Anderson Localization for the Almost Mathieu Equation, III. Semi-Uniform Localization, Continuity of Gaps, and Measure of the Spectrum

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Abstract

We show that the almost Mathieu operator, $(H_{\omega,\lambda,\theta}\Psi)(n) = \Psi(n+1) + \Psi(n-1) + \lambda\cos(\pi\omega n + \theta)\Psi(n)$, has semi-uniform (and thus dynamical) localization for $\lambda > 15$ and a.e. ω, θ. We also obtain a new estimate on gap continuity (in ω) for this operator with $\lambda > 29$ (or $\lambda < 4/29$), and use it to prove that the measure of its spectrum is equal to $|4 - 2|\lambda||$ for λ in this range and *all* irrational ω's.

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1 Introduction

In this paper we study localization for the almost Mathieu operator $H_{\omega,\lambda,\theta}$ acting on $\ell^2(\mathbb{Z})$:

$$(H_{\omega,\lambda,\theta}\Psi)(n) = \Psi(n+1) + \Psi(n-1) + v_n\Psi(n), \tag{1.1}$$

where the potential v_n is given by

$$v_n = \lambda\cos(\pi\omega n + \theta). \tag{1.2}$$

For background and some recent results on the almost Mathieu operator see [26, 18, 19, 15].

An often used notion of localization is that of pure point spectrum with exponentially decaying eigenfunctions. It can be expressed by the following definition:

Definition 1. H exhibits localization if there exists a constant $\gamma > 0$ such that for any eigenfunction ψ_s one can find a constant $C(s) > 0$ and a site $n(s) \in \mathbb{Z}$ (center of localization) so that $|\psi_s(k)| \le C(s)e^{-\gamma|n(s)-k|}$ for any $k \in \mathbb{Z}$.

Localization, in this sense, for the almost Mathieu operator has been proven for $\lambda > 15$ and ω, θ satisfying certain arithmetic conditions [32, 13, 16].

At the same time, the physical understanding of localization is connected with what is often called dynamical localization—the non-spreading of initially localized wave-packets under the Schrödinger time evolution. It can be expressed by, for example, boundedness in the time t of $\|xe^{itH}\delta_0\|^2 = (e^{-itH}\delta_0, x^2e^{-itH}\delta_0)$. An even stronger condition that we prefer to adapt here is the following:

Definition 2. H exhibits *dynamical localization* if there exists a constant $\tilde{\gamma} > 0$ such that for any $\ell \in \mathbb{Z}$, there exists $\tilde{C}(\ell) > 0$ so that

$$\sup_t |e^{-itH}(n,\ell)| \le \tilde{C}(\ell)e^{-\tilde{\gamma}|n-\ell|}. \tag{1.3}$$

Both of the above definitions (as well as Definitions 3 and 4 below), with $\mathbb{Z}$ replaced by $\mathbb{Z}^d$, work for operators H acting on $\ell^2(\mathbb{Z}^d)$.

Dynamical localization has been established for the d-dimensional Anderson model in [2, 10] (and, in a restricted form, [28]); it remained an open question for the almost Mathieu operator.

While dynamical localization implies pure point spectrum [23, 5], the converse is not true in general. There exist operators H with localization, but, nevertheless, with $\limsup_{t\to\infty} \|xe^{itH}\delta_0\|^2/t^\alpha = \infty$ for any $\alpha < 2$ [7] (also see [27]). In fact, t^α can be replaced by an arbitrary function $f(t) = o(t^2)$. This example shows that Simon's theorem on the absence of ballistic motion for operators with pure point spectrum [31] is optimal, and that mere "exponential localization" of eigenfunctions is not sufficient to determine the dynamics.

Indeed, localization might not have much physical meaning if there is no control on the dependence of C on n (or, equivalently, on the eigenenergy E_n). In particular, if the $C(m)$'s are allowed to grow arbitrarily fast with m, then eigenvectors may be "extended" over arbitrarily large length-scales and one cannot effectively define a "localization length" corresponding to a typical size of the "essential support" of the eigenfunction. An appropriate level of control over the $C(m)$'s is given by the following definition, introduced in [7]:

Definition 3. H has SULE (*semi-uniformly localized eigenvectors*) if there exists a constant $\gamma > 0$ such that for any $b > 0$, there exists a constant $C(b) > 0$ such that for any eigenfunction ψ_s one can find $n(s) \in \mathbb{Z}$ so that $|\psi_s(k)| \le C(b)e^{b|n(s)|-\gamma|k-n(s)|}$ for any $k \in \mathbb{Z}$.

The notion of SULE, which strengthens mere localization, also has a dynamical counterpart, which strengthens mere dynamical localization:

Definition 4. H has SUDL (*semi-uniform dynamical localization*) if there exists a constant $\tilde{\gamma} > 0$ such that for any $b > 0$, there exists a constant $\tilde{C}(b) > 0$ so that

$$\sup_t |e^{-itH}(n,\ell)| \le \tilde{C}(b)e^{b|\ell|-\tilde{\gamma}|n-\ell|}. \tag{1.4}$$

It is shown in [7] that SULE implies SUDL (and thus dynamical localization), and that SUDL, along with a simple spectrum, implies SULE.

There is another direction from which the notion of localization had been challenged recently, when Gordon [14] and del Rio-Makarov-Simon [9] have shown that localization can be destroyed by infinitesimally small localized rank-one perturbations. However, SULE appears to be a condition that implies certain semistability (or physical stability) of localization [7]. Specifically, if H has SULE and H' is obtained from H by adding a localized rank-one perturbation, then all the spectral measures of H' are supported on a set of zero Hausdorff dimension; and, more importantly, while $\|xe^{-itH'}\delta_0\|$ may be unbounded, it never grows faster than $C\ln^2(t)$ [7].

That makes it particularly interesting to establish SULE for systems with localization. For the d-dimensional Anderson model, SULE had been derived in [7] from the dynamical estimates of Aizenman [2] ([10] in the one-dimensional case). More precisely, the dynamical estimates imply SUDL, from which SULE follows by the SULE $\Leftrightarrow$ SUDL relation. In the present paper, we obtain SULE for the almost Mathieu operator by direct analysis of eigenfunctions, and so we deduce SUDL (and, in particular, dynamical localization) for this operator by using the SULE $\Leftrightarrow$ SUDL relation in the other direction. (It should be pointed out that we do not know any other way to prove dynamical localization for the almost Mathieu operator.)

Throughout the paper we will often assume ω to be Diophantine; that is, that there exist $c(\omega) > 0$ and $1 < r(\omega) < \infty$ such that

$$|\sin \pi j\omega| > \frac{c(\omega)}{|j|^{r(\omega)}} \tag{1.5}$$

for all $j \neq 0$. Our main result is the following:

THEOREM 1.1 *The almost Mathieu operator has SULE for any Diophantine ω, $\lambda > 15$, and a.e. θ.*

As discussed above this immediately implies:

COROLLARY 1.2 *The almost Mathieu operator with ω, λ, θ as in Theorem 1 has dynamical localization.*

Remarks.

1. The set of parameters ω, λ in Theorem 1.1 is exactly the set for which localization has been proven [16].
2. We will, in fact, obtain a polynomial bound on $C(n(s))$. See (2.5).
3. The set of θ's for which we show SULE is smaller than the set of θ's for which localization has been proven. One can show that there is a zero measure set of θ's for which there is localization but not SULE.
4. Similar techniques can be applied to non-Diophantine ω's with exponential rate of approximation by rationals, for which localization is proven for large λ [21].
5. One can study local SULE, that is, semi-uniform localization of eigenfunctions with corresponding eigenvalues belonging to a certain interval. Such local SULE can be shown to imply dynamical localization for the spectral projection of H onto this interval. Our method allows us to establish local SULE for all energy intervals where localization has been shown so far. That includes certain intervals in the center of the spectrum for $\lambda > 5.6$ [16].
6. One can think that a more natural control on the $C(n)$'s is given by *uniform localization*, UL, defined as localization with a uniform bound $C(n) < c$. However, a large class of ergodic operators, including the almost Mathieu operator, has been shown not to have UL [20, 7]. Indeed, for large families of ergodic potentials, $v_n = v(T^n x)$, UL implies phase-stability of pure point spectrum (see [20]), that is, localization for all phases x [20, 7]. We would like to repeat a remark from [20] that for the almost Mathieu operator, SULE is accompanied by certain phase-semistability of pure point spectrum, as the spectral measures for all θ are zero-dimensional [19, 22]. This suggests that the general relation between SULE and such semistability, or even a stronger dynamical statement, could be similar to the general relation between UL and the stability. While both phase-stability of pure point spectrum and UL, as in the Maryland model [29, 12, 11, 30], are deeply connected with the lack of resonances (see [20]), both SULE and zero-dimensional spectral measures for the almost Mathieu operator are implied by certain control over the resonances.

The estimates we obtain allow us, in addition, to answer some other almost Mathieu questions. In particular, we establish a strong version of the continuity of gaps theorem [3, 24], and use it to extend the result of [24] on the measure of the spectrum to the case of an arbitrary irrational frequency ω. Let $\sigma(\omega, \lambda)$ denote the spectrum of $H_{\omega,\lambda,\theta}$, which is known to be completely θ-independent for any irrational ω [5]. We shall prove the following:

THEOREM 1.3 *For all irrational ω's and $|\lambda| > 29$ or $|\lambda| < 4/29$,*

$$|\sigma(\omega, \lambda)| = |4 - 2|\lambda||,$$

where $|\cdot|$ denotes Lebesgue measure.

Remark. The equality $|\sigma(\omega, \lambda)| = |4-2|\lambda||$ was conjectured by Aubry and Andre [1] to hold for every λ and irrational ω, and was studied by Thouless [33] and by Avron, van Mouche, and Simon [3], who proved the lower bound. Last [24] obtained the equality for every λ and a.e. ω.

In Section 2 we prove Theorem 1.1 up to some lemmas that we prove in Sections 3 and 4. In Section 5 we prove a result about continuity of gaps, which we use in Section 6 to prove Theorem 1.3. The Appendix provides a proof for a somewhat technical lemma that we use in Section 4.

2 Proof of Theorem 1.1

In this section we prove Theorem 1.1 up to some lemmas that will be proven later. Our proof consists of two parts:

1. Obtaining uniform estimates in the proof of localization for the nonresonant regime.
2. Studying the statistics of resonances, and, particularly, the dependence of the number of resonances on the position of the center of localization.

We introduce the sets of resonant phases:

$$\Theta_{j,k}^s = \left\{ \theta : (k+1)^{-s} < \left| \sin 2\pi \left(\theta + \frac{j}{2}\omega \right) \right| < k^{-s} \right\},\ k \in \mathbb{N},$$

$$\Theta_k^s = \cup_{|j|=0}^{k} \Theta_{j,k}^s,$$

$$\Theta^s = \overline{\lim_{k\to\infty}} \Theta_k^s,$$

and

$$\Theta = \cap_{s>r(\omega)} \Theta^s.$$

Note that

$$\Theta = \{\theta : \text{for every } s > r(\omega) \text{ the relation } |\sin 2\pi(\theta + (j/2)\omega)| < k^{-s} \text{ for some } |j| \le k$$
$$\text{holds for infinitely many } k'\text{s}\}.$$

Since $|\Theta^s_k| = O(k^{-s})$, it follows from the Borel-Cantelli lemma that every $\Theta^s, s > r(\omega)$, has zero Lebesgue measure and so does Θ. For $\theta \notin \Theta$, we define the resonant rate as $s(\theta) \equiv \inf\{s > r(\omega) : \theta \notin \Theta^s\} + 1 \ge r(\omega) + 1$. For $s > s(\theta) - 1$ we define the s-resonant number of θ, $k(\theta, s) = \#\{m \in \mathbb{N} : \theta \in \Theta^s_m\}$. Let $n_1(\theta, s) < \cdots < n_{k(\theta,s)}(\theta, s)$ be the positions of resonances: all $m \in \mathbb{N}$ such that $\theta \in \Theta^s_m$. For a fixed θ, the numbers $k(\theta, s)$ and $n_{k(\theta,s)}(\theta, s)$ decrease with s. Also, $s(\theta)$ is an invariant function: $s(\theta + \omega) = s(\theta)$, and Θ is an invariant set. Let $k(\theta) = k(\theta, s(\theta))$. We put $n_i(\theta) = n_i(\theta, s(\theta)),\ i = 1, \ldots, k(\theta)$, and will sometimes write $n_{k(\theta)}$ for $n_{k(\theta)}(\theta)$.

We first obtain some elementary information on the sparseness of the sequence $n_i(\theta)$.

LEMMA 2.1 *Suppose $\theta \in \Theta^s_n$, ω satisfies* (1.5), *and $s > r(\omega)$. Then there exists a positive constant $c_1(\omega)$ such that $\theta \notin \Theta^s_m, n < m < c_1(\omega) n^{\frac{s}{r(\omega)}}$.*

Proof: $|\sin 2\pi(\theta + j/2\omega)| \le n^{-s}$, some $|j| \le n$, and $|\sin 2\pi(\theta + \ell/2\omega)| \le m^{-s}$, some $|\ell| \le m$, $|\ell| > |j|$, imply $|\sin \pi(\ell - j)\omega)| < 2n^{-s}$, and so, by (1.5), we have $m \ge |\ell| \ge |\ell - j| - |j| \ge (c(\omega) n^s/2)^{\frac{1}{r(\omega)}} - n$. ∎

For $\theta \notin \Theta, \lambda > 15$ and ω satisfying (1.5) the localization has been proven in [16]. Thus every eigenfunction Ψ_E with eigenvalue E, attains its maximal value at no more than finitely many points. We define $n(E)$ to be the position of the leftmost maximum of Ψ_E.

Our key technical result is

LEMMA 2.2 *Let $\theta \notin \Theta, \lambda > 15$ and ω satisfies* (1.5). *Then there exist $C = C(\theta, \omega, \lambda) < \infty$ and $\gamma = \gamma(\lambda) > 0$ such that for any eigenfunction Ψ_E of H_θ, we have $|\Psi_E(m)| < 2|\Psi_E(n(E))| e^{-\gamma(\lambda)|m - n(E)|}$ for $|m - n(E)| > C(\theta, \omega, \lambda) \ln n_{k(\theta + n(E)\omega)}$.*

Lemma 2.2 will be proven in Section 4. For the continuity of gaps and measure of the spectrum parts, we will also need a similar, slightly more detailed statement, Lemma 5.3, asserting the exponential decay between the resonances.

In order to relate the number of resonances to the position of $n(E)$, we will need the following:

LEMMA 2.3 *Fix $0 < r < \infty$. Then for a.e. $\theta \notin \Theta^s$ with $s > r+1$, there exists $q(\theta, r, s) < \infty$ such that for every eigenvalue E of H_θ, we have $n(E) > n^r_{k(\theta + n(E)\omega)}$ if $n_{k(\theta + n(E)\omega)} > q(\theta, r, s)$.*

This lemma will be proven in Section 3.

Proof of Theorem 1.1: Suppose $\lambda > 15$ and ω satisfies (1.5). For a.e. $\theta \notin \Theta$, as in Lemma 2.3, we obtain, using Lemma 2.2, that for any eigenfunction Ψ_E of $H_{\omega,\lambda,\theta}$ and any $m \in \mathbb{Z}$,

$$|\Psi_E(m)| < 2|\Psi_E(n_E)|(n_{k(\theta+n(E)\omega)})^{C(\theta,\omega,\lambda)\gamma(\lambda)}e^{-\gamma(\lambda)|m-n(E)|)}. \tag{2.6}$$

Thus, if we normalize Ψ by $\Psi_E(n(E)) = 1$ and fix $0 < r < s(\theta) - 1$, Lemma 2.3 yields that

$$|\Psi_E(m)| < 2(\max(n(E)^{\frac{1}{r}}, q(\theta, r, s(\theta))))^{C(\theta,\omega,\lambda)\gamma(\lambda)}e^{-\gamma(\lambda)|m-n(E)|)} \tag{2.7}$$

which, in particular, proves the theorem. ∎

3 Resonant sets: Proof of Lemma 2.3

Let $p_n^s(\theta) = \min\{|m| : \theta + 2\pi m\omega \in \Theta_n^s\}$.

LEMMA 3.1 *Fix $0 < r < s - 1$. Then for a.e. θ, there exists $q(\theta, r, s) < \infty$ such that for $n > q(\theta, r, s)$, we have $p_n^s(\theta) > n^r$.*

Proof: Since $|\Theta_n^s| = O(n^{-s})$, we have $|\{\theta : p_n^s(\theta) < n^r\}| \leq 2n^{r+1-s}$. Thus, the Borel-Cantelli lemma implies the result. ∎

Proof of Lemma 2.3: Since $\theta + n(E)\omega \in \Theta^s_{n_{k(\theta+n(E)\omega)}}$, we have by the definition of $p_n^s(\theta)$ that $n(E) \geq p^s_{n_{k(\theta+n(E)\omega)}}(\theta)$. So, by Lemma 3.1, we obtain the needed statement. ∎

4 Uniform decay: Proof of Lemma 2.2

Throughout this section we assume that ω satisfies (1.5). We start with recalling the main definitions and lemmas from the proof of localization in [16]. We will use the notation $G_{[x_1,x_2]}(E)$ for the Green's function $(H-E)^{-1}$ of the operator $H_{\omega,\lambda,\theta}$ restricted to the interval $[x_1, x_2]$ with zero boundary conditions at $x_1 - 1$ and $x_2 + 1$. Let us denote

$$P_k(\theta, E) = \det\left[(H(\theta) - E)\Big|_{[0,k-1]}\right].$$

We now fix $E \in R$; $1 < m_1 < \frac{\lambda}{2}, \frac{1}{m_1} < m_2 < 1, \epsilon > 1$. We will need the following quantities:

$$M(E, \lambda) = \frac{1}{\sqrt{3}}|E + i + \sqrt{(E+i)^2 - \lambda^2}|$$

where by $\sqrt{(E+i)^2-\lambda^2}$ we understand the value with positive imaginary part;

$$C(E,\lambda)=\frac{\ln\frac{\lambda}{2}}{\ln M(E,\lambda)}-\frac{3}{4}$$

and $c_{\lambda,\epsilon}=\frac{\ln(m_1m_2)}{\ln(\epsilon M(E,\lambda))}$. Given $k>0$, let us denote

$$A_k(E)=\{x\in\mathbb{Z}:|P_k(\theta+x\omega,E)|>m_1^k\}.$$

We will sometimes drop the E-dependence, assuming E is fixed.

Definition. Fix $E\in\mathbb{R}$. A point $y\in\mathbb{Z}$ will be called (m_2,k)-regular if there exists an interval $[x_1,x_2]$ containing y such that

$$|G_{[x_1,x_2]}(y,x_i)|<m_2^k,\ \text{and}\ \operatorname{dist}(y,x_i)\leq k;\ i=1,2.$$

Otherwise y will be called (m_2,k)-singular.

The Poisson formula

$$\Psi(y)=G_{[x_1,x_2]}(y,x_1)\Psi(x_1-1)+G_{[x_1,x_2]}(y,x_2)\Psi(x_2+1) \tag{4.1}$$

implies that if Ψ_E is a generalized eigenfunction, then every point y with $\Psi_E(y)\neq 0$ is (m_2,k)-singular for k sufficiently large: $k>k(E,m_2,\theta,y)$. Of course, in general there is no uniform bound on $k(E,m_2,\theta,y)$. It turns out though that if we pick $H_{\omega,\lambda,\theta}$ with an eigenvalue E and take y to be $n(E)$, such a bound becomes immediate.

LEMMA 4.1 *$n(E)$ is (m,k)-singular for $k>-\frac{\ln 2}{\ln m}$.*

Proof: Obvious from (4.1). ∎

We will need to recall several statements from the proof of localization in [16].

LEMMA 4.2 (Proposition 1 of [16]) *For any $\epsilon>1$, there exists $k(\epsilon,E)$ such that for $k>k(\epsilon,E)$ and all θ we have*

$$|P_k(\theta,E)|<(\epsilon M(E,\lambda))^k.$$

LEMMA 4.3 (Proposition 2 of [16]) *Suppose $y\in\mathbb{Z}$ is (m_2,k)-singular. Then for any x such that $k(1-c_{\lambda,\epsilon})\leq y-x\leq kc_{\lambda,\epsilon}$, we have that x does not belong to A_k.*

The following lemma is proven in the proof of Lemma 3 in [16]

LEMMA 4.4 *Let $\frac{2m_1}{\lambda} < b < 1$. Then there exists $k_1(b) < \infty$ such that for $k > k_1(b)$, if the two points $x_1, x_2 \in \mathbb{Z}$ are such that*
1) $x_i, x_i+1, ..., x_i + [\frac{k+1}{2}] \notin A_k$, $i = 1, 2$,
2) $\mathrm{dist}(x_1, x_2) > [\frac{k+1}{2}]$,
then

$$\left|\cos\left(2\pi\left(\theta + \left(\frac{k-1}{2} + x_1 + j_1\right)\omega\right)\right) - \cos\left(2\pi\left(\theta + \left(\frac{k-1}{2} + x_1 + j_2\right)\omega\right)\right)\right| \le b^{\frac{k}{4}} \quad (4.2)$$

for some $j_1,\ j_2 \in [0, [\frac{k+1}{2}]] \cup [x_2 - x_1, x_2 - x_1 + k - 1 - [\frac{k+1}{2}]]$.

Let $E(\theta)$ be a generalized eigenvalue of $H_{\omega,\lambda,\theta}$, $\Psi(x)$ the corresponding generalized eigenfunction. To finish the proof of Lemma 2.2 we will need E-independent bounds on how large the scale k should be in Lemmas 4.1–4.4. Since Lemmas 4.1, 4.3, 4.4 are already E-independent, we will only have to take care of Lemma 4.2. It turns out that Lemma 4.2, although formulated and proven in [16] in a non-uniform way, is, in fact, a uniform statement:

LEMMA 4.5 *There exists $k(\epsilon) < \infty$ such that $k(\epsilon, E) \le k(\epsilon)$ for any $E \in [-\lambda - 2, \lambda + 2]$.*

Lemma 4.5 will be proven in the appendix.

Proof of Lemma 2.2: Let $\lambda > 15$. It was shown in [16] that in such case, $C(E, \lambda) > C(\lambda + 2, \lambda) > 0$ for any $E \in [\lambda - 2, \lambda + 2]$. Thus, there exist ($E$-independent) $1 < m_1 < \frac{\lambda}{2}$, $m_2 < 1$ and $\epsilon > 1$ such that $2c_{\lambda,\epsilon} - 1 > \frac{1}{2}$. Fix $\frac{2m_1}{\lambda} < b < 1$. Let $k = |x - n(E)| > \max[k(\epsilon), k_1(b), -\frac{\ln 2}{\ln m_2}]$. Suppose x is (m_2, k)-singular. Since, by Lemma 4.1, $n(E)$ is also (m_2, k)-singular, we can, by Lemma 4.3, apply Lemma 4.4 with $x_1 = x - [c_{\lambda,\epsilon}|x - n(E)|]$ and $x_2 = n(E) - [c_{\lambda,\epsilon}|x - n(E)|]$. We then obtain, by (1.5), (4.2),

$$\left|\sin\left(2\pi\left(\theta + \left(\frac{k-1}{2} + x_1\right)\omega + \frac{j_1 + j_2}{2}\omega\right)\right)\right| < \frac{b^{k/4}(3k/2)^{r(\omega)}}{2c(\omega)} < b^{k/5},\ k > \hat{k}(\omega, b)$$

with some $j_1,\ j_2 \in [0, [\frac{k+1}{2}]] \cup [x_2 - x_1, x_2 - x_1 + k - 1 - [\frac{k+1}{2}]]$. Noting that $\theta + n(E)\omega \notin \Theta_n^{s(\theta)}$ for $n > n_{k(\theta+n(E)\omega)}$, we obtain that either $k < \frac{-5s(\theta)\ln n_{k(\theta+n(E)\omega)}}{\ln b}$ or $|k - 1 - 2c_{\lambda,\epsilon}k + j_1 + j_2| > b^{\frac{-k}{5s(\theta)}}$. Since for any allowed j_1, j_2 we have $|k - 1 - 2c_{\lambda,\epsilon}k + j_1 + j_2| < 5/2k$, the second inequality is contradictory for $k > \hat{k}_1(b, s(\theta))$. Thus any x with

$$\begin{aligned} k = |x - n(E)| &> k_0(\epsilon, m_2, b, \omega, \theta) \\ &= \max\left(k(\epsilon), k_1(b), -\frac{\ln 2}{\ln m_2}, \hat{k}(\omega, b), \hat{k}_1(b, s(\theta)), \frac{-5s(\theta)\ln n_{k(\theta+n(E)\omega)}}{\ln b}\right) \end{aligned}$$

is $(m_2, |x - n(E)|)$-regular. Repeating the argument of [16], we have that there exists an interval $[y_1, y_2]$ containing x such that

$$|y_i - x| \le |x - n(E)|,\ \ |G_{[y_1,y_2]}(x, y_i)| \le m_2^{|x-n(E)|},\ i = 1, 2.$$

Using (4.1), we obtain the estimate: $|\Psi(x)| \leq 2|\Psi(n(E))|e^{-\gamma(\lambda)|x-n(E)|}$, $\gamma(\lambda) = -\ln m_2$, for any x with $|x - n(E)| > C_1(\omega, \theta, \lambda) \ln n_{k(\theta+n(E)\omega)}$, since our choice of ϵ, m_2, b was dependent on λ only. ∎

5 Continuity of gaps

Let $\sigma(\omega, \lambda, \theta)$ denote the spectrum of $H_{\omega,\lambda,\theta}$ which depends on θ only if ω is a rational. We denote $S(\omega, \lambda) = \cup_\theta \sigma(\omega, \lambda, \theta)$. In this section we establish the following continuity property of the set $S(\omega, \lambda)$:

THEOREM 5.1 *For every $\lambda > 29$, there are constants $C(\lambda), D(\lambda) > 0$, such that if $\omega, \omega' \in \mathbb{R}$ satisfy $|\omega - \omega'| < C(\lambda)$, then for every $E \in S(\omega, \lambda)$ there is $E' \in S(\omega', \lambda)$ with*

$$|E - E'| < F(|\omega - \omega'|),$$

where $F(x) = -D(\lambda) x \ln x$.

Remarks.

1. This theorem with $F(x) = 6(\lambda x)^{1/2}$ was proven for any $\lambda > 0$ by Avron, van Mouche, and Simon [3].
2. The constant $D(\lambda)$ can be effectively estimated. Namely, as can be seen from the proof, it can be shown that $D(\lambda) < \frac{264}{3 \ln \frac{\lambda}{2M(\lambda+2,\lambda)^{5/6}}}\lambda$. This estimate explodes for λ approaching the root of $\lambda = 2M(\lambda+2, \lambda)^{5/6}$ which is slightly smaller than 29. However, as λ grows, the estimate becomes increasingly better.

Theorem 5.1 immediately implies the following corollary:

COROLLARY 5.2 (i) *For every $\lambda > 29$, there are constants $C(\lambda), D(\lambda) > 0$, such that if $|\omega - \omega'| < C(\lambda)$, then for every gap in $S(\omega, \lambda)$ with midpoint E_g, and measure $|g|$ larger than $-2D(\lambda)|\omega - \omega'| \ln |\omega - \omega'|$, there is a corresponding (containing E_g) gap in $S(\omega', \lambda)$ with measure larger than $|g| + 2D(\lambda)|\omega - \omega'| \ln |\omega - \omega'|$.*

(ii) *The same continuity as in* (i) *also holds for the extreme edges of $S(\omega, \lambda)$ namely, for $|\omega - \omega'| < C(\lambda)$:*

$$|{}^{\max}_{\min} S(\omega, \lambda) - {}^{\max}_{\min} S(\omega', \lambda)| < -D(\lambda)|\omega - \omega'| \ln |\omega - \omega'|.$$

For the proof of Theorem 5.1, we will need a statement asserting decay between the resonances. Fix θ. Let $j(n)$ be the position of the resonance of order $n : |\sin 2\pi(\theta + \frac{j(n)}{2}\omega)| < n^{-s}, |\sin 2\pi(\theta + \frac{j}{2}\omega)| > n^{-s}, \ |j| < |j(n)|$. By the proof of Lemma 2.1, we have, for $m > n$:

$$|j(m)| > c_1(\omega)|j(n)|^{\frac{s}{r(\omega)}}. \tag{5.1}$$

LEMMA 5.3 *Let $\lambda > 29$, ω satisfies* (1.5)*, $\theta \notin \Theta$, E is an eigenvalue of H_θ and $\theta + n(E)\omega \in \Theta_n^s \cap \Theta_m^s$, $n < m$; $\theta + n(E)\omega \notin \Theta_i^s$, $n < i < m$. Then there exists $k(\lambda, \omega) = \max(c_2(\lambda), -c_3(\lambda)\ln(2c(\omega)))$, such that for $|m - n(E)| > k(\lambda, \omega)$ we have*

$$|\Psi_E(m)| < 2|\Psi_E(n(E))|e^{-\gamma(\lambda)|m-n(E)|}$$

whenever $3j(n) < |m - n(E)| < 3/8j(m)$, where $\gamma(\lambda)$ is the same as in Lemma 2.2.

For the proof of Theorem 5.1, we will need the following elementary lemma:

LEMMA 5.4 *For any Borel set $S \subset [0,1]$ with $|S| > 0$, there exist $\omega \in S$ for which* (1.5) *holds with $r(\omega) = 3$ and $c(\omega) > \frac{|S|}{3}$.*

Proof: If not, then for every $\omega \in S$ with $r(\omega) = 3$ (such ω's form a set of full measure) there would exist $j \neq 0$ such that

$$|\sin \pi j\omega| < \frac{|S|}{3|j|^3}. \tag{5.2}$$

If we denote by A_j the set of $\omega \in S$ for which (5.2) is satisfied, we obtain $|S| \le \sum |A_j| < |S|$, and the contradiction proves the lemma. ∎

Proof of Theorem 5.1: Fix $\lambda > 29$. By Lemma 5.4, we can find $\omega_1 \in (\omega, \omega')$ satisfying (1.5) with $r(\omega_1) = 3$, such that $c(\omega) > \frac{|\omega - \omega'|}{3}$. Take $E \in \sigma(\omega_1, \lambda)$ and $L > k(\lambda, \omega_1)$ (from Lemma 5.3). Pick $\theta \notin \Theta$ so that $H_{\omega_1,\lambda,\theta}$ has pure point spectrum. Pick E_1, an eigenvalue of $H_{\omega_1,\lambda,\theta}$ with $|E_1 - E| < e^{-3/4\gamma(\omega_1)L}$. Let Ψ_{E_1} be the corresponding eigenfunction. Let $j(n_i)$ be a sequence of resonance positions for the phase $\theta + n(E_1)\omega_1$. Let i be such that $j(n_i) < L < j(n_{i+1})$. Then by (2.5), one can find a constant $3/8 < a < 3$ such that $3j(n_i) < aL < 3/8j(n_{i+1})$. Let Φ_{E_1} be Ψ_{E_1} restricted to $[n(E_1) - aL, n(E_1) + aL]$ and normalized by $||\Phi_{E_1}|| = 1$. Then we have

$$\begin{aligned} ||(H_{\omega',\lambda,\theta} - E)\Phi_{E_1}|| &\le ||(H_{\omega_1,\lambda,\theta} - H_{\omega',\lambda,\theta})\Phi_{E_1}|| + ||(H_{\omega_1,\lambda,\theta} - E_1)\Phi_{E_1}|| + |E - E_1|||\Phi_{E_1}|| \\ &\le 2a\lambda|\omega - \omega'|L + 4e^{-\gamma(\lambda)aL} + e^{-3/8\gamma(\lambda)L} \\ &\le 6\lambda|\omega - \omega'|L + 5e^{-3/4\gamma(\lambda)L}. \end{aligned}$$

Here we used Lemma 5.3 to estimate the second term. By taking $L = -D_1(\lambda)\ln|\omega - \omega'|$, where $D_1(\lambda) = \max(2c_3(\lambda), 4/3\gamma(\lambda)^{-1})$, we obtain the needed statement for $\lambda > 29$. ∎

Proof of Lemma 5.3: The constant 29 was chosen so that $\frac{\ln \frac{\lambda}{2}}{\ln M(31,29)} > 5/6$. This implies that for $\lambda > 29$, one can choose $1 < m_1 < \lambda/2$, $m_2 < 1$, $\epsilon > 1$ so that $2c_{\lambda,\epsilon} - 1 > 2/3$. That means, by Lemma 4.3, that every (m_2, k)-singular point "produces" $[2k/3]$ points not belonging to A_k. We will now formulate a version of Lemma 4.4:

LEMMA 5.5 *Let* $\frac{2m_1}{\lambda} < b < 1$. *Then there exists* $k_2(b) < \infty$ *such that for* $k > k_2(b)$, *if the two points* $x_1, x_2 \in \mathbb{Z}$ *are such that*
1) $x_i, x_i+1, ..., x_i + 2[\frac{k+1}{3}] \notin A_k$, $i = 1, 2$,
2) $\mathrm{dist}(x_1, x_2) > [\frac{k+1}{2}]$,
then

$$\left|\cos\left(2\pi\left(\theta + \left(\frac{k-1}{2} + x_1 + j_1\right)\omega\right)\right) - \cos\left(2\pi\left(\theta + \left(\frac{k-1}{2} + x_1 + j_2\right)\omega\right)\right)\right| \leq b^{\frac{k}{4}} \quad (5.3)$$

for some $j_1,\ j_2 \in [0, 2[\frac{k+1}{3}]] \cup [x_2 - x_1, x_2 - x_1 + [\frac{k+1}{3}]]$.

The proof of this lemma is identical to that of Lemma 4.4 and is given in [16].

Assume now that $\theta + n(E)\omega \in \Theta_n^s \cap \Theta_m^s$, $n < m$; $\theta \notin \Theta_i^s$, $n < i < m$. Assume x is (m_2, k)-singular, with $k = |x - n(E)|$. We will consider several cases depending on the position of x with respect to the resonance.

- If $x < n(E) < n(E) + j(n)$, we can apply Lemma 5.5 with $x_1 = x - 5k/6$ and $x_2 = n(E) - 5k/6$. As in the proof of Lemma 2.2 this together with (1.5) will imply a resonance condition
$$\sin\left(2\pi\left(\theta + n(E) + \frac{j}{2}\omega\right)\right) < b^{k/5}, \quad (5.4)$$
for $j = -8k/3 - 1 + j_1 + j_2$, some $j_1,\ j_2 \in [0, 2[\frac{k+1}{3}]] \cup [k, k + [\frac{k+1}{3}]]$, and k satisfying
$$\frac{b^{k/4}(4k/3)^{r(\omega)}}{2c(\omega)} < b^{k/5}. \quad (5.5)$$
We get a contradiction provided $j \neq j(n)$ or $j(m)$ for all choices of j_1, j_2. Since for any allowed j_1, j_2 we have $-8k/3 - 1 \leq j \leq 0$, the contradiction follows from $k < 3/8j(m)$.

- If $x > n(E) > n(E) + j(n)$, we apply Lemma 5.5 with $x_1 = x - 5/6k$ and $x_2 = n(E) - \frac{k}{2}$. We get a resonant condition (5.4) with $j = 4k/3 - 1 + j_1 + j_2$, with $j_1 + j_2 \in [-4k/3, 4k/3]$. By the same argument we obtain a contradiction from $k < 3/8j(m)$ if k obeys (5.5).

- If $x > n(E) + j(n) > n(E)$, we apply Lemma 5.5 with $x_1 = x - 5/6k$ and $x_2 = n(E) - 5/6k$. The possible values for j in (5.4) are now $j \in [-2k/3, 0] \cup [k/3, 8k/3]$. For the contradiction, we need $3j(n) < k < 3/8j(m)$ and
$$\frac{b^{k/4}(5k/3)^{r(\omega)}}{2c(\omega)} < b^{k/5}. \quad (5.6)$$

- Similarly, if $x < n(E) + j(n) < n(E)$, we apply Lemma 5.5 with $x_1 = x - 5/6k$ and $x_2 = n(E) - \frac{k}{2}$. Again, the contradiction follows if $3j(n) < k < 3/8j(m)$ and k obeys (5.6).

In the same way as in the proof of Lemma 2.2, we obtain, using (5.5),(5.6), that for $|x - n(E)| > \max(k(\epsilon), k_2(b), \frac{-\ln 2}{\ln m_2}, \frac{20\ln(2c(\omega))}{\ln b} + 3/2)$ and satisfying $3j(n) < |x - n(E)| < 3/8j(m)$ we have that x is $(m_2, |x - n(E)|)$-regular which, as before, proves the statement of Lemma 5.3. Here, in estimating how large k should be to satisfy (5.5), (5.6), we used that $c(\omega) \leq 1$ for all ω obeying (1.5). ∎

6 Measure of the spectrum: Proof of Theorem 1.3

Once we have established the strong version of continuity of gaps, Theorem 5.1, the proof of Theorem 1.3 simply follows the lines of the measure-of-the-spectrum theorem in [24, 25]. We present the argument here for the reader's convenience.

Let $G(\omega, \lambda)$ be the union of the gaps in $S(\omega, \lambda)$, so that

$$|S(\omega, \lambda)| = \max S(\omega, \lambda) - \min S(\omega, \lambda) - |G(\omega, \lambda)|. \tag{6.1}$$

If $\omega = p/q$ is a rational, then $S(\omega, \lambda)$ consists of no more than q bands, and $G(\omega, \lambda)$ of no more than $q - 1$ intervals. It is well known that for any irrational ω, there exists a sequence of rationals $p_n/q_n \to \omega$ such that

$$|\omega - p_n/q_n| < \frac{1}{q_n^2}. \tag{6.2}$$

Avron, van Mouche, and Simon [3] had proven that for every λ and every sequence $\{p_n/q_n\}$ with p_n and q_n relatively prime and $q_n \to \infty$,

$$\lim_{n\to\infty} |S(p_n/q_n, \lambda)| = |4 - 2|\lambda||. \tag{6.3}$$

(6.3) along with (any) gap continuity implies (see [3, 25]) the upper bound

$$|\sigma(\omega, \lambda)| \geq |4 - 2|\lambda|| \tag{6.4}$$

for any irrational ω. We now obtain from (i) of Corollary 5.2:

$$|G(\omega, \lambda)| > |G(p_n/q_n, \lambda)| - 2D(\lambda)(q_n - 1)\lambda|\omega - p_n/q_n| \ln|\omega - p_n/q_n|. \tag{6.5}$$

By (6.1) and (ii) of Corollary 5.2, this implies:

$$|S(\omega, \lambda)| < |S(p_n/q_n, \lambda)| + 2D(\lambda)q_n\lambda|\omega - p_n/q_n| \ln|\omega - p_n/q_n|.$$

By (6.2) and (6.3), we obtain $|\sigma(\omega, \lambda)| = |S(\omega, \lambda)| \leq |4 - 2|\lambda||$, which together with (6.4), completes the proof of Theorem 1.3 for $\lambda > 29$. Since $S(\omega, \lambda)$ is independent of the sign of λ, it is enough to have $|\lambda| > 29$. The result for $|\lambda| < 4/29$ follows from duality: $S(\omega, \lambda) = (\lambda/2)S(\omega, 4/\lambda)$. ∎

7 Appendix: Proof of Lemma 4.5

We denote

$$B(\theta, E) = \begin{pmatrix} E - \lambda\cos\theta & 1 \\ -1 & 0 \end{pmatrix}, \quad B_k(\theta, E) = B(\theta + k\pi\omega, E).$$

It was shown in the proof of Proposition 1 in [16] that for any $k > 0$, we have

$$|P_k(\theta, E)| \leq \left(\frac{2}{\sqrt{3}}\right)^{k+1} \prod_{i=0}^{k} \|B_j(\theta, E)\|,$$

where for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we use $||A|| = \max(\sqrt{a^2 + c^2}, \sqrt{b^2 + d^2})$. We now want to find $k(\epsilon)$ (not dependent on E!) such that for any $k \geq k(\epsilon)$,

$$\left(\frac{2}{\sqrt{3}}\right)^{k+1} \prod_{j=0}^{k} \|B_j(\theta, E)\| \leq \epsilon^k \left(\frac{2}{\sqrt{3}}\right)^k e^{\frac{k}{2\pi}\int_0^{2\pi} \ln\|B(\theta,E)\|d\theta} \tag{7.1}$$

which can be rewritten as:

$$\ln\left(\frac{2}{\sqrt{3}}\right) + \sum_{j=0}^{k} \ln\|B_j(\theta, E)\| \leq k\ln\epsilon + \frac{k}{2\pi}\int_0^{2\pi} \ln\|B(\theta, E)\|d\theta. \tag{7.2}$$

This will prove Lemma 4.5 since $\int_0^{2\pi} \ln\|B(\theta, E)\|d\theta = \ln(\frac{\sqrt{3}}{2}M(E, \lambda))$.

Let p_n/q_n be the sequence of continued fraction approximants of ω. Let $n(k)$ be such that $q_{n(k)} \leq k < q_{n(k)+1}$. We will write r for $r(\omega)$ and c for $c(\omega)$.

PROPOSITION 1. *For any $f \in C[0, 2\pi), k > 0$ we have:*

$$\left|\sum_{j=0}^{k-1} f(\theta + j\pi\omega) - \frac{k}{2\pi}\int_0^{2\pi} f(\theta)d\theta\right| \leq k(c^{-1/r}\frac{n(k)+1}{k^{1/r}})\mathrm{Var}(f). \tag{7.3}$$

Proof: Writing $k = b_n q_n + b_{n-1}q_{n-1} + \cdots + b_1 q_1 + b_0$ and using the Denjoy-Koksma inequality (see, e.g., Lemma 4.1, Ch. 3 [4]), we get

$$\left|\sum_{j=0}^{k-1} f(\theta + j\alpha) - \frac{k}{2\pi}\int_0^{2\pi} f(\theta)\|d\theta\right| \leq (b_0 + \cdots + b_n)\mathrm{Var}(f) \leq \left(\sum_{i=0}^{n}\left[\frac{q_{i+1}}{q_i}\right]\right)\mathrm{Var}(f). \tag{7.4}$$

Since (1.5) implies $q_{i+1} < \frac{q_i^r}{c}$, we have $\frac{q_{i+1}}{q_i} < \frac{q_{i+1}}{(cq_{i+1})^{1/r}} = \frac{q_{i+1}^{1-1/r}}{c^{1/r}}$. The right-hand side of (7.4) can now be estimated as

$$\leq \left(c^{-1/r}\sum_{i=1}^{n(k)} q_i^{1-1/r} + \frac{k}{q_{n(k)}}\right)\mathrm{Var}(f) < \left(c^{-1/r}n(k)q_{n(k)}^{1-1/r} + \frac{k}{q_{n(k)}}\right)\mathrm{Var}(f).$$

Since $k < q_{n(k)+1} \leq \frac{q^r_{n(k)}}{c}$, we have $q_{n(k)} \geq (ck)^{1/r}$, and $\frac{k}{q_{n(k)}} \leq c^{-1/r}k^{1-1/r}$. Thus we can continue our estimate as

$$\leq (c^{-1/r}(n(k)+1)k^{1-1/r})\mathrm{Var}(f). \quad \blacksquare$$

Proposition 1 implies that

$$\left| \sum_{j=0}^{k} \ln \|B_k(\theta, E)\| - \frac{k}{2\pi} \int_0^{2\pi} \ln \|B(\theta, E)\| d\theta \right|$$
$$\leq \; k \cdot \frac{c^{-1/r} n(k)}{k^{1/r}} \max_{-\lambda-2<E<\lambda+2} \mathrm{Var}(\ln((E - \lambda \cos \theta)^2 + 1)).$$

Denote $\max_{-\lambda-2<E<\lambda+2} \mathrm{Var}(\ln((E - \lambda \cos \theta)^2 + 1)) = A(\lambda)$. Since $q_n \geq (\sqrt{2})^n$, $n \geq 2$, we have that $n(k) \leq \frac{\ln k}{\ln \sqrt{2}}$, and we can find $k(\epsilon)$ such that for any $k > k(\epsilon)$ we have

$$\frac{c^{-1/r} n(k)}{k^{1/r}} A(\lambda) + \frac{\ln(2/\sqrt{3})}{k} \leq \ln \epsilon$$

which implies (7.2). This completes the proof of Lemma 4.5. $\blacksquare$

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