscr{E}$ is continuously and densely embedded in $\mathscr{H}$ and $\mathscr{H}$ is continuously embedded in $\mathscr{F}$; then one clearly has a canonical continuous embedding $B(\mathscr{H}) \subset B(\mathscr{E};\mathscr{F})$. *The main point of the conjugate operator theory is that it provides natural candidates for the spaces $\mathscr{E}$ and $\mathscr{F}$*: they can be constructed with the help of the "conjugate operator" A. But the A-regularity condition on $\{R(z)\}$, although important, is not sufficient in order to obtain interesting results. Indeed, the set of real λ for which $R(\lambda \pm i0)$ can be given a sense in this framework depends on A and could be empty. So, for the theory to be non-trivial, the resolvent family has to satisfy a certain non-degeneracy condition with respect to A. Such a condition has been isolated by Mourre in [M1] and for this reason it is called the *(strict) Mourre estimate.*

1.2. Before going to the general theory we think it worthwile to discuss in some detail a particular case which explains and motivates many of the later developments. More precisely we consider here the case where H is the operator of multiplication by the bounded Borel function $h : \mathbb{R} \to \mathbb{R}$ in the Hilbert space $L^2(\mathbb{R})$ and A is the usual self-adjoint realization of $i(d/dx)$ in $L^2(\mathbb{R})$. Later on we shall refer to this situation as the classical case. Then the A-regularity condition (1.2) is equivalent to the fact that h belongs to the Besov space $B_\infty^{1,1}(\mathbb{R})$. Let us remark that in this paper we follow the conventions of Peetre [P] and denote by $B_q^{s,p}$ the Besov spaces associated to the translation group in L^q, but that in the cases $q = 2$ and $q = \infty$ we use the following special notations:

$$B_2^{s,p} = \mathscr{H}^{s,p},\ \mathscr{H}^{s,2} = \mathscr{H}^s; \quad B_\infty^{s,p} = \Lambda^{s,p},\ \Lambda^{s,\infty} = \Lambda^s.$$

So (1.2) means that $h \in \Lambda^{1,1}$. We recall that $\Lambda^{1,1}(\mathbb{R}) \subset BC^1(\mathbb{R})$ (space of bounded functions with bounded and continuous derivative) and that this embedding is optimal on the scale $\{\Lambda^{s,p}\}$. More precisely there are functions in $\cap_{p>1}\Lambda^{1,p}$ which are not Lipschitz on any interval; this explains the degree of precision of the condition (1.2).

The behaviour of $\langle g, (H-\lambda-i\mu)^{-1}f\rangle$ as $\mu \to \pm 0$ is intimately related to the spectral properties of the operator H. The spectrum of H is equal to the closure $\overline{h(\mathbb{R})}$ of the image of h, so it is a compact real interval (h being continuous). It is easily shown that the singular spectrum of H is included in the closure of the set of critical values of h (real numbers λ such that $\lambda = h(x)$ for some x with $h'(x) = 0$). Now, even if h is of class C^∞, this set could be a rather arbitrary real set of Lebesgue measure zero. In order to be able to say something interesting we have to restrict ourselves to the set of non-critical values of h, i.e. the set of numbers λ such that $h'(x) \neq 0$ if $h(x) = \lambda$ (this is the non-degeneracy condition alluded to at the end of §1.1). For simplicity we shall assume that $h'(x) > 0$ for all $x \in \mathbb{R}$. Then, for $f, g \in L^2(\mathbb{R})$ and $z = \lambda + i\mu$ with $\lambda \in \mathbb{R}$ and $\mu > 0$ one has

(1.3)
$$\langle g, (H-z)^{-1}f\rangle = \int_I \frac{\overline{g(h^{-1}(x))}f(h^{-1}(x))}{x-\lambda-i\mu}\frac{dx}{h'(h^{-1}(x))} \equiv \int_I \frac{u(x)dx}{x-\lambda-i\mu}.$$

Here $I = h(\mathbb{R})$ is an open real interval whose closure is the spectrum $\sigma(H)$ of the operator H. We are interested in the behaviour of the resolvent $(H-z)^{-1}$

as $\mu \to +0$. Since outside $\sigma(H)$ the resolvent is a holomorphic function, we consider only values $\lambda \in I$ (the ends of the interval I should be considered as critical values of h and are not treated here). If we set $u(x) = 0$ for $x \in \mathbb{R} \setminus I$, so that $u \in L^1(\mathbb{R})$, and if we denote by $\tilde{u}$ the Hilbert transform of u, then the limit of (1.3) as $\mu \to +0$ exists for (Lebesgue) almost every real λ and we have

$$(1.4) \quad \lim_{\mu \to +0} \langle g, (H - \lambda - i\mu)^{-1} f \rangle = \pi(-\widetilde{u}(\lambda) + iu(\lambda)) = 2i \int_0^\infty e^{i\lambda x} \hat{u}(x) dx.$$

Here $\hat{u}(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-ixy} u(y) dy$ is the Fourier transform of u.

By using the first equality in (1.4) and well-known results concerning the Hilbert transformation one can now easily deduce various properties of the holomorphic function $z \mapsto \langle g, (H - z)^{-1} f \rangle$ and of its boundary value function $I \ni \lambda \mapsto \langle g, (H - \lambda - i0)^{-1} f \rangle$ (which is the first member of (1.4)). For example, if u is locally of class Λ^α on I for some real $\alpha > 0$ then $\tilde{u}$ has the same property, and so the limit in (1.4) exists locally uniformly in λ on I and the function $\lambda \mapsto \langle g, (H - \lambda - i0)^{-1} f \rangle$ is (locally on I) of the same class Λ^α. Moreover, the holomorphic function $z \mapsto \langle g, (H - z)^{-1} f \rangle$ extends to a function of class Λ^α in each rectangle $a \le \lambda \le b$, $0 \le \mu \le c$ with $[a, b] \subset I$ and $c > 0$. From the second inequality in (1.4) one can also see that only the behaviour of the Fourier transform $\hat{u}(x)$ as $x \to +\infty$ really matters.

When so formulated this type of results cannot be extended to a general and abstract framework, like that of §1.1. Indeed, u has been obtained from f, g by a procedure which has no meaning in a purely Hilbert space setting. However, it is not difficult to find simple and natural conditions on f, g and h which make sense in the context of §1.1 and which in the present context imply that u is locally of class Λ^α. Let us set $s = \alpha + 1/2$, so that s is a real number strictly larger than $1/2$. Then we can apply the Sobolev embedding theorem and get:

$$\mathscr{H}^s \equiv B_2^{s,2}(\mathbb{R}) \subset B_\infty^{\alpha,2}(\mathbb{R}) \equiv \Lambda^{\alpha,2} \subset \Lambda^\alpha.$$

So if f, g belong to $\mathscr{H}^s$ then they belong to Λ^α, which is an algebra, hence $\bar{f} g \in \Lambda^\alpha$. Moreover, h' is of class Λ^α if and only if h is of class $\Lambda^{\alpha+1}$, and then the local class Λ^α is stable under composition with h^{-1}.

In conclusion, if $f, g \in \mathscr{H}^s$ with $s = \alpha + 1/2 > 1/2$ and if $h \in \Lambda^{\alpha+1}$, then u is locally of class Λ^α on I, hence the assertions made above in connection with the function $z \mapsto \langle g, (H - z)^{-1} f \rangle$ remain true. The condition on f, g now makes sense in the context of §1.1: for example $f \in \mathscr{H}^s$ means that f belongs to the domain of the operator $|A|^s$ or, equivalently, that the vector valued function $\tau \mapsto W_\tau f \in \mathscr{H}$ is of class $\Lambda^{s,2}$. The condition $h \in \Lambda^{\alpha+1}$ can be expressed in abstract terms too: this means precisely that the operator valued function $\tau \mapsto W_\tau^* H W_\tau \in B(\mathscr{H})$ is of class $\Lambda^{\alpha+1}$. It turns out that this point of view is convenient not only because it can be immediately considered at an abstract level but also because it allows one to study some limit cases by using the general classes $\mathscr{H}^{s,p}$ and $\Lambda^{\alpha,q}$. For example, if f, g belong to $\mathscr{H}^{1/2,1}$ and $h \in \Lambda^{1,1}$ (which is, formally, the limit case $\alpha = 0$), then the limit (1.4) still exists, locally uniformly in $\lambda \in \Lambda$.

Now let us consider the last member of (1.4): we see that the local regularity properties of the boundary value function $\lambda \mapsto \langle g, (H - \lambda - i0)^{-1} f \rangle$

depend only on the behaviour of $\hat{u}$ at $+\infty$. If we set $f_0(x) = f(h^{-1}(x))$ and $g_0(x) = g(h^{-1}(x))(h'(h^{-1}(x)))^{-1}$ and if we extend these functions by zero outside I, we get $\hat{u}(x) = (2\pi)^{1/2} \int_{-\infty}^{\infty} \hat{f}_0(x+y)\hat{g_0}^* dy$. So if $\hat{f}_0$ decays rapidly enough at $+\infty$ *and* $-\infty$, then $\hat{u}$ will decay as rapidly as we wish at $+\infty$ provided that $\hat{g_0}$ decays suficiently rapidly *at* $-\infty$ *only*. It is not so easy to control the decay at $-\infty$ of $\hat{g_0}$ in terms of that of $\hat{g}$ (in particular this shows that the methods that we shall use in the proof of the general theorems give better results even in the classical case). But if we take $h(x) \equiv x$ (we have assumed, until now, h bounded only for the simplicity of the presentation) and if $\operatorname{supp} \hat{g} \subset [0, \infty)$, then one can show quite easily that $\int_0^{\infty} (1 + |x|)^{\alpha} |\hat{u}(x)| dx < \infty$ provided that $h \in \mathscr{H}^s$ and $g \in \mathscr{H}^{1-s+\alpha}$ for some $s > \alpha + 1/2 > 1/2$ (before we had to put the much stronger condition $g \in \mathscr{H}^s$). Clearly, the preceding relation satisfied by $\hat{u}$ implies that $\lambda \mapsto \langle g, (H - \lambda - i0)^{-1} f \rangle$ is of class Λ^{α}. Of course,we have proved this only for a very special function h, but it will turn out that the result remains true under very general hypotheses. We note that the condition $\operatorname{supp} \hat{g} \subset [0, \infty)$ can be very simply stated in terms of the operator A: if Π_- is the spectral projection of A associated to the interval $(-\infty, 0]$ then $\operatorname{supp} \hat{g} \subset [0, \infty)$ is equivalent to $\Pi_- g = g$.

The last point that we want to discuss is the abstract analogue of the condition $h'(x) > 0$, which played such an important role in the preceding analysis (we could as well assume that $h'(x) < 0$ for all x; however, this trivial modification amounts to the replacement of the operator A by $-A$ below). Note that $[H, iA]$ is just the operator of multiplication by the function h' in $L^2(\mathbb{R})$ (we recall the choice $A = i(d/dx)$), so we see that we could express the condition $h' > 0$ in terms of the strict positivity of the operator $[H, iA]$. Now it turns out that such a global condition is rather difficult to check and is also quite restricitve in many applications. On the other hand, one may easily convince himself that in order to study (by the preceding methods) the behaviour of $(H - \lambda - i\mu)^{-1}$ as $\mu \to \pm 0$ for λ in some open real set J it suffices to have $h'(x) > 0$ if $h(x) \in J$, i.e. $h'(x) > 0$ on the open set $h^{-1}(J)$. The characteristic function of the set $h^{-1}(J)$, when considered as multiplication operator in $L^2(\mathbb{R})$, is just the spectral projection $E(J)$ associated to the self-adjoint operator H and to the set J. So the condition $h'(x) > 0$ if $h(x) \in J$ can be expressed in terms of the "strict positivity" of the self-adjoint operator $E(J)[H, iA]E(J)$, where E is the spectral measure of H. A rather natural ormulatio of this "strict positivity" condition is: there is a strictly positive real number a such that $E(J)[H, iA]E(J) \geq aE(J)$. This is exactly the strict Mourre estimate.

1.3. The general theory which will be developed from now on should be considered as an abstract version of the "model" discussed in §1.2. Although our framework and our assumptions are simple and natural generalizations of those from §1.2, it turns out that they cover quite large classes of hamiltonian operators that appear in applications and that our results give optimal answers to several questions important in spectral and scattering theory. We also stress the fact that several consequences of the general theory in the classical case (i.e. the case stated in §1.2) are rather difficult to obtain by classical means (cf. a comment in §1.2; see also Theorem 7.6.2 in [ABG 2]).

Our decision to treat general resolvent families (which could not be associated to densely defined self-adjoint operators in $\mathscr{H}$) could be justified by esthetical and practical reasons as well. Our attitude towards the requirement of density in $\mathscr{H}$ of the domain of a self-adjoint operator is the same as that of A.P. Morse and J.P. Randolph regarding the measurability assumptions for planar sets: it is "usually a luxury, rarely a convenience, never a necessity" (Trans. Amer. Math. Soc. 55 (1944)). It was observed already in [BG5] that the conjugate operator method can be extended to situations where the domain of the hamiltonian is not dense in $\mathscr{H}$ and the usefulness of this fact was shown in [BGS] by studying in detail a class of N-body hamiltonians with hard-core interactions. There are other examples in which the possibility of analysing a hamiltonian H with the help of an operator A which generates a unitary group in a larger Hilbert space is very convenient.

1.4. In order to describe the most general resolvent family we find convenient to generalize the standard notion of self-adjoint operator. A linear operator $H : D(H) \subset \mathscr{H} \to \mathscr{H}$ will be called <u>*self-adjoint*</u> if $\langle Hf, g\rangle = \langle f, Hg\rangle$ for all f, $g \in D(H)$ and $(H \pm i)D(H) = \overline{D(H)}$ (closure of $D(H)$ in $\mathscr{H}$). In other terms, H sends $D(H)$ into its closure in $\mathscr{H}$ and, when considered as operator in $\overline{D(H)}$, H is self-adjoint in the usual sense. So a densely defined (i.e. $\overline{D(H)} = \mathscr{H}$) self-adjoint operator is a self-adjoint operator in the usual sense. To a self-adjoint operator H in $\mathscr{H}$ we associate the resolvent family defined as follows: if $f \in \overline{D(H)}$ then $R(z)f = (H - z)^{-1}f$, where $(H - z)^{-1}$ is the inverse in the Hilbert space $\overline{D(H)}$ of the bijective operator $H - z : D(H) \to \overline{D(H)}$, and $R(z)f = 0$ if f is orthogonal to $D(H)$. Then each resolvent family is obtained by this procedure from a uniquely defined self-adjoint operator.

In the context of this paper it is natural to call *spectral measure* a countable additive orthogonal projection valued function E defined on the σ-algebra of the real Borel sets (so we do *not* assume $E(\mathbb{R}) = 1$). Then the formulas $H = \int_{\mathbb{R}} \lambda E(d\lambda)$ and $R(z) = \int_{\mathbb{R}} (\lambda - z)^{-1} E(d\lambda)$ establish bijective correspondances between self-adjoint operators, spectral measures and resolvent families; note that $E(\mathbb{R})$ is the orthogonal projection of $\mathscr{H}$ onto $\overline{D(H)}$. The spectral measure E allows one to speak about various spectral properties of the operator H or of the resolvent family $\{R(z)\}$. In particular, the spectrum $\sigma(H)$ of H (or of its resolvent family) is the support of the measure E. Moreover, we can define functions of H by using E: if $\varphi : \mathbb{R} \to \mathbb{C}$ is continuous and convergent to zero at infinity, we set $\varphi(H) = \int_{\mathbb{R}} \varphi(\lambda) E(d\lambda)$.

¿From now on we fix a self-adjoint operator H in $\mathscr{H}$ with resolvent family $\{R(z)\}$ and a *densely defined* self-adjoint operator A in $\mathscr{H}$. The commutator $[H, iA]$ is then well defined as a symmetric sesquilinear form with domain $D(A) \cap D(H)$ by the formula $\langle f, [H, iA]f\rangle = 2\,\mathrm{Re}\langle Hf, iAf\rangle$.

We say that H *is of class* $C^1(A)$ if there is a number $z \in \mathbb{C} \setminus \sigma(H)$ such that the map $\tau \mapsto W_\tau^* R(z) W_\tau \in B(\mathscr{H})$ is strongly C^1 (this property is independent of z; recall that $W_\tau = \exp(i\tau A)$). If this is the case then $D(A) \cap D(H)$ is a core for H (i.e. it is dense in $D(H)$ for the graph topology), for each $\varphi \in C_0^\infty(\mathbb{R})$ we have $\varphi(H)D(A) \subset D(A) \cap D(H)$, and the densely defined symmetric sesquilinear form $\varphi(H)^\star[H, iA]\varphi(H)$ (with domain $D(A)$) is continuous for the topology induced by $\mathscr{H}$ on $D(A)$ (see §4.5). We shall denote by the same symbol $\varphi(H)^*[H, iA]\varphi(H)$ the bounded (everywhere defined) symmetric operator in $\mathscr{H}$ associated to the sesquilinear form $\varphi(H)^\star[H, iA]\varphi(H)$ (in §4.5 we adopt a more pedantic notation).

If H is of class $C^1(A)$ we associate to it two subsets of the real line defined as follows:

(i) *the strict Mourre set* $\mu^A(H)$ *of* H *with respect to* A is the set of real numbers λ with the property that there are a real function $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(\lambda) \neq 0$ and a strictly positive real number a such that $\varphi(H)[H, iA]\varphi(H) \geq a\varphi(H)^2$;

(ii) *the Mourre set* $\tilde{\mu}^A(H)$ *of* H *with respect to* A is the set of real numbers λ such that a real function $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(\lambda) \neq 0$, a strictly positive real number a and a compact operator K in $\mathscr{H}$ such that $\varphi(H)[H, iA]\varphi(H) \geq a\varphi(H)^2 + K$.

The next result is a rather straightforward consequence of the definitions (see §4.5).

Theorem A. *Assume that H is of class $C^1(A)$. Then $\mu^A(H)$ and $\tilde{\mu}^A(H)$ are open real sets, $\mu^A(H) \subset \tilde{\mu}^A(H)$, and the set $\tilde{\mu}^A(H) \setminus \mu^A(H)$ does not have accumulation points in $\tilde{\mu}^A(H)$. Moreover, $\tilde{\mu}^A(H) \setminus \mu^A(H)$ consists of eigenvalues of H of finite multiplicity and the spectrum of H in $\mu^A(H)$ is purely continuous.*

We do not know whether the $C^1(A)$ regularity property suffices for the absence of the singularly continuous spectrum of H in $\mu^A(H)$ (this is true in the classical case and also in some rather non-significative generalizations). However, we know that a very natural version of the limiting absorption principle breaks down if H is only $C^1(A)$ (see Section 7.B in [ABG2] for a precise statement). For such matters the regularity of H expressed in (1.1), plays an important role. Before stating our next results we have to introduce several regularity classes of functions, vectors and operators.

1.5. Let E be a Banach space and $\phi : \mathbb{R} \to E$ a bounded continuous function. For each integer $m \geq 1$ we define the *modulus of continuity* (or *smoothness*) of order m of ϕ as the function ω_m given by $\omega_m(\varepsilon) = \sup_{x\in\mathbb{R}} \| \sum_{k=0}^m (-1)^k \binom{m}{k} \phi(x + k\varepsilon)\|_E$ for $\varepsilon > 0$. Let $s > 0$ be a real number and $p \in [1, \infty]$. Then ϕ *is of class* $\Lambda^{s,p}$ if there is an integer $m > s$ such that $[\int_0^1 (\varepsilon^{-s}\omega_m(\varepsilon))^p \varepsilon^{-1} d\varepsilon]^{1/p} < \infty$ (if $p = \infty$ this means $\omega_m(\varepsilon) \leq c\varepsilon^s$ for a finite constant c). It is not difficult to show that this condition is independent of m. Moreover, one has $\Lambda^{s,p} \subset \Lambda^{t,q}$ if and only if $s > t$ or $s = t$ but $p \leq q$. The classes $\Lambda^{s,\infty} \equiv \Lambda^s$ are called *Lipschitz-Zygmund* (or *Hölder-Zygmund*) *classes*.

If $k \geq 1$ is an integer we can also consider the classes BC^k and $\mathrm{Lip}^{(k)}$ defined as follows: $\phi \in BC^k$ means that the derivatives of order $\leq k$ of ϕ exist and are bounded and norm continuous; $\phi \in \mathrm{Lip}^{(k)}$ means that $\omega_k(\varepsilon) \leq c\varepsilon^k$ for a constant c and all $\varepsilon > 0$ (this is a k-th order Lipschitz condition).

We shall point out now several relations between the preceding spaces. If $k \geq 1$ is an integer then $\Lambda^{k,1} \subset BC^k \subset \mathrm{Lip}^{(k)} \subset \Lambda^k$, all embeddings being strict and optimal on the scale $\Lambda^{s,p}$ (the spaces $\Lambda^{k,p}$ are not comparable with BC^k and $\mathrm{Lip}^{(k)}$ if $1 < p < \infty$). Now assume that $s = k + \sigma$ where $k \geq 1$ is an integer and $\sigma > 0$ is a real number. Then $\phi \in \Lambda^{s,p}$ if and only if $\phi \in BC^k$ and $\phi^{(k)} \in \Lambda^{\sigma,p}$. Note that one can always choose $0 < \sigma \leq 1$. Let $p = \infty$. If $0 < \sigma < 1$, then $\phi \in \Lambda^s$ means $\phi \in BC^k$ and $\|\phi^{(k)}(x+\varepsilon) - \phi^{(k)}(x)\|_E \leq C|\varepsilon|^\sigma$; if $\sigma = 1$, then $\phi \in \Lambda^s$ means that $\phi \in BC^k$ and $\|\phi^{(k)}(x+\varepsilon) + \phi^{(k)}(x-\varepsilon) - 2\phi^{(k)}(x)\|_E \leq C|\varepsilon|$, i.e. $\phi^{(k)}$ has to verify a Zygmund condition. All these results are consequences of the general theory developed in [BB] or in Chapter 3 of [ABG2].

There are natural classes associated to the preceding ones. For example, if Ω is an open real set and $\phi : \Omega \to E$ is a continuous function, then ϕ is locally of class $\Lambda^{s,p}$ if $\theta\phi \in \Lambda^{s,p}$ for each $\theta \in C_0^\infty(\Omega)$.

The classes $\Lambda^{s,p}$ can be defined for functions of several variables. We shall give just one example that we shall need later on. Let $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and let J be an open real set. Assume that ϕ is a continuous map from $\mathbb{C}_+ \cup J$ to E and let s be a strictly positive real number. We shall say that ϕ is locally of class Λ^s (on $\mathbb{C}_+ \cup J$) if for each rectangle $K = \{z \mid a \leq \operatorname{Re} z \leq b, 0 \leq \operatorname{Im} z \leq c\}$, with $[a,b] \subset J$ and $c > 0$, there are real numbers $\delta > 0$, $M > 0$ and an integer $m > s$ such that $\|\sum_{k=0}^m (-1)^k \binom{m}{k} \phi(z + k\varepsilon)\| \leq M|\varepsilon|^s$ for all $z \in K$ and $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \leq \delta$ and $\operatorname{Im}\varepsilon \geq 0$ (this property is, of course, independent of m).

One should notice one more fact concerning the classes Λ^s and $\mathrm{Lip}^{(k)}$: they do not really depend on the topology of E, but rather on its bornology (i.e. the family of bounded sets). More precisely, if we have a new vector space topology on E, and if the bounded sets associated to this topology are the same as those of the initial E, then the classes Λ^s and $\mathrm{Lip}^{(k)}$ are also the same. For example the weak Λ^s class (obvious definition) coincides with the norm Λ^s class; or if E is an adjoint space, then the weak* Λ^s class coincides with the norm Λ^s class. Moreover, if $E = B(\mathscr{E};\mathscr{F})$ (resp. $E = B(\mathscr{E};\mathscr{F}^*)$) for some Banach spaces $\mathscr{E}$, $\mathscr{F}$, then the norm and the weak (resp. weak*) operator topology on E give the same classes Λ^s and $\mathrm{Lip}^{(k)}$.

1.6. One can use the regularity classes $\Lambda^{s,p}$ in order to treat in a unified way various Besov type spaces of vectors $f \in \mathscr{H}$ or operators $S \in B(\mathscr{H})$ naturally associated to the densely defined self-adjoint operator A (a detailed discussion and other equivalent characterizations of these spaces may be found in Sections 2 and 3).

Let s be a strictly positive real number and $p \in [1,\infty]$. Then $\mathscr{H}_{s,p}$ is the set of vectors $f \in \mathscr{H}$ such that the function $\tau \mapsto W_\tau f \in \mathscr{H}$ is of class $\Lambda^{s,p}$. Similarly, $\mathscr{C}^{s,p}(A)$ is the set of operators $S \in B(\mathscr{H})$ such that the map $\tau \mapsto W_\tau^* S W_\tau \in B(\mathscr{H})$ is of class $\Lambda^{s,p}$. We thus get two scales of spaces which are totally ordered in the following sense: $\mathscr{H}_{s,p} \subset \mathscr{H}_{t,q}$ and $\mathscr{C}^{s,p}(A) \subset \mathscr{C}^{t,q}(A)$ if $s > t$ or if $s = t$ but $p \leq q$. There are natural topologies on $\mathscr{H}_{s,p}$ and $\mathscr{C}^{s,p}(A)$ for which these spaces become topological vector spaces, and these topologies can be defined by (complete) norms.

It is possible to extend the scale $\{\mathscr{H}_{s,p}\}$ to $s \leq 0$ by the following procedure (a more convenient method is presented in Section 2; it is also possible to define the spaces $\mathscr{C}^{s,p}(A)$ for $s \leq 0$, but we shall not need them). Let $\mathscr{H}_\infty = \cap_{s>0}\mathscr{H}_{s,p}$ and let $\mathscr{H}_{s,p}^o$ be the closure of $\mathscr{H}_\infty$ in $\mathscr{H}_{s,p}$.

Then $\mathscr{H}_\infty$ is dense in $\mathscr{H}$ and $\mathscr{H}^o_{s,p} = \mathscr{H}_{s,p}$ if $1 \le p < \infty$. Now we set $\mathscr{H}_{-s,p'} = [\mathscr{H}^o_{s,p}]^*$ (adjoint space, i.e. the space of continuous anti-linear forms on $\mathscr{H}^o_{s,p}$ equipped with the strong topology). This defines $\mathscr{H}_{t,q}$ for all $t \in \mathbb{R} \setminus \{0\}$ and $1 \le q \le \infty$. We identify $\mathscr{H}^* = \mathscr{H}$ with the help of Riesz's lemma and so we get continuous embeddings $\mathscr{H}_{s,p} \subset \mathscr{H} \subset \mathscr{H}_{t,q}$ if $s > 0$ and $t < 0$. Finally, if $s = 0$ we define $\mathscr{H}_{0,p}$ for $1 \le p \le \infty$ by real interpolation (see (2.11)).

A rather remarkable fact happens for $p = 2$ (this is due to the Hilbert space geometry of $\mathscr{H}$, the unitarity of W_τ plays no role; cf. Section 3.7 in [ABG2]). Let $k \ge 1$ be an integer and let $f \in \mathscr{H}$. Then the map $\tau \mapsto W_\tau f \in \mathscr{H}$ is of class $\Lambda^{k,2}$ if and only if it is of class BC^k and also if and only if it is of class $\mathrm{Lip}^{(k)}$ (see §2.5 for an elementary proof). In particular, $\mathscr{H}_{k,2} = D(A^k)$. We set $\mathscr{H}_s = \mathscr{H}_{s,2}$ for all $s \in \mathbb{R}$. Then $\mathscr{H}_s = D(|A|^s)$ for $s \ge 0$, in particular $\mathscr{H}_0 = \mathscr{H}$.

Geometrically speaking, the Banach space $B(\mathscr{H})$ looks like a space L^∞. For this reason the value $p = \infty$ plays a rather special role in this case, so we set $\mathscr{C}^s(A) = \mathscr{C}^{s,\infty}(A)$. Let $s = k$ be an integer ≥ 1. Then one can also introduce the space $C^k(A)$ of operators $S \in B(\mathscr{H})$ such that the map $\tau \mapsto W^*_\tau S W_\tau \in B(\mathscr{H})$ is of class $\mathrm{Lip}^{(k)}$. One can prove that $C^k(A) \subset \mathscr{C}^k(A)$ *strictly*. Moreover, S belongs to $C^k(A)$ if and only if the function $\tau \mapsto W^*_\tau S W_\tau \in B(\mathscr{H})$ is of class BC^k in the strong (or weak) operator topology; if this map is of class BC^k in the norm topology, we write $S \in C^k_u(A)$. One has the following relation between these spaces: $\mathscr{C}^{k,1}(A) \subset C^k_u(A) \subset C^k(A) \subset \mathscr{C}^k(A)$ strictly (see Section 3). We set $C^\infty(A) = \cap_{k\in\mathbb{N}} C^k(A)$.

Let H be a self-adjoint operator in $\mathscr{H}$ and $\{R(z)\}$ its resolvent family. We say that H, or $\{R(z)\}$, *is of class* $\mathscr{C}^{s,p}(A)$ ($\mathscr{C}^s(A)$ if $p = \infty$) if there is $z \in \mathbb{C} \setminus \sigma(H)$ such that $R(z) \in \mathscr{C}^{s,p}(A)$ (this property is independent of z). We see that $\{R(z)\}$ *is a regular (in the sense of* (1.1)*) resolvent family if and only if H is of class $\mathscr{C}^{1,1}(A)$, and then H is of class $C^1(A)$.*

Assume that H is of class $\mathscr{C}^{s,p}(A)$ for some $s > 0$ and let $\varepsilon \in (0, s)$ be an arbitrary small real number. Then for each $z \in \mathbb{C} \setminus \sigma(H)$ the operator $R(z) : \mathscr{H} \to \mathscr{H}$ has a unique extension to a continuous operator $R(z) : \mathscr{H}_{-s+\varepsilon} \to \mathscr{H}_{-s+\varepsilon}$ (see §3.9). Moreover, this extension has the property $R(z)\mathscr{H}_{t,q} \subset \mathscr{H}_{t,q}$ for each real t with $|t| < s - \varepsilon$ and each $q \in [1, \infty]$. Since ε is arbitrary we see that *for each real $t \in (-s, s)$ and each $q \in [1, \infty]$ the operator $R(z)$ ($z \in \mathbb{C} \setminus \sigma(H)$) induces a continuous operator (that we shall continue to denote by $R(z)$) $R(z) : \mathscr{H}_{t,q} \to \mathscr{H}_{t,q}$.* Clearly the family of operators $\{R(z) \mid z \in \mathbb{C} \setminus \sigma(H)\} \subset B(\mathscr{H}_{t,q})\}$ satisfies the first resolvent identity, hence it is the resolvent family of an operator in the (Banach) space $\mathscr{H}_{t,q}$ naturally associated to the initial operator H in $\mathscr{H}$.

There are two interesting (for us) topologies on the space $B(\mathscr{H}_{s,p}; \mathscr{H}_{t,q})$, where s, t are arbitrary real numbers and p, $q \in [1, \infty]$. The first one is the norm topology, obtained by fixing admissible norms on the spaces $\mathscr{H}_{s,p}$ and $\mathscr{H}_{t,q}$. The second one is more interesting: it is the topology defined by the family of seminorms $S \mapsto |\langle g, Sf \rangle|$ with $f \in \mathscr{H}_{s,p}$ and $g \in \mathscr{H}_{-t,q'}$. We shall call it w-*topology* (see §2.6). If $q < \infty$ then the w-topology is just the weak operator topology on $B(\mathscr{H}_{s,p}; \mathscr{H}_{t,q})$. If $q = \infty$, then $\mathscr{H}_{t,\infty} =$

$[\mathscr{H}_{-t,1}]^* \equiv \mathscr{H}^*_{-t,1}$ and the w-topology is the weak* operator topology on $B(\mathscr{H}_{s,p};\mathscr{H}^*_{-t,1})$.

1.7. We are now able to describe the behaviour of the resolvent $R(\lambda+i\mu)$ as $\mu\to\pm 0$ at points λ from the strict Mourre set of H with respect to A (see §1.4). As we explained in §1.1, the operators $R(\lambda+i\mu)$ cannot have a (weak) limit in $B(\mathscr{H})$ as $\mu\to\pm 0$ if $\lambda\in\sigma(H)$, but they could have a limit in a "larger space". We also said that the conjugate operator theory furnishes natural candidates for these "larger spaces": this should be clear already from the discussion of §1.2. Now we shall state a precise result in this direction.

Let us recall that $\mathscr{H}_{1/2,1}$ is a Banach space continuously and densely embedded in $\mathscr{H}$ (cf. §2.4). We shall always identify $\mathscr{H}$ with its adjoint $\mathscr{H}^*$ with the help of the Riesz isomorphism, and this gives us canonical embeddings:

$$\mathscr{H}_{1/2,1}\subset\mathscr{H}\subset\mathscr{H}^*_{1/2,1}\subset\mathscr{H}_{-1/2,\infty}.$$

¿From the density of the first embedding here we get another continuous embedding $B(\mathscr{H})\subset B(\mathscr{H}_{1/2,1};\mathscr{H}_{-1/2,\infty})$. In particular the resolvent family $\{R(z)\}$ can be viewed as a holomorphic map $z\mapsto R(z)$ defined on $\mathbb{C}\setminus\sigma(H)$ and with values in $B(\mathscr{H}_{1/2,1};\mathscr{H}_{-1/2,\infty})$. We shall see that this map has boundary values on the real set $\mu^A(H)$.

We say that the self-adjoint operator H (and its resolvent family) has a *spectral gap* if $\sigma(H)\neq\mathbb{R}$. We also recall that $\mathbb{C}_+$ and $\mathbb{C}_-$ are the set of complex numbers z such that $\operatorname{Im} z>0$ and $\operatorname{Im} z<0$ respectively.

Theorem B. *Assume that $\{R(z)\}$ is an A-regular resolvent family having a spectral gap (i.e. H is of class $\mathscr{C}^{1,1}(A)$ and $\sigma(H)\neq\mathbb{R}$). Then the limit $\lim_{\mu\to+0}R(\lambda+i\mu)\equiv R(\lambda+i0)$ exists in the* w*-topology of $B(\mathscr{H}_{1/2,1};\mathscr{H}_{-1/2,\infty})$ locally uniformly in $\lambda\in\mu^A(H)$.*

In other terms, the holomorphic map $\mathbb{C}_+\ni z\mapsto R(z)\in B(\mathscr{H}_{1/2,1};\mathscr{H}^*_{1/2,1})$ extends to a weak* continuous map on $\mathbb{C}_+\cup\mu^A(H)$. So the boundary value $R(\lambda+i0)$ is a well defined continuous operator $\mathscr{H}_{1/2,1}\to\mathscr{H}_{-1/2,\infty}$ if $\lambda\in\mu^A(H)$ and the map $\lambda\to R(\lambda+i0)\in B(\mathscr{H}_{1/2,1};\mathscr{H}^*_{1/2,1})$ is weak* continuous on $\mu^A(H)$. Of course, similar assertions hold if $\mu\to+0$ and $\mathbb{C}_+$ are replaced by $\mu\to-0$ and $\mathbb{C}_-$ respectively, and this gives a meaning to the operator $R(\lambda-i0)$ and to the derivative $dE_\lambda/d\lambda=(2\pi i)^{-1}[R(\lambda+i0)-R(\lambda-i0)]$ of the spectral measure E of H (see (7.1.17) in [ABG2]). We expect that $R(\lambda+i0)\neq R(\lambda-i0)$ (or $dE_\lambda/d\lambda\neq 0$) if $\lambda\in\mu^A(H)\cap\sigma(H)$ (note that $\mu^A(H)$ contains $\mathbb{R}\setminus\sigma(H)$).

Theorem B is optimal in the following sense:

(i) The space $\mathscr{H}_{1/2,1}$ cannot be replaced by any other strictly larger space from the Besov scale $\{\mathscr{H}_{s,p}\}$, so $R(\lambda+i0)$ does not have any meaning in general if $f\in\mathscr{H}_{1/2,p}$ for some $p>1$; and this even if we are in the classical case (§1.2) and the function h is of class C^∞ (indeed, there are functions in $\cap_{p>1}\mathscr{H}^{1/2,p}(\mathbb{R})$ that are unbounded on each interval, see §1.1 in [BG3]).

(ii) The class $\mathscr{C}^{1,1}(A)$ cannot be replaced by $C^1_u(A)\cap_{p>1}\mathscr{C}^{1,p}(A)$, even if we are in the classical case and $\mathscr{H}_{1/2,1}$ is replaced by $\mathscr{H}_\infty$ (cf. Section 7.B in [ABG2]).

(iii) The weak* topology cannot be replaced by the strong operator topology of $B(\mathscr{H}_{1/2,1};\mathscr{H}^*_{1/2,1})$, even if we are in the classical case and h is of class C^∞ (cf. the argument which follows (iv)).

(iv) The range $R(\lambda+i0)\mathscr{H}_{1/2,1}$ of the operator $R(\lambda+i0)$ is contained in $\mathscr{H}_{-1/2,\infty}$ but generally is not contained in $\mathscr{H}^o_{-1/2,\infty}$; moreover, generally $R(\lambda+i0)\mathscr{H}_\infty$ is not included in $\mathscr{H}^o_{-1/2,\infty}$ (even in the classical case and with h of class C^∞; see the remarks after Theorem 4.13 in [BGM] or the next argument).

By (iii) above the space $\mathscr{H}_{1/2,1}$ is so large that it contains vectors f such that the weak* continuous map $\lambda \mapsto R(\lambda+i0)f \in \mathscr{H}_{-1/2,\infty}$ is not norm continuous, even if H is of a high order of A-regularity. To see this it suffices to check that in the classical case considered in §1.2 one has $(dE_\lambda/d\lambda)f = f(x)(h'(x))^{-1}\delta_x$ for $\lambda \in I$, where $x = h^{-1}(\lambda)$ and δ_x is the Dirac measure at x, and to observe that there is a constant $C>0$ such that $\|\delta_x - \delta_y\|_{\mathscr{H}^{-1/2,\infty}(\mathbb{R})} \geq C$ for $x,y \in \mathbb{R}$, $x \neq y$ (since δ_x is obtained from δ_0 by a translation, this also proves that δ_x does not belong to the closure of $\mathscr{H}^\infty(\mathbb{R})$ in $\mathscr{H}^{-1/2,\infty}(\mathbb{R})$). If $s>1/2$ then $x \mapsto \delta_x \in \mathscr{H}^{-s}(\mathbb{R})$ is Hölder continuous, but the map $\lambda \mapsto (dE/d\lambda)f \in \mathscr{H}^{-s}(\mathbb{R})$ cannot be expected to be more than norm continuous for a general $f \in \mathscr{H}^{1/2,1}(\mathbb{R})$ or $h \in \Lambda^{1,1}(\mathbb{R})$ due to the factors $f(x)$ or $h'(x)$.

Assume now that $\mathscr{E}$, $\mathscr{F}$ are Banach spaces such that $\mathscr{E} \subset \mathscr{H}_{1/2,1}$ continuously and densely and $\mathscr{H}_{-1/2,\infty} \subset \mathscr{F}$ continuously. Then $B(\mathscr{H}_{1/2,1};\mathscr{H}_{-1/2,\infty})$ is continuously embedded in $B(\mathscr{E};\mathscr{F})$ and the map $\lambda \mapsto R(\lambda+i0)$ could have better continuity properties when considered as $B(\mathscr{E};\mathscr{F})$ valued. According to the preceding discussion this also depends on the A-regularity class of H. It is not difficult to show that *under the hypotheses of Theorem B the map* $\lambda \mapsto R(\lambda+i0) \in B(\mathscr{H}_s;\mathscr{H}_{-s})$ *is norm continuous if* $s>1/2$ (see the proof of Theorem 4.13 in [BGM]). The next theorem contains a more subtle result in this direction.

Theorem C. *Let $\{R(z)\}$ be a resolvent family having a spectral gap.*

(a) *If $\{R(z)\}$ is of class $\mathscr{C}^{s+1/2}(A)$ for some real $s>1/2$, then the map $\lambda \mapsto R(\lambda+i0) \in B(\mathscr{H}_{s,\infty};\mathscr{H}_{-s,1})$ is locally of class $\Lambda^{s-1/2}$ on $\mu^A(H)$.*

(b) *Assume that $\{R(z)\}$ is of class $\mathscr{C}^{s+1/2,1}(A)$ for a real number s such that $s-1/2$ is an integer ≥ 1. Then the function $\lambda \mapsto R(\lambda+i0) \in B(\mathscr{H}_{s,1};\mathscr{H}_{-s,\infty})$ is of class $C^{s-1/2}$ in the weak* topology and its derivatives of order $k=0,1,2,\cdots,s-1/2$ are given by*

$$\frac{d^k}{d\lambda^k}R(\lambda+i0) = \lim_{\mu\to+0} k!R(\lambda+i\mu)^{k+1}, \tag{1.5}$$

where the limit exists weakly in $B(\mathscr{H}_{s,1};\mathscr{H}_{-s,\infty})$, locally uniformly in $\lambda \in \mu^A(H)$.*

Part (a) of the theorem can be reformulated as follows: *the holomorphic function $\mathbb{C}_+ \ni z \mapsto R(z) \in B(\mathscr{H}_{s,\infty};\mathscr{H}_{-s,1})$ extends to a norm continuous function on $\mathbb{C}\cup\mu^A(H)$ which is locally of class $\Lambda^{s-1/2}$* (see §1.5). Indeed, note first that, due to the uniform boundedness principle and to the polarization identity, it suffices to show that for each $f \in \mathscr{H}_{s,\infty}$ the complex holomorphic function $\mathbb{C}_+ \ni z \mapsto \phi(z) = \langle f, R(z)f\rangle$ extends to a function of class $\Lambda^{s-1/2}$

on $\mathbb{C}_+ \cup \mu^A(H)$. By Theorem B the limit $\lim_{\mu\to+0} \phi(\lambda+i\mu) = \phi(\lambda+i0)$ exists locally uniformly in $\lambda \in \mu^A(H)$ and by Theorem C (a) the function $\lambda \to \phi(\lambda+i0)$ is locally $\Lambda^{s-1/2}$. Now by using classical results concerning holomorphic (or harmonic) functions having boundary values with a certain degree of regularity, one gets the desired result. The same result can be obtained directly with the techniques of Section 6.

It is possible to reformulate part (b) of the Theorem C in a similar way: *the holomorphic function* $\mathbb{C}_+ \ni z \mapsto R(z) \in B(\mathscr{H}_{s,1}; \mathscr{H}_{-s,\infty})$ *extends to a function of class* $C^{s-1/2}$ *in the weak* topology on the set* $\mathbb{C}_+ \cup \mu^A(H)$; or, with the terminology introduced at the end of §2.6, *extends to a* w-$C^{s-1/2}$ *function on* $\mathbb{C}_+ \cup \mu^A(H)$.

The optimality of Theorem C can again be checked in the classical situation (see [BG3] for details). More precisely, if $f \in \mathscr{H}_{s,\infty}$ then the function $\phi(\lambda+i0) = \langle f, R(\lambda+i0)f\rangle$ is generally not of class $\Lambda^{s-1/2,p}$ with $p<\infty$, even if H is of class $C^\infty(A)$; and if H is of class $\mathscr{C}^{s+1/2}$, then $\phi(\lambda+i0)$ is generally not of class $\Lambda^{s-1/2,p}$ with $p<\infty$, even if $f \in \mathscr{H}_\infty$. In particular, if $s-1/2=k$ is an integer, then $\phi(\lambda+i0)$ is not of class $\mathrm{Lip}^{(k)}$, and so not of class C^k. Part (b) of the theorem gives, however, sufficient conditions for this to hold: it suffices to have $f \in \mathscr{H}_{s,1}$ and $H \in \mathscr{C}^{s+1/2,1}$. Moreover, these conditions are optimal in the following sense: if $f \in \mathscr{H}_{s,p}$ (or $H \in \mathscr{C}^{s+1/2,p}(A)$) for some $p>1$, then generally $\phi(\lambda+i0)$ is not of class $\mathrm{Lip}^{(k)}$, even if $H \in C^\infty(A)$ (resp. $f \in \mathscr{H}_\infty$).

We make a comment in connection with the relation (1.5). According to the discussion and the example mentioned after Theorem B, the composition of two or more operators $R(\lambda+i0)$ has not a direct meaning (since $R(\lambda+i0)\mathscr{H}_\infty$ is not even included in $\mathscr{H}^0_{-1/2,\infty}$ and $R(\lambda+i0)f$ has no sense for f in a space larger than $\mathscr{H}_{1/2,1}$). However, according to (1.5) one may interpret $\frac{1}{k!}(d/d\lambda)^k R(\lambda+i0)$ as a regularized version of the formal product $R(\lambda+i0)^{k+1}$.

Finally, we point out another version of part (a) of Theorem C (this follows from the description of the class $\Lambda^{s-1/2}$ in terms of differentiability properties, see §1.5). Let us write $s = 1/2+k+\sigma$ where $k \geq 0$ is an integer and $0<\sigma\leq 1$ is a real number. Recall that for $\operatorname{Im} z>0$ we have $(d/dz)^k R(z) = k! R(z)^{k+1}$. Then for each compact subset K of $\mu^A(H)$ there is a constant $C<\infty$ such that

(i) if $0<\sigma<1$, then for all z_1, z_2 with $\operatorname{Re} z_j \in K$ and $\operatorname{Im} z_j > 0$ one has

$$\|R(z_1)^{k+1} - R(z_2)^{k+1}\|_{\mathscr{H}_{s,\infty}\to\mathscr{H}_{-s,1}} \leq C|z_1-z_2|^\sigma;$$

(ii) if $\sigma=1$, then for all z, ε with $\operatorname{Re}(z+j\varepsilon) \in K$ if $j=0,1,2$ and $\operatorname{Im} z>0$, $\operatorname{Im}\varepsilon \geq 0$ one has

$$\|R(z+2\varepsilon)^{k+1} - 2R(z+\varepsilon)^{k+1} + R(z)^{k+1}\|_{\mathscr{H}_{s,\infty}\to\mathscr{H}_{-s,1}} \leq C|\varepsilon|.$$

In particular $\lim_{\mu\to+0} R(\lambda+i\mu)^n \equiv R^n(\lambda+i0)$ *exists in norm* in $B(\mathscr{H}_{s,\infty}; \mathscr{H}_{-s,1})$, locally uniformly in $\lambda \in \mu^A(H)$, for each $n=1,\cdots,k,k+1$; the map $\lambda \mapsto R(\lambda+i0) \in B(\mathscr{H}_{s,\infty}; \mathscr{H}_{-s,1})$ is of class C^k in norm and $(d/d\lambda)^n R(\lambda+i0) = n! R^{n+1}(\lambda+i0)$ if $n=0,1,\cdots,k$; the map $\lambda \mapsto R^{k+1}(\lambda+i0) \in B(\mathscr{H}_{s,\infty}; \mathscr{H}_{-s,1})$ is locally Hölder continuous of order σ on $\mu^A(H)$ if $0<\sigma<1$ and locally of class Λ^1 on $\mu^A(H)$ if $\sigma=1$.

1.8. We saw before (cf. (iv) after Theorem B) that generally we have $R(\lambda+i0)\mathscr{H}_\infty \not\subset \mathscr{H}^o_{-1/2,\infty}$. In other terms, vectors of the form $R(\lambda+i0)f$ do not decay at infinity in the spectral representation of A. On the other hand, the classical case suggests that there should be an asymmetry between the behaviour in the region where $A \to -\infty$ and that where $A \to +\infty$. This is clear from the second equality in (1.4) in the case $h(x) \equiv x$; alternatively, one may consider the case when $\mathscr{H} = L^2(\mathbb{R})$, $H = P \equiv -i(d/dx)$ and $A = Q$ (the operator of multiplication by x).

Let us denote by Π_+ and Π_- the spectral projections of the self-adjoint operator A associated to the intervals $[0,\infty)$ and $(-\infty, 0]$ respectively. Clearly $\Pi_\pm$ induce continuous linear operators in $\mathscr{H} = 8B_{t,q}$ for each $t \in \mathbb{R}$ and $q \in [1,\infty]$, so the products $\Pi_\pm R(\lambda+i0)$ are well defined elements of $B(\mathscr{H}_{1/2,1}; \mathscr{H}_{-1/2,\infty})$ for each $\lambda \in \mu^A(H)$ (under the hypotheses of Theorem A). We show that these operators have a much better behaviour than $R(\lambda \pm i0)$. We shall treat only the case of the operator $\Pi_- R(\lambda+i0)$, the other one being entirely similar.

Theorem D. *Let $\{R(z)\}$ be a resolvent family with a spectral gap and of class $\mathscr{C}^{s+1/2,p}(A)$ for some real $s > 1/2$ and some $p \in [1,\infty]$. Then one has $\Pi_- R(\lambda + i0)\mathscr{H}_{s,p} \subset \mathscr{H}_{s-1,p}$ for all $\lambda \in \mu^A(H)$, and the map $\lambda \mapsto \Pi_- R(\lambda + i0) \in B(\mathscr{H}_{s,p}; \mathscr{H}_{s-1,p})$ is* w*-continuous.*

Here the w-continuity corresponds to the w-topology, introduced at the end of §1.6; see also §2.6. Note that, as explained in the last part of §1.6, if $\operatorname{Im} z \neq 0$ then $R(z) \in B(\mathscr{H}_{s,p}) \subset B(\mathscr{H}_{s,p}; \mathscr{H}_{s-1,p})$ and depends holomorphically on z. By using standard facts concerning holomorphic (or rather harmonic) functions with continuous boundary values on a real interval, one gets the following version of Theorem D: *the holomorphic function $\mathbb{C}_+ \ni z \mapsto \Pi_- R(z) \in B(\mathscr{H}_{s,p}; \mathscr{H}_{s-1,p})$ extends to a* w*-continuous function on $\mathbb{C}_+ \cup \mu^A(H)$.* We also remark that we can replace $B(\mathscr{H}_{s,p}; \mathscr{H}_{s-1,p})$ by $B(\mathscr{H}_{t,p}; \mathscr{H}_{t-1,q})$ if $1/2 < t < s$ (then $1 \le q \le \infty$ is arbitrary), or $t = s$ but $q \ge p$. Indeed, this follows from the fact that under these conditions $\mathscr{C}^{s+1/2,p} \subset \mathscr{C}^{t+1/2,q}$.

We shall now prove the optimality of two assertions of Theorem D (the optimality of the regularity condition $H \in \mathscr{C}^{s+1/2,p}(A)$ is a consequence of the topics treated in §1.9):

(i) for each $s > 1/2$ and $1 \le p \le \infty$ the space $\mathscr{H}_{s-1,p}$ is the smallest on the scale $\{\mathscr{H}_{t,q}\}$ which contains $\Pi_- R(\lambda + i0)\mathscr{H}_{s,p}$;

(ii) the map $\lambda \mapsto \Pi_- R(\lambda + i0) \in B(\mathscr{H}_s; \mathscr{H}_{s-1})$ (so $p = 2$) is not norm continuous in general.

For this purpose we consider the following example: $\mathscr{H} = L^2(\mathbb{R})$, $H \equiv P$ =usual self-adjoint extension of $-i(d/dx)$ in $\mathscr{H}$, $A = Q$ =operator of multiplication by the independent variable x in $\mathscr{H}$. Note that $P \in C^\infty(Q)$ and $[P, iQ] = 1$ (the spectral gap condition is not satisfied, but we could take $H = h(P)$ with $h : \mathbb{R} \to \mathbb{R}$ a C^∞ function with $h'(x) > 0$ $(\forall x)$ and such that $h(x) = x$ on a neighbourhood of a compact interval K; then the resolvents $(P - \lambda - i\mu)^{-1}$ and $(h(P) - \lambda - i\mu)^{-1}$ have a similar behaviour as $\mu \to \pm 0$ if $\lambda \in K$). Then for all $\lambda \in \mathbb{R}$ one has $R(\lambda + i0) = ie^{i\lambda Q} I e^{-i\lambda Q}$, where $(If)(x) = \int_{-\infty}^{x} f(y)dy$. Now it is possible to show by elementary computations that $\Pi_- I \mathscr{H}_{s,p} \subset \mathscr{H}_{t,q}$ if and only if $\mathscr{H} = 8B_{s-1,p} \subset \mathscr{H} = 8B_{t,q}$.

We show that $\lambda \mapsto \Pi_- R(\lambda + i0) = ie^{i\lambda Q}\Pi_- I e^{-i\lambda Q} \in B(\mathscr{H}_s; \mathscr{H}_{s-1})$ is not norm continuous if $s > 1/2$. For this it suffices to prove the following fact: *let $\mathscr{I}$ be the bounded operator in $L^2(\mathbb{R}_-)$, where $\mathbb{R}_- = (-\infty, 0)$, defined by $(\mathscr{I} f)(x) = \int_{-\infty}^x f(y)y^{-1}dy$; then the map $\lambda \to e^{i\lambda Q}\mathscr{I} e^{-i\lambda Q} \in B(L^2(\mathbb{R}_-))$ is not norm continuous* (now Q is the operator of multiplication by x in $L^2(\mathbb{R}_-)$). Observe first that

$$(\mathscr{I} f)(x) = \int_1^\infty f(xt)t^{-1}dt = \int_0^\infty f(xe^\tau)d\tau = \int_0^\infty (e^{i\tau D}f)(x)e^{-\tau/2}d\tau,$$

where the generator D of the dilation group is the usual self-adjoint realization of $(PQ+QP)/2$ in $L^2(\mathbb{R}_-)$ and $(e^{i\tau D}f)(x) = e^{\tau/2}f(xe^\tau)$, so $\{e^{i\tau D}\}_{\tau\in\mathbb{R}}$ is the unitary dilation group in $L^2(\mathbb{R}_-)$. Then $\mathscr{I} = \int_0^\infty \exp(i\tau D - \tau/2)d\tau = i(D + i/2)^{-1}$, hence $\|\mathscr{I}\| = 2$. Observe that $e^{\pm i\lambda Q} = e^{i\varepsilon D}e^{\pm iQ}e^{-i\varepsilon D}$ if $\lambda > 0$ and $e^\varepsilon = \lambda$. Since $\mathscr{I}$ commutes with the dilation group we get $e^{i\lambda Q}\mathscr{I} e^{-i\lambda Q} - \mathscr{I} = e^{i\varepsilon D}[e^{iQ}\mathscr{I} e^{-iQ} - \mathscr{I}]e^{-i\varepsilon D}$, hence $\|e^{i\lambda Q}\mathscr{I} e^{-i\lambda Q} - \mathscr{I}\| = \|e^{iQ}\mathscr{I} e^{-iQ} - \mathscr{I}\| = \text{const.} \neq 0$ if $\lambda > 0$, which finishes the proof.

One may improve the continuity properties of the map $\lambda \mapsto \Pi_- R(\lambda+i0)$ if we consider it values with in a larger space and if H has better A-regularity properties.

Theorem E. *Let $\{R(z)\}$ be a resolvent family with a spectral gap and of class $\mathscr{C}^{s+1/2}(A)$ for some $s > 1/2$ and let α be a real number such that $0 < \alpha < s - 1/2$. Then the map $\lambda \mapsto \Pi_- R(\lambda + i0) \in B(\mathscr{H}_{s,\infty}; \mathscr{H}_{s-1-\alpha,1})$ is locally of class Λ^α on $\mu^A(H)$. If, moreover, α is an integer and $\{R(z)\}$ is of class $\mathscr{C}^{s+1/2,p}(A)$ for some $p \in [1,\infty]$, then $\lambda \mapsto \Pi_- R(\lambda + i0) \in B(\mathscr{H}_{s,p}; \mathscr{H}_{s-1-\alpha,p})$ is of class C^α in the* w*-topology.*

Let us consider the case $\mathscr{H} = L^2(\mathbb{R})$, $H = Q$ and $A = i(d/dx)$. Let $f \in \mathscr{H}^s(\mathbb{R})$ and $g \in \mathscr{H}^{1-s+\alpha}(\mathbb{R})$ be defined as follows: the Fourier transform $\hat{f}$ (resp. $\hat{g}$) of f (resp. g) is equal to zero if $x \leq 1$ and is equal to $x^{-s-1/2-\varepsilon}$ (resp. $x^{s-1-\alpha-1/2-\varepsilon}$) if $x > 1$. Here ε is a strictly positive number such that $\alpha + \varepsilon + 1/2 < s$. Then by a straightforward calculation (see [S1]) we have $\int_{|x|>r} \hat{u}(x)dx \sim r^{-\alpha-2\varepsilon}$ as $r \to \infty$, where $u = \overline{g}f$. On the other hand, we have (see (1.4)):

$$\langle \Pi_- g, (H - \lambda - i0)^{-1}f \rangle = 2i \int_0^\infty e^{i\lambda x}\hat{u}(x)dx.$$

Since $\hat{u}$ is a positive function one may use a wellknown result of Boas and deduce that $\lambda \mapsto \langle \Pi_- g, (H - \lambda - i0)^{-1}f \rangle$ is precisely of class $\Lambda^{\alpha+2\varepsilon}$ (not more !). We thus see that the map $\lambda \mapsto \Pi_- R(\lambda + i0) \in B(\mathscr{H}_s; \mathscr{H}_{s-1-\alpha})$ cannot be of class Λ^β for some $\beta > \alpha$, and this even if H is of class $C^\infty(A)$. The optimality of the condition $H \in \mathscr{C}^{s+1/2}(A)$ will be discussed later on (cf. end of §1.9).

1.9. We shall now mention two simple but useful consequences of Theorems C and E (see [BGSh] for other result of this nature). Let H be a regular self-adjoint operator and $\varphi \in C_0^\infty(\mu^A(H))$. Then $\varphi(H)$ leaves invariant the space $\mathscr{H}_{1/2,1}$ (because $\varphi(H)$ is of class $\mathscr{C}^{1,1}(A)$) hence, if H has a spectral gap, we may use Theorem B in order to obtain for all f, $g \in \mathscr{H}_{1/2,1}$ and all real $t>0$:

$$\langle g, e^{-iHt}\varphi(H)f\rangle = (2\pi i)^{-1}\int_{\mathbb{R}} e^{-it\lambda}\langle g, R(\lambda+i0)\varphi(H)f\rangle d\lambda.$$

Now we use the following elementary fact: if the function $u:\mathbb{R}^n\to\mathbb{C}$ belongs to the Besov space $B_1^{\alpha,\infty}(\mathbb{R}^n)$ for some real $\alpha>0$, then its Fourier transform $\hat{u}$ is a continuous function such that $|\hat{u}(t)|\le \text{const.}\langle t\rangle^{-\alpha}$, where $\langle t\rangle = (1+|t|^2)^{1/2}$. So the Theorems C and E and the uniform boundedness principle give:

Theorem F. *Let H be a self-adjoint operator with a spectral gap and of class $\mathscr{C}^{s+1/2}(A)$ for some real $s>1/2$. Let α be a real number such that $0<\alpha<s-1/2$ and let $\varphi\in C_0^\infty(\mu^A(H))$. Then there is a constant $C<\infty$ such that*

$$\|e^{-itH}\varphi(H)\|_{\mathscr{H}_{s,\infty}\to\mathscr{H}_{-s,1}} \le C\langle t\rangle^{-(s-1/2)}, \forall t\in\mathbb{R}; \tag{1.6}$$

$$\|\Pi_- e^{-itH}\varphi(H)\|_{\mathscr{H}_{s,\infty}\to\mathscr{H}_{s-1-\alpha,1}} \le C\langle t\rangle^{-\alpha}\ \forall t\ge 0. \tag{1.7}$$

The estimate (1.6) is a *local decay* (or *dispersion*) *property* of H relatively to A, while (1.7) expresses a *propagation property*. It is possible to deduce from (1.6) and (1.7) new interesting estimates by interpolation (see [BGSh]). In particular, if H is of class $C^\infty(A)$ then for all real $s>\sigma>0$ and each $\varepsilon>0$ there is a constant $C<\infty$ such that

$$\|e^{-itH}\varphi(H)\|_{\mathscr{H}_s\to\mathscr{H}_{-s}} \le C\langle t\rangle^{-s+\varepsilon}, \forall t\in\mathbb{R};$$

$$\|\Pi_- e^{-itH}\varphi(H)\|_{\mathscr{H}_s\to\mathscr{H}_{s-\sigma}} \le C\langle t\rangle^{-\sigma+\varepsilon}, \forall t\ge 0.$$

We see that *the rates of decay (1.6) and (1.7) cannot be optimal in all parameters*: if H is of higher regularity class then they can be significantly improved (see the last section in [BGSh] for an optimal result in the case of "homogeneous" hamiltonians).

However, *Theorem F is optimal: the rates of decay connot be improved if H is only of class $\mathscr{C}^{s+1/2}(A)$.* To see this, we again consider the classical case of §1.2: $H=h(Q)$, $A=i(d/dx)$ and $\mathscr{H}=L^2(\mathbb{R})$. Then

$$\langle g, e^{-itH}\varphi(H)f\rangle = \int e^{-it\lambda}\varphi(\lambda)\overline{g(k(\lambda))}f(k(\lambda))k'(\lambda)d\lambda,$$

where $k=h^{-1}$ is the reciprocal function of h. Let $\alpha\in(0,1)$ and construct h such that the derivative of its reciprocal function k be equal to a constant plus the Weierstrass function $\sum_{n=1}^\infty 2^{-n\alpha}\cos 2^n x$ if $|x|\le 1$. Then $h\in\Lambda^{\alpha+1}$ and it is possible to find $f\in C_0^\infty(\mathbb{R})$ and $\varphi\in C_0^\infty(\mathbb{R})$ such that $\limsup_{t\to\infty} t^\alpha|\langle f, e^{-itH}\varphi(H)f\rangle|>0$. Moreover, one can construct $f\in C_0^\infty(\mathbb{R})$, $g\in\mathscr{S}(\mathbb{R})$ (with $\operatorname{supp}\hat{g}$ a compact in $[0,\infty)$, hence $\Pi_- g=g$) and $\varphi\in C_0^\infty(\mathbb{R})$ such that $\limsup_{t\to\infty} t^\alpha|\langle \Pi_- g, e^{-itH}\varphi(H)f\rangle|>0$. The details can be found in [S1].

Incidentaly, this proves (in the range $0<\alpha<1$) the optimality of the regularity class $\mathscr{C}^{s+1/2}(A)$ in Theorem E ($s+1/2=\alpha+1$ with the preceding notations). Indeed, although the elements f, g constructed above belong to $\mathscr{S}(\mathbb{R})$, hence to $\mathscr{H}_\infty$, the map $\lambda\mapsto\langle\Pi_- g, R(\lambda+i0)\varphi(H)f\rangle$ is of class Λ^α

and not better (otherwise its Fourier transform would decay more rapidly than $\langle t\rangle^{-\alpha}$). Note that $\varphi(H)f \in \mathscr{H}_{\alpha+1,\infty}$.

1.10. The operator $\Pi_- R(\lambda+i0)\Pi_+$ has still better properties than $\Pi_- R(\lambda+i0)$, for example it can be defined on spaces $\mathscr{H}_{s,p}$ with $s < 1/2$ (note that $\Pi_- R(\lambda + i0)f$ generally has no meaning if $\lambda \in \mu^A(H)$ is a spectral value of H and $f \notin \mathscr{H}_{1/2,1}$). We state here only one result in this direction (see Theorems 6.5 and 6.8 for other informations).

Theorem G. *Let $\{R(z)\}$ be a spectral family with a spectral gap and of class $\mathscr{C}^{2+t-s}(A)$, where s, t are real numbers such that $s < 1/2$, $t > -1/2$. Let τ be a real number such that $0 < \tau < t + 1/2$. Then $\Pi_- R(\lambda + i0)\Pi_+$ has a unique* w-*continuous extension to an operator $\mathscr{H}_{s,\infty} \to \mathscr{H}_{t-\tau,1}$ and the map $\lambda \mapsto \Pi_- R(\lambda + i0)\Pi_+ \in B(\mathscr{H}_{s,\infty}; \mathscr{H}_{t-\tau,1})$ is locally of class Λ^τ on $\mu^A(H)$.*

This is a particular case of Theorem 6.8 (case (iii)). We require $\Pi_- R(\lambda + i0)\Pi_+$ to be w-continuous because $\mathscr{H}_{1/2,1}$ is not norm dense in $\mathscr{H}_{s,\infty}$.

2. Besov Classes of Operators

2.1. Throughout this paper we consider a fixed pair $(\mathscr{H}, A)$ consisting of a (complex) Hilbert space $\mathscr{H}$ and a (densely defined) self-adjoint operator A in $\mathscr{H}$. We shall identify $\mathscr{H}$ with its adjoint space $\mathscr{H}^*$ (space of anti-linear continuous functionals on $\mathscr{H}$) with the help of the Riesz isomorphism and we shall denote by $\|\cdot\|$ the norm in $\mathscr{H}$ and in $B(\mathscr{H})$; the scalar product $\langle\cdot,\cdot\rangle$ in $\mathscr{H}$ is linear in the second variable. The operator A is assumed unbounded (otherwise most of our statements are trivial) and we denote by E_A its spectral measure. Furthermore, we set $\langle A\rangle = (1 + A^2)^{1/2}$ and for $s \in \mathbb{R}$, $f \in \mathscr{H}$:

$$\|f\|_s = \|\langle A\rangle^s f\| = \Big[\int_{\mathbb{R}} (1 + \lambda^2)^s \|E_A(d\lambda)f\|^2\Big]^{1/2}. \tag{2.1}$$

If $s > 0$ then one may have $\|f\|_s = \infty$. Now we define the Sobolev scale associated to A as the family $\{\mathscr{H}_s\}_{s\in\mathbb{R}}$ of Hilbert spaces given by the following rules:

(i) if $s \geq 0$ then $\mathscr{H}_s$ is the vector subspace of $\mathscr{H}$ consisting of the vectors f such that $\|f\|_s < \infty$;

(ii) if $s < 0$ then $\mathscr{H}_s$ is the completion of $\mathscr{H}$ for the norm $\|\cdot\|_s$.

In both cases $\mathscr{H}_s$ is equipped with the (Hilbert space) norm naturally induced by (2.1). Note that $\mathscr{H}_0 = \mathscr{H}$.

We recall that the Sobolev scale is totally ordered, more precisely for $s \leq t$ we have a canonical norm-decreasing and dense embedding $\mathscr{H}_t \subset \mathscr{H}_s$. Moreover, the identification $\mathscr{H}^* = \mathscr{H}$ induces a canonical identification $(\mathscr{H}_s)^* = \mathscr{H}_{-s}$ for all $s \in \mathbb{R}$. Indeed, $|\langle f, g\rangle| \leq \|f\|_s\|g\|_{-s}$ holds for f, $g \in \mathscr{H}$ and $s \in \mathbb{R}$, hence the scalar product of $\mathscr{H}$ extends to a continuous sesquilinear form on $\mathscr{H}_s \times \mathscr{H}_{-s}$ which will be denoted by the same symbol $\langle\cdot,\cdot\rangle$ and which provides us with the desired identification of the adjoint space of $\mathscr{H}_s$ with $\mathscr{H}_{-s}$. Hence for $0 < s < t$ we have norm-decreasing dense embeddings

$$\mathscr{H}_t \subset \mathscr{H}_s \subset \mathscr{H} \subset \mathscr{H}_s^* = \mathscr{H}_{-s} \subset \mathscr{H}_t^* = \mathscr{H}_{-t} \tag{2.2}$$

which induce norm-decreasing (non-dense) embeddings

$$B(\mathscr{H}) \subset B(\mathscr{H}_s;\mathscr{H}_{-s}) \subset B(\mathscr{H}_t;\mathscr{H}_{-t}). \tag{2.3}$$

Note that there is a canonical identification of $B(\mathscr{H}_s;\mathscr{H}_{-s})$ with the space $B(\mathscr{H}_s;\mathscr{H}_s^*)$ of continuous sesquilinear forms on $\mathscr{H}_s$. An operator $S \in B(\mathscr{H}_s;\mathscr{H}_{-s})$ with $s > 0$ belongs to $B(\mathscr{H})$ if and only if the sesquilinear form on $\mathscr{H}_s$ associated to it is continuous for the topology induced by $\mathscr{H}$ on $\mathscr{H}_s$. In such a case we say that S extends to a bounded operator in $\mathscr{H}$.

We set $\mathscr{H}_\infty = \cap_{s\in\mathbb{R}}\mathscr{H}_s$ and $\mathscr{H}_{-\infty} = \cup_{s\in\mathbb{R}}\mathscr{H}_s$ and we extend $\|\cdot\|_s$ to all of $\mathscr{H}_{-\infty}$ by setting $\|\cdot\|_s = \infty$ if $f \notin \mathscr{H}_s$. The space $\mathscr{H}_\infty$ has a natural Fréchet space topology and there is a canonical identification of its adjoint space with $\mathscr{H}_{-\infty}$; we shall equip $\mathscr{H}_{-\infty}$ with the strong adjoint topology. Notice that if $\varphi : \mathbb{R} \to \mathbb{C}$ is a Borel function such that $|\varphi(x)| \leq c\langle x\rangle^\sigma$ for some $c > 0$ and $\sigma \in \mathbb{R}$ then $\varphi(A)$ has a unique extension to a continuous operator $\varphi(A) : \mathscr{H}_{-\infty} \to \mathscr{H}_{-\infty}$ and this operator sends each $\mathscr{H}_s$ continuously into $\mathscr{H}_{s-\sigma}$.

2.2. If $S \in B(\mathscr{H}_s;\mathscr{H}_{-s})$ and $k \geq 0$ is an integer then we define $\operatorname{ad}_A^k S \in B(\mathscr{H}_{s+k};\mathscr{H}_{-s-k})$ by induction: $\operatorname{ad}_A^0 S = S$, $\operatorname{ad}_A S \equiv \operatorname{ad}_A^1 S \equiv [A,S] = AS - SA$, and $\operatorname{ad}_A^{k+1} S = \operatorname{ad}_A(\operatorname{ad}_A^k S)$. The following formula holds

$$\operatorname{ad}_A^k S = \sum_{i+j=k} \frac{k!}{i!j!}(-1)^j A^i S A^j. \tag{2.4}$$

If $S \in B(\mathscr{H})$ then it is more convenient to interpret this formula in terms of sesquilinear forms, namely $\operatorname{ad}_A^k S$ is the continuous sesquilinear form on $\mathscr{H}_k (\equiv D(A^k)$ domain of A^k in $\mathscr{H})$ given for $f,\, g \in D(A^k)$ by

$$\langle f, (\operatorname{ad}_A^k S)g\rangle = \sum_{i+j=k} \frac{k!(-1)^j}{i!j!}\langle A^i f, S A^j g\rangle.$$

The following observation is useful. *Let $S \in B(\mathscr{H})$ such that for some integer $m \geq 2$ the sesquilinear form $\operatorname{ad}_A^m S$ on $D(A^m)$ is continuous for the topology induced by $\mathscr{H}$ on $D(A^m)$. Then for each integer $k \in [0,m]$ the sesquilinear form $\operatorname{ad}_A^k S$ is continuous for the topology induced by $\mathscr{H}$ on $D(A^k)$.* For the proof note that if $f,\, g \in D(A^m)$ and $W_x = e^{iAx}$ then $x \mapsto \langle W_x f, S W_x g\rangle$ is a function of class C^m on $\mathbb{R}$. Let $\psi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\psi \subset [0,1]$, $\int x^k \psi(x)dx = i^k k!$ and $\int x^j\psi(x)dx = 0$ if $0 \leq j < m$, $j \neq k$. Then by using a Taylor expansion of order m we get

$$\begin{aligned}\langle W_x f, S W_x g\rangle = &\sum_{0\leq j<m} \frac{(-ix)^j}{j!}\langle f, (\operatorname{ad}_A^j S)g\rangle \\ &+ \frac{(-ix)^m}{(m-1)!}\int_0^1 \langle W_{\tau x} f, (\operatorname{ad}_A^m S) W_{\tau x} g\rangle (1-\tau)^{m-1} d\tau.\end{aligned}$$

If we multiply this by $\psi(x)$ and we integrate with respect to x we obtain

$$\begin{aligned}\langle f, (\operatorname{ad}_A^k S)g\rangle = &\int_{\mathbb{R}} \Big[\langle W_x f, S W_x g\rangle \\ &+ \frac{(ix)^m}{m!}\int_0^1 \langle W_{\tau x} f, (\operatorname{ad}_A^m S) W_{\tau x} g\rangle d(\tau-1)^m\Big]\psi(x)dx\end{aligned}$$

which clearly implies the assertion that we want to prove.

2.3. For real τ we set $E_A(\tau) = E_A([-2\tau, -\tau] \cup [\tau, 2\tau])$, in particular $E_A(0)^\perp = 1 - E_A(\{0\})$ is the orthogonal projection onto the orthogonal complement of the kernel of A. Then we have the following version of the (continuous) Littlewood-Paley decomposition (or Calderón identity):

$$\begin{aligned}
(2.5) \qquad E_A(0)^\perp &= \frac{1}{\log 2} \int_{\to 0}^{\to\infty} E_A(\tau)\tau^{-1} d\tau \\
&\equiv \operatorname*{s-lim}_{a\to+0, b\to+\infty} \frac{1}{\log 2} \int_a^b E_A(\tau)\tau^{-1} d\tau
\end{aligned}$$

For the proof of this identity it suffices to insert in the last integral the expression $E_A(\tau) = \chi(A/\tau) = \int \chi(\lambda/\tau) E_A(d\lambda)$, where χ is the characteristic function of the set $[-2, -1] \cup [1, 2]$, to invert the order of integration, and to take the limit with the help of the dominated convergence theorem.

It is now natural to introduce, for each $s \in \mathbb{R}$ and $p \in [1, \infty]$, the (non-homogeneous) *Besov gauge* $\|\cdot\|_{s,p} : \mathscr{H}_{-\infty} \to [0, \infty]$ by the following formula:

$$(2.6) \qquad \|f\|_{s,p} \equiv \|E_A([-2, 2])f\| + \Big[\int_1^\infty \|\tau^s E_A(\tau)f\|^p \tau^{-1} d\tau\Big]^{1/p}$$

If $p = \infty$ then the second term on the r.h.s. must be read $\sup_{\tau\geq 1} \|\tau^s E_A(\tau)f\|$. It is easily shown that the gauges $\|\cdot\|_s$ and $\|\cdot\|_{s,2}$ are equivalent on $\mathscr{H}_{-\infty}$.

An important role in our arguments is played by the following result.

Theorem. *Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a locally bounded Borel function and α a real number, and assume that there are real numbers $\nu > 0$ and c such that $|\varphi(x)| \leq c|x|^\alpha \min(|x|^\nu, |x|^{-\nu})$ for all $x \in \mathbb{R}$. Then there is a constant $C < \infty$ such that for all $s \in \mathbb{R}$, all $p \in [1, \infty]$ and all $f \in \mathscr{H}_{-\infty}$:*

$$(2.7) \qquad \Big[\int_0^1 \|\varepsilon^{-\alpha}\varphi(\varepsilon A)f\|_{s,1}^p \varepsilon^{-1} d\varepsilon\Big]^{1/p} \leq C\|f\|_{s+\alpha,p}.$$

In particular $\|\varphi(\varepsilon A)f\|_{s,1} \leq C\varepsilon^\alpha \|f\|_{s+\alpha,\infty}$ for all $\varepsilon \in (0, 1)$ and $f \in \mathscr{H}_{-\infty}$.

Proof. (i) Set $\rho(u) = \min(u^\nu, u^{-\nu})$ for $u > 0$ and observe that for $0 < a < x < 2a$ we have $\rho(x) \leq 2^\nu \rho(a)$. We show that for all $\varepsilon > 0$, $\tau > 0$ and $f \in \mathscr{H}_{-\infty}$ one has

$$(2.8) \qquad \|E_A(\tau)\varphi(\varepsilon A)f\| \leq c2^\nu \max(1, 2^\alpha)(\varepsilon\tau)^\alpha \rho(\varepsilon\tau)\|E_A(\tau)f\|$$

Indeed, we have by hypothesis $|\varphi(x)| \leq c|x|^\alpha \rho(|x|)$ and, if we denote by χ the characteristic function of the set $[-2, -1] \cup [1, 2]$, then the l.h.s. of (2.8) can be estimated as follows

$$\begin{aligned}
\|\chi(A/\tau)\varphi(\varepsilon A)f\| &\leq c\|\chi(A/\tau)|\varepsilon A|^\alpha \rho(\varepsilon|A|)f\| \\
&\leq c\max(1, 2^\alpha)(\varepsilon\tau)^\alpha \|\chi(A/\tau)\rho(\varepsilon|A|)f\| \\
&\leq c\max(1, 2^\alpha)(\varepsilon\tau)^\alpha \sup_{\tau<x<2\tau} \rho(\varepsilon x)\|\chi(A/\tau)f\|
\end{aligned}$$

which clearly implies (2.8).

(ii) Let us set $C_1 = c2^\nu \max(1, 2^\alpha)$. Then from (2.6) and (2.8) we get

$$\|\varepsilon^{-\alpha}\varphi(\varepsilon A)f\|_{s,1} =$$
$$\|E_A([-2,2])\varepsilon^{-\alpha}\varphi(\varepsilon A)f\| + \textstyle\int_1^\infty \|\tau^s E_A(\tau)\varepsilon^{-\alpha}\varphi(\varepsilon A)f\|\frac{d\tau}{\tau} \le$$
$$\|E_A([-2,2])\varepsilon^{-\alpha}\varphi(\varepsilon A)f\| + C_1 \textstyle\int_1^\infty \rho(\varepsilon\tau)\|\tau^{s+\alpha}E_A(\tau)f\|\frac{d\tau}{\tau} =$$
$$\|E_A([-2,2])\varepsilon^{-\alpha}\varphi(\varepsilon A)f\| + C_1 \textstyle\int_0^\infty \rho(\varepsilon\tau)\|\chi_1(\tau/\varepsilon)(\tau/\varepsilon)^{s+\alpha}E_A(\tau/\varepsilon)f\|\frac{d\tau}{\tau}$$

where χ_1 is the characteristic function of the interval $(1,\infty)$. Now observe that if g is a positive Borel function on $(0,\infty)$ then for $\tau \ge 1$ we have

$$\Big[\int_0^1 g(\tau/\varepsilon)^p\varepsilon^{-1}d\varepsilon\Big]^{1/p} = \Big[\int_\tau^\infty g(\sigma)^p\sigma^{-1}d\sigma\Big]^{1/p} \le \Big[\int_1^\infty g(\sigma)^p\sigma^{-1}d\sigma\Big]^{1/p}.$$

Since $\int_0^\infty \rho(\tau)\tau^{-1}d\tau = 2/\nu$ we see that the l.h.s. of (2.7) is smaller than

$$\Big[\int_0^1 \|E_A([-2,2])\varepsilon^{-\alpha}\varphi(\varepsilon A)f\|^p\frac{d\varepsilon}{\varepsilon}\Big]^{1/p} + \frac{2C_1}{\nu}\Big[\int_1^\infty \|\sigma^{s+\alpha}E_A(\sigma)f\|^p\frac{d\sigma}{\sigma}\Big]^{1/p}.$$

If $\alpha < 0$ then our hypotheses imply that φ is a bounded function and , by using the inequality $(1-\alpha p)^{-1/p} \le 1$, we see that the first term above is bounded by $\sup|\varphi| \cdot \|E_A([-2,2])f\|$. If $\alpha \ge 0$ we use the inequality $|\varphi(x)| \le c|x|^{\alpha+\nu}$ and we bound this term by

$$c2^{\alpha+\nu}\Big[\int_0^1 \|\varepsilon^\nu E_A([-2,2])f\|^p\varepsilon^{-1}d\varepsilon\Big]^{1/p} \le c2^{\alpha+\nu}(1+\nu p)^{-1/p}\|E_A([-2,2])f\|.$$

$\square$

. We mention that if φ is a more regular function (e.g. $\varphi \in C^\infty_{\mathrm{pol}}(\mathbb{R})$) then, by using the techniques of Section 3.5 from [ABG2], one can generalize the preceding theorem to the case where the unitary group $W_\sigma = e^{iA\sigma}$ is replaced by a strongly continuous polynomially bounded one-parameter group on an arbitrary Banach space.

2.4. We now define the Besov scale associated to A: this is the family of Banach spaces $\mathscr{H}_{s,p}$ defined, for $s \in \mathbb{R}$ and $p \in [1,\infty]$, by $\mathscr{H}_{s,p} = \{f \in \mathscr{H}_{-\infty} \mid \|f\|_{s,p} < \infty\}$ equipped with the norm obtained by restricting $\|\cdot\|_{s,p}$ to it. Note that $\mathscr{H}_{s,2} = \mathscr{H}_s$ as topological vector spaces.

An elementary treatment of the Besov scale may be found in Appendix A of [BGM]. For a deeper study of a more general situation one may consult Chapter 3 of [ABG2]. We shall summarize here only the most important facts. Note that all the non-trivial assertions are easy consequences of the Theorem from §2.3.

We introduce a total order relation on the set $\mathbb{R} \times [1,\infty]$ by the following rule: $(s,p) > (t,q)$ means either that $s > t$ (then p, q are arbitrary) or that $s = t$ but $p < q$. Then $\mathscr{H}_{s,p} \subset \mathscr{H}_{t,q}$ strictly if and only if $(s,p) > (t,q)$. In this case the embedding is continuous; the embedding is dense if and only if $q \neq \infty$. Moreover, $\mathscr{H}_\infty$ is a dense subspace of $\mathscr{H}_{t,q}$ if and only if $q \neq \infty$. We denote by $\mathscr{H}^o_{t,\infty}$ the closure of $\mathscr{H}_\infty$ in $\mathscr{H}_{t,\infty}$.

Recall that $\mathscr{H}^*_\infty = \mathscr{H}_{-\infty}$. If $p \neq \infty$ then we have a continuous dense embedding $\mathscr{H}_\infty \subset \mathscr{H}_{s,p}$, hence we get a canonical embedding $(\mathscr{H}_{s,p})^* \subset \mathscr{H}_{-\infty}$. One can show that

$$[\mathscr{H}_{s,p}]^* = \mathscr{H}_{-s,p'} \text{ if } 1 \le p < \infty \text{ and } p^{-1} + p'^{-1} = 1. \tag{2.9}$$

The adjoint of the space $\mathscr{H}_{s,\infty}$ is not a Besov space (in fact it can not be realized as a subspace of $\mathscr{H}_{-\infty}$). However we have $\mathscr{H}_\infty \subset \mathscr{H}^o_{s,\infty}$ continuously and densely, so the space adjoint to $\mathscr{H}^o_{s,\infty}$ can be realized as a subspace of $\mathscr{H}_{-\infty}$, and in fact one can show that

$$[\mathscr{H}^o_{s,\infty}]^* = \mathscr{H}_{-s,1}. \tag{2.10}$$

The Besov scale is stable under real interpolation. More precisely, let s, $t \in \mathbb{R}$ and p, $q \in [1,\infty]$. Then for each $\theta \in (0,1)$ and $r \in [1,\infty]$ we have

$$(\mathscr{H}_{s,p}, \mathscr{H}_{t,q})_{\theta,r} = \mathscr{H}_{(1-\theta)s+\theta t,r} \tag{2.11}$$

as topological vector spaces.

2.5. The theorem proved in §2.3 allows us to give other descriptions of the Besov spaces $\mathscr{H}_{s,p}$. Let $\psi : \mathbb{R} \to \mathbb{C}$ be a locally bounded Borel function and assume that there are numbers $b > a > 0$ and $c > 0$ such that $|\psi(x)| \geq c^{-1}$ on $[a,b]$. Let n be the first integer such that $(b/a)^{n+1} \geq 2$. Then for $1 \leq x \leq 2$ we have $\sum_{0\leq k\leq n} c|\psi(a^{k+1}b^{-k}x)| \geq 1$ hence, if we denote by χ_{12} the characteristic function of the interval $[1,2]$, then we have

$$\Big[\int_1^\infty \|\tau^s\chi_{12}(A/\tau)f\|^p\frac{d\tau}{\tau}\Big]^{1/p} \leq \sum_{k=0}^n \frac{ca^{(k+1)s}}{b^{ks}}\Big[\int_0^{a^{k+1}b^{-k}} \|\varepsilon^{-s}\psi(\varepsilon A)f\|^p\frac{d\varepsilon}{\varepsilon}\Big]^{1/p}.$$

If $|\psi|$ is also bounded from below by a strictly positive constant on an interval $\subset (-\infty,0)$ then we shall have a similar estimate but with $\chi_{12}(A/\tau)$ replaced by $\chi_{12}(-A/\tau)$. This will give us an upper bound for $\|f\|_{s,p}$ in terms of the function ψ. In order to get a lower bound we use (2.7) with $s=0$ and take into account that $\|g\| \leq c_1\|g\|_{0,2} \leq c_2\|g\|_{0,1}$. We finally obtain the following result. *Let $\psi : \mathbb{R} \to \mathbb{C}$ be a locally bounded Borel function such that $|\psi(x)| \geq const. > 0$ for $x \in J$, where J is an open set with $J \cap (-\infty,0) \neq \emptyset$ and $J \cap (0,\infty) \neq \emptyset$; let $s \in \mathbb{R}$ and $p \in [1,\infty]$ and assume that $|\psi(x)| \leq c|x|^s \cdot \min(|x|^\nu, |x|^{-\nu})$ for some constants $c, \nu > 0$. Then there is a constant $C > 0$ such that for all $f \in \mathscr{H}_{-\infty}$:*

(2.12)
$$C^{-1}\|f\|_{s,p} \leq \|E_A([-2,2])f\| + [\int_0^1 \|\varepsilon^{-s}\psi(\varepsilon A)f\|^p\varepsilon^{-1}d\varepsilon]^{1/p} \leq C\|f\|_{s,p}.$$

We mention the following possible choices:

(i) if $s < 0$ then one may take ψ equal to the characteristic function of the interval $[-1,1]$;

(ii) if $s > 0$ then ψ can be chosen as the charcteristic function of the set $(-\infty,-1] \cup [1,\infty)$;

(iii) if $s > 0$ and if m is an integer strictly larger than s, then one may take $\psi(x) = [x(x+i)^{-1}]^m$.

A more interesting example in the case $s > 0$ is obtained by choosing $\psi(x) = (e^{ix}-1)^m$ with $m > s$ integer. Let us set $W_\sigma = e^{iA\sigma}$ for $\sigma \in \mathbb{R}$. Now let s be a strictly positive real number, $m > s$ an integer, and $p \in [1,\infty]$. Then there is a constant $C > 0$ such that

$$C^{-1}\|f\|_{s,p} \leq \|f\| + [\int_0^1 \|\varepsilon^{-s}(W_\varepsilon - 1)^m f\|^p\varepsilon^{-1}d\varepsilon]^{1/p} \leq C\|f\|_{s,p}. \tag{2.13}$$

Note the following difference between the description of $\mathscr{H}_{s,p}$ given by the gauge (2.6) and that associated to the gauge which appears in the middle term of (2.13) (for $s > 0$): the first one gives a characterization of the property $f \in \mathscr{H}_{s,p}$ in terms of the behaviour of f at infinity in a spectral representation of A, while the second one describes the property $f \in \mathscr{H}_{s,p}$ in terms of local regularity conditions on the function $\mathbb{R} \ni \sigma \mapsto W_\sigma f \in \mathscr{H}$.

Finally, we explain how one may obtain the description of $\mathscr{H}_{s,p}$ in terms of moduli of continuity of the function $\sigma \mapsto W_\sigma f$ (see Section 1). If $f \in \mathscr{H}$ and $m \geq 1$ is an integer we set $\omega_m(\varepsilon) = \sup_{|\sigma| \leq \varepsilon} \|(W_\sigma - 1)^m f\|$ and $\omega(\varepsilon) \equiv \omega_1(\varepsilon)$. Let $s \in (0, m)$ and $p \in [1, \infty]$. Then the function $\sigma \mapsto W_\sigma f \in \mathscr{H}$ is of class $\Lambda^{s,p}$ if and only if $\left[\int_0^1 [\varepsilon^{-s}\omega_m(\varepsilon)]^p \varepsilon^{-1} d\varepsilon\right]^{1/p} < \infty$. It is clear that this implies $f \in \mathscr{H}_{s,p}$ (see (2.13)). The reciprocal assertion is not so obvious: we shall prove it here in the case $m = 1$ (for the general case see the remarks after the proof of Theorem 3.4.6 in [ABG2]). Observe that if σ, ε are non-zero real numbers then

$$\begin{aligned} e^{iA\sigma} - 1 &= (e^{iA\sigma} - 1)\frac{e^{iA\varepsilon} - 1}{iA\varepsilon} + (e^{iA\sigma} - 1)[1 - \frac{e^{iA\varepsilon} - 1}{iA\varepsilon}] \\ &= \frac{\sigma}{\varepsilon}\frac{e^{iA\sigma}}{iA\sigma}(e^{iA\varepsilon} - 1) + (e^{iA\sigma} - 1)\int_0^1 [1 - e^{iA\varepsilon\tau}]d\tau. \end{aligned}$$

This clearly gives for $f \in \mathscr{H}$:

$$\|(e^{iA\sigma} - 1)\| \leq |\sigma/\varepsilon| \cdot \|(e^{iA\varepsilon} - 1)f\| + 2\int_0^1 \|(e^{iA\varepsilon\tau} - 1)f\| d\tau.$$

In particular we have

$$\omega(\varepsilon) \leq \|(W_\varepsilon - 1)f\| + 2\int_0^1 \|(W_{\varepsilon\tau} - 1)f\| d\tau.$$

It is now straightforward to show that $\left[\int_0^1 [\varepsilon^{-s}\omega(\varepsilon)]^p \varepsilon^{-1} d\varepsilon\right]^{1/p} < \infty$ if $f \in \mathscr{H}_{s,p}$ (with $0 < s < 1$) .

We make one more remark concerning the case $s = k \geq 1$ integer and $p = 2$. We know that $\mathscr{H}_{k,2} = \mathscr{H}_k = D(A^k)$. The preceding assertion gives $\int_0^1 [\varepsilon^{-k}\omega_m(\varepsilon)]^2 \varepsilon^{-1} d\varepsilon < \infty$ as a necessary and sufficient condition in order to have $f \in \mathscr{H}_k$, where $m > k$ is an integer. A new description of $\mathscr{H}_k$ can be obtained in terms of the modulus of continuity ω_k of order k by taking into account the identity

$$\|\varepsilon^{-k}(W_\varepsilon - 1)^k f\|^2 = \int_{\mathbb{R}} \left|\frac{e^{i\varepsilon\lambda} - 1}{\varepsilon}\right|^{2k} \|E_A(d\lambda)f\|^2.$$

By using Fatou lemma we see that $f \in \mathscr{H}_k$ if and only if $f \in \mathscr{H}$ and $\liminf_{\varepsilon \to 0} \|\varepsilon^{-k}(W_\varepsilon - 1)^k f\| < \infty$, and the second condition is in fact equivalent to $\omega_k(\varepsilon) \leq c\varepsilon^k$ and also to the aparently much stronger condition that the function $\sigma \mapsto W_\sigma f \in \mathscr{H}$ be strongly of class C^k.

2.6. Besides the norm topology it will be convenient to consider on $\mathscr{H}_{s,p}$ the topology defined by the family of seminorms $f \mapsto |\langle f, g\rangle|$ where $g \in \mathscr{H}_{-s,p'}$; we shall call it w-*topology.* If $1 < p < \infty$ then $\mathscr{H}_{s,p}$ is a reflexive space and the w-topology is just the weak topology. If $p = 1$ then the adjoint of $\mathscr{H}_{s,1}$ is $\mathscr{H}_{-s,\infty}$, so the w-topology is again just the weak topology of $\mathscr{H}_{s,1}$. Note, however, that $\mathscr{H}_{s,1}$ is also the adjoint of $\mathscr{H}^0_{-s,\infty}$ (see (2.10)) so we could consider on it the associated weak* topology; but we are not going to do it. Finally, if $p=\infty$ then $\mathscr{H}_{s,\infty}$ is the adjoint of $\mathscr{H}_{-s,1}$ and the w-topology is the weak* topology.

The following property is important : *the space $\mathscr{H}_{s,p}$ is sequentially complete in the* w*-topology.* So, if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathscr{H}_{s,p}$ such that $\lim_{n\to\infty}\langle f_n, g\rangle$ exists for all $g\in\mathscr{H}_{-s,p'}$, then there is $f\in\mathscr{H}_{s,p}$ such that the limit is equal to $\langle f,g\rangle$. If $1<p<\infty$ then this follows from the reflexivity of the Banach space $\mathscr{H}_{s,p}$ and from the fact that the w-topology is equal to the weak topology. If $p=\infty$ then the w-topology is the weak* topology, which again implies the sequential completeness of the w-topology. The only non-trivial case is $p=1$. Now we obtain the desired result by using (2.11) (with $p=q=2$ and $r=1$) and the next proposition.

Proposition. *Let X, Y be Hilbert spaces such that $Y\subset X$ continuously and densely. Then for each $\theta\in(0,1)$ the space $(X,Y)_{\theta,1}$, defined by real interpolation, is weakly sequentially complete.*

Proof. (i) By a theorem of Friedrichs, there is a positive self-adjoint operator Λ in X with domain Y. If we denote F the spectral measure of Λ and if we set $F_0=F([0,1])$, $F_j=F((2^{j-1},2^j])$ for $j\geq 1$, then $(X,Y)_{\theta,1}$ is the space of $x\in X$ such that $\|x\|_{\theta,1}\equiv\sum_{j\in\mathbb{N}}\|2^{j\theta}F_jx\|_X<\infty$, the topology of $(X,Y)_{\theta,1}$ being defined by the norm $\|\cdot\|_{\theta,1}$ (see, for example, Proposition 3.3 in [BG4]). Let Z be the Hilbert space defined as follows: $Z=\oplus_{j\in\mathbb{N}}X_j$ (Hilbert direct sum) where $X_j=F_jX$ with the norm $\|x\|_j=2^{j\theta}\|x\|_X$. Then let $\ell^1(Z)$ be the Banach space consisting of sequences $f:\mathbb{N}\to Z$ such that $\|f\|_{l^1}\equiv\sum_{j\in\mathbb{N}}\|f(j)\|_Z<\infty$. We see that $(X,Y)_{\theta,1}$ can be identified with the closed subspace of $\ell^1(Z)$ consisting of sequences f such that $f(j)\in X_j$ for all $j\in\mathbb{N}$. A closed subspace of a weakly sequentially complete Banach space is weakly sequentially complete (because of the Hahn-Banach theorem and of the fact that a closed subspace of a Banach space is weakly closed). At last we see that it suffices to show that $\ell^1(Z)$ is a weakly sequentially complete Banach space.

(ii) Let $\{f_n\}_{n\in\mathbb{N}}$ be a weakly Cauchy sequence in $\ell^1(Z)$. Since $\ell^1(Z)^*=l^\infty(Z^*)$, it is easily shown that there is $f\in\ell^1(Z)$ such that $\lim_{n\to\infty}f_n(j)=f(j)$ for all $j\in\mathbb{N}$. The sequence $\{f_n\}$ is bounded in $\ell^1(Z)$ so in order to get $f_n\to f$ weakly in $\ell^1(Z)$ (i.e. $\lim_{n\to\infty}\sum_{j=0}^\infty\langle f_n(j),\varphi(j)\rangle=\sum_{j=0}^\infty\langle f(j),\varphi(j)\rangle$ for all $(\varphi(j))_{j\in\mathbb{N}}\in l^\infty(Z^*)$) it suffices to prove that: for each $\varepsilon>0$ there is a finite set $K\subset\mathbb{N}$ such that $\sum_{j\notin K}\|f_n(j)\|_Z<\varepsilon$ for all $n\in\mathbb{N}$. For this we shall use Theorem 4 on page 104 of [DU] where the measure space (Ω,μ) is $\mathbb{N}$ with the counting measure. However, in order to make it sure that the boundedness of the measure μ is not needed for the result of the theorem to hold, we indicate a modification of the first few lines of the proof of that theorem.

(iii) Let $\mathscr{L}$ be a bounded subset of $\ell^1(Z)$ and assume that there is $\varepsilon>0$ such that for each finite $K\subset\mathbb{N}$ there is $f\in\mathscr{L}$ with the property $\sum_{j\notin K}\|f(j)\|_Z>\varepsilon$. Then for each $n\in\mathbb{N}$ there is $g_n\in\mathscr{L}$ and $k_n>n$ such that $\sum_{n<j<k_n}\|g_n(j)\|_Z>\varepsilon/2$. Now let $h_1=g_1$, $m_1=1$, $n_1=k_1$; $h_2=g_{n_1}$,$m_2=n_1$, $n_2=k_{m_2}$; $h_3=g_{n_2}$, $m_3=n_2$, $n_3=k_{m_3}$, etc. Then we get a sequence $\{h_i\}_{i\geq 1}$ of elements $h_i \in \mathscr{L}$ and two sequences of integers $\{m_i\}$, $\{n_i\}$ with $m_1 < n_1 = m_2 < n_2 = m_3 < n_3 = \dots$, such that $\sum_{m_i<j<n_i} \|h_i(j)\|_Z > \varepsilon/2$. Now we may apply Rosentahl's lemma as in [DU] in order to find an infinite set $\mathscr{A}$ of finite pair-wise disjoint subsets of $\mathbb{N}$ and a family $\{f_A\}_{A\in\mathscr{A}}$ of elements $f_A \in \mathscr{L}$ such that $\sum_{j\in A} \|f_A(j)\|_Z > \varepsilon/2$ and $\sum\{\|f_A(j)\|_Z \mid j \in B \text{ for some } B \in \mathscr{A} \setminus \{A\}\} < \varepsilon/4$. Finally, as it is shown on page 104 of [DU], there are constants $\beta > \alpha > 0$ such that for all complex sequences $\{c_A\}_{A\in\mathscr{A}}$:

$$\alpha \sum_{A\in\mathscr{A}} |c_A| \leq \| \sum_{A\in\mathscr{A}} c_A f_A\|_{l^1} \leq \beta \sum_{A\in\mathscr{A}} |c_A|.$$

This clearly shows that we cannot take $\mathscr{L} = \{f_n \mid n \in \mathbb{N}\}$ with $\{f_n\}_{n\in\mathbb{N}}$ a weakly Cauchy sequence in $\ell^1(Z)$. □

We shall similarly consider, besides the norm topology, the w-*topology on the space* $B(\mathscr{H}_{s,p};\mathscr{H}_{t,q})$. This is the topology of simple convergence when $\mathscr{H}_{t,q}$ is equiped with the w-topology. In other terms, it is the topology associated to the family of seminorms $T \mapsto |\langle g, Tf\rangle|$ with $f \in \mathscr{H}_{s,p}$ and $g \in \mathscr{H}_{-t,q'}$. If $q < \infty$ then the w-topology is just the weak operator topology on $B(\mathscr{H}_{s,p};\mathscr{H}_{t,q})$. If $q = \infty$ then the w-topology is the weak* operator topology of the space $B(\mathscr{H}_{s,p};\mathscr{H}^*_{-t,1})$.

Let Ω be a subset of $\mathbb{R}^n$ and F a function $\Omega \to B(\mathscr{H}_{s,p};\mathscr{H}_{t,q})$. We shall say that F is w-*continuous* if it is continuous in the w-topology, i.e. if $x \mapsto \langle g, F(x)f\rangle$ is continuous for each $f \in \mathscr{H}_{s,p}$ and $g \in \mathscr{H}_{-t,q'}$. If Ω is an open set (or, more generally, a submanifold) in $\mathbb{R}^n$ and $k \geq 0$ is an integer, then F *is of class C^k in the* w-*topology*, shortly F is w-C^k, if $x \mapsto \langle g, F(x)f\rangle$ is of class C^k for each f, g as above. If, moreover, all the derivatives of order $\leq k$ of these functions extend to continuous functions on $\Omega \cup J$, where J is a subset of the boundary of Ω, then we say that F *extends to a* w-C^k *function on* $\Omega \cup J$. In this case, due to the sequential completeness of $\mathscr{H}_{t,q}$ in the w-topology, there is a unique w-continuous function from $\Omega \cup J$ to $B(\mathscr{H}_{s,p};\mathscr{H}_{t,q})$ which extends F and, if J is a submanifold of $\mathbb{R}^n$, then the restriction of this function to J is of class w-C^k.

3. Besov Classes of Operators

Our purpose in this section is to develop a theory parallel to that of Section 2 in the case where the Hilbert space $\mathscr{H}$ and the operator A are replaced by the Banach space $B(\mathscr{H})$ and by the operator formally given by $\mathscr{A} = -\operatorname{ad}_A$ respectively (see §2.2). In Section 2 we took advantage of the fact that the self-adjoint operator A has a very rich functional calculus and so we were able to start with a rather natural (Littlewood-Paley type) definition of the Besov spaces $\mathscr{H}_{s,p}$ for each $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. Moreover, in the case where $s \equiv k$ was a positive integer, we had from the start a simple alternative definition $\mathscr{H}_{k,2} \equiv \mathscr{H}_k$ as the domain of the k-th power A^k of A. In turn this gives us a third definition of $\mathscr{H}_k$ in terms of regularity properties of the function $\tau \mapsto W_\tau f$, namely $f \in \mathscr{H}_k$ if and only if $f \in \mathscr{H}$ and the map $\tau \mapsto W_\tau f \in \mathscr{H}$ is of class C^k (or, equivalently, its k-th order modulus of continuity ω_k satisfies $\omega_k(\varepsilon) \leq c\varepsilon^k$). In §2.5 we saw that for all real $s > 0$

and all $p \in [1,\infty]$ the condition $f \in \mathscr{H}_{s,p}$ can be expressed in terms of regularity properties of the map $\tau \mapsto W_\tau f \in \mathscr{H}$.

It is not possible to follow this procedure in the case of the operator $\mathscr{A} = -\operatorname{ad}_A$ since we do not have a functional calculus to start with. Our first task will be to give a precise definition of $\mathscr{A}$, i.e. to specify its domain as operator in $B(\mathscr{H})$. It turns out then that $\mathscr{A}$ is the infinitesimal generator of a weak (cf. [BB]) one-parameter group in $B(\mathscr{H})$, namely $\mathscr{W}_\tau \equiv \exp(i\mathscr{A}\tau)$ acts as $\mathscr{W}_\tau[S] = W_\tau^* S W_\tau$. It is now possible to characterize the domain $C^k(A)$ of the power $\mathscr{A}^k$ of $\mathscr{A}$ in terms of the regularity properties of the map $\tau \mapsto \mathscr{W}_\tau S \in B(\mathscr{H})$: let ω_k be the k-th order modulus of continuity of this function; then an operator $S \in B(\mathscr{H})$ belongs to $C^k(A)$ if and only if $\omega_k(\varepsilon) \leq c\varepsilon^k$. Moreover, this is also equivalent to the fact that $\tau \mapsto W_\tau S$ is of class C^k in the strong operator topology. One gets strictly smaller classes $C^k_u(A)$ by requiring that the function $\tau \mapsto W_\tau S$ be of class C^k in norm (i.e. in the uniform operator topology, as suggested by the sub-index u). Now the extension to regularity of fractional order is quite natural: if $s > 0$ is a real number and $p \in [1,\infty]$, then $S \in \mathscr{C}^{s,p}(A)$ means that $S \in B(\mathscr{H})$ and $[\int_0^1 (\varepsilon^{-s}\omega_m(\varepsilon))^p \varepsilon^{-1} d\varepsilon]^{1/p} < \infty$ for some integer $m > s$. The final step is to give a Littlewood-Paley version of the last condition: this is now possible since with the help of the group $\{\mathscr{W}_\tau\}$ one may associate to $\mathscr{A}$ a sufficiently large functional calculus. The Littlewood-Paley characterization of the classes $\mathscr{C}^{s,p}(A)$ plays a fundamental role in the proof of our main results.

3.1. We shall denote by $\mathscr{A}$ the operator acting in the Banach space $B(\mathscr{H})$ according to the following rule: an element $S \in B(\mathscr{H})$ belongs to the domain of $\mathscr{A}$ if and only if the sesquilinear form $\langle f, SAg\rangle - \langle Af, Sg\rangle$ (with domain $D(A)$) is continuous for the topology induced by $\mathscr{H}$; and then $\mathscr{A}[S] \equiv \mathscr{A}S$ is the unique element of $B(\mathscr{H})$ such that $\langle f, \mathscr{A}[S]g\rangle = \langle f, SAg\rangle - \langle Af, Sg\rangle$ for all $f,\, g \in D(A)$. We find convenient to denote $C^1(A)$ the domain of $\mathscr{A}$ and we set $C^0(A) = B(\mathscr{H})$.

If S is an arbitrary element of $B(\mathscr{H})$ then $[S, A] = SA - AS \equiv -\operatorname{ad}_A S$ is a well defined element of $B(\mathscr{H}_1; \mathscr{H}_{-1})$ (see §2.2). Since we make the identification (2.3) we see that $S \in C^1(A) \Leftrightarrow [S, A] \in B(\mathscr{H})$, and then $\mathscr{A}S = [S, A]$. However, for the clarity of the exposition, we shall in general distinguish between $[S, A]$ (sesquilinear form on $D(A)$) and $\mathscr{A}S$ (bounded operator in $\mathscr{H}$ associated to it), although this is rather pedantic (because of the identification (2.3)).

We shall denote by $C^k(A)$ the domain of the power $\mathscr{A}^k$ of $\mathscr{A}$ for each $k \in \mathbb{N}$. According to an observation made in §2.2 we have $S \in C^k(A)$ if and only if $S \in B(\mathscr{H})$ and the sesquilinear form $\operatorname{ad}_A^k S$ with domain $D(A^k)$ (see (2.4)) is continuous for the topology induced by $\mathscr{H}$ on $D(A^k)$. Under this condition one has

$$\mathscr{A}^k[S] \equiv \mathscr{A}^k S = (-1)^k \operatorname{ad}_A^k S = \sum_{i+j=k} \frac{k!}{i!j!}(-1)^i A^i S A^j \tag{3.1}$$

with the same comment as in the case $k = 1$ (i.e. $\mathscr{A}^k S$ is the element of $B(\mathscr{H})$ associated to the sesquilinear form on $D(A^k)$ defined by the last member of (3.1)).

3.2. Our purpose now is to show that $\mathscr{A}$ can be interpreted as the infinitesimal generator of a one-parameter group of automorphisms of $B(\mathscr{H})$. For each real τ we define $\mathscr{W}_\tau : B(\mathscr{H}) \to B(\mathscr{H})$ by $\mathscr{W}_\tau[S] \equiv \mathscr{W}_\tau S = W_\tau^* S W_\tau$. Then $\mathscr{W}_0 = 1$, $\mathscr{W}_{\tau+\sigma} = \mathscr{W}_\tau \mathscr{W}_\sigma$ for all τ, $\sigma \in \mathbb{R}$, and the function $\tau \mapsto \mathscr{W}_\tau S \in \mathbb{R}$ is strongly continuous (but not norm continuous in general).

Proposition. *For each $S \in B(\mathscr{H})$ and each integer $k \geq 1$ the following conditions are equivalent:*
(a) $S \in C^k(A)$*;*
(b) $\liminf_{\varepsilon \to +0} \varepsilon^{-k} \|(\mathscr{W}_\varepsilon - 1)^k S\| < \infty$*;*
(c) $\|(\mathscr{W}_\tau - 1)^k S\| \leq c|\tau|^k$ *for some number c and all real τ;*
(d) *the function $\tau \mapsto \mathscr{W}_\tau S \in B(\mathscr{H})$ is strongly of class C^k on $\mathbb{R}$.*
Under these conditions one has

$$\mathscr{A}^k S = (-id/d\tau)^k \mathscr{W}_\tau S \,|_{\tau=0} = \lim_{\varepsilon \to 0} (i\varepsilon)^{-k} (\mathscr{W}_\varepsilon - 1)^k S \tag{3.2}$$

where the derivative and the limit exist in the strong operator topology.

Proof. For each $s > 0$ and $\tau \in \mathbb{R}$ the operator W_τ sends $\mathscr{H}_s$ continuously into itself and, if $f \in \mathscr{H}_s$, the map $\tau \mapsto W_\tau f \in \mathscr{H}_s$ is continuous. So the action of $\mathscr{W}_\tau$ can be extended in an obvious way to $B(\mathscr{H}_s; \mathscr{H}_{-s})$ (see (2.3)) and for $T \in B(\mathscr{H}_s; \mathscr{H}_{-s})$ we have in $B(\mathscr{H}_{s+1}; \mathscr{H}_{-s-1})$ the identities $\mathscr{W}_\tau \operatorname{ad}_A T = \operatorname{ad}_A \mathscr{W}_\tau T$ and

$$(\mathscr{W}_\varepsilon - 1)T = -i\varepsilon \int_{\mathbb{R}} \mathscr{W}_{\varepsilon\sigma} \operatorname{ad}_A T \chi_1(\sigma) d\sigma), \tag{3.3}$$

where χ_1 is the characteristic function of $[0,1]$. If we set $\chi_k = \chi_1 \star \cdots \star \chi_1$ (convolution product of k factors) then by iterating k times the identity (3.3) we get for $S \in B(\mathscr{H})$:

(3.4)
$$\begin{aligned}(\mathscr{W}_\varepsilon - 1)^k S &= (-i\varepsilon)^k \int \cdots \int \mathscr{W}_{\varepsilon(\sigma_1 + \cdots + \sigma_k)} \operatorname{ad}^k S \chi_1(\sigma_1) \ldots \chi_k(\sigma_k) d\sigma_1 \ldots d\sigma_k \\ &= (-i\varepsilon)^k \int \mathscr{W}_{\varepsilon\sigma} \operatorname{ad}_A^k S \chi_k(\sigma) d\sigma.\end{aligned}$$

This holds strongly in $B(\mathscr{H}_k; \mathscr{H}_{-k})$, or in the sense of sesquilinear forms on $D(A^k)$. Since $0 \leq \chi_k \leq 1$, $\operatorname{supp} \chi_k = [0,k]$ and $\int \chi_k(\sigma) d\sigma = 1$, we have for all f, $g \in D(A^k)$:

$$\lim_{\varepsilon \to 0} \langle f, \Big[\frac{\mathscr{W}_\varepsilon - 1}{i\varepsilon}\Big]^k g\rangle = (-1)^k \langle f, \operatorname{ad}_A^k S g\rangle. \tag{3.5}$$

Now the implication $(b) \Rightarrow (a)$ is clear. On the other hand $(c) \Rightarrow (b)$ is obvious and $(d) \Rightarrow (c)$ is a consequence of the Taylor formula. Finally, $(a) \Rightarrow (d)$ follows from the argument of §2.2, i.e. Taylor formula again. (3.2) is a consequence of (3.4). □

The case $k = 1$ of the preceding proposition justifies the notation $\mathscr{W}_\tau = \exp(i\mathscr{A}\tau)$ and the interpretation of $\mathscr{A}$ as the infinitesimal generator of the "weak" one-parameter group $\{\mathscr{W}_\tau\}_{\tau \in \mathbb{R}}$ in the Banach space $B(\mathscr{H})$. For $S \in B(\mathscr{H})$ we have $S \in C^1(A)$ (= domain of $\mathscr{A}$) if and only if $\lim_{\varepsilon \to 0} (i\varepsilon)^{-1} (\mathscr{W}_\varepsilon -$

$1)S$ exists in $B(\mathscr{H})$ in the weak or strong operator topology, and then the limit is just $\mathscr{A}S$ and one has for all $\tau \in \mathbb{R}$

$$\mathscr{W}_\tau S = S + i \int_0^\tau \mathscr{W}_\sigma \mathscr{A} S d\sigma. \tag{3.6}$$

In particular the operator $\mathscr{A}$ is sequentially closed in the weak (hence strong and norm) operator topology.

It is easy to prove (by using the preceding proposition) that $C^k(A)$ is a full *-subalgebra of $B(\mathscr{H})$ (a subalgebra $\mathscr{B}$ of $B(\mathscr{H})$ is full if for each $S \in \mathscr{B}$ which is invertible in $B(\mathscr{H})$ one has $S^{-1} \in \mathscr{B}$).

Later on we shall need the following fact:

Lemma. *Let $[a,b]$ be a real interval and $\{S_x\}_{a \le x \le b}$ a family of bounded operators on $\mathscr{H}$ having the following properties:*

(i) *$x \mapsto S_x \in B(\mathscr{H})$ is strongly of class C^1 (with derivative $S'_x = \frac{d}{dx} S_x$);*

(ii) *S_x and S'_x are of class $C^1(A)$ for all $x \in [a,b]$;*

(iii) *$x \mapsto \mathscr{A} S'_x \in B(\mathscr{H})$ is strongly continuous.*

Then the map $x \mapsto \mathscr{A} S_x \in B(\mathscr{H})$ is strongly C^1 and its derivative is given by $\frac{d}{dx} \mathscr{A} S_x = \mathscr{A} S'_x$.

Proof. We have $S_x = S_a + \int_a^x S'_y dy$ hence for $\varepsilon \neq 0$:

$$\frac{\mathscr{W}_\varepsilon - 1}{i\varepsilon} S_x = \frac{\mathscr{W}_\varepsilon - 1}{i\varepsilon} S_a + \int_a^x \frac{\mathscr{W}_\varepsilon - 1}{i\varepsilon} S'_y dy.$$

Now we make $\varepsilon \to 0$ and use the dominated convergence theorem by taking into account the inequality $\varepsilon^{-1} \|(\mathscr{W}_\varepsilon - 1) S'_y\| \le \|\mathscr{A} S'_y\|$, which follows from (3.6). $\square$

3.3. For $k \in \mathbb{N}$ we denote by $C^k_u(A)$ the space of $S \in B(\mathscr{H})$ such that the map $\tau \mapsto \mathscr{W}_\tau S \in B(\mathscr{H})$ is of class C^k in the norm (or uniform) topology. Clearly $C^0_u(A)$ is a C*-subalgebra of $B(\mathscr{H})$ and it is strongly dense in $B(\mathscr{H})$ as will follow from the functional calculus which will be constructed below. For $k \ge 1$ we obviously have $C^k_u(A) \subset C^k(A) \subset C^{k-1}_u(A)$. We shall set $C^\infty(A) = \cap_{k=0}^\infty C^k(A)$. All these classes are full *-subalgebras of $B(\mathscr{H})$ stable under $\mathscr{W}_\tau$.

If $k \ge 1$ and $S \in B(\mathscr{H})$ then the following assertions are equivalent:

(i) $S \in C^k_u(A)$;

(ii) $S \in C^k(A)$ and $\mathscr{A}^k S \in C^0_u(A)$;

(iii) $\lim_{\varepsilon \to 0} \varepsilon^{-k} (\mathscr{W}_\tau - 1)^k S$ exists in the norm topology.

For the proof of (iii)$\Rightarrow$(ii) note that we shall have $S \in C^k(A)$ (by the Proposition of §3.2), hence $S \in C^0_u(A)$ and so $\mathscr{A}^k S \in C^0_u(A)$ by the norm closedness of $C^0_u(A)$. Then (ii)$\Rightarrow$(i) by Taylor's formula.

3.4. We say that $S \in B(\mathscr{H})$ is an A-*analytic operator* if the function $\zeta \mapsto e^{\zeta \mathscr{A}}[S] = e^{-\zeta A} S e^{\zeta A} \in B(\mathscr{H})$, which is defined for purely imaginary numbers, has a holomorphic extension to a neighbourhood of the origin in $\mathbb{C}$. If this neighbourhood is equal to $\mathbb{C}$ we say that S is A-*entire*. Clearly S is A-analytic (resp. A-entire) if and only if $S \in C^\infty(A)$ and the radius of convergence of the series $\sum_{k=0}^\infty (k!)^{-1} \zeta^k \mathscr{A}^k[S]$ in $B(\mathscr{H})$ is strictly positive (resp. is equal to infinity). It is straightforward to show that for each A-analytic operator S there are strictly positive numbers a, b and there is a function $\zeta \mapsto S(\zeta) \in B(\mathscr{H})$ holomorphic on the strip $-a < \operatorname{Re}\zeta < b$ such that $S(i\tau) = \mathscr{W}_\tau S$ for $\tau \in \mathbb{R}$. The numbers a, b and the function $S(\cdot)$ are uniquely defined if we require a, b to be maximal with respect to the stated property. Then we say that $-a < \operatorname{Re}\zeta < b$ is the *analyticity strip* of S. Moreover, one has $S(\zeta + i\tau) = e^{i\tau\mathscr{A}}[S(\zeta)]$ for all $\tau \in \mathbb{R}$ and all ζ in the analyticity strip of S. We shall set $S(\zeta) \equiv e^{\zeta\mathscr{A}}[S]$.

The formal relation $e^{\zeta\mathscr{A}}[S] = e^{-\zeta A} S e^{\zeta A}$ can be given a precise meaning as follows. First one shows that a vector f belongs to the domain of $e^{-\sigma A}$ for all $\sigma \in (-a, b)$ if and only if there is a function $\zeta \mapsto f(\zeta) \in \mathscr{H}$ holomorphic on the strip $-a < \operatorname{Re}\zeta < b$ and such that $f(i\tau) = e^{-i\tau A} f$ for real τ; and in fact one has $f(\zeta) \equiv e^{-\zeta A} f$. Then clearly Sf has the same property and $e^{-\zeta A} S f = S(\zeta) e^{-\zeta A} f$. In particular we get $e^{-\zeta A} S e^{\zeta A} g = S(\zeta) g$ for each A-entire vector g (i.e. $g \in D(e^{\sigma A})\ \forall \sigma \in \mathbb{R}$).

The class of A*-analytic operators is a full *-subalgebra of* $B(\mathscr{H})$ (we shall see later on that it *is norm dense in* $C_u^0(A)$). Note that $S^*(\zeta) = S(-\overline{\zeta})^*$, so that S and S^* do not have the same analyticity strip in general. "Full" means that S^{-1} is $\mathscr{A}$-analytic if S is $\mathscr{A}$-analytic and invertible in $B(\mathscr{H})$. This property is easy to prove but we have to stress the fact that the analyticity strip of S^{-1} can be quite different from that of S. Assume that $-a < \operatorname{Re}\zeta < b$ is the analyticity strip of S and let a_0 (resp. b_0) be the upper bound of the numbers $\sigma \geq 0$ such that $S(\sigma)$ (resp. $S(-\sigma)$) is invertible in $B(\mathscr{H})$. Then $0 < a_0 \leq a$, $0 < b_0 \leq b$, the analyticity strip of S^{-1} contains $-a_0 < \operatorname{Re}\zeta < b_0$, and for such ζ one has $S^{-1}(\zeta) = S(\zeta)^{-1}$. In other terms, one has the simple relation $e^{\zeta A}[S^{-1}] = (e^{\zeta\mathscr{A}}[S])^{-1}$ if $-a_0 < \operatorname{Re}\zeta < b_0$.

3.5. We shall define a functional calculus for the operator $\mathscr{A}$ with the help of the group $\{\mathscr{W}_\tau\}_{\tau\in\mathbb{R}}$. Let $\mathscr{M} = \mathscr{M}(\mathbb{R})$ be the unital subalgebra of the algebra $BC(\mathbb{R})$ (bounded continuous complex functions on $\mathbb{R}$) consisting of Fourier transforms of bounded Borel measures. The algebraic operations in $\mathscr{M}$ are those inherited from the embedding $\mathscr{M} \subset BC(\mathbb{R})$, but we take as norm in $\mathscr{M}$ of $\varphi(t) = \int_{\mathbb{R}} e^{it\tau}\mu(d\tau)$ the total variation of the measure μ. This makes $\mathscr{M}$ an abelian Banach algebra with unit. If φ is given by the preceding expression we define a linear continuous operator $\varphi(\mathscr{A}) : B(\mathscr{H}) \to B(\mathscr{H})$ by setting $\varphi(\mathscr{A})S \equiv \varphi(\mathscr{A})[S] \equiv \int_{\mathbb{R}} W_\tau^* S W_\tau \mu(d\tau)$; the integral exists in the strong operator topology. Observe that the notation $\mathscr{W}_\tau = e^{i\mathscr{A}\tau}$ is consistent with the functional calculus, i.e. $\varphi(\mathscr{A}) = \mathscr{W}_\tau$ if φ is the function $\varphi(t) = e^{it\tau}$. It is easily checked that the map $\varphi \mapsto \varphi(\mathscr{A})$ is a unital homomorphism such that $(\varphi(\mathscr{A})[S])^* = \varphi^+(\mathscr{A})[S^*]$ if $\varphi^+(t) = \overline{\varphi(-t)}$, and that the norm of the operator $\varphi(\mathscr{A})$ (acting in the Banach space $B(\mathscr{H})$) is $\leq \|\varphi\|_{\mathscr{M}}$. Finally, observe that the restriction of $\varphi(\mathscr{A})$ to a bounded part of $B(\mathscr{H})$ is continuous for the weak and strong operator topology.

It is clear that $\varphi(\mathscr{A})$ commutes with $\mathscr{A}$, in fact if $k \in \mathbb{N}$ then $\varphi(\mathscr{A})C^k(A) \subset C^k(A)$ and $\mathscr{A}^k \varphi(\mathscr{A}) S = \varphi(\mathscr{A})\mathscr{A}^k S$ for $S \in C^k(A)$ (use the second equality from (3.2), for example). Our purpose now is to show that if φ decays at infinity then it improves regularity with respect to $\mathscr{A}$. If $m \geq 1$ is an integer we define $\varphi_{(m)}$ by $\varphi_{(m)}(x) = x^m \varphi(x)$.

Proposition. *If $\varphi \in \mathscr{M}$ has the property $\varphi_{(1)} \in \mathscr{M}$, then $\varphi(\mathscr{A})C^k(A) \subset C^{k+1}(A)$ for each $k \in \mathbb{N}$. Furthermore, one has $\mathscr{A}\varphi(\mathscr{A}) = \varphi_{(1)}(\mathscr{A})$. In particular, if $S \in B(\mathscr{H})$ then $\varphi(\mathscr{A})S \in C^1(A)$ and $\mathscr{A}\varphi(\mathscr{A})S = \varphi_{(1)}(\mathscr{A})S$.*

Proof. It is clearly sufficient to prove the last assertion of the proposition. We shall use the following easily proven fact: if μ is a bounded measure on $\mathbb{R}$ and if its distributional derivative μ' is also a bounded measure, then for each bounded function $h : \mathbb{R} \to \mathbb{C}$ with bounded continuous derivative one has $\int h'(x)\mu(dx) = -\int h(x)\mu'(dx)$.

For $f,\, g \in \mathscr{H}$ and $\tau \in \mathbb{R}$ we have

$$\langle W_\tau f, \varphi(\mathscr{A})[S]W_\tau g\rangle = \int_{\mathbb{R}} \langle W_{x+\tau} f, SW_{x+\tau} g\rangle \mu(dx).$$

Here μ is the bounded measure on $\mathbb{R}$ such that $\varphi(t) = \int e^{itx}\mu(dx)$ and the hypothesis $\varphi_{(1)} \in \mathscr{M}$ is equivalent to the fact that the distributional derivative μ' is a bounded measure too. Now assume that $f,\, g \in D(A)$. Then by taking the derivative with respect to τ at $\tau = 0$ in the preceding identity and by using the dominated convergence theorem, we get:

$$\langle iAf, \varphi(\mathscr{A})[S]g\rangle + \langle f, \varphi(\mathscr{A})[S]iAg\rangle = \int_{\mathbb{R}} \frac{d}{dx}\langle W_x f, SW_x g\rangle \mu(dx).$$

We integrate by parts in the r.h.s. by using the observation that we made at the beginning of the proof, and then we notice that μ' is just the measure associated to $-\varphi_{(1)}$. So we obtain $\langle f, \varphi(\mathscr{A})[S]Ag\rangle - \langle Af, \varphi(\mathscr{A})[S]g\rangle$ $= \langle f, \varphi_{(1)}(\mathscr{A})[S]g\rangle$. Hence the sesquilinear form $\mathrm{ad}_A(\varphi(\mathscr{A})[S])$ extends to the bounded operator $-\varphi_{(1)}(\mathscr{A})[S]$. $\square$

If $\varphi \in \mathscr{M}$, $m \geq 1$ is an integer, and $\varphi_{(m)} \in \mathscr{M}$, then for each integer $j \in [0, m]$ one has $\varphi_{(j)} \in \mathscr{M}$. Hence by the preceding proposition we have $\varphi(\mathscr{A})C^k(A) \subset C^{k+m}(A)$ for all $k \in \mathbb{N}$ and $\mathscr{A}^m\varphi(\mathscr{A}) = \varphi_{(m)}(\mathscr{A})$. In particular, if φ is the Fourier transform of a function $\psi \in C^\infty(\mathbb{R})$ such that $\psi^{(m)} \in L^1(\mathbb{R})$ $\forall m \in \mathbb{N}$, we get $\varphi(\mathscr{A})B(\mathscr{H}) \subset C^\infty(A)$. Finally, if $\varphi \in C_0^\infty(\mathbb{R})$ then $\varphi(\mathscr{A})S$ is an A-entire operator and for each $\zeta \in \mathbb{C}$ one has $e^{\zeta\mathscr{A}}[\varphi(\mathscr{A})S] = (e_\zeta\varphi)(\mathscr{A})$, where $e_\zeta(x) = \exp(\zeta x)$.

3.6. The main purpose of the functional calculus introduced above is to allow us to construct operators of class $C^\infty(A)$ which approximate a given operator $S \in B(\mathscr{H})$ rapidly enough, in a sense that we shall make precise in this subsection.

Note first that if $\varphi \in \mathscr{M}$ and $\varepsilon \in \mathbb{R}$ then the function $x \mapsto \varphi(\varepsilon x)$, denoted φ^ε, belongs to $\mathscr{M}$ and $\|\varphi^\varepsilon\|_{\mathscr{M}} \leq \|\varphi\|_{\mathscr{M}}$ (the equality holds if $\varepsilon \neq 0$). We set $\varphi(\varepsilon\mathscr{A}) = \varphi^\varepsilon(\mathscr{A})$, in other terms $\varphi(\varepsilon\mathscr{A})S = \int_{\mathbb{R}} W_{\varepsilon\tau}^* SW_{\varepsilon\tau}\mu(d\tau)$. So for each $S \in B(\mathscr{H})$ the map $\varepsilon \mapsto \varphi(\varepsilon\mathscr{A})S \in B(\mathscr{H})$ is strongly continuous and $\varphi(0\mathscr{A})S = \varphi(0)S$. In particular, if $\varphi(0) = 1$ and $\varphi_{(m)} \in \mathscr{M}$ for some integer $m \geq 1$, then by what we have seen before we have $\varphi(\varepsilon\mathscr{A})S \in C^m(A)$ for $\varepsilon \neq 0$ and $\varphi(\varepsilon\mathscr{A})S \to S$ in the strong operator topology as $\varepsilon \to 0$. Operators of the form $S_\varepsilon = \varphi(\varepsilon\mathscr{A})S$ with $\varphi \in C_0^\infty(\mathbb{R})$ and $\varphi(0) = 1$ will be called *regularizations* of S ($\varepsilon \neq 0$). Observe that the regularizations of S are A-entire operators.

The rapidity of the convergence of $\varphi(\varepsilon\mathscr{A})S$ to S is determined by the degree of regularity of S with respect to $\mathscr{A}$. We explain this fact here in rather rough terms. A quite precise result will be presented later on. Assume that $S \in C^k(A)$ for some integer $k \geq 1$ and let $\varphi \in \mathscr{M}$ be of the form $\varphi(x) = 1 + x^k\eta(x)$ for some $\eta \in \mathscr{M}$. Then by using the Proposition from §3.5 we get for $\varepsilon \neq 0$:

$$\varphi(\varepsilon\mathscr{A})S = S + \varepsilon^k\mathscr{A}^k\eta(\varepsilon\mathscr{A})S = S + \varepsilon^k\eta(\varepsilon\mathscr{A})\mathscr{A}^kS$$

hence $\|\varphi(\varepsilon\mathscr{A})S - S\| \leq |\varepsilon|^k\|\eta\|_{\mathscr{M}}\|\mathscr{A}^kS\|$. So if $\varphi \in C_0^\infty(\mathbb{R})$ and $\varphi(x) - 1 = O(x^k)$ as $x \to 0$, then for each $S \in C^k(A)$ we have $\|\varphi(\varepsilon\mathscr{A})S - S\| = O(\varepsilon^k)$ as $\varepsilon \to 0$. We shall see in §3.7 that this behaviour characterizes the class of operators $\mathscr{C}^k(A)$, which is slightly larger than $C^k(A)$.

The explicit integral formula which gives $\varphi(\varepsilon\mathscr{A})S$ shows that $S \in C_u^0(\mathscr{H})$ if and only if $\|\varphi(\varepsilon\mathscr{A})S - \varphi(0)S\| \to 0$ as $\varepsilon \to 0$ for all $\varphi \in \mathscr{M}$. On the other hand, assume that $\lim_{\varepsilon\to 0}\|\varphi(\varepsilon\mathscr{A})S - S\| = 0$ holds for *one* function $\varphi \in \mathscr{M}$ such that $\varphi(0) = 1$ and $\varphi_{(1)} \in \mathscr{M}$; then $S \in C_u^0(\mathscr{H})$. Indeed, $\varphi(\varepsilon\mathscr{A})S \in C^1(A) \subset C_u^0(A)$ if $\varepsilon \neq 0$ and $C_u^0(A)$ is norm closed. In particular, $C^\infty(A)$ and the set of A-entire operators are dense subspaces of the Banach space $C_u^0(A)$ and the regularizations of S are norm convergent to S if and only if $S \in C_u^0(A)$.

Now let $\varphi \in \mathscr{M}$ be a function of class C^1. If φ' is the derivative of φ we set $\tilde{\varphi}(x) = x\varphi'(x)$ and we assume that φ' and $\tilde{\varphi}$ belong to $\mathscr{M}$. Then for each $S \in B(\mathscr{H})$ the function $\varepsilon \mapsto \varphi(\varepsilon\mathscr{A})S \in B(\mathscr{H})$ is strongly C^1 on $\mathbb{R}\setminus\{0\}$, for $\varepsilon \neq 0$ the operator $\varphi'(\varepsilon\mathscr{A})S$ is of class $C^1(A)$, and we have

$$\frac{d}{d\varepsilon}\varphi(\varepsilon\mathscr{A})S = \varepsilon^{-1}\tilde{\varphi}(\varepsilon\mathscr{A})S = \mathscr{A}\varphi'(\varepsilon\mathscr{A})S. \tag{3.7}$$

Indeed, by using the dominated convergence theorem we obtain for f, $g \in D(A)$:

$$\begin{aligned}\frac{d}{d\varepsilon}\langle f, \varphi(\varepsilon\mathscr{A})[S]g\rangle &= \int \frac{d}{d\varepsilon}\langle W_{\varepsilon\tau}f, SW_{\varepsilon\tau}g\rangle\mu(d\tau) \\ &= \int(\langle W_{\varepsilon\tau}f, SW_{\varepsilon\tau}Ag\rangle - \langle W_{\varepsilon\tau}Af, SW_{\varepsilon\tau}g\rangle)i\tau\mu(d\tau) \\ &= \frac{1}{\varepsilon}\int\frac{d}{d\tau}\langle W_{\varepsilon\tau}f, SW_{\varepsilon\tau}g\rangle\tau\mu(d\tau).\end{aligned} \tag{3.8}$$

We can integrate by parts in the last member by using the observation made at the beginning of the proof of the Proposition from §3.5. This gives the first equality in (3.7). For the second one observe that the last but one expression in (3.8) is equal to $-\langle f, \mathrm{ad}_A[\varphi(\varepsilon\mathscr{A})S]g\rangle$.

We shall need (3.7) only for $\varphi \in C_0^\infty(\mathbb{R})$. Clearly we then have $\varphi^{(k)}(\varepsilon\mathscr{A})S \in C^\infty(A)$ if $\varepsilon \neq 0$ and $k \in \mathbb{N}$, and the map $\varepsilon \mapsto \varphi(\varepsilon\mathscr{A})S \in B(\mathscr{H})$ is of class C^∞ on $\mathbb{R}\setminus\{0\}$ and satisfies $(d/d\varepsilon)^k\varphi(\varepsilon\mathscr{A})S = \mathscr{A}^k\varphi^{(k)}(\varepsilon\mathscr{A})S$.

3.7. We define Besov classes of operators in terms of regularity properties of the function $\tau \mapsto W_\tau S \in B(\mathscr{H})$. Let s be a strictly positive real number and let $p \in [1,\infty]$. If $S \in B(\mathscr{H})$ then its *modulus of continuity of order m*

(integer ≥ 1) with respect to A is the function $\omega_m = \omega_m(\cdot; S, A)$ defined on $\mathbb{R}_+ = (0, \infty)$ by

$$\omega_m(\varepsilon) = \sup_{|\tau|<\varepsilon} \|(\mathscr{W}_\tau - 1)^m S\| = \sup_{|\tau|<\varepsilon} \left\| \sum_{j=0}^{m} (-1)^j \binom{m}{j} W_{\tau j}^* S W_{\tau j} \right\|.$$

We say that *S is of class $\mathscr{C}^{s,p}(A)$* if there is an integer $m > s$ such that one of the following equivalent conditions is satisfied:

$$\left[\int_0^1 \|\varepsilon^{-s}(\mathscr{W}_\varepsilon - 1)^m S\|^p \frac{d\varepsilon}{\varepsilon}\right]^{1/p} < \infty \quad \text{or} \quad \left[\int_0^1 [\varepsilon^{-s}\omega_m(\varepsilon)]^p \frac{d\varepsilon}{\varepsilon}\right]^{1/p} < \infty.$$

If $p = \infty$ then these conditions must be read $\omega_m(\varepsilon) \leq c\varepsilon^s$ for some number c and $\varepsilon > 0$. We shall write $\mathscr{C}^s(A)$ for $\mathscr{C}^{s,\infty}(A)$.

The equivalence of the two conditions stated above is easy to prove if $0 < s < 1 = m$: the argument is the same as that given in §2.5 for the case of the group W_τ in $\mathscr{H}$. For general s, m see [ABG2]. The fact that the class $\mathscr{C}^{s,p}$ is independent of the integer m will be a consequence of the theorem that we shall state below (see [ABG2] for a direct proof).

By using the fact that m can be chosen as large as we wish it is easily shown that each *$\mathscr{C}^{s,p}(A)$ is a full *-subalgebra of $B(\mathscr{H})$.*

The condition $S \in \mathscr{C}^{s,p}(A)$ is clearly equivalent to the fact that the function $\tau \mapsto \mathscr{W}_\tau S \in B(\mathscr{H})$ is of class $\Lambda^{s,p}$. In particular we obtain the following description of the class $\mathscr{C}^{s,p}$ in terms of differentiability properties of the function $\tau \mapsto \mathscr{W}_\tau S$. *Let k be an integer such that $0 \leq k < s$ and set $s = k + \sigma$. Then $S \in \mathscr{C}^{s,p}(A)$ if and only if $S \in C^k(A)$ and $\mathscr{A}^k S \in \mathscr{C}^{\sigma,p}(A)$.*

If $k \geq 1$ is an integer, then we have strict and optimal embeddings $\mathscr{C}^{k,1} \subset C_u^k \subset C^k \subset \mathscr{C}^{k,\infty}$ (the spaces $\mathscr{C}^{k,p}$ with $1 < p < \infty$ are not comparable with C_u^k and C^k). The last embedding is an obvious consequence of the Proposition from §3.2; the inclusion $\mathscr{C}^{k,1} \subset C_u^k$ is more subtle and we shall prove it later on.

If (s,p) and (t,q) are two couples from $(0,\infty) \times [1,\infty]$ then we have $\mathscr{C}^{s,p}(A) \subset \mathscr{C}^{t,q}(A)$ if and only if $(s,p) \geq (t,q)$ in the sense defined in §2.4. This is a straightforward consequence of the theorem which follows.

We come now to an important characterization of the property $S \in \mathscr{C}^{s,p}(A)$ in terms of the rapidity of the convergence of the regularizations S_ε of S.

Theorem. *Let $S \in B(\mathscr{H})$, s a strictly positive real number, and $p \in [1,\infty]$. If there is a function $\theta \in \mathscr{M}(\mathbb{R})$, which is not identically zero on $(0,\infty)$ and on $(-\infty, 0)$, such that*

$$\left[\int_0^1 \|\varepsilon^{-s}\theta(\varepsilon\mathscr{A})S\|^p \varepsilon^{-1} d\varepsilon\right]^{1/p} < \infty \tag{3.9}$$

then $S \in \mathscr{C}^{s,p}(A)$. Reciprocally, if $S \in \mathscr{C}^{s,p}$ and if $m > s$ is an integer, then (3.9) holds for each θ such that $\theta^{(k)} \in \mathscr{M}(\mathbb{R})$ for $0 \leq k \leq m$ and $\theta^{(k)}(0) = 0$ for $0 \leq k \leq m-1$.

The proof of this result can be found in [BG3]. One may take $\theta(x) = (e^{ix} - 1)^m$, but this choice is not useful for our purposes in Section 5.

As a first application of the preceding theorem we shall give a simple proof of the embedding $\mathscr{C}^{s,p}(A) \subset \mathscr{C}^{t,q}(A)$ for $(s,p) > (t,q)$. The fact that $\mathscr{C}^{s,\infty}(A) \subset \mathscr{C}^{t,1}(A)$ if $s > t > 0$ is an immediate consequence of the definition, so it suffices to consider the case $s = t$ and $1 \le p < q \le \infty$. Since a function which is of class $L^p \cap L^\infty$ is of class L^q for all $q \in (p, \infty)$, we have to consider only the case $q = \infty$. Let $\theta,\ \varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ with $\theta(x) > 0$ if and only if $1 < |x| < 2$ and $\varphi(x) = 1$ if $1 < |x| < 4$. Then $\theta(x) = \theta(x)\varphi(\nu x)$ for all $x \in \mathbb{R}$ and $1 \le \nu \le 2$, hence $\theta(\varepsilon x) = \theta(\varepsilon x)\varphi(\tau x)$ for all $x \in \mathbb{R}$ and $0 < \varepsilon \le \tau \le 2\varepsilon$, in particular $\theta(\varepsilon\mathscr{A}) = \theta(\varepsilon\mathscr{A})\varphi(\tau\mathscr{A})$ for such $\varepsilon,\ \tau$. This implies $\|\theta(\varepsilon\mathscr{A})S\| \le \|\theta\|_{\mathscr{M}}\|\varphi(\tau\mathscr{A})S\|$ for $S \in B(\mathscr{H})$, so there is a constant C (depending only on s and p) such that for all $\varepsilon > 0$

$$\|\varepsilon^{-s}\theta(\varepsilon\mathscr{A})S\| \le C\left[\int_\varepsilon^{2\varepsilon} \|\tau^{-s}\varphi(\tau\mathscr{A})S\|^p\tau^{-1}d\tau\right]^{1/p}.$$

If $\mathscr{C}^{s,p}(A)$ then the r.h.s. above is uniformly bounded in $\varepsilon > 0$ (by the preceding theorem) hence $S \in \mathscr{C}^{s,\infty}(A)$ (by a new application of the theorem).

Finally, let us prove that $\mathscr{C}^{k,1}(A) \subset C_u^k(A)$ if $k \ge 1$ is an integer. Let $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(x) = 1$ on a neighbourhood of the origin, and let us set $S_\varepsilon = \varphi(\varepsilon\mathscr{A})S$ for some $S \in \mathscr{C}^{k,1}(A)$. Then $S_\varepsilon \in C^\infty(A)$ for $\varepsilon \ne 0$ and $\lim_{\varepsilon\to0} S_\varepsilon = S$ in the strong operator topology. Now for $j = 0, \ldots, k$ and $\varepsilon \ne 0$ let us set $S_{j\varepsilon} = \mathscr{A}^j S_\varepsilon$. Then we have

$$S'_{j\varepsilon} \equiv \frac{d}{d\varepsilon}S_{j\varepsilon} = \mathscr{A}^{j+1}\varphi'(\varepsilon\mathscr{A})S = \varepsilon^{-j-1}\varphi_j(\varepsilon\mathscr{A})S,$$

where $\varphi_j(x) = x^{j+1}\varphi'(x)$. An application of the theorem stated before in this paragraph gives: $S \in \mathscr{C}^{j,1}(A)$ if and only if $\int_0^1 \|S'_{j\varepsilon}\|d\varepsilon < \infty$. Since S is of class $\mathscr{C}^{k,1}(A)$ we shall have this property for each $j = 0, 1, \cdots, k$. In particular $\lim_{\varepsilon\to0} \mathscr{A}^j S_\varepsilon$ exists in norm for $j = 0, 1, \ldots, k$. But clearly $C_u^k(A)$ is a Banach space for the norm $\sum_{j=0}^k \|\mathscr{A}^j T\|$. Hence $S \in C_u^k(A)$.

3.8. We have mentioned before that the spaces $C^k(A)$, $C_u^k(A)$ and $\mathscr{C}^{s,p}(A)$ are full involutive subalgebras of $B(\mathscr{H})$. We shall now prove that they are stable under a much larger class of operations. The following result will be needed.

Lemma. *For each integer $m \ge 1$ there is a number C_m such that for any bounded operator B and any bounded self-adjoint operator S the next estimate holds*

$$\| \operatorname{ad}_B^m(e^{iS})\| \le C_m \sum \| \operatorname{ad}_B^{m_1}(S)\| \ldots \| \operatorname{ad}_B^{m_k}(S)\|. \tag{3.10}$$

The sums runs over all decompositions $m = m_1 + \cdots + m_k$ of m into a sum of integers $m_1, \ldots, m_k \ge 1$.

Proof. For $m = 1$ we use

$$\begin{aligned}\operatorname{ad}_B(e^{iS}) = (B - e^{iS}Be^{-iS})e^{iS} &= -\int_0^1 \frac{d}{dt}e^{itS}Be^{i(1-t)S}dt \\ &= \int_0^1 e^{itS}\operatorname{ad}_B(iS)e^{i(1-t)S}dt,\end{aligned}$$

which gives $\|\operatorname{ad}_B e^{iS}\| \leq \|\operatorname{ad}_B S\|$. Now assume that (3.10) has been proved for all integers $\leq m$ and all S. From the preceding identity we obtain

$$\begin{aligned}\operatorname{ad}_B^{m+1}(e^{iS}) &= \int_0^1 \operatorname{ad}_B^m[e^{itS}\operatorname{ad}_B(iS)e^{i(1-t)S}]dt \\ &= \sum_{a+b+c=m,a,b,c\geq 0}\int_0^1 \frac{m!}{a!b!c!}\operatorname{ad}_B^a(e^{itS})\operatorname{ad}_B^{b+1}(iS)\operatorname{ad}_B^c(e^{i(1-t)S})dt.\end{aligned}$$

Then the induction hypothesis gives

$$\begin{aligned}\|\operatorname{ad}_B^{m+1}(e)\| \leq C(m)\int_0^1 dt \sum \|\operatorname{ad}_B^{a_1}(tS)\|\ldots\|\operatorname{ad}_B^{a_n}(tS)\|\cdot\|\operatorname{ad}_B^{b+1}(iS)\|\cdot \\ \|\operatorname{ad}_B^{c_1}((1-t)S\|\ldots\|\operatorname{ad}_B^{c_l}((1-t)S\|\end{aligned}$$

where the sum runs over all decompositions of a and c into sum of integers $a_1,\ldots,a_n$ and $c_1,\ldots,c_l$ respectively, with $a_j \geq 1$, $c_j \geq 1$. This clearly implies (3.10) with m replaced by $m+1$. $\square$

Let us fix now an integer $n \geq 1$ and a family $S_1,\ldots,S_n$ of bounded self-adjoint operators on $\mathscr{H}$. We set $\mathbb{S} = (S_1,\ldots,S_n)$ and for $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ we denote $x\mathbb{S} = x_1S_1 + \cdots + x_nS_n$. We shall define a bounded operator $\phi(\mathbb{S})$ for each bounded continuous function $\phi : \mathbb{R}^n \to \mathbb{C}$ such that the Fourier transform $\hat{\phi}$ is a bounded measure by setting $\phi(\mathbb{S}) = \int_{\mathbb{R}^n}\exp(ix\mathbb{S})\hat{\phi}(x)dx$. Here $\phi(y) = \int \exp(ixy)\hat{\phi}(dx)$, with a slightly formal notation. We clearly have $\|\phi(\mathbb{S})\| \leq \int|\hat{\phi}(x)|dx$ (=total variation of the measure $\hat{\phi}$). If the operators $S_1,\cdots,S_n$ are pairwise commuting, then there is a unique spectral measure F on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} y_jF(dy) = S_j$ for each j (F is the joint spectral measure of the family $\mathbb{S}$). In this case one clearly has $\phi(\mathbb{S}) = \int_{\mathbb{R}^n}\phi(y)F(dy)$; note that this depends only on the restriction of ϕ to the joint spectrum of $\mathbb{S}$. In particular, if $S = S_1 + iS_2$ is a normal (e.g. unitary) operator, then the operator $\phi(\mathbb{S})$ defined by the preceding procedure coincides with that defined by standard functional calculus.

Proposition. *Assume that ϕ satisfies $\int_{\mathbb{R}^n}\langle x\rangle^m|\hat{\phi}(x)|dx < \infty$ for some integer $m \geq 1$. If the bounded self-adjoint operators $S_1,\ldots,S_n$ are of class $C^m(A)$, or $C^m_u(A)$, or $\mathscr{C}^{s,p}(A)$ for some $0 < s < m$, then the operator $\phi(\mathbb{S})$ belongs to the same class.*

Proof. (i) For $\varepsilon \neq 0$ real we set $A_\varepsilon = (W_\varepsilon - 1)(i\varepsilon)^{-1}$. Then for an arbitrary bounded operator T one has $(i\tau)^{-m}(\mathscr{W}_\varepsilon - 1)^mT = (-1)^mW^*_{m\varepsilon}\operatorname{ad}^m_{A_\varepsilon}(T)$. It follows that $T \in C^m(A)$ if and only if $\|\operatorname{ad}^m_{A_\varepsilon} T\| \leq$ const. $< \infty$ for $0 < \varepsilon < 1$, and $T \in \mathscr{C}^{s,p}(A)$ if and only if $[\int_0^1 \|\varepsilon^{m-s}\operatorname{ad}^m_{A_\varepsilon} T\|^p\varepsilon^{-1}d\varepsilon]^{1/p} < \infty$ for some integer $m > s$.

(ii) An application of the lemma proved above in this paragraph with the choices $B = A_\varepsilon$ and $S = x\mathbb{S}$ will clearly give us

$$\begin{aligned}\|\operatorname{ad}^m_{A_\varepsilon}(e^{ix\mathbb{S}})\| &\leq C_m \sum_{m_1+\cdots+m_k=m,m_1,\ldots,m_k\geq 1}\ \sum_{1\leq j_1,\ldots,j_k\leq n}\|x_{j_1}\operatorname{ad}^{m_1}_{A_\varepsilon} S_{j_1}\|\ldots\|x_{j_k}\operatorname{ad}^{m_k}_{A_\varepsilon} S_{j_k}\| \\ &\leq C\langle x\rangle^m \sum \|\operatorname{ad}^{m_1}_{A_\varepsilon} S_{j_1}\|\ldots\|\operatorname{ad}^{m_k}_{A_\varepsilon} S_{j_k}\|.\end{aligned}$$

(iii) Assume first that $S_1,\dots,S_n$ are of class $C^m(A)$. Then, for example, $\|\operatorname{ad}^{m_1}_{A_\varepsilon} S_{j_1}\| \le \text{const.} < \infty$ if $m_1 \le m$ and $0<\varepsilon<1$. Hence we shall have

$$\|\operatorname{ad}^m_{A_\varepsilon} \phi(\mathbb{S})\| \le \int_{\mathbb{R}^n} \|\operatorname{ad}^m_{A_\varepsilon} e^{ix\mathbb{S}}\| \cdot |\hat{\phi}(x)| \le C' \int_{\mathbb{R}^n} \langle x\rangle^m |\hat{\phi}(x)|dx,$$

from which we get $\phi(\mathbb{S}) \in C^m(A)$. If the operators S_j are of class $C^m_u(A)$, then the map $\varepsilon \mapsto W^*_\varepsilon \exp(ix\mathbb{S})W_\varepsilon \equiv \exp[ixW^*_\varepsilon \mathbb{S} W_\varepsilon]$ is clearly of class C^m in norm, for each $x\in\mathbb{R}^n$. Moreover, the estimate proved at the step (ii) implies the following bound for its derivative of order $k=0,1,\dots,m$ with respect to ε at $\varepsilon=0$: $\|\operatorname{ad}^k_A[\exp(ix\mathbb{S})]\| \le c_k\langle x\rangle^k$. So by using the dominated convergence theorem we see that $\varepsilon\mapsto W^*_\varepsilon \phi(\mathbb{S})W_\varepsilon$ is norm C^m, i.e. $\phi(\mathbb{S}) \in C^m_u(A)$.

(iv) Finally, let us assume that $S_1,\dots,S_n$ are of class $\mathscr{C}^{s,p}(A)$. By the estimate obtained at (ii) we have

$$\|\varepsilon^{m-s}\operatorname{ad}^m_{A_\varepsilon} e^{ix\mathbb{S}}\| \le C\langle x\rangle^m \sum \|\varepsilon^{m_1-s_1}\operatorname{ad}^{m_1}_{A_\varepsilon} S_{j_1}\| \dots \|\varepsilon^{m_k-s_k}\operatorname{ad}^{m_k}_{A_\varepsilon} S_{j_k}\|.$$

Here, for each decomposition $m=m_1+\dots+m_k$ with $m_j\ge 1$ integer we have chosen a decomposition $s=s_1+\dots+s_k$ with $0<s_j<m_j$. Then let $p_1,\dots,p_k\in[1,\infty]$ with $p^{-1}=p_1^{-1}+\dots+p_k^{-1}$. By estimating each term in the sum in the r.h.s. above with the help of the Holder inequality and by taking into account that, for example, $S_{j_1}\in\mathscr{C}^{s_1,p_1}(A)$ we see that

$$\left[\int_0^1 \|\varepsilon^{m-s}\operatorname{ad}^m_{A_\varepsilon} e^{ix\mathbb{S}}\|^p \varepsilon^{-1}d\varepsilon\right]^{1/p} \le \text{const.}\langle x\rangle^m.$$

This obviously implies that $[\int_0^1 \|\varepsilon^{m-s}\operatorname{ad}^m_{A_\varepsilon}\phi(\mathbb{S})\|^p\varepsilon^{-1}d\varepsilon]^{1/p} \le \infty$. □

If the condition $\int_{\mathbb{R}^n}\langle x\rangle^m|\hat{\phi}(x)|dx < \infty$ is satisfied, then clearly ϕ is of class C^m. One can show that, reciprocally, if $\phi\in\Lambda^{m+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon>0$ and $\hat{\phi}\ge 0$, then the condition is fulfilled. Another sufficient condition is $\phi\in\mathscr{H}^{m+n/2+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon>0$.

3.9. Let $S\in B(\mathscr{H})$ be an operator of class $\mathscr{C}^{s,p}(A)$ for some $s>0$ and $p\in[1,\infty]$. One can show quite easily that $S\mathscr{H}_{s,p}\subset\mathscr{H}_{s,p}$ (see Lemma 5.3.2 in [ABG2]) and that the operator S_0 induced by S in $\mathscr{H}_{s,p}$ is continuous (closed graph theorem). Now assume that $1\le p<\infty$. Since S^* is of the same class as S, by considering the adjoint of the restriction of S^* to $\mathscr{H}_{s,p}$ we get a continuous extension $\widehat{S}$ of S to $\mathscr{H}_{-s,p'}$. If $1<p<\infty$ then $\widehat{S}$ is uniqely determined by the fact that it is continuous ($\mathscr{H}$ being dense in $\mathscr{H}_{-s,p'}$ if $p'<\infty$); if $p=1$ we have to require that $\widehat{S}$ be continuous in the w-topology. By interpolating between $S_0:\mathscr{H}_{s,p}\to\mathscr{H}_{s,p}$ and $\widehat{S}:\mathscr{H}_{-s,p'}\to\mathscr{H}_{-s,p'}$, we obtain for each $t\in(-s,s)$ and $q\in[1,\infty]$ an operator in $B(\mathscr{H}_{t,q})$ which is naturally associated to S and which we shall denote again by S (in fact we have $\widehat{S}\mathscr{H}_{t,q}\subset\mathscr{H}_{t,q}$ and we denote by S the restriction $\widehat{S}\mid\mathscr{H}_{t,q}$ for any t, q). If S is invertible in $B(\mathscr{H})$ then its inverse is of the same class as S, hence the operator induced by S in $\mathscr{H}_{t,q}$ is invertible too.

Now assume that S is a symmetric operator in $B(\mathscr{H})$ and let $\sigma(S)$ be its spectrum. Then the operator induced by S in $\mathscr{H}_{t,q}$ has the same spectrum. Indeed, by the preceding remark, the resolvent set of S in $\mathscr{H}$ is included in the resolvent set of S in $\mathscr{H}_{t,q}$. Reciprocally, if λ is a real number in the resolvent set of S in $\mathscr{H}_{t,q}$ then $S-\lambda$ is invertible in $\mathscr{H}_{t,q}$ hence (by taking adjoints) $S-\lambda$ is invertible in $\mathscr{H}_{-t,q'}$ too, hence (by interpolation) $S-\lambda$ is invertible in $\mathscr{H}$ (it suffices to consider $1<q<\infty$ since t can be chosen arbitrarily close to s).

3.10. In this paragraph we set $\Pi_- = E_A((-\infty,0))$ and $\Pi_+ = E_A([0,\infty))$, but the next theorem clearly remains true if $(-\infty,0)$ and $[0,\infty)$ are replaced by any intervals of the form $(-\infty,a]$ and $[b,\infty)$ with $a,\, b\in\mathbb{R}$. Observe that the direct sum decomposition $\mathscr{H} = \Pi_-\mathscr{H} \bigoplus \Pi_+\mathscr{H}$ allows us to identify each operator $S:\mathscr{H}\to\mathscr{H}$ with a 2×2 matrix with diagonal elements $S^{--}=\Pi_-S\Pi_-$, $S^{++}=\Pi_+S\Pi_+$ and off-diagonal elements $S^{-+}=\Pi_-S\Pi_+$, $S^{+-}=\Pi_+S\Pi_-$. We shall prove that the off-diagonal elements improve regularity on the Besov scale associated to A if S belongs to a regularity class $\mathscr{C}^{\alpha,p}(A)$.

Theorem. *If $S\in\mathscr{C}^{\alpha,p}(A)$ for some real $\alpha>0$ and some $p\in[1,\infty]$, then $\Pi_\mp S\Pi_\pm\mathscr{H}\subset\mathscr{H}_{\alpha,p}$. In particular, if $S\in\mathscr{C}^{\alpha,2}(A)$ then $\Pi_\mp S\Pi_\pm\in B(\mathscr{H}_s;\mathscr{H}_{s+\alpha})$ for all real s such that $-\alpha\le s\le 0$.*

Proof. (i) We first prove a weak-type estimate, namely we show that $S_0\equiv \Pi_-S\Pi_+$ sends $\mathscr{H}$ into $\mathscr{H}_{m,\infty}$ if $S\in C^m(A)$ for some integer $m\ge 1$. Let χ be the characteristic function of the real set defined by $1\le|x|\le 2$. Then it suffices to show that $\|\chi(\varepsilon A)S_0\|\le C\varepsilon^m$ for some constant C and all $0<\varepsilon<1$. Set $S_\tau=\exp(\tau A)S_0\exp(-\tau A)$ for $\tau\ge 0$. Then $\tau\mapsto S_\tau$ is strongly of class C^m on $[0,\infty)$ and its k-th order derivative $(0\le k\le m)$ is equal to $\operatorname{ad}_A^k S_\tau=\exp(\tau A)\Pi_-(\operatorname{ad}_A^k S)\Pi_+\exp(-\tau A)$. By making a Taylor expansion up to order m we get (see (6.1)):

$$S_0=\sum_{k=0}^{m-1}\frac{(-1)^k}{k!}\operatorname{ad}_A^k S_1+\frac{(-1)^m}{(m-1)!}\int_0^1 \operatorname{ad}_A^m S_\tau\cdot\tau^{m-1}d\tau.$$

The operators $\operatorname{ad}_A^k S_1$ clearly send $\mathscr{H}_{-\infty}$ into $\mathscr{H}_{+\infty}$, so it suffices to consider the contribution of the integral term. We have :

$$\begin{aligned}\int_0^1\|\chi(\varepsilon A)\operatorname{ad}_A^m S_\tau\|\tau^{m-1}d\tau &\le \|\operatorname{ad}_A^m S_0\|\int_0^1\|\chi(\varepsilon A)\Pi_-e^{\tau A}\|\tau^{m-1}d\tau\\ &\le \|\operatorname{ad}_A^m S_0\|\int_0^1\sup_{x>0}\chi(\varepsilon x)e^{-\tau x}\tau^{m-1}d\tau\\ &= \|\operatorname{ad}_A^m S_0\|\int_0^1 e^{-\tau/\varepsilon}\tau^{m-1}d\tau\le C\varepsilon^m,\end{aligned}$$

which is the desired estimate.

(ii) Let $\mathscr{P}:B(\mathscr{H})\to B(\mathscr{H})$ be the linear continuous operator given by $\mathscr{P}S=\Pi_-S\Pi_+$. Then $\|\mathscr{P}\|=1$ and $\mathscr{P}C^m(A)\subset B(\mathscr{H};\mathscr{H}_{m,\infty})$ (by what we have shown above and the closed graph theorem). On the space $C^m(A)$ there is a natural Banach space structure such that the embedding $C^m(A)\subset B(\mathscr{H})$ be continuous. Then one can obtain the spaces $\mathscr{C}^{\alpha,p}(A)$ by real interpolation: $\mathscr{C}^{\alpha,p}(A)=(C^m(A),B(\mathscr{H})_{\theta,p}$ with $\theta=1-\alpha/m$ if $0<\alpha<m$ (see (5.2.22) in [ABG2]). Now fix a vector $f\in\mathscr{H}$ and consider the map $S\mapsto\mathscr{P}(S)f$. It sends $B(\mathscr{H})$ into $\mathscr{H}$ continuously and $C^m(A)$ into $\mathscr{H}_{m,\infty}$ continuously. By interpolation (use (2.11)) it will send $\mathscr{C}^{\alpha,p}$ into

$\mathscr{H}_{\alpha,p}$. For the last assertion of the theorem note that S^* is of the same class as S. □

4. Resolvent Families and Mourre Estimate

4.1. In this paper we shall call (self-adjoint) *resolvent family* a family $\{R(z) \mid z \in \mathbb{C} \setminus \mathbb{R}\}$ of bounded operators in $\mathscr{H}$ depending on the complex non-real number z and satisfying the conditions $R(z_1) - R(z_2) = (z_1 - z_2)R(z_1)R(z_2)$ (first resolvent identity) and $R(z)^* = R(z^*)$. It follows easily from these relations that the map $R(\cdot) : \mathbb{C} \setminus \mathbb{R} \to B(\mathscr{H})$ is holomorphic and $(d/dz)^k R(z) = k!R(z)^{k+1}$; moreover, $R(z)$ is a normal operator for each z. The *spectrum* of the resolvent family is the set of real numbers λ such that the function $R(\cdot)$ has no holomorphic extension to any neighbourhood of λ. Clearly, the first resolvent identity holds for all complex numbers z_1, z_2 not in the spectrum of the resolvent family. Note also that we have $\|R(z)\| \leq |\operatorname{Im} z|^{-1}$ as a consequence of this identity; moreover, one has that $\lim_{\varepsilon\to 0} \|R(\lambda + i\varepsilon)\| = \infty$ if and only if λ belongs to the spectrum of $\{R(z)\}$.

It is more convenient to think of resolvent families in terms of possibly non-densely defined self-adjoint operators in $\mathscr{H}$. To be precise, we shall generalize the standard notion of self-adjoint operator: for us a *self-adjoint operator* in $\mathscr{H}$ is a linear operator H defined on a linear subspace $D(H)$ of $\mathscr{H}$ with values in $\mathscr{H}$, such that $HD(H) \subset \overline{D(H)}$ (=closure of $D(H)$ in $\mathscr{H}$) and such that, when considered as operator in the Hilbert space $\overline{D(H)}$, H is self-adjoint in the usual sense (so, in our terminology, a "usual" self-adjoint operator is a *densely defined self-adjoint operator*). The resolvent family associated to such an operator is defined by $R(z)f = (H - z)^{-1}f$ if $f \in \overline{D(H)}$ (note that $H - z : D(H) \to \overline{D(H)}$ is bijective if $\operatorname{Im} z \neq 0$) and $R(z)f = 0$ if f is orthogonal to $D(H)$. Reciprocally, if $\{R(z)\}$ is a resolvent family then the range of the operator $R(z)$ is independent of z and if we denote it by $D(H)$ then the restriction of $R(z)$ to $\overline{D(H)}$ is the resolvent of a densely defined self-adjoint operator in $\overline{D(H)}$. It will be convenient to set $R(z) = (H - z)^{-1}$, although $R(z)$ is, in fact, the extension by zero on $\overline{D(H)}^{\perp}$ of the r.h.s. (so H should be thought as being equal to ∞ on $\overline{D(H)}^{\perp}$). Note that the spectrum of the densely defined self-adjoint operator H in the Hilbert space $\overline{D(H)}$ coincides with the spectrum of the resolvent family $\{R(z)\}$. From now on, when we speak about some spectral property of the self-adjoint operator H we refer in fact to a property of the densely defined self-adjoint operator H in $\overline{D(H)}$.

4.2. If φ is a complex continuous function on $\mathbb{R}$ which tends to zero at infinity, then $\varphi(H)$ is a well-defined bounded operator in $\overline{D(H)}$ (by the functional calculus associated to the densely defined self-adjoint operator H in $\overline{D(H)}$). We extend $\varphi(H)$ to a bounded operator on $\mathscr{H}$ by the rule: $\varphi(H)f = 0$ if f is orthogonal to $D(H)$. Then clearly we have

$$\varphi(H) = \operatorname*{w\text{-}lim}_{\varepsilon\to+0} \frac{1}{\pi} \int_{\mathbb{R}} \varphi(\lambda) \operatorname{Im} R(\lambda + i\varepsilon) d\lambda \tag{4.1}$$

where the integral exists in the weak topology. This formula expresses $\varphi(H)$ directly in terms of the resolvent family but it is not convenient for our purposes here.

We shall now present a very simple proof of a version of the Helffer-Sjöstrand formula. Let r be a strictly positive number. We shall use Taylor's formula for the function $\mu \mapsto R(\lambda+i\mu)$ on the interval $[\varepsilon, r]$ with $0<\varepsilon<r$. Since, by holomorphy, we have $(d/d\mu)R(\lambda+i\mu) = (id/d\lambda)R(\lambda+i\mu)$, we get for any integer $m \geq 1$:

$$\begin{aligned} R(\lambda+i\varepsilon) = & \sum_{k=0}^{m-1} \frac{(r-\varepsilon)^k}{k!}(-id/d\lambda)^k R(\lambda+ir) \\ & + \frac{1}{(m-1)!}\int_\varepsilon^r (-id/d\lambda)^m R(\lambda+i\mu)(\mu-\varepsilon)^{m-1} d\mu. \end{aligned}$$

So if φ is of class $C_0^m(\mathbb{R})$ we get after an integration by parts:

$$\begin{aligned} \int_{\mathbb{R}} \varphi(\lambda)R(\lambda+i\varepsilon)d\lambda = & \sum_{k=0}^{m-1} \frac{(r-\varepsilon)^k}{k!}\int_{\mathbb{R}} i^k \varphi^{(k)}(\lambda)R(\lambda+ir)d\lambda \\ & + \frac{i^m}{(m-1)!}\int_{\mathbb{R}}\int_\varepsilon^r \varphi^{(m)}(\lambda)R(\lambda+i\mu)\mu^{m-1}d\mu d\lambda. \end{aligned}$$

We clearly have $\|R(\lambda+i\mu)\| \leq |\mu|^{-1}$, hence if $m \geq 2$ then the double integral exists in norm even if $\varepsilon = 0$. Hence for each $m \geq 2$ the next limit exists in norm and we have

$$\begin{aligned} \text{(4.2)} \quad \lim_{\varepsilon\to+0}\int_{\mathbb{R}} \varphi(\lambda)R(\lambda+i\varepsilon)d\lambda = & \sum_{k=0}^{m-1} \frac{(ir)^k}{k!}\int_{\mathbb{R}} \varphi^{(k)}(\lambda)R(\lambda+ir)d\lambda \\ & + \frac{i^m}{m!}\int_{\mathbb{R}}\int_0^r \varphi^{(m)}(\lambda)R(\lambda+i\mu)d\mu^m d\lambda, \end{aligned}$$

where the integrals exist in norm and $d\mu^m = m\mu^{m-1}d\mu$. By inserting this into (4.1) (assuming φ real to begin with) one gets an expression for $\varphi(H)$ first obtained in [BG1] (by a slightly more complicated method) and then developed in [ABG2] (for a quite general class of functions φ). The difference with respect to the Helffer-Sjostrand formula (see [D1, D2] and references therein) is that no almost analytic extension of φ is needed.

In the preceding argument one can allow r to be a function of λ (e.g. $r(\lambda) = |\lambda|$ for $|\lambda| > 1$). Then the final formula, although more complicated, is sometimes more useful. Let us set $R^{(k)}(z) = (d/dz)^k R(z) = k!R(z)^{k+1}$. Then the Taylor expansion of the resolvent used before can be written as follows:

$$R(\lambda+i\varepsilon) = \sum_{k=0}^{m-1} \frac{i^k}{k!}(\varepsilon-r)^k R^{(k)}(\lambda+ir) + \frac{(-i)^m}{(m-1)!}\int_\varepsilon^r \partial_\lambda^m R(\lambda+i\mu)\cdot(\mu-\varepsilon)^{m-1}d\mu.$$

This holds for an arbitrary $r > \varepsilon$. Now assume that a function $\lambda \mapsto r(\lambda)$ of class C^m is given on $\mathbb{R}$ such that $\inf_{\lambda\in\mathbb{R}} r(\lambda) > 0$ and let $0 < \varepsilon < \inf r(\lambda)$. The following formula is easily verified by induction

$$\partial_\lambda^m \int_\varepsilon^{r(\lambda)} F(\lambda,\mu)d\mu = \sum_{j+k=m-1} \partial_\lambda^j[r'(\lambda)F^{(k)}(\lambda, r(\lambda))] + \int_\varepsilon^{r(\lambda)} F^{(m)}(\lambda,\mu)d\mu,$$

where $F^{(k)}(\lambda,\mu)=\partial_\lambda^k F(\lambda,\mu)$. In particular, the last term in the expansion of $R(\lambda+i\varepsilon)$ (with $r=r(\lambda)$) is equal to

$$\frac{(-i\partial_\lambda)^m}{(m-1)!}\int_\varepsilon^{r(\lambda)} R(\lambda+i\mu)(\mu-\varepsilon)^{m-1}d\mu$$
$$-\sum_{j+k=m-1}\frac{(-i)^m}{(m-1)!}\partial_\lambda^j[r'(\lambda)R^{(k)}(\lambda+ir(\lambda))(r(\lambda)-\varepsilon)^{m-1}].$$

So if $\varphi\in C_0^m(\mathbb{R})$ we obtain for $\int_{\mathbb{R}}\varphi(\lambda)R(\lambda+i\varepsilon)d\lambda$ the following expression

$$\sum_{k=0}^{m-1}\int_{\mathbb{R}}\Big[\frac{(-i)^k}{k!}(r(\lambda)-\varepsilon)^k\varphi(\lambda)$$
$$+\frac{i^{m-k}(-i)^k}{(m-1)!}r'(\lambda)(r(\lambda)-\varepsilon)^{m-1}\varphi^{(m-1-k)}(\lambda)\Big]\cdot R^{(k)}(\lambda+ir(\lambda))d\lambda$$
$$+\frac{i^m}{(m-1)!}\int_{\mathbb{R}}\int_\varepsilon^{r(\lambda)}\varphi^{(m)}(\lambda)R(\lambda+i\mu)(\mu-\varepsilon)^{m-1}d\mu d\lambda.$$

Now let us assume $m\geq 2$ (this assures the convergence in norm of the integrals below; in the case $m=1$ the next formulas are to be interpreted in the weak topology, cf. Section 6.1 in [ABG2]). Then the next limit clearly exists in norm and we have

$$\lim_{\varepsilon\to+0}\int_{\mathbb{R}}\varphi(\lambda)R(\lambda+i\varepsilon)d\lambda= \tag{4.3}$$
$$\sum_{k=0}^{m-1}\int_{\mathbb{R}}\Big[\frac{1}{k!}\varphi(\lambda)+\frac{ir'(\lambda)}{(m-1)!}(ir(\lambda))^{m-1-k}\varphi^{(m-1-k)}(\lambda)\Big]\cdot$$
$$(-ir(\lambda))^kR^{(k)}(\lambda+ir(\lambda))d\lambda$$
$$+\frac{i^m}{m!}\int_{\mathbb{R}}\int_0^{r(\lambda)}\varphi^{(m)}(\lambda)R(\lambda+i\mu)d\mu^m d\lambda.$$

By using (4.1) as before one gets a new expression for $\varphi(H)$ which has the advantage that it will hold for a larger class of functions φ. Indeed, we assumed until now that φ has compact support, but the final formula will clearly remain valid for all φ such that the integrals from the r.h.s. of the formula are norm convergent. Assume, for example, that $\operatorname{supp}\varphi\subset[1,\infty)$ and take $r(\lambda)=\lambda$ for $\lambda>1$. Then

$$\lim_{\varepsilon\to+0}\int_{\mathbb{R}}\varphi(\lambda)R(\lambda+i\varepsilon)d\lambda=$$
$$\sum_{k=0}^{m-1}\int_{\mathbb{R}}\Big[\frac{1}{k!}\varphi(\lambda)+\frac{i(i\lambda)^{m-1-k}}{(m-1)!}\varphi^{(m-1-k)}(\lambda)\Big]\cdot(-i\lambda)^kR^{(k)}(\lambda+i\lambda)d\lambda$$
$$+\frac{i^m}{m!}\int_{\mathbb{R}}\int_0^{\lambda}\varphi^{(m)}(\lambda)R(\lambda+i\mu)d\mu^m d\lambda.$$

We have $\|\lambda^kR^{(k)}(\lambda+i\lambda)\|\leq\lambda^{-1}$ so the integrals from the sum above are convergent provided that $\int_1^\infty|\lambda^j\varphi^{(j)}(\lambda)|\lambda^{-1}d\lambda<\infty$ for $0\leq j\leq m-1$. Since $\int_0^\lambda\|R(\lambda+i\mu)\|d\mu^m\leq m\int_0^\lambda\mu^{m-2}d\mu=m(m-1)^{-1}\lambda^{m-1}$ if $m\geq 2$, we see that the convergence of the last integral is assured by the condition $\int_1^\infty |\lambda^m \varphi^{(m)}(\lambda)|\lambda^{-1} d\lambda < \infty$. If these conditions are satisfied then by taking into account that $\partial_\lambda^k R(\lambda + i\lambda) = (1+i)^k R^{(k)}(\lambda + i\lambda)$ we also obtain

$$\text{(4.4)} \qquad \lim_{\varepsilon \to +0} \int_{\mathbb{R}} \varphi(\lambda) R(\lambda + i\varepsilon) d\lambda = \int_{\mathbb{R}} \sum_{k=0}^{m-1} (1+i)^{-k} \partial_\lambda^k \Big[\frac{(i\lambda)^k}{k!} \varphi(\lambda) +$$

$$\frac{i(i\lambda)^{m-1}}{(m-1)!} \varphi^{(m-1-k)}(\lambda) \Big] \cdot R(\lambda + i\lambda) d\lambda + \frac{i^m}{m!} \int_{\mathbb{R}} \int_0^\lambda \varphi^{(m)}(\lambda) R(\lambda + i\mu) d\mu^m d\lambda.$$

4.3. Let us denote by $\mathscr{S}(\mathscr{H})$ the set of all self-adjoint resolvent families on $\mathscr{H}$ (or, equivalently, the set of all self-adjoint operators in $\mathscr{H}$) and by $\mathscr{U}(\mathscr{H})$ the set of all unitary operators in $\mathscr{H}$. The *Cayley transform* of a resolvent family $\{R(z)\}$ (or of a self-adjoint operator H) is the unitary operator $U = 1 - 2iR(-i)$ (or $U = (H-i)(H+i)^{-1}$ with the convention $Uf = f$ if f is orthogonal to $\overline{D(H)}$). The unitarity of U follows from $U^* = 1 + 2iR(i)$, which gives $U^*U = UU^* = 1$. We obtain in this way a map $\mathscr{S}(\mathscr{H}) \to \mathscr{U}(\mathscr{H})$ which is bijective: the injectivity follows from the fact that $R(-i)$ determines $R(\cdot)$ (because $R(\cdot)$ is holomorphic and all its derivatives at $z = -i$ are known once $R(-i)$ is known), while the surjectivity follows from the formula $Hf = i(1+U)(1-U)^{-1}$ for $f \in D(H) \equiv$ range of $(U-1)$, where $(1-U)^{-1}$ is the inverse of the restriction of $1-U$ to the orthogonal complement of its kernel. Observe also that by taking $z_0 = -i$ in (4.5) one gets the following expression for $R(z)$ in terms of U: if $\zeta = (z-i)(z+i)^{-1}$ then $(z+i)R(z) = (1-U)(U-\zeta)^{-1}$.

$\mathscr{U}(\mathscr{H})$ is a complete metric space for the metric induced by the norm. So if we equip $\mathscr{S}(\mathscr{H})$ with the metric

$$\delta(H_1, H_2) = \|(H_1+i)^{-1} - (H_2+i)^{-1}\| = \frac{1}{2}\|U_1 - U_2\|,$$

where U_j is the Cayley transform of the self-adjoint operator H_j, *we provide $\mathscr{S}(\mathscr{H})$ with a complete metric space structure.* We have $\lim_{n\to\infty} H_n = H$ in $\mathscr{S}(\mathscr{H})$ if and only if H_n converges to H in norm-resolvent sense.

The set of densely defined self-adjoint operators is dense in $\mathscr{S}(\mathscr{H})$. Indeed, $H \in \mathscr{S}(\mathscr{H})$ is densely defined if and only if its Cayley transform U has not 1 as eigenvalue, so it suffices to show that a unitary operator U which has 1 as eigenvalue is a norm limit of unitary operators U_λ which do not have 1 as eigenvalue. Let P be the orthogonal projection of $\mathscr{H}$ onto $ker(U-1)$ and let V be the unitary operator in $(P\mathscr{H})^\perp$ such that $U = V \oplus P$ relatively to the decomposition $\mathscr{H} = (P\mathscr{H})^\perp \oplus P\mathscr{H}$. Then we set $U_\lambda = V \oplus \lambda P$ with $|\lambda| = 1, \lambda \neq 1$. Since $V - 1$ is injective, $U_\lambda - 1 = (V-1) \oplus (\lambda - 1)P$ is injective too and $\|U_\lambda - U\| = |\lambda - 1| \to 0$ as $\lambda \to 1$.

The operator $H \in \mathscr{S}(\mathscr{H})$ is everywhere defined (hence bounded) if and only if the number 1 does not belong to the spectrum of its Cayley transform U. One can easily prove as above that *the set of (bounded) everywhere defined self-adjoint operators is dense in $\mathscr{S}(\mathscr{H})$.*

For each function $\varphi : \mathbb{R} \to \mathbb{C}$ continuous and convergent to zero at infinity the map $\mathscr{S}(\mathscr{H}) \ni H \mapsto \varphi(H) \in B(\mathscr{H})$ is norm continuous. Indeed, the set of functions φ that have this property is stable for addition, multiplication and conjugation, and contains the function $\varphi(x) = (x+i)^{-1}$; then we apply the Stone-Weierstrass theorem. Note that $\mathscr{S}(\mathscr{H})$ has the weakest topology for which this property holds.

4.4. Assume now that a densely defined self-adjoint operator A is given in $\mathscr{H}$ and let $\{R(z)\}$ be the resolvent family associated to a self-adjoint operator H. We shall say that $\{R(z)\}$ (or H) *is of class* $C^k(A)$, $C^k_u(A)$, *or* $\mathscr{C}^{s,p}(A)$ if there is a complex number z_0 outside the spectrum of H such that the bounded operator $R(z_0)$ is of class $C^k(A)$, $C^k_u(A)$ or $\mathscr{C}^{s,p}(A)$ respectively. Note that if this property holds for some z_0 then it holds for all complex numbers z outside the spectrum of H. Indeed, the operator $1-(z-z_0)R(z_0)$ will then be invertible in $\mathscr{H}$ with inverse equal to $1+(z-z_0)R(z)$, and so

$$R(z) = R(z_0)[1-(z-z_0)R(z_0)]^{-1}. \tag{4.5}$$

Hence the assertion follows from the fact that $C^k(A)$, $C^k_u(A)$ and $\mathscr{C}^{s,p}(A)$ are full subalgebras of $B(\mathscr{H})$. In particular, H belongs to one of the preceding regularity classes with respect to A if and only if its Cayley transform U belongs to the same class.

The map $\lambda \mapsto (\lambda - i)(\lambda + i)^{-1}$ extends to a homeomorphism of the one point compactification $\mathbb{R} \cup \{\infty\}$ of the real line $\mathbb{R}$ onto the unit circle $\Sigma = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$ which sends ∞ into 1. So the rule $\varphi(\lambda) = \varphi^{\#}((\lambda - i)(\lambda+i)^{-1})$ will give us a bijective correspondence between complex functions φ on $\mathbb{R}$ which are continuous and tend to zero at infinity and functions $\varphi^{\#} : \Sigma \to \mathbb{C}$ continuous and such that $\varphi^{\#}(1) = 0$. Then clearly we have $\varphi(H) = \varphi^{\#}(U)$ if U is the Cayley transform of the self-adjoint operator H. This fact allows us to use the Proposition from §3.8 in order to show that the bounded operator $\varphi(H)$ is of the same regularity class with respect to A as H if the function φ is sufficiently smooth. For example, by taking into account the remark made after the proof of the quoted proposition we see that *if the function* $\varphi : \mathbb{R} \to \mathbb{C}$ *has compact support and is of Sobolev class* $\mathscr{H}^{m+1+\varepsilon}(\mathbb{R})$ *for some integer* $m \geq 1$ *and some real* $\varepsilon > 0$ *then* $\varphi(H)$ *is of class* $C^m(A)$, *or* $C^m_u(A)$, *or* $\mathscr{C}^{s,p}(A)$ *if* H *is of class* $C^m(A)$, *or* $C^m_u(A)$, *or* $\mathscr{C}^{s,p}(A)$ *respectively and if* $0 < s < m$. The behaviour at infinity of φ is unnecessarily restricted here. For example, denote by $C^\infty(\mathbb{R} \cup \{\infty\})$ the set of φ such that $\varphi^{\#}$ is of class C^∞ on Σ (this class of functions is discussed in detail in [ABG3]). Let $\varphi \in C^\infty(\mathbb{R} \cup \{\infty\})$ with $\varphi(\infty) = 0$; then $\varphi(H)$ is always of the same class as H.

These results can be improved by using the Helffer-Sjostrand formula as presented in [D1, D2], or, equivalently, formula (4.3), or the simpler case (4.4). Let us give one exmaple. If $A_\varepsilon = (i\varepsilon)^{-1}(W_\varepsilon - 1)$ for $\varepsilon \neq 0$, then by starting with $R(z) = R(z_0) + (z-z_0)R(z)R(z_0)$ one gets:

$$[A_\varepsilon, R(z)](1-(z-z_0)R(z_0)) = (1+(z-z_0)R(z))[A_\varepsilon, R(z_0)].$$

The operator $1+(z_0-z)R(z_0)$ is invertible and has $1+(z-z_0)R(z)$ as inverse. Hence by right multiplying the preceding identity by $1+(z-z_0)R(z)$ and then by making $\varepsilon \to 0$ (under the assumption $H \in C^1(A)$ we get

$$\mathscr{A}[R(z)] = (1+(z-z_0)R(z))\mathscr{A}[R(z_0)](1+(z-z_0)R(z)).$$

If we take here $z_0 = i$ and $z = \lambda + i\mu$ we see that there is a constant C such that $\|\mathscr{A}[R(\lambda+i\mu)]\| \leq C(1+\lambda^2\mu^{-2})$. Now consider the identity (4.4) with $m=3$ (in fact it is possible, but slightly more difficult, to take $m=2$, which would give a better result below; see the proof of Theorem 6.2.5 from [ABG2]). It follows immediately that $\varphi(H)\in C^1(A)$ *if* H *is of class* $C^1(A)$ *and* $\sum_{k=0}^{3}\int_{\mathbb{R}}\langle\lambda\rangle^k|\varphi^{(k)}(\lambda)|d\lambda<\infty$. (Note that (4.4) is valid if and only if $\operatorname{supp}\varphi\subset[1,\infty)$; to get the result one has to decompose $\varphi=\varphi_-+\varphi_0+\varphi_+$ with $\varphi_-(x)=0$ if $x>-1$, φ_0 of compact support, and $\varphi_+(x)=0$ if $x<1$).

4.5. The commutator $[H,iA]$ is defined as the symmetric sesquilinear form on the domain $D(A)\cap D(H)$ given by $\langle f,[H,iA]f\rangle=2\operatorname{Re}\langle Hf,iAf\rangle$. Now assume that H is of class $C^1(A)$; then $D(A)\cap D(H)$ is a dense subspace of $D(H)$ (for the graph topology). Indeed, we have $R(z)D(A)\subset D(A)$ (see §3.9) and $R(z)$ is a continuous surjective map $\mathscr{H}$ onto $D(H)$ (z does not belong to the spectrum of H); since $D(A)$ is dense in $\mathscr{H}$ we see that $R(z)D(A)$ is a dense subspace of $D(H)$ and $R(z)D(A)\subset D(A)\cap D(H)$.

Let φ, $\psi\in C_0^\infty(\mathbb{R})$ real and such that $x\varphi(x)=\psi(x)\varphi(x)$. Then $\varphi(H)\in C^1(A)$, hence for $f\in D(A)$ we have $\varphi(H)f\in D(A)\cap D(H)$ (see §3.9) and

$$\begin{aligned}\langle\varphi(H)f,[H,iA]\varphi(H)f\rangle&=2\operatorname{Re}\langle H\varphi(H)f,iA\varphi(H)\rangle\\&=2\operatorname{Re}\langle\psi(H)\varphi(H)f,iA\varphi(H)f\rangle=\langle\varphi(H)f,[\psi(H),iA]\varphi(H)f\rangle.\end{aligned}$$

In other terms we have $\varphi(H)D(A)\subset D(A)\cap D(H)$ and

$$\varphi(H)[H,iA]\varphi(H)=\varphi(H)[\psi(H),iA]\varphi(H) \tag{4.6}$$

as sesquilinear forms on $D(A)$. But $\psi(H)\in C^1(A)$, so the r.h.s. of (4.6) extends to the bounded operator $\varphi(H)i\mathscr{A}[\psi(H)]\varphi(H)$ on $\mathscr{H}$, hence the sesquilinear form $\varphi(H)[H,iA]\varphi(H)$ with domain $D(A)$ (dense in $\mathscr{H}$) extends to a bounded operator, denoted $\varphi(H)i\mathscr{A}[H]\varphi(H)$, on $\mathscr{H}$ and we have $\varphi(H)\mathscr{A}[H]\varphi(H)=\varphi(H)\mathscr{A}[\psi(H)]\varphi(H)$.

We can now define the *strict Mourre set* $\mu^A(H)$ *of* H *with respect to* A as the set of real numbers λ such that there are a real function $\varphi\in C_0^\infty(\mathbb{R})$ with $\varphi(\lambda)\neq0$ and a strictly positive real number a such that $\varphi(H)i\mathscr{A}[H]\varphi(H)\geq a\varphi(H)^2$. This is clearly an open subset of $\mathbb{R}$.

In non-trivial practical situations it is impossible to find explicitly the set $\mu^A(H)$. For this reason it is useful to introduce the *Mourre set* $\tilde{\mu}^A(H)$ *of* H *with respect to* A, defined as the set of real numbers λ for which there are a real function $\varphi\in C_0^\infty(\mathbb{R})$ with $\varphi(\lambda)\neq0$, a strictly positive real number a and a compact operator K on $\mathscr{H}$ such that $\varphi(H)i\mathscr{A}[H]\varphi(H)\geq a\varphi(H)^2+K$. It turns out that in many interesting cases one can describe $\tilde{\mu}^A(H)$ rather explicitly. For this reason the next result is important. Note that $\tilde{\mu}^A(H)$ is an open set and $\mu^A(H)\subset\tilde{\mu}^A(H)$.

Theorem. *The set* $\tilde{\mu}^A(H)\setminus\mu^A(H)$ *does not have accumulation points inside* $\tilde{\mu}^A(H)$ *and it consists of eigenvalues of* H *of finite multiplicity. The spectrum of* H *in* $\mu^A(H)$ *is purely continuous.*

Proof. (i) We first show that the *Virial Theorem* holds true, namely that if $f\in D(H)$ is an eigenvector of H then $\langle f,\varphi(H)\mathscr{A}[H]\varphi(H)f\rangle=0$ for all $\varphi\in C_0^\infty(\mathbb{R})$ real. Let $\psi\in C_0^\infty(\mathbb{R})$ real such that $x\varphi(x)=\psi(x)\varphi(x)$. Then

$$\begin{aligned}\langle f,\varphi(H)\mathscr{A}[H]\varphi(H)f\rangle&=\langle f,\varphi(H)\mathscr{A}[\psi(H)]\varphi(H)f\rangle\\&=\varphi(\lambda)^2\lim_{\varepsilon\to0}\langle f,[\psi(H),A_\varepsilon]f\rangle\end{aligned}$$

where λ is the eigenvalue of H associated to f and $A_\varepsilon = (i\varepsilon)^{-1}(W_\varepsilon - 1)$. But

$$\begin{aligned}\langle f, [\psi(H), A_\varepsilon]f\rangle &= \langle \psi(H)f, A_\varepsilon f\rangle - \langle f, A_\varepsilon \psi(H)f\rangle \\ &= \psi(\lambda)\langle f, A_\varepsilon f\rangle - \psi(\lambda)\langle f, A_\varepsilon f\rangle = 0,\end{aligned}$$

so the virial theorem is proved.

(ii) Now assume that φ is a real function of class $C_0^\infty(\mathbb{R})$, $a > 0$ is a real number and K is compact operator such that $\varphi(H)i\mathscr{A}[H]\varphi(H) \geq a\varphi(H)^2 + K$. If $f \in D(H)$ is an eigenvector of H associated to the eigenvalue λ then $0 \geq a\varphi(\lambda)^2\|f\|^2 + \langle f, Kf\rangle$. It follows that for each $\varepsilon > 0$ there is at most a finite number of eigenvalues λ of H with $|\varphi(\lambda)| \geq \varepsilon$ and each has finite multiplicity. Otherwise there is an infinite orthonormal sequence $\{f_n\}$ consisting of eigenvectors with eigenvalues λ_n such that $|\varphi(\lambda_n)| \geq \varepsilon$, hence $\langle f_n, Kf_n\rangle \leq -a\varepsilon < 0$; but $\lim\langle f_n, Kf_n\rangle = 0$ due to the compacity of K, so we have a contradiction. If $K = 0$ then clearly there are no eigenvalues λ of H with $\varphi(\lambda) \neq 0$.

(iii) At this stage we have shown that there are no eigenvalues in $\mu^A(H)$ and that the eigenvalues in $\tilde{\mu}^A(H)$ are of finite multiplicity and do not have accumulation points inside $\tilde{\mu}^A(H)$. It remains to be shown that the points from $\tilde{\mu}^A(H) \setminus \mu^A(H)$ are eigenvalues of H. For this it suffices to prove the following assertion: if λ is not an eigenvalue of H and if there are $\varphi_0 \in C_0^\infty(\mathbb{R})$ real with $\varphi_0(\lambda) \neq 0$ a real number $a_0 > 0$ and a compact operator K such that $\varphi_0(H)i\mathscr{A}[H]\varphi_0(H) \geq a_0\varphi_0(H)^2 + K$, then for each $a < a_0$ there is $\varphi \in C_0^\infty(\mathbb{R})$ real with $\varphi(\lambda) \neq 0$ and such that $\varphi(H)i\mathscr{A}[H]\varphi(H) \geq a\varphi(H)^2$.

(iv) Let us choose a real function $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(x) = x$ on $\operatorname{supp}\varphi_0$, and let us set $B = i\mathscr{A}[\psi(H)]$, so that B is a bounded self-adjoint operator. Then we have $\varphi_0(H)i\mathscr{A}[H]\varphi_0(H) = \varphi_0(H)B\varphi_0(H)$ hence $\varphi_0(H)B\varphi_0(H) \geq a\varphi_0(H)^2 + K$. We can assume that $\varphi_0(x) = 1$ on a neighbourhood of λ (otherwise we left and right multiply the preceding inequality by $\eta(H)$, where $\eta \in C_0^\infty(\mathbb{R})$ is real and such that $\eta(x) = \varphi_0(x)^{-1}$ on a neighbourhood of λ; hence it suffices to replace φ_0 by $\varphi_0\eta$ and K by $\eta(H)K\eta(H)$). Now let $\varphi_n \in C_0^\infty(\mathbb{R})$ such that $0 \leq \varphi_n \leq 1$, $\varphi_n(x) = 1$ if $|x - \lambda| \leq 2^{-n}$ and $\varphi_n(x) = 0$ if $|x - \lambda| \geq 2^{-n+1}$. Then for $n \in \mathbb{N}$ large enough we have $\varphi_n(H)B\varphi_n(H) \geq a_0\varphi_n(H)^2 + \varphi_n(H)K\varphi_n(H)$. Since λ is not an eigenvalue of H, we have $\lim_{n\to\infty}\varphi_n(H) = 0$ strongly, hence $\|\varphi_n(H)K\varphi_n(H)\| \to 0$ as $n \to \infty$. So there is $n \in \mathbb{N}$ such that $\varphi_n(H)K\varphi_n(H) \geq a - a_0$, in particular $\varphi_n B\varphi_n(H) \geq a_0\varphi_n(H)^2 + a - a_0$. Upon pre-and post multiplication of this inequality by $\varphi_{n+1}(H)$ we get $\varphi_{n+1}(H)B\varphi_{n+1}(H) \geq a_0\varphi_{n+1}(H)^2$, and so it suffices to take $\varphi = \varphi_{n+1}$. $\square$

4.6. We shall say that the self-adjoint operator H (or the resolvent family $\{R(z)\}$ associated to it) *has a spectral gap* if its spectrum is not equal to $\mathbb{R}$. This class of operators is convenient because the study of its resolvent can easily be reduced to the study of the resolvent of a bounded, everywhere defined self-adjoint operator. Indeed, let λ_0 be a real number outside the spectrum of H and let $R = -R(\lambda_0) = (\lambda_0 - H)^{-1}$. Then R is a bounded self-adjoint operator $R : \mathscr{H} \to \mathscr{H}$ and for $\operatorname{Im} z \neq 0$

$$R(z) = (\lambda_0 - z)^{-1}R[R - (\lambda_0 - z)^{-1}]^{-1}. \tag{4.7}$$

In the rest of this paragraph we shall keep the notations introduced above and we shall explain how one may reduce the proof of the theorems stated in Section 1 to the proof of the corresponding results with H replaced by R. First we have to relate the (strict) Mourre set of H to that of R.

Proposition. *H is of class $C^1(A)$ if and only if R is of class $C^1(A)$. A real number $\lambda \neq \lambda_0$ belongs to $\mu^A(H)$ (resp. $\tilde{\mu}^A(H)$) if and only if $(\lambda_0 - \lambda)^{-1}$ belongs to $\mu^A(R)$ (resp. $\tilde{\mu}^A(R)$).*

The proof of this result is straightforward and will be not given; see Proposition 8.3.4 in [ABG2] and note that in our context one can replace the class C^1_u by the class C^1 (cf. Propositions 7.2.5 and 7.2.7 of [ABG2] for the case of densely defined operators).

Let us set $\zeta = (\lambda_0 - z)^{-1}$. Then $z \mapsto \zeta$ is a holomorphic diffeomorphism of $\mathbb{C} \setminus \{\lambda_0\}$ onto $\mathbb{C} \setminus \{0\}$ which leaves the upper and the lower half-planes invariant and restricts to a C^∞ diffeomorphism of $\mu^A(H) \setminus \{\lambda_0\}$ onto $\mu^A(R) \setminus \{0\}$. For f, $g \in \mathscr{H}$ and $\operatorname{Im} z \neq 0$ (so $\operatorname{Im} \zeta \neq 0$) we have as a consequence of (4.7):

$$\langle g, R(z) f\rangle = \zeta \langle g, (R - \zeta)^{-1} R f\rangle. \tag{4.8}$$

We can now prove Theorems B and C from §1.7 assuming that they are known in the case of the bounded everywhere defined self-adjoint operator R. Note that H is of the same class as R and that it suffices to assume that λ is outside a neighbourhood of λ_0 (because λ_0 is in the resolvent set of H, so $R(\cdot)$ is holomorphic on a neighbourhood of λ_0). If H is of class $\mathscr{C}^{1,1}(A)$, then $R \in \mathscr{C}^{1,1}(A)$, hence it leaves invariant $\mathscr{H}_{1/2,1}$ (by §3.9). So if, f, $g \in \mathscr{H}_{1/2,1}$ then Rf belongs to $\mathscr{H}_{1/2,1}$ and (4.8) clearly shows that Theorem B holds for H if it holds for R. If H is of class $\mathscr{C}^{s+1/2}(A)$ for some $s > 1/2$, then $R \in \mathscr{C}^{s+1/2}(A)$, hence $Rf \in \mathscr{H}_{s,p}$ if $f \in \mathscr{H}_{s,p}$ ($p = \infty$ and $p = 1$ are of interest). So we obtain Theorem C by using (4.8) again (for the class $\Lambda^{s-1/2}$ we may use the uniform boundedness principle, cf. the end of §1.5).

Now assume that we are in the conditions of Theorem D of §1.8. By (4.7) and by what we have seen before we clearly have for $\lambda \in \mu^A(H) \setminus \{\lambda_0\}$:

$$\Pi_- R(\lambda + i0) = (\lambda_0 - \lambda)^{-1} \Pi_- [R - (\lambda_0 - \lambda)^{-1} - i0)]^{-1} R.$$

Since $R \in B(\mathscr{H}_{s,p})$ we clarly obtain the result stated in Theorem D (assuming that it holds for R), and similarly for Theorem E of §1.8.

The argument is slightly more involved in the case of Theorem G of §1.10 (or, more generally, Theorems 6.5 and 6.8). We write (here $\Pi_\pm$ are as in §3.10):

$$\begin{aligned} \langle \Pi_- g, R(z) \Pi_+ f\rangle &= \zeta \langle \Pi_- g, (R - \zeta)^{-1} R \Pi_+ f\rangle \\ &= \zeta \langle \Pi_- g, (R - \zeta)^{-1} \Pi_+ R \Pi_+ f\rangle + \zeta \langle \Pi_- g, (R - \zeta)^{-1} \Pi_- R \Pi_+ f\rangle. \end{aligned}$$

The first term in the last member here is treated exactly as before (i.e. Theorem G applies directly, because $R\Pi_+ f \in \mathscr{H}_{s,\infty}$ if $f \in \mathscr{H}_{s,\infty}$). For the last term we first use the theorem from §3.10. The operator R belongs to $\mathscr{C}^{2+t-s}(A)$ with $s < 1/2$, $t > -1/2$ (cf. Theorem G). Since $-(2+t-s) < s < 2+t-s$ we have $\Pi_- R \Pi_+ \in B(\mathscr{H}_{s,\infty}; \mathscr{H}_{s+\alpha})$ if $\alpha < 2+t-s$ and $s+\alpha < 2+t-s$ (see §3.10). In any case, if $f \in \mathscr{H}_{s,\infty}$ then $f_0 = \Pi_- R \Pi_+ f \in \mathscr{H}_{t+3/2}$. Now we can apply Theorem D to $\langle \Pi_- g, (R-\zeta)^{-1} f_0\rangle$: we see that if τ is as in Theorem G and $g \in \mathscr{H}_{\tau-t,\infty}$, then its boundary value function is locally of class $\Lambda^{\tau+1/2}$, which is more than needed.

5. The Twisted Hamiltonian

In this section we consider a bounded everywhere defined self-adjoint operator H in $\mathscr{H}$, we denote by E its spectral measure, and we assume that H is of class $C^1(A)$. Furthermore, we fix a real open set J and a real number $a>0$ such that the following condition is satisfied: there is an open set J_0 with $\operatorname{dist}(J, \mathbb{R}\setminus J_0) \equiv \inf\{|x-y| \mid x \in J, y \notin J_0\} = \delta > 0$ and there is a number $a_0 > a$ such that $E(J_0) i\mathscr{A}[H]E(J_0) \geq a_0 E(J_0)$.

Our first result contains a version of the so-called quadratic estimate of Mourre; see [M1], [ABG1, ABG2], [BG3].

Proposition 5.1. *Let $\{H_\varepsilon\}_{\varepsilon\geq 0}$ be a family of bounded operators in $\mathscr{H}$ such that $H_0 = H$, $\|H_\varepsilon - H\| \to 0$ and $\|\varepsilon^{-1}\operatorname{Im} H_\varepsilon + i\mathscr{A}[H_\varepsilon]\| \to 0$ as $\varepsilon \to 0$. Then there are strictly positive numbers ε_0, b such that, for each $\varepsilon \in [0,\varepsilon_0]$ and each $z \in \mathbb{C}$ with $\operatorname{Re} z \in J$ and $\operatorname{Im} z > -a\varepsilon$, the operator $H_\varepsilon - z : \mathscr{H} \to \mathscr{H}$ is bijective and its inverse $G_\varepsilon = G_\varepsilon(z) = (H_\varepsilon - z)^{-1} \in B(\mathscr{H})$ satisfies the estimates*

$$(5.1)\quad \|G_\varepsilon^{(\pm)} f\|^2 \leq \pm\frac{1}{a\varepsilon + \operatorname{Im} z}\operatorname{Im}\langle f, G_\varepsilon f\rangle + \frac{b\varepsilon}{(a\varepsilon+\operatorname{Im} z)[\delta^2 + (\operatorname{Im} z)^2]}\|f\|^2$$

for all $f \in \mathscr{H}$. We have set $G_\varepsilon^{(+)} = G_\varepsilon$, $G_\varepsilon^{(-)} = G_\varepsilon^*$. In particular, one has

$$(5.2)\qquad \|G_\varepsilon(z)\| \leq \frac{1}{a\varepsilon + \operatorname{Im} z} + \left[\frac{b\varepsilon}{(a\varepsilon+\operatorname{Im} z)[\delta^2 + (\operatorname{Im} z)^2]}\right]^{1/2}.$$

The following consequences of the inequalities (5.1) and (5.2) will be especially useful later on: if $\operatorname{Im} z \geq 0$ then for $0 < \varepsilon \leq \varepsilon_0$ one has

$$(5.3)\qquad \|G_\varepsilon^{(\pm)} f\|^2 \leq \pm\frac{1}{a\varepsilon}\operatorname{Im}\langle f, G_\varepsilon f\rangle + \frac{b}{a\delta^2}\|f\|^2,$$

$$(5.4)\qquad \|G_\varepsilon\| \leq \frac{1}{a\varepsilon} + \left(\frac{b}{a\delta^2}\right)^{1/2}.$$

Proof of Proposition 5.1. (i) We first establish a preliminary estimate involving the bounded everywhere defined self-adjoint operator $S = i\mathscr{A}[H]$. Let ν be a strictly positive real number and let us set $P = 1 - E(J_0)$. Since $\pm 2\operatorname{Re} C \leq \nu + \nu^{-1}C^*C$ holds for all bounded operators C, we have

$$a_0 E(J_0) \leq (1-P)S(1-P) = S - 2\operatorname{Re}(SP) + PSP \leq S + \nu + P(S+\nu^{-1}S^2)P.$$

Hence

$$a_0 - \nu = a_0 E(J_0) - \nu + a_0 P \leq S + P(a_0 + S + \nu^{-1}S^2)P \leq S + \|a_0 + S + \nu^{-1}S^2\|P.$$

If $\operatorname{Re} z \in J$ then $P|H-z|^{-2}$ is a bounded operator with norm smaller than $d(z)^{-2}$, where $d(z) = \operatorname{dist}(z, \mathbb{R}\setminus J_0) \geq [\delta^2 + (\operatorname{Im} z)^2]^{1/2}$. So, by writing $P = P|H-z|^{-2}|H-z|^2$, we get for all $f \in \mathscr{H}$:

$$(5.5)\qquad (a_0 - \nu)\|f\|^2 \leq \langle f, Sf\rangle + \frac{\|a_0 + S + \nu^{-1}S^2\|}{\delta^2 + (\operatorname{Im} z)^2}\|(H-z)f\|^2$$

(ii) Now let us set $z = \lambda + i\mu$ with $\lambda \in J$ and $\mu \in \mathbb{R}$ and let $C = C(\nu, \mu) = \|a_0 + S + \nu^{-1} S^2\| (\delta^2 + \mu^2)^{-1}$. Then for an arbitrary operator $K \in B(\mathscr{H})$ and an arbitrary self-adjoint operator $T \in B(\mathscr{H})$ we have as a consequence of (5.5):

$$(a_0 - \nu)\|f\|^2 \leq \langle f, Tf\rangle + \langle f, (S-T)f\rangle + 2C\|(K-z)f\|^2 + 2C\|(H-K)f\|^2.$$

Since $C(\nu, \mu) \leq C(\nu, 0)$, we get

$$\big[a_0 - \nu - \|S-T\| - 2\delta^{-2}\|a_0 + S + \nu^{-1}S^2\| \cdot \|H-K\|\big] \cdot \|f\|^2 \leq \langle f, Tf\rangle + 2C\|(K-z)f\|^2.$$

We define $H_\varepsilon = H^*_{-\varepsilon}$ if $\varepsilon < 0$ and we take $K = H_\varepsilon$ and $T = \varepsilon^{-1} \operatorname{Im} H^*_\varepsilon$ with $\varepsilon \neq 0$. Since $H_\varepsilon \to H$ and $\varepsilon^{-1} \operatorname{Im} H^*_\varepsilon \to S$ in norm as $\varepsilon \to 0$, by choosing first a small enough number $\nu > 0$ and then $\varepsilon_0 > 0$, we obtain for $\varepsilon \in \mathbb{R}$, $0 < |\varepsilon| \leq \varepsilon_0$:

$$a\|f\|^2 \leq \varepsilon^{-1} \operatorname{Im}\langle H_\varepsilon f, f\rangle + 2C\|(H_\varepsilon - z)f\|^2. \tag{5.6}$$

For $0 \leq \varepsilon \leq \varepsilon_0$ we then get

$$(a\varepsilon + \mu)\|f\|^2 \leq \operatorname{Im}\langle (H_\varepsilon - z)f, f\rangle + 2C\varepsilon\|(H_\varepsilon - z)f\|^2. \tag{5.7}$$

Now let us consider (5.6) with ε replaced by $-\varepsilon$ and μ by $-\mu$. Then, again for $0 \leq \varepsilon \leq \varepsilon_0$, we obtain:

$$(a\varepsilon + \mu)\|f\|^2 \leq -\operatorname{Im}\langle (H_\varepsilon - z)^* f, f\rangle + 2C\varepsilon\|(H_\varepsilon - z)^* f\|^2. \tag{5.8}$$

Until now μ was arbitrary. If $a\varepsilon + \mu > 0$, then (5.7) implies that $\|(H_\varepsilon - z)f\| \geq \text{const.}\|f\|$ for some strictly positive constant and all $f \in \mathscr{H}$, so $H_\varepsilon - z$ is injective with closed range. Since by (5.8), its adjoint operator is also injective, we get that $H_\varepsilon - z : \mathscr{H} \to \mathscr{H}$ is bijective and bounded, so its inverse G_ε is also bounded. We obtain (5.1) with $b = 2\|a_0 + S + \nu^{-1}S^2\|$ if we replace f in (5.7) and (5.8) by $G_\varepsilon f$ and $G^*_\varepsilon f$ respectively. Finally, (5.1) implies (5.2) because from (5.1) we get

$$\|G_\varepsilon\|^2 \leq \frac{1}{a\varepsilon + \mu}\|G_\varepsilon\| + \frac{b\varepsilon}{(a\varepsilon + \mu)(\delta^2 + \mu^2)}.$$

□

Now let us assume that the family $\{H_\varepsilon\}$ from Proposition 5.1 has two more properties:

(1) H_ε is of class $C^1(A)$ if $0 < \varepsilon < \varepsilon_0$;

(2) the map $\varepsilon \mapsto H_\varepsilon \in B(\mathscr{H})$ is strongly C^1 on $(0, \varepsilon_0)$.

Let z be a complex number with $\operatorname{Re} z \in J$ and $\operatorname{Im} z \geq 0$ and let $0 < \varepsilon < \varepsilon_0$. Then $G_\varepsilon \in C^1(A)$ and $\mathscr{A}[G_\varepsilon] = -G_\varepsilon \mathscr{A}[H_\varepsilon] G_\varepsilon$. Indeed, if for $\tau \neq 0$ we set $A_\tau = (i\tau)^{-1}(e^{iA\tau} - 1)$ then we clearly have $[A_\tau, G_\varepsilon] = G_\varepsilon [H_\varepsilon, A_\tau] G_\varepsilon$ and the result follows by taking the limit as $\tau \to 0$ and by using, for example, the fact that $[H_\varepsilon, A_\tau] \to \mathscr{A}[H_\varepsilon]$ strongly as $\tau \to 0$. Furthermore, the map $\varepsilon \mapsto G_\varepsilon \in B(\mathscr{H})$ is strongly C^1 on $(0, \varepsilon_0)$ and its derivative is given by $G'_\varepsilon \equiv \frac{d}{d\varepsilon} G_\varepsilon = -G_\varepsilon H'_\varepsilon G_\varepsilon$ (this is an easy consequence of (5.4)). In particular we get

$$G'_\varepsilon = \mathscr{A}[G_\varepsilon] + G_\varepsilon(\mathscr{A}[H_\varepsilon] - H'_\varepsilon)G_\varepsilon. \tag{5.9}$$

This equation plays a fundamental role in the theory.

In this paper we shall choose H_ε of the form $H = \xi(\varepsilon\mathscr{A})H$ where $\xi : \mathbb{R} \to \mathbb{C}$ is a function such that the preceding expression makes sense, at least for small enough ε (notice that only the behaviour of H_ε as $\varepsilon \to 0$ matters). Other choices for H_ε are sometimes convenient but will not be considered here (see [ABG1], [BGM]). Let us see what conditions should ξ satisfy for H_ε to have the properties required in Proposition 5.1. For $H_0 = H$ we demand that $\xi(0) = 1$. Then, at least formally, we have $H_\varepsilon^* = \xi^+(\varepsilon\mathscr{A})H$, hence $2i \operatorname{Im} H_\varepsilon^* = [\overline{\xi}(-\epsilon\mathscr{A}) - \xi(\varepsilon\mathscr{A})]H$. So the condition $\lim_{\varepsilon\to 0} \varepsilon^{-1} \operatorname{Im} H_\varepsilon^\star = i\mathscr{A}H$ is formally satisfied if ξ is of class C^1 and $\operatorname{Re}\xi'(0) = 1$. For simplicity we would also like to have $H_\varepsilon^* = H_{-\varepsilon}$ (see the Proposition 5.1), which formally follows from $\xi^+(x) = \xi(-x)$, i.e. ξ should be real.

In conclusion, if we take $H_\varepsilon = \xi(\varepsilon\mathscr{A})H$, then the function ξ on $\mathbb{R}$ has to be real and to satisfy $\xi(0) = \xi'(0) = 1$. Then, again formally, we have

$$\mathscr{A}H_\varepsilon - H'_\varepsilon = \mathscr{A}\xi(\varepsilon\mathscr{A})H - \frac{d}{d\varepsilon}\xi(\varepsilon\mathscr{A})H = \mathscr{A}\xi(\varepsilon\mathscr{A})H - \mathscr{A}\xi'(\varepsilon\mathscr{A})H = \frac{1}{\varepsilon}\eta(\varepsilon\mathscr{A})H,$$

where $\eta(x) \equiv x\xi(x) - x\xi'(x) = O(x^2)$ as $x \to 0$. So (5.9) becomes

$$G'_\varepsilon = \mathscr{A}[G_\varepsilon] + \varepsilon^{-1} G_\varepsilon \eta(\varepsilon\mathscr{A})[H]G_\varepsilon. \tag{5.10}$$

We shall now give three examples, which are not relevant for our approach, but explain the constructions from [M1, M2] and [JMP].

(i) Assume $H \in C^2(A)$. Then one may take $\xi(x) = 1 + x$, which gives $\eta(x) = x^2$, hence

$$G'_\varepsilon = [G_\varepsilon, A] + \varepsilon G_\varepsilon [A, [A, H]]G_\varepsilon. \tag{5.11}$$

(ii) Let $H \in C^k(A)$ for some integer $k \geq 2$. Then one can take $\xi(x) = \sum_{j=0}^{k-1} x^j/j!$, hence $\eta(x) = x^k/(k-1)!$, so

$$G'_\varepsilon = [G_\varepsilon, A] + \frac{\varepsilon^{k-1}}{(k-1)!} G_\varepsilon \mathscr{A}^k[H]G_\varepsilon. \tag{5.12}$$

Notice that for $k = 3$ the second term in the r.h.s. of the preceding identity is bounded as $\varepsilon \to 0$, while for $k > 3$ it is an $O(\varepsilon^{k-3})$, so it vanishes as $\varepsilon \to 0$. Moreover, and this is the main fact, these estimates are independent of z (with $\operatorname{Re} z \in J$ and $\operatorname{Im} z \geq 0$) as follows from (5.4).

(iii) The best choice can be made if H is A-analytic: then we take $\xi(x) = e^x$, so that $\eta = 0$. Note that this time the expression $H_\varepsilon = e^{\varepsilon\mathscr{A}}H = e^{-\varepsilon\mathscr{A}}He^{\varepsilon\mathscr{A}}$ has a meaning only if $|\varepsilon|$ is small enough (unless H is A-entire). Then we have $G'_\varepsilon = \mathscr{A}G_\varepsilon$. This situation appears in the theory of dilation-analytic hamiltonians [AC], [BC].

Our choice for H_ε is related to (iii): the point is that H being non-analytic in general, we shall have to regularize it first according to the general procedure described in §3.6. Let $\theta \in C_0^\infty(\mathbb{R})$ be a real even function with $\theta(x) = 1$ on a neighbourhood of zero. From now on in this section we take $\xi(x) = e^x\theta(x)$ and $H_\varepsilon = \xi(\varepsilon\mathscr{A})H$ for all $\varepsilon \in \mathbb{R}$. Note that the operator H_ε is not self-adjoint in general, but we have $H_\varepsilon^* = H_{-\varepsilon}$. The function η which appears in (5.10) is now given by $\eta(x) = x(\xi(x) - \xi'(x)) = -e^x x\theta'(x)$, so that $\eta \in C_0^\infty(\mathbb{R} \setminus \{0\})$. The fact that 0 does not belong to the support of η is quite important for what follows.

It is convenient to have in mind a slightly different expression for H_ε. Set $H^\varepsilon = \theta(\varepsilon\mathscr{A})H$ for $\varepsilon \in \mathbb{R}$. Then H^ε is a self-adjoint operator and for $\varepsilon \neq 0$

the operators H^ε and H_ε are A-entire and are related by $H_\varepsilon = e^{\varepsilon\mathscr{A}}H^\varepsilon$ (see the end of §3.5).

It is not yet clear whether the so-called family $\{H_\varepsilon\}$ satisfies or not the hypotheses of Proposition 5.1. In fact it does not if H is only of class $C^1(A)$, as we explain in the next proposition:

Proposition 5.2. *The family $\{H_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ defined above satisfies the hypotheses of Proposition* 5.1 *if and only if the operator H is of class $C^1_u(A)$. Assume that $H \in C^1_u(A)$ and let $z \in \mathbb{C}$ with $\operatorname{Re} z \in J$ and $\operatorname{Im} z > 0$.*

(a) *For $0 \le \varepsilon \le \varepsilon_0$ one has $G_\varepsilon \in C^1_u(A)$ and $\mathscr{A}[G_\varepsilon] = -G_\varepsilon \mathscr{A}[H_\varepsilon]G_\varepsilon$; if $0 < \varepsilon < \varepsilon_0$ then $G_\varepsilon \in C^\infty(A)$.*

(b) *The map $\varepsilon \mapsto H_\varepsilon$ is of class C^1 in norm on $\mathbb{R}$ and is of class C^∞ on $\mathbb{R}\setminus\{0\}$. The map $\varepsilon \mapsto G_\varepsilon$ is of class C^1 in norm on the closed interval $[0,\varepsilon_0]$, where its derivative is given by $G'_\varepsilon = -G_\varepsilon H'_\varepsilon G_\varepsilon$, and is of class C^∞ on $(0,\varepsilon_0]$.*

(c) *Set $K_\varepsilon = \varepsilon^{-1}\eta(\varepsilon\mathscr{A})H$ for $\varepsilon \neq 0$, where $\eta \in C_0^\infty(\mathbb{R}\setminus\{0\})$ is given by $\eta(x) = -e^x x\theta'(x)$. Then $K_\varepsilon \in C^\infty(A)$, $\varepsilon \mapsto K_\varepsilon$ is of class C^∞ on $\mathbb{R}\setminus\{0\}$, and for each $0 < \varepsilon \le \varepsilon_0$ one has*

$$G'_\varepsilon = \mathscr{A}[G_\varepsilon] + G_\varepsilon K_\varepsilon G_\varepsilon. \tag{5.13}$$

(d) *Set $K^{(j)}_\varepsilon = (d/d\varepsilon)^j K_\varepsilon$ and let $\alpha > -1$ real and $p \in [1,\infty]$. Then H is of class $\mathscr{C}^{1+\alpha,p}(A)$ if and only if the condition*

$$\left[\int_0^1 \|\varepsilon^{-\alpha+j}K^{(j)}_\varepsilon\|^p \varepsilon^{-1}d\varepsilon\right]^{1/p} < \infty \tag{5.14}$$

holds for $j = 0$. If this is the case then (5.14) *holds for each integer $j \ge 0$.*

Proof. We define a real even function $\rho \in C_0^\infty(\mathbb{R})$ by $\rho(0) = 1$ and $\rho(x) = x^{-1}\sinh x \cdot \theta(x)$ if $x \neq 0$. Then for an arbitrary bounded self-adjoint operator H we have $\varepsilon^{-1}\operatorname{Im} H^*_\varepsilon = i\mathscr{A}\rho(\varepsilon\mathscr{A})H \equiv S_\varepsilon$ (see the Proposition from §3.5).

Assume first that $\lim_{\varepsilon\to 0} S_\varepsilon$ exists in norm in $B(\mathscr{H})$ and denote by S the limit. Since $C^\infty(A)$ is a subspace of the norm-closed space $C^0_u(A)$ and $S_\varepsilon \in C^\infty(A)$ if $\varepsilon \neq 0$, we get $S \in C^0_u(A)$. For $f \in D(A)$ we have

$$\langle f, S_\varepsilon f\rangle = \langle f, [\rho(\varepsilon\mathscr{A})H, iA]f\rangle = 2\operatorname{Re}\langle(\rho(\varepsilon\mathscr{A})H)f, iAf\rangle$$

which converges to $2\operatorname{Re}\langle Hf, iAf\rangle$ as $\varepsilon \to 0$. So we have $2\operatorname{Re}\langle Hf, iAf\rangle = \langle f, Sf\rangle$ for $f \in D(A)$, i.e. $i\mathscr{A}H = S \in C^0_u(A)$. This clearly means $H \in C^1_u(A)$ (see §3.3).

Reciprocally, if $H \in C^1_u(A)$ then H is of class $C^0_u(A)$ hence $\|H_\varepsilon - H\| \to 0$ as $\varepsilon \to 0$. Moreover, we shall also have $S_\varepsilon = i\rho(\varepsilon\mathscr{A})\mathscr{A}H$ (see §3.5) and $\mathscr{A}H \in C^0_u(A)$, so $\|S_\varepsilon - i\mathscr{A}H\| \to 0$ as $\varepsilon \to 0$. Hence the family $\{H_\varepsilon\}_{\varepsilon\ge 0}$ satisfies the hypotheses of Proposition 5.1.

The proof of the assertions (a), (b) and (c) is easy, see the arguments which follow the proof of Proposition 5.1. For part (d) we use the Theorem from §3.7. Observe that $\eta \in C_0^\infty(\mathbb{R}\setminus\{0\})$ is not identically zero on $(-\infty,0)$ and on $(0,\infty)$, so if (5.14) holds with $j = 0$ then $H \in \mathscr{C}^{1+\alpha,p}(A)$. Reciprocally, if H has this property then we have (5.14) for all j because $\varepsilon^j K^{(j)}_\varepsilon = \eta_j(\varepsilon\mathscr{A})H$ for some $\eta_j \in C_0^\infty(\mathbb{R}\setminus\{0\})$. □

The next proposition contains one of the basic estimates of the theory. In the proof we use Mourre's method of differential inequalities (see [M1], [M2]) together with a version of the Gronwall lemma obtained in [BGM].

We first introduce some notations and make some conventions for the rest of this section. We denote by $|||\cdot|||$ either the norm in the Banach space $\mathscr{K}=\mathscr{H}_{1/2,1}$ or the norm associated to it in $B(\mathscr{K};\mathscr{K}^*)$, and we recall that we have continuous embeddins $\mathscr{K}\subset\mathscr{H}\subset\mathscr{K}^*$ and $B(\mathscr{H})\subset B(\mathscr{K};\mathscr{K}^*)$. *From now on we assume that H is (at least) of class $\mathscr{C}^{1,1}(A)$.* We write $z=\lambda+i\mu$ and the numbers ε, λ, μ are supposed to verify $0<\varepsilon<\varepsilon_0$, $\lambda\in J$, $\mu>0$. One shoukd think of μ rather as a parameter, but it is important that the various constants that appear below are independent of μ. If F is a function of $(\lambda,\varepsilon)\in J\times(0,\varepsilon_0)$ we denote by $F^{(k,m)}\equiv\partial_\lambda^k\partial_\varepsilon^m F$ its derivative of order k with respect to λ and of order m with respect to ε. We also set $F^{(m)}=F^{(0,m)}$. The operator $G_\varepsilon=G_\varepsilon(z)=G_\varepsilon(\lambda+i\mu)$ will be considered as a function of $(\lambda,\varepsilon)\in J\times(0,\varepsilon_0)$; we clearly have for $k\in\mathbb{N}$:

$$G_\varepsilon^{(k,0)}=\partial_\lambda^k G_\varepsilon(\lambda+i\mu)=k!G_\varepsilon^{k+1}. \tag{5.15}$$

Proposition 5.3. *If H is of class $\mathscr{C}^{1,1}(A)$ then for each k, $m\in\mathbb{N}$ there is a number $C<\infty$, independent of $\varepsilon\in(0,\varepsilon_0)$, $\lambda\in J$ and $\mu>0$, such that*

$$|||G_\varepsilon^{(k,m)}|||\le C\varepsilon^{-k-m}, \tag{5.16}$$

$$\|G_\varepsilon^{(k,m)}\|_{\mathscr{K}\to\mathscr{H}}+\|G_\varepsilon^{(k,m)}\|_{\mathscr{H}\to\mathscr{K}^*}\le C\varepsilon^{-k-m-1/2}. \tag{5.17}$$

Proof. (i) We first prove (5.16), (5.17) in the case $k=m=0$. Fix a number $\varepsilon_1\in[0,\varepsilon_0)$ and a family $\{F_\varepsilon\}_{\varepsilon_1<\varepsilon\le\varepsilon_0}$ of vectors in $D(A)$ such that the function $\varepsilon\mapsto f_\varepsilon\in\mathscr{H}$ is of class C^1. We set $F_\varepsilon=\langle f_\varepsilon,G_\varepsilon f_\varepsilon\rangle$ for $\varepsilon_1<\varepsilon\le\varepsilon_0$ and we get by using (5.13):

$$F'_\varepsilon=\langle f'_\varepsilon-Af_\varepsilon,G_\varepsilon f_\varepsilon\rangle+\langle G^*_\varepsilon f_\varepsilon,f'_\varepsilon+Af_\varepsilon\rangle+\langle G^*_\varepsilon f_\varepsilon,K_\varepsilon G_\varepsilon f_\varepsilon\rangle.$$

Denote $l_\varepsilon=\|f'_\varepsilon\|+\|Af_\varepsilon\|$. Then (5.3) implies

$$|F'_\varepsilon|\le l_\varepsilon(\|G_\varepsilon f_\varepsilon\|+\|G^*_\varepsilon f_\varepsilon\|)+\|K_\varepsilon\|\cdot\|G_\varepsilon f_\varepsilon\|\cdot\|G^*_\varepsilon f_\varepsilon\|$$

$$\le 2l_\varepsilon a^{-1/2}(\varepsilon^{-1/2}|F_\varepsilon|^{1/2}+b^{1/2}\delta^{-1}\|f_\varepsilon\|)+\|K_\varepsilon\|a^{-1}(\varepsilon^{-1}|F_\varepsilon|+b\delta^{-2}\|f_\varepsilon\|^2).$$

So there is a constant $c>0$, depending only on a, b and δ, such that for $\varepsilon_1<\varepsilon\le\varepsilon_0$:

$$c^{-1}|F'_\varepsilon|\le l_\varepsilon\|f_\varepsilon\|+\|K_\varepsilon\|\cdot\|f_\varepsilon\|^2+l_\varepsilon\varepsilon^{-1/2}|F_\varepsilon|^{1/2}+\|K_\varepsilon\|\varepsilon^{-1}|F_\varepsilon|.$$

According to Proposition 3.1 from [BGM] the preceding estimate implies

$$|F_{\varepsilon_1}|\le 2\Big\{|F_{\varepsilon_0}|+c\int_{\varepsilon_1}^{\varepsilon_0}[l_\tau\|f_\tau\|+\|K_\tau\|\cdot\|f_\tau\|^2]d\tau + c^2\Big[\int_{\varepsilon_1}^{\varepsilon_0}l_\tau\tau^{-1/2}d\tau\Big]^2\Big\}\exp\int_{\varepsilon_1}^{\varepsilon_0}c\|K_\tau\|\tau^{-1}d\tau. \tag{5.18}$$

By Proposition 5.2 (d) we have $\int_0^{\varepsilon_0}\|K_\tau\|\tau^{-1}d\tau$ if and only if $H\in\mathscr{C}^{1,1}(A)$. Now let $f_\varepsilon\in\mathscr{H}_{1/2,1}$ and $f=\theta((\varepsilon-\varepsilon_1)A)f$, with the same function θ as in the definition of H_ε. If we set $\tilde\theta(x)=x\theta'(x)$ and $\theta_{(1)}(x)=x\theta(x)$, then

$$\int_{\varepsilon_1}^{\varepsilon_0}l_\tau\tau^{-1/2}d\tau=\int_0^{\varepsilon_0-\varepsilon_1}(\|\tilde\theta(\sigma A)f\|+\|\theta_{(1)}(\sigma A)f\|)\frac{d\sigma}{\sigma(\sigma+\varepsilon_1)^{1/2}}$$

$$\leq c'\|f\|_{\mathscr{H}_{1/2,1}} = c'|||f|||$$

where c' is a finite constant depending only on ε_0 and θ. Now by using (5.18) we easily see that there is a constant $c'' < \infty$ such that $|\langle f, G_\varepsilon f\rangle| \leq c''|||f|||^2$ for $0 < \varepsilon \leq \varepsilon_0$, $\lambda \in J$, $\mu > 0$ and $f \in \mathscr{K}$. The polarization identity will then give $|||G||| \leq$ const. Finally the estimate (5.17) with $k = m = 0$ is an immediate consequence of the preceding one and of (5.3).

(ii) Now we treat the case where one of the numbers k, m is not zero. If $m = 0$ then the estimates follow easily from those with $k = m = 0$ by taking into account (5.4) and (5.15), so we can assume $m \geq 1$. Then by Proposition 5.2 (b) the operator $G_\varepsilon^{(m)}$ is a linear combination of terms of the form

$$G_\varepsilon H_\varepsilon^{(m_1)} G_\varepsilon H_\varepsilon^{(m_2)} \ldots G_\varepsilon H_\varepsilon^{(m_n)}$$

with $m_1, \ldots, m_n \geq 1$ integers and $m_1 + \cdots + m_n = m$. So from (5.15) it follows that $G_\varepsilon^{(k,m)}$ is a linear combination of terms of the form

$$G_\varepsilon^{k_0+1} H_\varepsilon^{(m_1)} G_\varepsilon^{k_1+1} H_\varepsilon^{(m_2)} G_\varepsilon^{k_2+1} \ldots H_\varepsilon^{(m_n)} G_\varepsilon^{k_n+1}$$

with $m_1, \ldots, m_n$ as above and $k_0, k_1, \ldots, k_n \in \mathbb{N}$ such that $k_0 + k_1 + \cdots + k_n = k$. The norm in $B(\mathscr{K}; \mathscr{K}^*)$ of such a term is bounded by

$$\|G_\varepsilon\|_{\mathscr{H} \to \mathscr{K}^*} \|G_\varepsilon\|^{k_0} \|H_\varepsilon^{(m_1)}\| \cdot \|G_\varepsilon\|^{k_1+1} \ldots \|H_\varepsilon^{(m_n)}\| \cdot \|G_\varepsilon\|^{k_n} \|G_\varepsilon\|_{\mathscr{K} \to \mathscr{H}}$$

$$\leq \text{const.} \varepsilon^{-1/2} \cdot \varepsilon^{-k_0} \|H_\varepsilon^{(m_1)}\| \cdot \varepsilon^{-k_1-1} \ldots \|H_\varepsilon^{(m_n)}\| \varepsilon^{-k_n} \cdot \varepsilon^{-1/2}$$

where we have used (5.17) with $k = m = 0$ and (5.4). Similarly, the norm in $B(\mathscr{K}; \mathscr{H})$ is bounded by

$$\|G_\varepsilon\|^{k_0+1} \|H_\varepsilon^{(m_1)}\| \cdot \|G_\varepsilon\|^{k_1+1} \ldots \|H_\varepsilon^{(m_n)}\| \cdot \|G_\varepsilon\|^{k_n} \|G_\varepsilon\|_{\mathscr{K} \to \mathscr{H}}$$

$$\leq \text{const.} \varepsilon^{-k_0-1} \|H_\varepsilon^{(m_1)}\| \cdot \varepsilon^{-k_1-1} \ldots \|H_\varepsilon^{(m_n)}\| \cdot \varepsilon^{-k_n} \cdot \varepsilon^{-1/2}.$$

We see that the assertions of the proposition are a consequence of the estimate $\|H_\varepsilon^{(m)}\| \leq c_m \varepsilon^{1-m}$ for $m \geq 1$ integer and $\varepsilon > 0$. But we have (see (3.7), the end of §3.6 and the Proposition from §3.5):

$$H_\varepsilon^{(m)} = \partial_\varepsilon^m \xi(\varepsilon\mathscr{A}) H = \mathscr{A}^m \xi^{(m)}(\varepsilon\mathscr{A}) H = \varepsilon^{1-m} (\varepsilon\mathscr{A})^{m-1} \xi^{(m)}(\varepsilon\mathscr{A}) \mathscr{A} H$$

$$= \varepsilon^{1-m} \varphi(\varepsilon\mathscr{A}) \mathscr{A} H.$$

where $\varphi(x) = x^{m-1}\xi^{(m)}(x)$ is a function of class $C_0^\infty(\mathbb{R})$. Hence

$$\|H_\varepsilon^{(m)}\| \leq \varepsilon^{1-m} \|\varphi\|_{\mathscr{M}} \|\mathscr{A} H\|.$$

□

Lemma 5.4. *Set* $\widetilde{G}_\varepsilon = G_\varepsilon K_\varepsilon G_\varepsilon$, *where* K_ε *is as in Proposition* 5.2 (c). *Then for each* k, $m \in \mathbb{N}$ *there is a finite constant* C, *independent of* $\varepsilon, \lambda, \mu$, *such that*

$$|||\widetilde{G}_\varepsilon^{(k,m)}||| \leq C \varepsilon^{-k-m-1} \sum_{j=0}^{m} \|\varepsilon^j K_\varepsilon^{(j)}\|. \tag{5.19}$$

In particular, if $H \in \mathscr{C}^{1+\alpha}(A)$ *for some* $\alpha > 0$, *then we have* $|||\widetilde{G}_\varepsilon^{(k,m)}||| \leq c\varepsilon^{\alpha-k-m-1}$.

Proof. By Leibnitz formula, and since K_ε does not depend on λ, $\widetilde{G}_\varepsilon^{(k,m)}$ is a linear combination of terms of the form $\widetilde{G}_\varepsilon^{(a,u)}\tilde{K}_\varepsilon^{(w)}\widetilde{G}_\varepsilon^{(b,v)}$, with a, b, u, v, $w \in \mathbb{N}$ and $a+b=k$, $u+v+w=n$. Then Proposition 5.3 implies

$$\begin{aligned}|||G_\varepsilon^{(a,u)}K_\varepsilon^{(w)}G_\varepsilon^{(b,v)}||| &\leq \|G_\varepsilon^{(a,u)}\|_{\mathscr{H}\to\mathscr{K}^*}\|K_\varepsilon^{(w)}\|\cdot\|G_\varepsilon^{(b,v)}\|_{\mathscr{K}\to\mathscr{H}}\\ &\leq \text{const.}\varepsilon^{-a-u-1/2}\|K_\varepsilon^{(w)}\|\cdot\varepsilon^{-b-v-1/2} = \text{const.}\varepsilon^{-k-m-1}\|\varepsilon^w K_\varepsilon^{(w)}\|.\end{aligned}$$

For the proof of the next estimates we need a generalization of the identity (5.13). Assume that we are under the hypotheses of Proposition 5.2 and let $\widetilde{G}_\varepsilon = G_\varepsilon K_\varepsilon G_\varepsilon$. Then for all l, $k \in \mathbb{N}$ with $k \geq 1$ and all $\varepsilon \in (0,\varepsilon_0)$, $z=\lambda+i\mu$, $\lambda \in J$, $\mu > 0$ we have

$$G_\varepsilon^{(l,k)} = l!\mathscr{A}^k[G_\varepsilon^{l+1}] + \sum_{r=0}^{k-1}\mathscr{A}^{k-r-1}[\widetilde{G}_\varepsilon^{(l,r)}]. \tag{5.20}$$

If $l=0$, $k=1$ this is just (5.13). (5.20) follows from this special case by taking successively derivatives with respect to ε and λ and by using the Lemma from §3.2. □

Now let us fix two functions φ, $\psi \in \mathscr{S}(\mathbb{R})$ and let us define the operator $L_\varepsilon \equiv L_\varepsilon(z) : \mathscr{H}_{-\infty} \to \mathscr{H}_{+\infty}$ by:

$$L_\varepsilon(z) = \varphi(\varepsilon A)G_\varepsilon(z)\psi(\varepsilon A) \tag{5.21}$$

for $0 < \varepsilon < \varepsilon_0$ and $z = \lambda + i\mu$ with $\lambda \in J$ and $\mu > 0$. Let l, $m \in \mathbb{N}$. By using Leibnitz formula and by taking into account the relation $\partial_\varepsilon^i\varphi(\varepsilon A) = A^i\varphi^{(i)}(\varepsilon A) = \varepsilon^{-i}\varphi_i(\varepsilon A)$ with $\varphi_i(x) = x^i\varphi^{(i)}(x)$ we obtain

$$L_\varepsilon^{(l,m)} = \sum_{i+j+k=m}\frac{m!}{i!j!k!}\varepsilon^{k-m}\varphi_i(\varepsilon A)G_\varepsilon^{(l,k)}\psi_j(\varepsilon A),$$

where the indices i, j, k run over $\mathbb{N}$. If we use (5.20) the expression in the r.h.s. above becomes

$$\begin{aligned}L_\varepsilon^{(l,m)} = &\sum_{i+j+k=m}\frac{l!m!}{i!j!k!}\varepsilon^{k-m}\varphi_i(\varepsilon A)\mathscr{A}^k[G_\varepsilon^{l+1}]\psi_j(\varepsilon A)\\ &+\sum_{i+j+k=m,k\geq 1,n+r=k-1}\frac{m!}{i!j!k!}\varepsilon^{k-m}\varphi_i(\varepsilon A)\mathscr{A}^n[\widetilde{G}_\varepsilon^{(l,r)}]\psi_j(\varepsilon A).\end{aligned}$$

Then by taking into account the identity (3.1) we get

(5.22)

$$\begin{aligned}\varepsilon^m L_\varepsilon^{(l,m)} = &\sum_{i+j+p+q=m}\frac{l!m!}{i!j!p!q!}(-\varepsilon A)^p(\varepsilon A)^i\varphi^{(i)}(\varepsilon A)G_\varepsilon^{l+1}(\varepsilon A)^{j+q}\psi^{(j)}(\varepsilon A)\\ +&\sum_{i+j+p+q+r=m-1}\frac{m!(p+q)!(-1)^p\varepsilon^{r+1}}{i!j!p!q!(m-i-j)!}(\varepsilon A)^{i+p}\varphi^{(i)}(\varepsilon A)\widetilde{G}_\varepsilon^{(l,r)}(\varepsilon A)^{j+q}\psi^{(j)}(\varepsilon A).\end{aligned}$$

Proposition 5.5. *Let φ, $\psi \in \mathscr{S}(\mathbb{R})$ and let $L_\varepsilon = L_\varepsilon(z)$ be defined by $L_\varepsilon = \varphi(\varepsilon A)G_\varepsilon\psi(\varepsilon A)$. Then for each $l, m \in \mathbb{N}$ there is a constant C, independent* *of $\varepsilon, \lambda, \mu$, such that for all $f, g \in \mathscr{H}_{-\infty}$:*

(5.23)

$$|\langle g, \varepsilon^{l+m} L_\varepsilon^{(l,m)} f\rangle| \leq C \sum_{a+b=m, 0\leq i\leq a, 0\leq j\leq b} |||\varphi_{i,a}(\varepsilon A)g||| \cdot |||\psi_{j,b}(\varepsilon A)f|||$$
$$+ C \sum_{a+b+c\leq m-1, 0\leq i\leq a, 0\leq j\leq b} |||\varphi_{i,a}(\varepsilon A)g||| \cdot |||\psi_{j,b}(\varepsilon A)f||| \cdot \|\varepsilon^c K_\varepsilon^{(c)}\|.$$

Here the functions $\varphi_{i,a}$ and $\psi_{j,b}$ are defined by $\varphi_{i,a}(x) = x^a \varphi^{(i)}(x)$ and $\psi_{j,b}(x) = x^b \psi^{(j)}(x)$.

Proof. We use (5.22) and the estimates $|||\varepsilon^l G_\varepsilon^{l+1}||| \leq C(l)$ and $|||\varepsilon^{l+r+1} \widetilde{G}_\varepsilon^{(l,r)}||| \leq C(l,r) \sum_{0\leq c\leq r} \|\varepsilon^c K_\varepsilon^{(c)}\|$ which have been obtained in Proposition 5.3 and Lemma 5.4. ☐

It is clear that the first sum from (5.22) becomes much more simpler if φ is a function such that $\varphi^{(i)}(x) = \varphi(x)$ for all x. But the only function which has this property is $\varphi(x) = e^x$ and it does not belong to $\mathscr{S}(\mathbb{R})$. However, one can circumvent this difficulty if in place of L_ε one considers the operator $\Pi_- L_\varepsilon$ where $\Pi_- = E_A((-\infty, 0])$ is the spectral projection of A associated with the interval $(-\infty, 0]$. Then we take a function $\varphi \in \mathscr{S}(\mathbb{R})$ such that $\varphi(x) = e^x$ if $x \leq 0$. Observe that for j, q fixed with $n = m - j - q > 0$ one has $\sum_{i+p=n} (i!p!)^{-1}(-x)^p x^j = 0$. Hence, after left multiplication by Π_- of (5.22), in the first sum on the r.h.s. will remain only terms with $j + q = m$, so $i = p = 0$. On the other hand:

$$\sum_{j+q=m} \frac{m!}{j!q!} x^{j+q} \psi^{(j)}(x) = x^m (1 + \frac{d}{dx})^m \psi(x) \equiv \zeta(x). \tag{5.24}$$

Hence we obtain :

$$\varepsilon^m \Pi_- L_\varepsilon^{(l,m)} = l! \Pi_- e^{\varepsilon A} G_\varepsilon^{l+1} \zeta(\varepsilon A) +$$
$$\sum_{i+j+p+q+r=m-1} \frac{m!(p+q)!(-1)^p \varepsilon^{r+1}}{i!j!p!q!(m-i-j)!} \Pi_-(\varepsilon A)^{i+p} e^{\varepsilon A} \widetilde{G}_\varepsilon^{(l,r)} (\varepsilon A)^{j+q} \psi^{(j)}(\varepsilon A).$$

By the same argument as in the proof of Proposition 5.5 we get, with a slight change of notation:

Proposition 5.6. *Let $\psi \in \mathscr{S}(\mathbb{R})$, define ζ by (5.24), and let us set $L_\varepsilon = \Pi_- e^{\varepsilon A} G_\varepsilon \psi(\varepsilon A)$. Then for each l, $m \in \mathbb{N}$ there is a constant C, independent of $\varepsilon, \lambda, \mu$, such that for all $f, g \in \mathscr{H}_{-\infty}$:*

$$|\langle g, \varepsilon^{l+m} L_\varepsilon^{(l,m)} f\rangle| \leq C |||\Pi_- e^{\varepsilon A} g||| \cdot |||\zeta(\varepsilon A) f||| + \tag{5.25}$$
$$C \sum_{a+b+c\leq m-1, 0\leq j\leq b} |||\Pi_-(\varepsilon A)^a e^{\varepsilon A} g||| \cdot \|(\varepsilon A)^b \psi^{(j)}(\varepsilon A) f||| \cdot \|\varepsilon^c K_\varepsilon^{(c)}\|.$$

This estimate can be further simplified by a special choice of ψ. Note that if $\psi(x) = e^{-x}$ then $\zeta = 0$. Of course this choice is not allowed by the condition $\psi \in \mathscr{S}(\mathbb{R})$. However, if we take ψ of class $\mathscr{S}(\mathbb{R})$ and such that $\psi(x) = e^{-x}$ if $x \geq 0$, then $\Pi_+ \zeta(\varepsilon A) f = 0$ for each $f \in \mathscr{H}_{-\infty}$. Hence Proposition 5.6 immediately implies the next one. Here $\Pi_+ = E_A([0, \infty))$.

Proposition 5.7. *Let $L_\varepsilon = \Pi_- e^{\varepsilon A} G_\varepsilon e^{-\varepsilon A} \Pi_+$. Then for each $l, m \in \mathbb{N}$ with $m \geq 1$ there is $C < \infty$, independent of $\varepsilon, \lambda, \mu$, such that for all $f, g \in \mathscr{H}_{-\infty}$:*

$$(5.26) \qquad |\langle g, \varepsilon^{l+m} L_\varepsilon^{(l,m)} f \rangle|$$
$$\leq C \sum_{a+b+c \leq m-1} |||\Pi_-(\varepsilon A)^a e^{\varepsilon A} g||| \cdot |||\Pi_+(\varepsilon A)^b e^{-\varepsilon A} f||| \cdot \|\varepsilon^c K_\varepsilon^{(c)}\|.$$

6. Resolvents of Bounded Regular Operators

This section contains the main results of the paper. For their proof we shall use the estimates obtained in Section 5 and the following elementary lemmas. The proof of Lemma 6.1 is quite easy and will not given; that of Lemma 6.2 can be found in [BG3].

Lemma 6.1. *Let $h : (0, \varepsilon_0] \to \mathbb{C}$ be a function of class C^m for some integer $m \geq 1$ and some real $\varepsilon_0 > 0$. Assume that $\int_0^{\varepsilon_0} |\varepsilon^{m-1} h^{(m)}(\varepsilon)| d\varepsilon < \infty$. Then $\lim_{\varepsilon \to 0} h(\varepsilon) \equiv h(0)$ exists and*

$$(6.1) \qquad h(0) = \sum_{k=0}^{m-1} \frac{(-\varepsilon_0)^k}{k!} h^{(k)}(\varepsilon_0) + \frac{(-1)^m}{(m-1)!} \int_0^{\varepsilon_0} h^{(m)}(\varepsilon) \varepsilon^{m-1} d\varepsilon.$$

Lemma 6.2. *Let $J \subset \mathbb{R}$ be an open set, $\varepsilon_0 > 0$ a real number and $\widetilde{J} = \{(\lambda, \varepsilon) \in \mathbb{R}^2 \mid \lambda \in J, 0 < \varepsilon < \varepsilon_0\}$. Let $F : \widetilde{J} \to \mathbb{C}$ be a function of class C^m for some integer $m \geq 1$ and assume that there are real numbers σ, M with $0 < \sigma < m$ and $M > 0$ such that $\sum_{l+k=m} |\partial_\lambda^l \partial_\varepsilon^k F(\lambda, \varepsilon)| \leq M \varepsilon^{\sigma - m}$ on $\widetilde{J}$. Then the limit $\lim_{\varepsilon \to 0} F(\lambda, \varepsilon) \equiv F_0(\lambda)$ exists uniformly in $\lambda \in J$ and the function $F_0 : J \to \mathbb{C}$ is locally of class Λ^σ. Moreover, there is a constant C_m (depending only on m) such that*

$$(6.2) \qquad |[(T_\nu - 1)^m F_0](\lambda)| \leq C_m M \sigma^{-1} |\nu|^\sigma$$

if $\lambda \in J$ and $\nu \in \mathbb{R}$ have the properties $|\nu| < \varepsilon_0$ and $\lambda + t\nu \in J$ for all $t \in [0, m]$. In (6.2) the translation operator T_ν acts according to $(T_\nu g)(\lambda) = g(\lambda + \nu)$.

We keep the notations and assumptions of the preceding section. In particular H is a bounded everywhere defined self-adjoint operator and J is an open real set such that the conditions stated at the beginning of Section 5 are fulfilled. Note that the regularity hypotheses that we make below imply $H \in \mathscr{C}^{1,1}(A)$, which in turn implies $H \in C_u^1(A)$. We recall that $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and we set $R(z) = (H - z)^{-1}$.

Theorem 6.3. *Assume that $H \in \mathscr{C}^{1+l,1}(A)$ for some integer $l \geq 0$ and set $s = l + 1/2$. Then for each $f \in \mathscr{H}_{s,1}$ the holomorphic map $\mathbb{C}_+ \ni z \mapsto \langle f, R(z) f \rangle$ extends to a function of class C^l on $\mathbb{C}_+ \cup J$, i.e. for each integer $0 \leq k \leq l$ the holomorphic function on $\mathbb{C}_+$ given by $(d/dz)^k \langle f, R(z) f \rangle = \langle f, k! R(z)^{k+1} f \rangle$ has a continuous extension to $\mathbb{C}_+ \cup J$. The limit $\lim_{\mu \to 0} \langle f, R(\lambda + i\mu) f \rangle \equiv \langle f, R(\lambda + i0) f \rangle$ exists uniformly in $\lambda \in J$, the boundary value function $\lambda \mapsto \langle f, R(\lambda + i0) f \rangle$ is of class C^l on J, and for $0 \leq k \leq l$ integer one* *has*

$$\frac{d^k}{d\lambda^k}\langle f, R(\lambda+i0)f\rangle = \lim_{\mu\to+0}\langle f, k!R(\lambda+i\mu)^{k+1}f\rangle \tag{6.3}$$

uniformly in $\lambda \in J$.

Proof. Let $L_\varepsilon = L_\varepsilon(z) = \varphi(\varepsilon A)G_\varepsilon(z)\varphi(\varepsilon A)$ where φ is a function in $\mathscr{S}(\mathbb{R})$ with $\varphi(0) = 1$ and $0 \le \varepsilon \le \varepsilon_0$, $z = \lambda + i\mu$ with $\lambda \in J$, $\mu > 0$. Clearly

$$L_\varepsilon^{(l,0)} = \partial_\lambda^l L_\varepsilon = (\frac{d}{dz})^l \varphi(\varepsilon A)G_\varepsilon(z)\varphi(\varepsilon A) = \varphi(\varepsilon A)l!G_\varepsilon(z)^{l+1}\varphi(\varepsilon A). \tag{6.4}$$

Note that by Proposition 5.2 (b) the map $\varepsilon \mapsto L_\varepsilon^{(l,0)} \in B(\mathscr{H})$ is strongly C^1 on the closed interval $[0, \varepsilon_0]$ and $L_0^{(l,0)} = \partial_z^l R(z) = l!R(z)^{l+1}$.

Now let us fix $f \in \mathscr{H}_{s,1}$ and define $h(\varepsilon) = \langle f, L_\varepsilon^{(l,0)} f\rangle$ for $0 \le \varepsilon \le \varepsilon_0$. Then for $\varepsilon > 0$ and $m \ge 0$ integer we have $h^{(m)}(\varepsilon) = \langle f, L_\varepsilon^{(l,m)} f\rangle$ which can be estimated as in (5.23). So there is $C < \infty$, independent of ε, λ, μ and f, such that

$$\begin{aligned} |\varepsilon^m h^{(m)}(\varepsilon)| \le C & \sum_{a+b=m, i\le a, j\le b} \varepsilon^{-l} |||\varphi_{i,a}(\varepsilon A)f||| \cdot |||\varphi_{j,b}(\varepsilon A)f||| \\ & + C|||f|||^2 \sum_{0\le j\le m-1} \varepsilon^{-l}\|\varepsilon^j K_\varepsilon^{(j)}\|. \end{aligned} \tag{6.5}$$

By Proposition 5.2 (d) the condition $H \in \mathscr{C}^{1+l,1}(A)$ is equivalent to the integrability with respect to the measure $\varepsilon^{-1}d\varepsilon$ on $(0, \varepsilon_0)$ of the second term on the r.h.s. of (6.5). We claim that if $m > 2l$ then each term of the sum from (6.5) is also integrable (with respect to the same measure). Indeed, if $a + b = m$ then either $a > l$ or $b > l$. In the first case we have

$$\int_0^1 \varepsilon^{-l}|||\varphi_{i,a}(\varepsilon A)f||| \cdot |||\varphi_{j,b}(\varepsilon A)f|||\varepsilon^{-1}d\varepsilon \le$$

$$C'|||f||| \int_0^1 \|\varepsilon^{-l}\varphi_{i,a}(\varepsilon A)f\|_{1/2,1}\varepsilon^{-1}d\varepsilon \le C''|||f||| \cdot \|f\|_{s,1}$$

due to the theorem from §2.3 (observe that $\varphi_{i,a}$ has a zero of order $\ge a > l$ at the origin).

Let us fix an integer $m > 2l$. We have seen that there is a function $\chi : (0, \varepsilon_0) \to \mathbb{R}$, independent of λ and μ, such that $|\varepsilon^m h^{(m)}(\varepsilon)| \le \chi(\varepsilon)$ and $\int_0^{\varepsilon_0} \chi(\varepsilon)\varepsilon^{-1}d\varepsilon < \infty$. So we can apply Lemma 6.1 and thus obtain

$$\begin{aligned} \langle f, \partial_z^l R(z)f\rangle = & \sum_{k=0}^{m-1} \frac{(-\varepsilon_0)^k}{k!}\langle f, L_{\varepsilon_0}^{(l,k)} f\rangle \\ & + \frac{(-1)^m}{(m-1)!}\int_0^{\varepsilon_0} \langle f, L_\varepsilon^{(l,m)} f\rangle \varepsilon^{m-1}d\varepsilon. \end{aligned} \tag{6.6}$$

According to Proposition 5.1, for each $\varepsilon \in [0, \varepsilon_0]$ the function $z \mapsto G_\varepsilon = (H_\varepsilon - z)^{-1}$ is holomorphic in the region $\lambda \in J$, $\mu > -a\varepsilon$, where $a > 0$. So each term in the sum from (6.6) extends to a holomorphic function of z below the real axis if $\operatorname{Re} z \in J$ (see (5.22) for example). For the integral in (6.6) we can use the dominated convergence theorem in order to deduce that its limit as $\mu \to +0$ exists uniformly in $\lambda \in J$.

We have shown that $\lim_{\mu\to 0}\langle f, \partial_z^l R(z) f\rangle$ exists uniformly in $\lambda \in J$. Clearly the arguments still work if l is replaced by a small integer. $\square$

It is convenient to reformulate Theorem 6.3 in slightly different terms. For an arbitrary self-adjoint operator H the map $z \mapsto R(z) \in B(\mathscr{H})$ is holomorphic on $\mathbb{C}_+$. Recall that we have continuous embeddings

$$B(\mathscr{H}) \subset B(\mathscr{K};\mathscr{K}^*) \subset B(\mathscr{H}_{s,1};\mathscr{H}_{-s,\infty}) \tag{6.7}$$

if $s \geq 1/2$. So, for example, $z \mapsto R(z) \in B(\mathscr{K};\mathscr{K}^*)$ is a holomorphic map on $\mathbb{C}_+$. Now assume that $H \in \mathscr{C}^{1,1}(A)$, i.e. the hypothesis of Theorem 6.3 holds with $l = 0$. Then the theorem says that the preceding function extends to a weak* continuous function on $\mathbb{C}_+ \cup J$, in fact $\lim_{\mu\to+0} R(\lambda+i\mu) \equiv R(\lambda+i0) \in B(\mathscr{K};\mathscr{K}^*)$ exists in the weak* topology of $B(\mathscr{K};\mathscr{K}^*)$, uniformly in $\lambda \in J$. So the boundary value function $\lambda \mapsto R(\lambda+i0) \in B(\mathscr{K};\mathscr{K}^*)$ is well defined and weak* continuous on J. According to (6.7), we may consider the map $\lambda \mapsto R(\lambda+i0) \in B(\mathscr{H}_{s,1};\mathscr{H}_{-s,\infty})$ for each $s \geq 1/2$; clearly it is a weak* continuous function (recall that $\mathscr{H}_{-s,\infty} = \mathscr{H}_{s,1}^*$, which defines the weak* topology of the preceding space). Now assume that $H \in \mathscr{C}^{1+l,1}(A)$ for some integer $l \geq 1$. Then the Theorem 6.3 says that the map $\lambda \mapsto R(\lambda+i0) \in B(\mathscr{H}_{s,1};\mathscr{H}_{-s,\infty})$ is of class C^l on J in the weak* topology if $s = l + 1/2$. Moreover its weak* derivatives are given by

$$\frac{d^k}{d\lambda^k} R(\lambda+i0) = \lim_{\mu\to+0} k! R(\lambda+i\mu)^{k+1} \equiv k! R^{k+1}(\lambda+i0) \tag{6.8}$$

where the limit exists in the weak* topology of $B(\mathscr{H}_{s,1};\mathscr{H}_{-s,\infty})$, uniformly in $\lambda \in J$.

$\mathscr{K}^* = \mathscr{H}_{-1/2,\infty}$ is the smallest space in the Besov scale associated to A which contains the set $R(\lambda+i0)\mathscr{H}_\infty$ (if $\lambda \in J$ is a spectral value of H). We show now that the operator $\Pi_- R(\lambda+i0)\mathscr{H}_\infty$ behaves much better. Here $\Pi_- = E_A((-\infty,0])$ extends to a continuous operator in $\mathscr{H}_{-\infty}$ which leaves invariant each $\mathscr{H}_{s,p}$; hence the product $\Pi_- R(\lambda+i0)$ is well defined and belongs to $B(\mathscr{K};\mathscr{K}^*)$. Observe that in the next theorems we implicitly use the facts established in §3.9. For example, under the conditions of Theorem 6.4 we have $R(z)\mathscr{H}_{s,p} \subset \mathscr{H}_{s,p}$, hence the r.h.s. of (6.9) makes sense.

Theorem 6.4. *Let $H \in \mathscr{C}^{s+1/2,p}(A)$ for some real number $s > 1/2$ and some $p \in [1,\infty]$. Then for all $\lambda \in J$ one has $\Pi_- R(\lambda+i0)\mathscr{H}_{s,p} \subset \mathscr{H}_{s-1,p}$. Let $l \geq 0$ be an integer such that $l < s - 1/2$. Then for each $f \in \mathscr{H}_{s,p}$ and $g \in \mathscr{H}_{l+1-s,p'}$ the function $\lambda \mapsto \langle g, \Pi_- R(\lambda+i0) f\rangle$ is of class C^l on J and one has*

$$\frac{d^l}{d\lambda^l}\langle g, \Pi_- R(\lambda+i0) f\rangle = \lim_{\mu\to+0}\langle \Pi_- g, l! R(\lambda+i\mu)^{l+1} f\rangle \tag{6.9}$$

where the limit exists uniformly in $\lambda \in J$.

Proof. Let L_ε be the operator introduced in Proposition 5.6, where ψ is assumed to have the property $\psi(0) = 1$. We set $h(\varepsilon) = \langle g, L_\varepsilon^{(l,0)} f\rangle$ for $0 < \varepsilon \leq \varepsilon_0$ and some given vectors $f \in \mathscr{H}_{s,p}$ and $g \in \mathscr{H}_{1+l-s,p'}$. Here p' is defined by $1/p + 1/p' = 1$. Then (5.25) gives

$$|\varepsilon^m h^{(m)}(\varepsilon)| \le C\varepsilon^{-l} |||\Pi_- e^{\varepsilon A} g||| \cdot |||\zeta(\varepsilon A) f||| + C|||f||| \sum_{i+j\le m-1} \varepsilon^{-l} |||\Pi_-(\varepsilon A)^i e^{\varepsilon A} g||| \cdot \|\varepsilon^j K_\varepsilon^{(j)}\| \tag{6.10}$$

where C is a constant independent of ε, λ, μ, f and g. We choose $m > \alpha \equiv s - 1/2$. Then the integral over the interval $(0,1)$ with respect to the measure $\varepsilon^{-1}d\varepsilon$ of the first term on the r.h.s. of (6.10) is bounded by

$$C\Big[\int_0^1 |||\varepsilon^{\alpha-l}\Pi_- e^{\varepsilon A} g|||^{p'} \varepsilon^{-1} d\varepsilon\Big]^{1/p'} \cdot \Big[\int_0^1 |||\varepsilon^{-\alpha}\zeta(\varepsilon A) f|||^p \varepsilon^{-1} d\varepsilon\Big]^{1/p} \le C'\|g\|_{1/2-\alpha+l,p'} \|f\|_{1/2+\alpha,p}$$

We have used the theorem from §2.3 which is allowed by the fact that $\alpha - l > 0$, $0 < \alpha < m$ and $\zeta(x) = O(x^m)$ as $x \to 0$. The integral over $(0,1)$ with respect to $\varepsilon^{-1}d\varepsilon$ of a typical term of the sum in the r.h.s. of (6.10) is similarly bounded by

$$C|||f|||\Big[\int_0^1 |||\varepsilon^{\alpha-l}\Pi_-(\varepsilon A)^i e^{\varepsilon A} g|||^{p'} \varepsilon^{-1} d\varepsilon\Big]^{1/p'} \Big[\int_0^1 \|\varepsilon^{-\alpha+j} K_\varepsilon^{(j)}\|^p \varepsilon^{-1} d\varepsilon\Big]^{1/p} \le C'|||f||| \cdot \|g\|_{1/2-\alpha+l,p'}.$$

The rest of the proof is similar to that of Theorem 6.3. The conclusion is that the function $z \mapsto \langle \Pi_- g, R(z) f\rangle$, which is holomorphic on $\mathbb{C}_+$, extends to a function of class C^l on $\mathbb{C}_+ \cup J$ (in a sense explained in the statement of Theorem 6.3). In particular, if we take $l = 0$ we see that $\lim_{\mu\to+0}\langle \Pi_- g, R(\lambda + i\mu) f\rangle$ exists (uniformly in $\lambda \in J$) for each $f \in \mathscr{H}_{s,p}$ and $g \in \mathscr{H}_{1-s,p'}$. Now recall that $\mathscr{H}_{1-s,p'} = (\mathscr{H}_{s-1,p})^*$ if $1 < p \le \infty$ and $\mathscr{H}_{1-s,1} = (\mathscr{H}^0_{s-1,\infty})^*$. Hence for each $f \in \mathscr{H}_{s,p}$ the limit $\lim_{\mu\to+0} \Pi_- R(\Lambda + i\mu) f$ exists in the weak* topology of $\mathscr{H}_{s-1,p}$ so $\Pi_- R(\lambda + i0)\mathscr{H}_{s,p} \subset \mathscr{H}_{s-1,p}$. ☐

We recall the definition of the w-topology on the space $B(\mathscr{H}_{s,p};\mathscr{H}_{t,q})$: it is the topology associated to the family of seminorms $S \mapsto |\langle g, Sf\rangle|$ with $f \in \mathscr{H}_{s,p}$ and $g \in \mathscr{H}_{-t,q'}$. So the second part of Theorem 6.4 can be expressed as follows: the map $\lambda \mapsto \Pi_- R(\lambda + i0) \in B(\mathscr{H}_{s,p};\mathscr{H}_{s-l-1,p})$ is of class C^l in the w-topology.

If $f \notin \mathscr{H}_{1/2,1}$ then $\Pi_- R(\lambda + i0) f$ has no meaning in general. However, one can give a sense to this expression if $E_A((-\infty, a)) f = 0$ for some $a \in \mathbb{R}$.

Theorem 6.5. *Assume that $H \in \mathscr{C}^{1+\alpha,r}(A)$ with $\alpha > 0$ real and $r \in [1,\infty]$. Let $l \in \mathbb{N}$ with $l < \alpha$, let s be a real number such that $1/2 - (\alpha - l) \le s \le 1/2$, and let us denote $t = s - 1 + (\alpha - l)$, so that $-1/2 \le t \le -1/2 + (\alpha - l)$. Finally, let $f \in \mathscr{H}_{s,p}$ and $g \in \mathscr{H}_{-t,q'}$ where p, $q \in [1,\infty]$ are such that*

(i) *if $s = 1/2 - (\alpha - l)$ then $p = r'$ and $q = \infty$;*
(ii) *if $s = 1/2$ then $p = 1$ and $q = r$;*
(iii) *if $1/2 - (\alpha - l) < s < 1/2$ then p, q are related by $1/q = 1/p + 1/r$.*

Then the holomorphic map $\mathbb{C}_+ \ni z \mapsto \langle \Pi_- g, R(z)\Pi_+ f\rangle$ extends to a function of class C^l on $\mathbb{C}_+ \cup J$ and one has

$$\frac{d^l}{d\lambda^l}\langle \Pi_- g, R(\lambda + i0)\Pi_+ f\rangle = \lim_{\mu\to+0}\langle \Pi_- g, l! R(\lambda + i\mu)^{l+1}\Pi_+ f\rangle \tag{6.11}$$

where the limit exists uniformly in $\lambda \in J$.

Proof. Let L_ε be as in Proposition 5.7 and let us set $h(\varepsilon) = \langle g, L_\varepsilon^{(l,0)} f\rangle$ with $f \in \mathscr{H}_{s,p}$ and $g \in \mathscr{H}_{-t,q'}$. Then, according to (5.26), for each integer $m \geq 1$ (in fact this time one can take $m = 1$, which simplifies the proof, but not significantly) we have

$$(6.12) \qquad |\varepsilon^m h^{(m)}(\varepsilon)| \leq$$

$$C \sum_{a+b+c \leq m-1} |||\varepsilon^{\alpha-l-\sigma}\Pi_-(\varepsilon A)^a e^{\varepsilon A} g||| \cdot |||\varepsilon^{\sigma}\Pi_+(\varepsilon A)^b e^{-\varepsilon A} f||| \cdot \|\varepsilon^{-\alpha+c} K_\varepsilon^{(c)}\|,$$

where σ is the real number defined by $s = 1/2 - \sigma$, so that $0 \leq \sigma \leq \alpha - l$. If $\sigma = 0$ (case (ii) of the theorem) then we bound a term in the sum from (6.12) by

$$C'|||\varepsilon^{\alpha-l}\Pi_-(\varepsilon A)^a e^{\varepsilon A} g||| \cdot |||\varepsilon^{-\alpha+c} K_\varepsilon^{(c)}\|.$$

Then by using the Theorem from §2.3 we obtain $|\varepsilon^m h^{(m)}(\varepsilon)| \leq \chi(\varepsilon)$, where $\chi(\varepsilon)$ is independent of λ and μ, and

$$\int_0^{\varepsilon_0} |\chi(\varepsilon)|\varepsilon^{-1} d\varepsilon \leq C''|||f||| \cdot \|g\|_{1/2-\alpha+l,r'}.$$

If $\sigma = \alpha - l$ (case (i) of the theorem) we estimate a typical term in the r.h.s. of (6.12) by

$$C'|||g||| \cdot |||\varepsilon^{\alpha-l}\Pi_+(\varepsilon A)^b e^{-\varepsilon A} f||| \cdot \|\varepsilon^{-\alpha+c} K_\varepsilon^{(c)}\|.$$

Then as above we get

$$\int_0^{\varepsilon_0} |\chi(\varepsilon)|\varepsilon^{-1} d\varepsilon \leq C''|||g||| \cdot \|g\|_{1/2-\alpha+l,r'}$$

Finally, if $0 < \sigma < \alpha - l$ we can use the Theorem established in §2.3 for each of the factors in (6.12) which contains f or g. So the relation $1/q' + 1/p + 1/r = 1$ and the Holder inequality with three factors will give

$$\int_0^{\varepsilon_0} |\chi(\varepsilon)\varepsilon^{-1} d\varepsilon \leq C''|||g|||_{1/2-\alpha+l+\sigma,q'} \|f\|_{1/2-\sigma,p'}$$

Since $1/2 - \alpha + l + \sigma = 1 - s - (\alpha - l) = -t$, we can finish the proof as usual (see the proof of Theorem 6.3). $\square$

We shall reformulate Theorem 6.5 as follows. We know (cf. §3.9) that for each real s with $|s| < 1 + \alpha$ and each $p \in [1, \infty]$ the operator $R(z)$ has a canonical extension to a bounded operator in $\mathscr{H}_{s,p}$ if $z \in \mathbb{C}_+$ and the map $z \mapsto R(z) \in B(\mathscr{H}_{s,p})$ is holomorphic. If s, p, t, q are as in Theorem 6.5 then $t < s$, so we have a holomorphic map

$$(6.13) \qquad \mathbb{C}_+ \ni z \mapsto \Pi_- R(z)\Pi_+ \in B(\mathscr{H}_{s,p}; \mathscr{H}_{t,q}).$$

Now the theorem says that if we equip $B(\mathscr{H}_{s,p}; \mathscr{H}_{t,q})$ with the w-topology then (6.13) extends to a map of class C^l on $\mathbb{C}_+ \cup J$. If $\lambda \in J$ then the operator $\Pi_- R(\lambda + i0)\Pi_+$ is a well defined element of $B(\mathscr{K}; \mathscr{K}^*)$ (by Theorem 6.3). Hence, according to Theorem 6.5, this operator sends $\mathscr{H}_{1/2,1}$ into $\mathscr{H}_{-1/2+\alpha,r}$ and, more generally, induces a continuous operator $\mathscr{H}_{s,p} \to \mathscr{H}_{t,q}$ (with s, p, t, q as in the theorem). Moreover, the map

$$(6.14) \qquad J \ni \lambda \mapsto \Pi_- R(\lambda + i0)\Pi_+ \in B(\mathscr{H}_{s,p}; \mathscr{H}_{t,q})$$

is of class C^l in the w-topology.

The three theorems proved until now in this section describe the differentiability properties of the operators $R(\lambda+i0)$, $\Pi_- R(\lambda+i0)$ and $\Pi_- R(\lambda+i0)\Pi_+$ as functions of the spectral parameter $\lambda\in J$. Our purpose now is to describe the regularity of these functions in terms of the classes Λ^α. For the proof of the next results we shall need the following particular case of the Theorem from §2.3: *if $\chi:\mathbb{R}\to\mathbb{C}$ is a bounded Borel function and $s=\alpha+1/2$ is a real number $>1/2$, and if χ has a zero of order $>\alpha$ at the origin (i.e. $|\chi(x)|\le c|x|^\beta$ for some $\beta>\alpha$), then there is a constant $C<\infty$ such that for all $\varepsilon>0$:*

$$\|\chi(\varepsilon A)\|_{\mathscr{H}_{s,\infty}\to\mathscr{H}_{1/2,1}}+\|\chi(\varepsilon A)\|_{\mathscr{H}_{-1/2,\infty}\to\mathscr{H}_{-s,1}}\le C\varepsilon^\alpha. \tag{6.15}$$

Theorem 6.6. *Let $H\in\mathscr{C}^{1+\alpha}(A)$ for some real $\alpha>0$ and let us set $s=\alpha+1/2$. Then the function*

$$J\ni\lambda\mapsto R(\lambda+i0)\in B(\mathscr{H}_{s,\infty};\mathscr{H}_{s,1}) \tag{6.16}$$

is locally of class Λ^α.

Proof. (i) Let L_ε be as in the proof of Theorem 6.3. We first prove that for each l, $m\in\mathbb{N}$ with $m>2\alpha$ we have

$$\|L_\varepsilon^{(l,m)}\|_{\mathscr{H}_{s,\infty}\to\mathscr{H}_{-s,1}}\le C(l,m)\varepsilon^{\alpha-l-m} \tag{6.17}$$

for a number $C(l,m)<\infty$ independent of $\varepsilon\in(0,\varepsilon_0)$, $\lambda\in J$ and $\mu>0$. For this purpose we use the Proposition 5.5. Note that for each term of the first sum on the r.h.s. of (5.23) we have either $a>\alpha$ or $b>\alpha$. If, for example $a>\alpha$, we use the estimate (6.15) with $\chi=\varphi_{i,a}$ and get that the corresponding term is bounded by a constant times $\varepsilon^\alpha\|g\|_{s,\infty}|||f|||$, and this is better than needed (because $s>1/2$). A typical term of the second sum on the r.h.s. of (5.23) is dominated by const.$|||g|||\cdot|||f|||\cdot\|\varepsilon^c K_\varepsilon^{(c)}\|$ and now we may use Proposition 5.2 (d).

(ii) Now let $f\in\mathscr{H}_{s,\infty}$ and $F(\lambda,\varepsilon)=\langle f,L_\varepsilon(\lambda+i\mu)f\rangle$. Then (6.17) gives

$$|\partial_\lambda^l\partial_\varepsilon^m F(\lambda,\varepsilon)|\le C(l,m)\|f\|_{s,\infty}^2\varepsilon^{\alpha-l-m}. \tag{6.18}$$

This implies the hypothesis of Lemma 6.2, namely $|\partial_\lambda^l\partial_\varepsilon^k F(\lambda,\varepsilon)|\le M\varepsilon^{\alpha-m}$ if $l+k=m$, with $M=\text{const.}\|f\|_{s,\infty}^2$. Indeed, if $l=0$ this is a particular case of (6.18). If $l\ge1$ we integrate (6.18) l times with respect to ε over an interval of the form (τ,ε_0) with $0<\tau<\varepsilon_0$; since $\alpha-m<0$ we shall get $|\partial_\lambda^l\partial_\tau^{m-l}F(\lambda,\tau)|\le M\tau^{\alpha-m}$, which is the estimate we were looking for. Now we use Lemma 6.2. Since $F_0=\langle f,R(z)f\rangle$ and $\mathscr{H}_{s,\infty}=(\mathscr{H}_{-s,1})^*$, the estimate (6.2) implies the assertion of the theorem. □

We remark that the proof gives more than stated in Theorem 6.6: the function $z\mapsto R(z)\in B(\mathscr{H}_{s,\infty};\mathscr{H}_{-s,1})$ is in fact of class Λ^α (and not only locally) on the set $\{z\in\mathbb{C}\mid \operatorname{Re}z\in J,\operatorname{Im}z\ge0\}$.

Theorem 6.7. *Let s, α be real numbers such that $0<\alpha<s-1/2$ and assume that $H\in\mathscr{C}^{s+1/2}(A)$. Then the map*

$$J\ni\lambda\mapsto\Pi_- R(\lambda+i0)\in B(\mathscr{H}_{s,\infty};\mathscr{H}_{s-1-\alpha,1}) \tag{6.19}$$

is locally of class Λ^α.

Proof. (i) We first show that the operator L_ε used in the proof of Theorem 6.4 satisfies the following estimates: for each $l, m \in \mathbb{N}$ with $m > s - 1/2 \equiv \beta$ there is a number $C(l,m)$, independent of $\varepsilon, \lambda, \mu$, such that

$$\|L_\varepsilon^{(l,m)}\|_{\mathscr{H}_{s,\infty}\to\mathscr{H}_{s-1-\alpha,1}} \le C(l,m)\varepsilon^{\alpha-l-m}. \tag{6.20}$$

We use Proposition 5.6. Then (6.15) with $\chi = \zeta$ (which vanishes of order $m > \beta$ at the origin, see (5.24)) implies $|||\zeta(\varepsilon A)f||| \le C'\varepsilon^\beta\|f\|_{s,\infty}$. On the other hand the Theorem from §2.3 implies for $\beta - \alpha > 0$

$$\varepsilon^{\beta-\alpha}|||\Pi_- e^{\varepsilon A} g||| \le C''\|g\|_{1/2-\beta+\alpha,\infty} = C''\|g\|_{-s+\alpha,\infty}. \tag{6.21}$$

Hence the first term on the r.h.s. of (5.25) is bounded by a constant times $\varepsilon^\alpha\|g\|_{1-s+\alpha,\infty}\|f\|_{s,\infty}$. Now we bound the terms of the sum from (5.25) by using $|||(\varepsilon A)^b\psi^{(j)}(\varepsilon A)f||| \le C'|||f||| \le C''\|f\|_{s,\infty}$ and Proposition 5.2(d). We shall get terms of the form $C'''\varepsilon^\beta|||\Pi_-(\varepsilon A)^a eg||| \cdot \|f\|_{s,\infty}$. By an estimate similar to (6.21) (use the Theorem from §2.3 again) we finally obtain

$$|\langle g, \varepsilon^{l+m}L_\varepsilon^{(l,m)}f\rangle| \le C\varepsilon^\alpha\|g\|_{1-s+\alpha,\infty}\|f\|_{s,\infty}.$$

This implies (6.20) because $\mathscr{H}_{1-s+\alpha,\infty} = (\mathscr{H}_{s-1-\alpha,1})^*$.

(ii) Let $F(\lambda,\varepsilon) = \langle g, L_\varepsilon(\lambda + i\mu)f\rangle$ with $f \in \mathscr{H}_{s,\infty}$ and $g \in \mathscr{H}_{1+\alpha-s,\infty}$. If l, $m \ge 0$ are integers and $m > \beta$ then (6.20) gives

$$|\partial_\lambda^l\partial_\varepsilon^m F(\lambda,\varepsilon)| \le C(l,m)\|f\|_{s,\infty}\|g\|_{1+\alpha-s,\infty}\varepsilon^{\alpha-l-m}.$$

Now the proof can be finished as in the case of Theorem 6.6. □

Theorem 6.8. *Assume that $H \in \mathscr{C}^{1+\alpha}(A)$ for some $\alpha > 0$. Let β, s, t be real numbers such that $0 < \beta < \alpha, 1/2 - (\alpha - \beta) \le s \le 1/2$ and $t = s - 1 + (\alpha - \beta)$, so that $-1/2 \le t \le -1/2 + (\alpha - \beta)$. Finally, let p, $q \in [1,\infty]$ be such that*

(i) *if $s = 1/2 - (\alpha - \beta)$ then $p = q = \infty$;*
(ii) *if $s = 1/2$ then $p = q = 1$;*
(iii) *if $1/2 - (\alpha - \beta) < s < 1/2$ then $p = \infty$, $q = 1$.*

Then the map (6.14) is locally of class Λ^β.

Proof. We assume Lemma 6.2 with $F(\lambda,\varepsilon) = \langle g, L_\varepsilon(\lambda + i\mu)f\rangle$ where L_ε is as in Proposition 5.7 and $f \in \mathscr{H}_{s,p}$, $g \in \mathscr{H}_{-t,q'}$. As in the proof of Theorems 6.6 and 6.7 we shall need the following estimate: for l, $m \in \mathbb{N}$ there is a constant $C(l,m)$, independent of $\varepsilon \in (0,\varepsilon_0)$, $\lambda \in J$ and $\mu > 0$ such that

$$\|L_\varepsilon^{(l,m)}\|_{\mathscr{H}_{s,p}\to\mathscr{H}_{t,q}} \le C(l,m)\varepsilon^{\beta-l-m}. \tag{6.22}$$

In order to prove this we use the inequality established in Proposition 5.7. Each term in the r.h.s. of (5.26) is of the form $|||\varphi(\varepsilon A)g||| \cdot |||\psi(\varepsilon A)f||| \cdot \|\varepsilon^c K_\varepsilon^{(c)}\|$ where $\varphi, \psi \in \mathscr{S}(\mathbb{R})$ but do not vanish at zero in general. By Proposition 5.2(d) such a term is bounded by a constant times

$$\varepsilon^\alpha|||\varphi(\varepsilon A)g||| \cdot |||\psi(\varepsilon A)f||| = \varepsilon^\beta|||\epsilon^{\alpha-\beta-\sigma}\varphi(\varepsilon A)g||| \cdot |||\varepsilon^\sigma\psi(\varepsilon A)f||| \tag{6.23}$$

where σ could be an arbitrary real number. If $0 < \sigma < \alpha - \beta$ then the r.h.s. of (6.23) can be estimated with the help of the Theorem from §2.3. We clearly get a bound of the form $c\varepsilon^\beta\|g\|_{1/2-\alpha+\beta+\sigma,\infty}\|f\|_{1/2-\sigma,\infty}$. We set $s = 1/2 - \sigma$ and we obtain (6.22) by a simple argument. The limit cases $\sigma = 0$ and $\sigma = \alpha - \beta$ are treated similarly. □

References

[AC] J. Aguilar and J.-M. Combes, A Class of Analytic Perturbations of One-Body Schrődinger Hamiltonians, *Comm. Math. Phys.* **22** (1971) 269–279.

[ABG1] W. Amrein, A. Boutet de Monvel, and V. Georgescu, On Mourre's approach to spectral theory, *Helv. Phys. Acta* **62** (1989) 1–20.

[ABG2] W. Amrein, A. Boutet de Monvel, and V. Georgescu, *C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Progress in Math. Ser., **135**, Birkhäuser, Bâle, 1996.

[ABG3] W. Amrein, A. Boutet de Monvel, and V. Georgescu, Notes on the N-Body Problem, II, Preprint Université de Genève, UGVA-DPT, 1991/04-178 (1991) 160–423.

[BB] P.L. Butzer and H. Berens, *Semi-Groups of Operators and Approximations*, Springer, Berlin, 1967.

[BC] E. Balslev and J.-M. Combes, Spectral Properties of Many-Body Schrődinger Operators with Dilation-Analytic Interactions, *Comm. Math. Phys.* **22** (1971) 280–294.

[BG1] A. Boutet de Monvel and V. Georgescu, Locally Conjugate Operators, Boundary Values of the Resolvent and Wave Operators, *C. R. Acad. Sci. Paris Sér. I Math.* **313** (1991) 13–18.

[BG2] A. Boutet de Monvel and V. Georgescu, On the Regularity of the Boundary Values of a Resolvent Family, *C. R. Acad. Sci. Paris Sér. I Math.* **321** (1995) 1299–1304.

[BG3] A. Boutet de Monvel and V. Georgescu, Boundary Values of the Resolvent of a Self-Adjoint Operator: Higher Order Estimates, in *Algebraic and Geometric Methods in Mathematical Physics*, Kaciveli 1993, Math. Physics Studies **19**, Kluwer Academic Publ. (1996), 9–52.

[BG4] A. Boutet de Monvel and V. Georgescu, Spectral and Scattering Theory by the Conjugate Operator Method, *Algebra and Analysis* **4** (3) (1992) 79–116 (=*St Petersburg Math. J.* **4** (3) (1993) 469–501).

[BG5] A. Boutet de Monvel and V. Georgescu, Some Developments and Applications of the Abstract Mourre Theory, *Astérisque* **210** (1992) 27–48.

[BGM] A. Boutet de Monvel, V. Georgescu, and M. Mantoiu, Locally Smooth Operators and the Limiting Absorption Principle for N-Body Hamiltonians, *Rev. Math. Phys.* **5** (1) (1993) 105–189.

[BGS] A. Boutet de Monvel, V. Georgescu, and A. Soffer, N-Body Hamiltonians with Hard Core Interactions, *Rev. Math. Phys.* **6** (4) (1994) 515–596.

[BGSh] A. Boutet de Monvel, V. Georgescu, J. Sahbani, Boundary Values of Resolvent Families and Propagation Properties, *C. R. Acad. Sci. Paris Sér. I Math.* **322** (1996) 289–294.

[CFKS] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrődinger Operators*, Springer-Verlag, Berlin, 1986.

[D1] E.B. Davies, *Spectral Theory and Differential Operators*, Cambridge Studies in Advanced Math. **42**, Cambridge, 1995.

[D2] E.B. Davies, The Functional Calculus, *J. London Math. Soc.* (1995).

[DU] J. Diestel and J.J. Uhl, Jr, *Vector Measures*, Math. Surveys **15**, Amer. Math. Soc., Providence, Rhode Island, 1977.

[J] A. Jensen, Propagation Estimates for Schrodinger-Type Operators, *Trans. Amer. Math. Soc.* **291** (1985) 129–144.

[JMP] A. Jensen, E. Mourre, and P. Perry, Multiple Commutator Estimates and Resolvent Smoothness in Quantum Scattering Theory, *Ann. Inst. H. Poincaré* **41** (1984) 207–225.

[JP] A. Jensen and P. Perry, Commutator Methods and Besov Space Estimates for Schrodinger Operators, *J. Op. Theory* **14** (1985) 181–189.

[M1] E. Mourre, Absence of Singular Continuous Spectrum for Certain Self-Adjoint Operators, *Comm. Math. Phys.* **78** (1981) 391–408.

[M2] E. Mourre, Opérateurs conjugués et propriétés de propagation, *Comm. Math. Phys.* **91** (1983) 279–300.

[P] J. Peetre, *New Thoughts on Besov Spaces*, Duke Univ. Math. Series I, Durham, 1976.

[PSS] P. Perry, I.M. Sigal, and B. Simon, Spectral Analysis of N-Body Schrődinger Operators, *Ann. Math.* **114** (1981) 519–567.

[S1] J. Sahbani, Théorèmes de propagation, hamiltoniens localement réguliers et applications, *Thèse Université Paris 7*, juillet 1996.

[S2] J. Sahbani, The Conjugate Operator Method for Locally Regular Hamiltonians, *J. Operator Theory*, to appear.

[W] R. Weder, "Spectral Analysis of Strongly Propagative Systems", *J. fűr Reine Angew. Math.*, 354, (1984), 95-122.