LIMIT BEHAVIOUR OF THE PARTITION FUNCTION OF SPIN GLASSES VIA STOCHASTIC CALCULUS

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ABSTRACT. This paper studies a martingale method introduced by Comets and Neveu for the Sherrington - Kirkpatrick model. We apply it here to a broad class of models to get a theorem that links the convergence in distribution of the partition function of a disordered model to the asymptotic behaviour of the partition function of its "ferromagnetic analogue ". A byproduct of this result is the deduction of the right rescaling for the Hamiltonian of the Sherrington Kirkpatrick model with external field. Lastly we find the equation for the critical surface delimiting the high temperature phase for a large class of mean field spin glass models.

1. Introduction

In their rigorous study of the high temperature phase for the Sherrington-Kirkpatrick model of spin-glass Aizenman, Lebowitz and Ruelle ([ALR]) were able to give the limit behaviour of the partition function by means of a carefully employed graph expansion.

Recently Comets and Neveu [CN] gave another derivation of the same result by defining a *cooling martingale* and making use of stochastic calculus.

In this work we investigate the extent of their method which proves to be very general for mean field spin-glasses.

The main result is stated and proven in Section 2. It links the convergence in distribution of the partition function of a disordered model to the asymptotic behaviour of the partition function of its "ferromagnetic analogue ".

Key words and phrases. martingale, spin glass, stochastic calculus.

As examples of models that verify our theorem we list the Sherrington Kirkpatrick model and the Random Energy Model. We also give an example of a model with nearest neighbour interaction (Edwards-Anderson), which instead does not satisfy our theorem. In this case the partition function exhibits much larger fluctuations, we expect this to be the standard behaviour for short range interaction models.

In Section 3, we discuss the Sherrington-Kirkpatrick model with external random field showing that our theorem applies to this case. As a consequence we show that the proper scaling for the external field is not what one would naively expect, but it is, instead, of order $N^{-\frac{1}{4}}$.

In Section 4, generalizing the ideas of Section 3, we present a whole class of mean field spin glass models with multibody interaction, that verify our theorem and we find the equation for the critical surface delimiting the high temperature phase for a large class of mean field spin glass models.

2. The Main Result

In this section we introduce the notation and the main result that will provide the framework for what follows.

Fixed $N \in \mathbb{N}$ and $\beta > 0$, let us consider an Hamiltonian, $H_N(\sigma)$, of the type

$$H_N(\sigma) = \sum_{\alpha \subseteq \{1,\ldots,N\}} f_\alpha(N) J_\alpha \sigma_\alpha, \tag{2.1}$$

with the corresponding partition function given by

$$Z_N(\beta) = \sum_{\sigma \in \{-1,1\}^N} e^{\beta H_N(\sigma)}, \tag{2.2}$$

where

- α varies over all subsets of $\{1, \dots, N\}$,
- $\{J_\alpha\}_{\alpha \subseteq \{1,\dots,N\}}$ is a family of 2^N independent $\mathcal{N}(0,1)$ random variables,
- $\{f_\alpha(N)\}_\alpha$ is a family of functions of N,
- $\sigma_\alpha = \prod_{i \in \alpha} \sigma_i$.

It is easy to see that choosing the f_α's in (2.1) above suitably, one gets all the spin-glass models with Ising type spins that are usually considered in the literature.

We define the deterministic (*ferromagnetic*) *analogue* of the model described in (2.1), (2.2) to be the model obtained by the following prescription:

replace all the J_α's with 1's and square the $f_\alpha(N)$'s, obtaining

$$\tilde{H}_N(\sigma) = \sum_{\alpha \subseteq \{1,\dots,N\}} f_\alpha^2(N)\sigma_\alpha \tag{2.3}$$

and

$$\tilde{Z}_N(\beta) = \sum_{\sigma \in \{-1,1\}^N} e^{\beta \tilde{H}_N(\sigma)}. \tag{2.4}$$

With this procedure, one gets the familiar Curie-Weiss model of a ferromagnet from the Sherrington-Kirkpatrick spin glass, or the usual Ising model from the Edwards-Anderson one.

Definition 2.1: *We define an Hamiltonian of type (2.1) to be mean field if*

$$f_\alpha(N) = f_{|\alpha|}(N), \qquad \alpha \subseteq \{1, \dots, N\}$$

that is to say the coefficients f_α's depend only on the cardinality of the set α.

With this notation we can write the following

Theorem 2.2: *Let $\sigma \in \{-1,1\}^N$ and $\beta > 0$.*

Let $H_N(\sigma)$ and $Z_N(\beta)$ be the Hamiltonian and the random partition function defined by (2.1) and (2.2), $\tilde{H}_N(\sigma)$ and $\tilde{Z}_N(\beta)$ their deterministic counterparts defined by (2.3) and (2.4) and ϕ a positive continuous increasing function from $\mathbb{R}$ to $\mathbb{R}$. If

$$\frac{\tilde{Z}_N(\beta^2)}{2^N} \xrightarrow[N\to\infty]{} e^{\phi(\beta^2)}, \tag{2.5}$$

increasingly, for each $\beta \in [0, \beta_\star)$, for some $\beta_\star \in \mathbb{R}_+$, then in the same interval

$$\frac{Z_N(\beta)}{2^N \exp\{\frac{\beta}{2}\sum_\alpha f_\alpha^2(N)\}} \Longrightarrow Z_\beta \qquad as \;\; N \to \infty, \tag{2.6}$$

where Z_β has a lognormal distribution of mean 1 and variance $e^{\phi(\beta)} - 1$.

(here $\Longrightarrow$ denotes convergence in law).

Proof: As σ varies in $\{-1,1\}^N$, let us define the processes

$$\bar{H}_N(t,\sigma) = \sum_{\alpha \subseteq \{1,\dots,N\}} f_\alpha(N)\sigma_\alpha B_\alpha(t), \tag{2.7}$$

with $t > 0$ and $\{B_\alpha\}_{\alpha\subseteq\{1,\dots,N\}}$ an independent family of Brownian motions defined on the same probability space $(\Omega, \mathcal{F}, P)$.

More precisely, for fixed N and σ, the process $\bar{H}_N(t,\sigma)$ is a martingale with respect to the filtration generated by B_α, that is

$$\mathcal{F}_t = \sigma(\{B_\alpha(s),\ \alpha \subseteq \{1,\dots,N\},\ 0 \le s \le t\}),$$

properly augmented of the P-null sets. It is then clear that

$$E(\bar{H}_N(t,\sigma)) = 0 \qquad \forall\, t > 0,\ \sigma \in \{-1,1\}^N$$

$$E(\bar{H}_N^2(t,\sigma)) = t \sum_{\alpha\subseteq\{1,\dots,N\}} f_\alpha^2(N),$$

hence with variance independent of the configuration σ.

Since the processes are continuous, it is easy to prove that the stochastic exponential of $\bar{H}_N(t,\sigma)$, denoted by $e_N(t,\sigma)$, is a positive martingale that may be written as

$$e_N(t,\sigma) = \exp\left(\bar{H}_N(t,\sigma) - \frac{1}{2}t \sum_{\alpha\subseteq\{1,\ldots,N\}} f_\alpha^2(N)\right). \tag{2.8}$$

For fixed σ, the process $e_N(t,\sigma)$ satisfies the linear stochastic differential equation

$$de_N(t,\sigma) = e_N(t,\sigma)d\bar{H}_N(t,\sigma), \qquad e_N(0,\sigma) = 1, \qquad t \geq 0. \tag{2.9}$$

On the other hand, for any two configurations σ, σ', the processes $\bar{H}_N(t,\sigma)$ and $\bar{H}_N(t,\sigma')$ are no longer independent, as a matter of fact their correlation matrix, indexed on σ, is given by

$$\begin{aligned} E(\bar{H}_N(t,\sigma)\bar{H}_N(t,\sigma')) &= E\left[\left(\sum_\alpha f_\alpha(N)\sigma_\alpha B_\alpha(t)\right)\left(\sum_\alpha f_\alpha(N)\sigma'_\alpha B_\alpha(t)\right)\right] \\ &= t\sum_\alpha f_\alpha^2(N)\sigma_\alpha\sigma'_\alpha, \end{aligned} \tag{2.10}$$

where the latter equality is obtained by virtue of the independence of the family of Brownian motions. By using integration by parts, it is easy to show that formulas (2.9) and (2.10) imply that

$$E(e_N(t,\sigma)e_N(t,\sigma')) = \exp\{E(\bar{H}_N(t,\sigma)\bar{H}_N(t,\sigma'))\}.$$

Finally let us define

$$\bar{Z}_N(t) = E_\sigma(e_N(t,\sigma)),$$

where $E_\sigma = \dfrac{1}{2^N}\displaystyle\sum_{\sigma\in\{-1,1\}^N}$. The process $\bar{Z}_N(t)$ is still a positive martingale, as linear combination of positive martingales. By linearity and martingale property we have that $E(\bar{Z}_N(t)) = 1$ all t, and we want to compute $E(\bar{Z}_N^2(t))$.

Applying the method of replicas and Fubini's theorem we have

$$
\begin{aligned}
E(\bar{Z}_N^2(t)) &= E([E_\sigma(e_N(t,\sigma)]^2) = E[E_{\sigma\sigma'}(e_N(t,\sigma)e_N(t,\sigma'))] \\
&= E_{\sigma\sigma'}[E(e_N(t,\sigma)e_N(t,\sigma'))] = E_{\sigma\sigma'}\left[\exp\{E(\bar{H}_N(t,\sigma)\bar{H}_N(t,\sigma'))\}\right] \\
&= E_{\sigma\sigma'}\left[\exp\left(t\sum_{\alpha\subseteq\{1,\ldots,N\}} f_\alpha^2(N)\sigma_\alpha\sigma'_\alpha\right)\right] \\
&= \frac{1}{2^{2N}}\sum_{\sigma,\sigma'\in\{-1,1\}^N}\left[\exp\left(t\sum_{\alpha\subseteq\{1,\ldots,N\}} f_\alpha^2(N)\sigma_\alpha\sigma'_\alpha\right)\right] \\
&= \frac{1}{2^N}\sum_{\sigma\in\{-1,1\}^N}\left[\exp\left(t\sum_{\alpha\subseteq\{1,\ldots,N\}} f_\alpha^2(N)\sigma_\alpha\right)\right],
\end{aligned}
$$

where the latter equality is true because for any real (or complex) valued function f on $\{-1,1\}^N$ the following equality holds

$$
\frac{1}{2^{2N}}\sum_{(\sigma,\sigma')\in\{-1,1\}^N\times\{-1,1\}^N} f(\sigma\star\sigma') = \frac{1}{2^N}\sum_{\sigma\in\{-1,1\}^N} f(\sigma)
$$

where $\star$ denotes the product component by component. Indeed the application which sends the couple (σ,σ') into $(\sigma,\sigma\star\sigma')$ is an automorphism of $\{-1,1\}^N\times\{-1,1\}^N$.

Taking $t=\beta^2$, we have that $E(\bar{Z}_N^2(t)) = \dfrac{\tilde{Z}_N(\beta^2)}{2^N}$, establishing the connection between the random and the deterministic model and, by hypothesis, we know that the right hand side converges to $\mathrm{e}^{\phi(t)}$ for all $t<\beta_\star^2$.

Consequently $E(\sup_{N,0\le t\le T}\bar{Z}_N^2(t))<+\infty$ for any $T<\beta_\star^2$. This implies that for each N, the process

$$
M_N(t) = \int_0^t \frac{1}{\bar{Z}_N(s)}d\bar{Z}_N(s),
$$

is a well defined martingale, as $\dfrac{1}{\bar{Z}_N(t)}$ is strictly positive and square integrable. (as a matter of fact we have $E(M_N^2(t)) \le \dfrac{t}{2}\sum_{\alpha\subseteq\{1,\ldots,N\}} f_\alpha^2(N)$).

Then, following the same proof as in [CN], it is possible to show that the quadratic variation of M_N converges in u.c.p. (uniformly on compacts in probability) to the function $\phi(t)$ previously introduced.

Indeed $[M_N, M_N]_t \xrightarrow[t\to+\infty]{} +\infty$, for each N. Hence by the characterization theorem of continuous martingales, we have that $M_N(t) = B_{[M_N,M_N]_t}$ and consequently the set

$$A^N_{a,\epsilon} = \{-M_N(t) \le a + (\epsilon/2)[M_N, M_N]_t,\ \forall\, t \ge 0\}$$

has probability $P(A^N_{a,\epsilon}) \ge 1 - \mathrm{e}^{-a\epsilon}$.

If we denote by $V_N(t) = \frac{d}{dt}([\bar{Z}_N, \bar{Z}_N]_t) - \bar{Z}^2_N(t)\phi'(t)$, the following inequality is satisfied on $A^N_{a,\epsilon}$

$$|\frac{d}{dt}([M_N, M_N]_t - \phi(t))| \le \mathrm{e}^{2a}|V_N(t)|\mathrm{e}^{(1+\epsilon)([M_N,M_N]_t-\phi(t))}\mathrm{e}^{(1+\epsilon)\phi(t)},$$

which implies, for any fixed $T < \beta^2_\star$ and all $t \le T$, that

$$\sup_{0\le t\le T}\left|\frac{1-\mathrm{e}^{(1+\epsilon)([M_N,M_N]_t-\phi(t))}}{1+\epsilon}\right| \le \mathrm{e}^{2a}\int_0^T |V_N(s)|\mathrm{e}^{(1+\epsilon)\phi(s)}ds$$

In order to prove the convergence in probability on $[0, T]$, it is then enough to show that $E(V_N(t)) \longrightarrow 0$, as all the other quantities are positive.

By Fubini's theorem and the method of replicas, we have

$$\begin{aligned}
E(V_N(t)) &= E\left(E_{\sigma\sigma'}\left(e_n(t,\sigma)e_N(t,\sigma')[\sum_\alpha f^2_\alpha(N)\sigma_\alpha\sigma'_\alpha - \phi'(t)]\right)\right)\\
&= E_{\sigma\sigma'}\left(E(e_n(t,\sigma)e_N(t,\sigma'))[\sum_\alpha f^2_\alpha(N)\sigma_\alpha\sigma'_\alpha - \phi'(t)]\right)\\
&= E_{\sigma\sigma'}(\exp\{t\sum_\alpha f^2_\alpha(N)\sigma_\alpha\sigma'_\alpha\}[\sum_\alpha f^2_\alpha(N)\sigma_\alpha\sigma'_\alpha - \phi'(t)])\\
&= E_\sigma(\mathrm{e}^{t\sum_\alpha f^2_\alpha(N)\sigma_\alpha}[\sum_\alpha f^2_\alpha(N)\sigma_\alpha - \phi'(t)]).
\end{aligned}$$

By what we said before, it is clear that the last term in the previous equality is such that

$$E_\sigma(\mathrm{e}^{t\sum_\alpha f_\alpha^2(N)\sigma_\alpha}\phi'(t)) \xrightarrow[N\to+\infty]{} \phi'(t)\mathrm{e}^{\phi(t)} = \frac{d}{dt}(\mathrm{e}^{\phi(t)}).$$

We want to show that the first term tends to $\frac{d}{dt}(\mathrm{e}^{\phi(t)})$ as well. In order to do so it suffices to show that it is possible to exchange the limits in t and N, that is we have to prove that one of them is uniform with respect to the other variable. For the time being, for convenience we will denote by X_N the random variable $\sum_\alpha f_\alpha^2(N)\sigma_\alpha$.

Of course we can write for fixed t

$$\begin{aligned} E_\sigma(X_N\mathrm{e}^{tX_N}) &= \frac{d}{dt}E_\sigma(\mathrm{e}^{tX_N}) = \lim_{h\to0}\frac{E_\sigma(\mathrm{e}^{(t+h)X_N}) - E_\sigma(\mathrm{e}^{tX_N})}{h} \\ &= \lim_{h\to0}\frac{1}{h}\left(E_\sigma(X_N\mathrm{e}^{tX_N})h + \frac{h^2}{2}E_\sigma(X_N^2\mathrm{e}^{(t+h')X_N})\right) \\ &= E_\sigma(X_N\mathrm{e}^{tX_N}) + \lim_{h\to0}\frac{h}{2}E_\sigma(X_N^2\mathrm{e}^{(t+h')X_N}) \end{aligned} \tag{2.11}$$

for some $h' < h$. On the other hand, by Hölder's inequality, applied to the argument of the last limit, for any conjugate pair $p, q > 1$ we have

$$\begin{aligned} E_\sigma(X_N^2\mathrm{e}^{(t+h')X_N}) &\le E_\sigma(X_N^{2q})^{1/q}E_\sigma(\mathrm{e}^{p(t+h')X_N})^{1/p} \\ &= (\int_0^{+\infty}\lambda^{2q-1}P_\sigma(|X_N|>\lambda)d\lambda\,)^{\frac{1}{q}}E_\sigma(\mathrm{e}^{p(t+h')X_N})^{1/p} \\ &= (\int_0^{+\infty}\lambda^{2q-1}P_\sigma(\mathrm{e}^{t|X_N|}>\mathrm{e}^{t\lambda})d\lambda\,)^{\frac{1}{q}}E_\sigma(\mathrm{e}^{p(t+h')X_N})^{1/p} \end{aligned}$$

and by Chebyshev's inequality we get

$$\begin{aligned} E_\sigma(X_N^2\mathrm{e}^{(t+h')X_N}) &\le (E_\sigma(\mathrm{e}^{t|X_N|})\int_0^{+\infty}\lambda^{2q-1}\mathrm{e}^{-t\lambda}d\lambda\,)^{\frac{1}{q}}E_\sigma(\mathrm{e}^{p(t+h')X_N})^{1/p} \\ &\le (2E_\sigma(\mathrm{e}^{tX_N})\frac{\Gamma(2q)}{t^{2q}})^{\frac{1}{q}}E_\sigma(\mathrm{e}^{p(t+h')X_N})^{1/p}, \end{aligned}$$

where the last inequality is due to the symmetry of the distribution of X_N.

Hence, taken any $T < \beta_\star^2$, we may choose p such that $1 < p < \frac{\beta_\star^2}{T}$. Then there must exist a T' such that $T < pT < T'$ and if $t < T$, eventually also $t + h' < T$, since h' is going to zero. Thus, from the last row of the previous inequality, using our assumption, we obtain

$$E_\sigma(X_N^2 \mathrm{e}^{(t+h')X_N}) \leq 2^{\frac{1}{q}}(\mathrm{e}^{\phi(T)})^{1/q}(\frac{\Gamma(2q)}{t^{2q}})^{1/q}(\mathrm{e}^{\phi(pT)})^{1/p}$$
$$\leq 2^{\frac{1}{q}}\frac{\Gamma(2q)^{1/q}}{t^2}\mathrm{e}^{\phi(T')}.$$

On the other hand $E_\sigma(X_N^2 \mathrm{e}^{tX_N})$ is increasing in t, so the above inequaliy can be improved to

$$E_\sigma(X_N^2 \mathrm{e}^{(t+h')X_N}) \leq 2^{\frac{1}{q}}\frac{\Gamma(2q)^{1/q}}{T^2}\mathrm{e}^{\phi(T')} \tag{2.12}$$

Since the right hand side of (2.12) does not depend on N, the limit in (2.11) goes to 0 uniformly in N.

This means that

$$\begin{aligned}\lim_{N\to+\infty} E_\sigma(\mathrm{e}^{t\sum_\alpha f_\alpha^2(N)\sigma_\alpha}[\sum_\alpha f_\alpha^2(N)\sigma_\alpha]) &= \lim_{N\to+\infty}\lim_{h\to 0}\frac{E_\sigma(\mathrm{e}^{(t+h)X_N}) - E_\sigma(\mathrm{e}^{tX_N})}{h}\\ &= \lim_{h\to 0}\lim_{N\to+\infty}\frac{E_\sigma(\mathrm{e}^{(t+h)X_N}) - E_\sigma(\mathrm{e}^{tX_N})}{h}\\ &= \lim_{h\to 0}\frac{\mathrm{e}^{\phi(t+h)} - \mathrm{e}^{\phi(t)}}{h} = \phi'(t)\mathrm{e}^{\phi(t)},\end{aligned}$$

which implies $E(V_n(t)) \xrightarrow[N\to+\infty]{} 0$ for all $t \leq T$.

By the Dominated Convergence theorem we therefore get

$$E(\mathbf{1}_{A_{a,\epsilon}^N}\sup_{0\leq t\leq T}\left|\frac{1-\mathrm{e}^{(1+\epsilon)([M_N,M_N]_t-\phi(t))}}{1+\epsilon}\right|) \xrightarrow[N\to+\infty]{} 0,$$

which implies convergence in probability as $P(A_{a,\epsilon}^N) \to 1$ for $N \to \infty$. By a theorem in [JSh] (see theorem 3.11, chapter 8), this implies that for any $T \in [0, \beta_\star^2)$, the martingale M_N converges in law to a centered Gaussian process M_∞ with independent increments, such that

$$E[(M_\infty(t) - M_\infty(s))^2] = \phi(t) - \phi(s) \qquad \text{for } 0 \leq s \leq t < T,$$

and the random variable $\bar{Z}_N(t)$ converges in law to the process

$$\exp\{M_\infty(t) - \frac{1}{2}\phi(t)\} \qquad \text{each } 0 < t \leq T.$$

for any fixed $T < \beta_\star^2$, that means, by the definition of $Z_N(\beta)$, taking square roots

$$\frac{Z_N(\beta)}{2^N \exp\{\frac{\beta}{2}\sum_\alpha f_\alpha^2(N)\}} \Longrightarrow Z \qquad \text{as } N \to \infty$$

each $\beta \in [0, \beta_\star)$ for some positive real $\beta_\star$. $\square$

Remark 2.3:

The opposite of theorem 2.2 is obviously true. As a matter of fact if there exists a bounded interval $[0, \beta_\star)$ and a positive continuous function such that

$$\frac{Z_N(\beta)}{2^N \exp\{\frac{\beta}{2}\sum_\alpha f_\alpha^2(N)\}} \Longrightarrow Z_\beta \qquad \text{as } N \to \infty,$$

for each $\beta \in [0, \beta_\star)$, where Z_β has a lognormal distribution of mean 1 and variance $\mathrm{e}^{\phi(\beta)} - 1$, then by the definition of weak convergence, we necessarily have that

$$\frac{\tilde{Z}_N(\beta^2)}{2^N} = \frac{1}{2^N} E\left(\frac{Z_N(\beta^2)}{\exp\{\beta^2 \sum_\alpha f_\alpha^2(N)\}}\right)$$

converges to $E(Z_{\beta^2}^2) = e^{\phi(\beta^2)}$ as $N \to \infty$, for $\beta < \beta_\star$.

As shown by Comets and Neveu, the Sherrington Kirkpatrick model verifies the hypothesis of theorem 2.2. Another such case is given by the Random Energy Model.

Example 2.4: *The Random Energy Model*

In this case the partition function is simple, as a matter of fact it is

$$Z_N(\beta) = \sum_\sigma \exp(\sqrt{N}\beta J_\sigma), \tag{2.13}$$

where the J_σ's are indipendent Gaussian random variables indexed on $\{-1,1\}^N$.

To show that the Random Energy Model satisfies the hypotheses of theorem 2.2, we use an alternative representation of (2.13) in terms of multibody interactions, (see [FI])

$$H_N(\sigma) = 2^{-\frac{N}{2}} \sum_{\alpha\subseteq\{1,\ldots,N\}} \sqrt{N} J_\alpha \sigma_\alpha, \qquad Z_N(\beta) = \sum_{\sigma\in\{-1,1\}^N} \mathrm{e}^{\beta H_N(\sigma)}. \tag{2.14}$$

By the prescription of theorem 2.2, the "deterministic analogue" of (2.14) is

$$\tilde{H}_N(\sigma) = 2^{-N} N \sum_{\alpha\subseteq\{1,\ldots,N\}} \sigma_\alpha, \qquad \tilde{Z}_N(\beta) = \sum_{\sigma\in\{-1,1\}^N} \mathrm{e}^{\beta \tilde{H}_N(\sigma)}. \tag{2.15}$$

Let us remark that if σ is such that $(\sigma_1, ..., \sigma_n) = (1, \ldots, 1)$, then $H(\sigma) = N$, while for all the other σ's we have $\tilde{H}(\sigma) = 0$. This can be proven by induction.

Therefore $\dfrac{\tilde{Z}_N(\beta^2)}{2^N} = \dfrac{1}{2^N} \displaystyle\sum_{\sigma\in\{-1,1\}^N} \mathrm{e}^{\beta\tilde{H}_N(\sigma)} = \dfrac{2^N - 1 + \mathrm{e}^{N\beta^2}}{2^N} \longrightarrow 1,$ as $N \longrightarrow \infty$, if $\beta < \sqrt{\ln 2}$.

For a rigorous treatment of the limit behaviour of the partition function of the REM see ([OP]).

Example 2.5: *The Edwards-Anderson model*

The Edwards - Anderson Model does not satisfy the hypothesis of theorem 2.2

In this case the particles are indexed on $\mathbb{Z}^d$, $d \geq 1$. For $N \in \mathbb{N}$ we choose the set

$$\Lambda_N^d = \{\mathbf{i} = (i_1, \ldots, i_d) : -N \leq i_k \leq N, \text{ for all } k = 1, \ldots, d\}$$

and we consider the partition function

$$Z_N(\beta) = \sum_{\sigma\in\{-1,1\}^{(2N)^d}} \exp\left\{\beta \sum_{\mathbf{i}\in\Lambda_N^d} \sum_{|\mathbf{j}-\mathbf{i}|=1} J_{\mathbf{ij}} \sigma_{\mathbf{i}} \sigma_{\mathbf{j}}\right\}, \tag{2.16}$$

where $\{J_{\mathbf{ij}}\}_{\mathbf{ij}}$ is an independent family of Gaussian random variables and we are assuming periodic boundary conditions.

Again we substitute $J_{\mathbf{ij}}$ with 1, we square the coefficients and we obtain the ferromagnetic analogue of (2.16)

$$\tilde{Z}_N(\beta) = \sum_{\sigma} \exp\left\{\beta \sum_{\mathbf{i}\in\Lambda_N^d} \sum_{|\mathbf{j}-\mathbf{i}|=1} \sigma_{\mathbf{i}}\sigma_{\mathbf{j}}\right\}, \tag{2.17}$$

which is exactly the partition function of the Ising model.

The function (2.17) can be computed exactly for $d = 1$ or $d = 2$, but it is easy to verify that $\frac{1}{2^{(2N)^d}}\tilde{Z}_N(\beta) \to \infty$, as $N \to \infty$, for any d, even at small positive β. Therefore we may conclude that the variance of the partition function does not converge.

3. Sherrington-Kirkpatrick model with random external field

In this section we want to give an example of a non trivial model that verifies our theorem, for which we can explicitly compute the limiting functions.

In the case of the Sherrington - Kirkpatrick model, the partition function is

$$Z_N(\beta) = \sum_{\sigma\in\{-1,1\}^N} \exp\left[\beta N^{-\frac{1}{2}} \sum_{1\le i<j\le N} J_{ij}\sigma_i\sigma_j\right],$$

where β is a positive parameter and the J_{ij}'s are independent Normal random variables. Here we want to consider the same Hamiltonian with the addition of a random external field, that is

$$Z_N(\beta) = \sum_{\sigma\in\{-1,1\}^N} \exp\left\{\beta\left(N^{-\frac{1}{2}} \sum_{1\le i<j\le N} J_{ij}\sigma_i\sigma_j + \gamma N^{-\frac{1}{4}} \sum_{k=1}^{N} I_k\sigma_k\right)\right\}, \tag{3.1}$$

where also the I_k's are indipendent Normal random variables, indipendent also of the J_{ij}'s and $\gamma \in \mathbb{R}$.

The $N^{-\frac{1}{4}}$ scaling for the external field may seem odd at first, but, as will be clear from the computation, is the only one that allows to take a non-trivial infinite volume limit.

The same scaling was used by Pastur and Scherbina in order to introduce a symmetry breaking external field in their analysis of the Sherrington-Kirkpatrick model.

As in the case of the REM, we follow the prescription of theorem 2.2 and we write the "deterministic analogue" of (3.1) as

$$\tilde{H}_N(\sigma) = \frac{1}{N} \sum_{1 \le i < j \le N} \sigma_i \sigma_j + \gamma^2 \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \sigma_k, \qquad \tilde{Z}_N(\beta) = \sum_{\sigma \in \{-1,1\}^N} \mathrm{e}^{\beta \tilde{H}_N(\sigma)}. \tag{3.2}$$

We may rewrite the Hamiltonian as

$$\tilde{H}_N(\sigma) = \frac{(\sum_k \sigma_k)^2 - N}{2N} + \frac{\gamma^2 \sum_{k=1}^N \sigma_k}{\sqrt{N}} = \frac{1}{2N} \left(\sum_{k=1}^{N} \sigma_k + \sqrt{N}\gamma^2 \right)^2 - \frac{\gamma^4 + 1}{2}.$$

This gives us

$$\frac{1}{2^N} \tilde{Z}_N(\beta) = \mathrm{e}^{-\frac{\beta}{2}(\gamma^4+1)} \frac{1}{2^N} \sum_{\sigma \in \{-1,1\}^N} \exp\{ \frac{\beta}{2N} (\sum_k \sigma_{k=1}^N + \sqrt{N}\gamma^2)^2 \}.$$

By a standard Gaussian trick, we obtain

$$\begin{aligned} \frac{1}{2^N} \tilde{Z}_N(\beta) &= \mathrm{e}^{-\frac{\beta}{2}(\gamma^4+1)} \frac{1}{2^N} \sum_{\sigma \in \{-1,1\}^N} \frac{1}{\sqrt{2\pi\beta}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{x^2}{2\beta}} \mathrm{e}^{x\gamma^2} \mathrm{e}^{\frac{x}{\sqrt{N}} \sum_k \sigma_k} dx \\ &= \mathrm{e}^{-\frac{\beta}{2}(\gamma^4+1)} \frac{1}{\sqrt{2\pi\beta}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{x^2}{2\beta}} \mathrm{e}^{x\gamma^2} \frac{1}{2^N} \sum_{k=0}^{N} \binom{N}{k} \mathrm{e}^{\frac{x}{\sqrt{N}}k} \mathrm{e}^{-\frac{x}{\sqrt{N}}(N-k)} dx \\ &= \mathrm{e}^{-\frac{\beta}{2}(\gamma^4+1)} \frac{1}{\sqrt{2\pi\beta}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{x^2}{2\beta}} \mathrm{e}^{x\gamma^2} \frac{1}{2^N} [\cosh(\frac{x}{\sqrt{N}})]^N dx \end{aligned}$$

As $N \longrightarrow \infty$, the factor $[\cosh(\frac{x}{\sqrt{N}})]^N$, under the integral sign, tends monotonically to $\mathrm{e}^{\frac{x^2}{2}}$. Rewriting the above equalities for β^2 and applying the Monotone Convergence theorem we then get

$$\begin{aligned} \frac{1}{2^N} \tilde{Z}_N(\beta^2) \xrightarrow[N \to \infty]{} &\mathrm{e}^{-\frac{\beta^2}{2}(\gamma^4+1)} \frac{1}{\sqrt{2\pi}\beta} \int_{\mathbb{R}} \mathrm{e}^{-\frac{x^2}{2\beta^2}} \mathrm{e}^{x\gamma^2} \mathrm{e}^{\frac{x^2}{2}} dx \\ =&\mathrm{e}^{-\frac{\beta^2}{2}(\gamma^4+1)} \frac{1}{\sqrt{2\pi}\beta} \int_{\mathbb{R}} \exp\left\{ -\frac{(x - \frac{\beta^2\gamma^2}{1-\beta^2})^2}{\frac{2\beta^2}{1-\beta^2}} \right\} \exp\left\{ \frac{\beta^2\gamma^4}{2(1-\beta^2)} \right\} dx \\ =&\frac{1}{\sqrt{1-\beta^2}} \exp\left\{ \frac{\beta^2[(1+\gamma^4)\beta^2 - 1]}{2(1-\beta^2)} \right\}, \end{aligned}$$

which makes sense and gives a finite limit if and only if $\beta < 1$. The function

$$\phi(t) = \frac{1}{2}\left[\ln(\frac{1}{1-t}) + \frac{t((1+\gamma^4)t-1)}{(1-t)}\right]$$

is continuous and increasing in the interval $[0,1)$. At this point we can apply again theorem 2.2 and obtain our result.

Note that if there is no mutual interaction, but only an external random field, then the right scaling is $N^{-\frac{1}{2}}$.

4. The multi-body interaction case

In this section we want to characterize a whole class of mean field spin glass models showing multi-body interaction up to any order that verify theorem 2.2.

Let us consider a mean field Hamiltonian (in the sense of Definition 2.1), where the coefficients $f_\alpha(N)$ are all identically equal to zero for all α such that $|\alpha| \geq l$, where l is a given integer, independent of N.

First we examine a particular form of the coefficients, namely

$$f_{|\alpha|}(N) = \begin{cases} \dfrac{\gamma_1}{N^{\frac{1}{4}}} & \text{if } |\alpha| = 1 \\ \dfrac{\gamma_j}{N^{\frac{j-1}{2}}} & \text{if } |\alpha| = j,\ 2 \leq j \leq l \\ 0 & \text{if } |\alpha| > l, \end{cases} \tag{4.1}$$

where the constants $\gamma_j \geq 0$ for all j.

We have the following result

Theorem 4.1: *Let $H_N(\sigma)$ be as in (2.1), with the $f_\alpha(N)$ given as in (4.1) and $\beta > 0$, then the corresponding partition function $Z_N(\beta)$ verifies theorem 2.2 if*

$$\beta \sum_{m=2}^{l} \frac{\gamma_m^2}{m!} < \frac{1}{2} \tag{4.2}$$

and diverges if

$$\beta \sum_{m=2}^{l} \frac{\gamma_m^2}{m!} > \frac{1}{2}. \tag{4.3}$$

PROOF: Following the prescription given in Section 2, we may write the "ferromagnetic analogue" of Z_N, obtaining

$$\begin{aligned} \tilde{H}_N(\sigma) &= \frac{\gamma_1^2}{\sqrt{N}} \sum_{i=1}^{N} \sigma_i + \sum_{m=2}^{l} \frac{\gamma_m^2}{N^{m-1}} \sum_{i_1<..<i_m}^{N} \sigma_{i_1}...\sigma_{i_m} \\ \tilde{Z}_N(\beta) &= \sum_{\sigma\in\{-1,1\}^N} \mathrm{e}^{\beta \tilde{H}_N(\sigma)}. \end{aligned} \tag{4.4}$$

The second sum in the Hamiltonian may be rewritten as

$$\sum_{i_1<..<i_m}^{N} \sigma_{i_1}...\sigma_{i_m} =$$
$$\left[(\sum_{i=1}^{N} \sigma_i)^m - \sum_{j=1}^{m-1} \sum_{\substack{i_1<\dots<i_j \\ k_1+\ldots+k_j=m}}^{N} \binom{m}{k_1}\binom{m-k_1}{k_2} \cdots \binom{m-k_1-..-k_j}{k_{j-1}} \sigma_{i_1}^{k_1}...\sigma_{i_j}^{k_j} \right].$$

On the other hand let us notice that for all $j = 1, \ldots, m-1$, with $k_1 + \cdots + k_j = m$, we have, since $j = 1, \ldots, m$,

$$\begin{aligned} &\left| \sum_{\substack{i_1<\dots<i_j \\ k_1+\ldots+k_j=m}} \binom{m}{k_1}\binom{m-k_1}{k_2} \cdots \binom{m-k_1-..-k_j}{k_{j-1}} \sigma_{i_1}^{k_1}...\sigma_{i_j}^{k_j} \right| \\ &\leq \sum_{\substack{i_1<\dots<i_j \\ k_1+\ldots+k_j=m}} \frac{m!}{k_1!\ldots k_j!} \leq m! N(N-1)\ldots(N-j+1) \leq m! N^{m-2}, \end{aligned}$$

hence

$$\frac{1}{N^{m-1}} \left| \frac{\gamma_m^2}{m!} \sum_{\substack{i_1<\dots<i_j \\ k_1+\ldots+k_j=m}} \binom{m}{k_1}\binom{m-k_1}{k_2} \cdots \binom{m-k_1-..-k_j}{k_{j-1}} \sigma_{i_1}^{k_1}...\sigma_{i_j}^{k_j} \right| \leq \frac{\gamma_m^2}{N}.$$

It is then clear that in order to study the convergence of $\frac{\tilde{Z}_N}{2^N}$, it is equivalent to analyze the convergence of

$$\frac{1}{2^N}\sum_{\sigma\in\{-1,1\}^N}\exp\{\frac{\gamma_1^2}{\sqrt{N}}\sum_{i=1}^N\sigma_i+\sum_{m=2}^l\frac{\gamma_m^2}{m!N^{m-1}}(\sum_{i=1}^N\sigma_i)^m\}. \tag{4.5}$$

The random variable $S_N=\sum_{i=1}^N\sigma_i$ is the sum of N independent Bernoulli($\frac{1}{2}$) random variables, thus, by the Central Limit Theorem, we know that

$$\frac{\sum_{i=1}^N\sigma_i}{\sqrt{N}}\Longrightarrow\mathcal{N}(0,1),\qquad \text{as } N\longrightarrow+\infty.$$

We want to show that, if we consider the sequence of functions

$$f_N(x)=\exp\{\beta\gamma_1^2x+N\beta\sum_{m=2}^l\frac{\gamma_m^2}{m!}\frac{x^m}{(\sqrt{N})^m}\}$$

then

$$E[f_N(\frac{S_N}{\sqrt{N}})-f_N(X)1_{\{|X|\leq\sqrt{N}\}}]\longrightarrow 0\quad \text{as } N\to\infty \tag{4.6}$$

where X is a standard Gaussian random variable.

This implies that the summation in (4.5) may be approximated by the integral

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_{-\sqrt{N}}^{\sqrt{N}}\exp\left\{\beta\gamma_1^2x+\beta N\sum_{m=2}^l\frac{\gamma_m^2}{m!}\left(\frac{x}{\sqrt{N}}\right)^m-\frac{x^2}{2}\right\}dx\\ =&\frac{\sqrt{N}}{\sqrt{2\pi}}\int_{-1}^{1}\exp\left\{\beta\gamma_1^2y\sqrt{N}+\beta N\sum_{m=2}^l\frac{\gamma_m^2}{m!}y^m-N\frac{y^2}{2}\right\}dy,\end{aligned} \tag{4.7}$$

where we applied the change of variable $y=\frac{x}{\sqrt{N}}$.

We will prove (4.6), but assuming it holds, then it is enough to study the convergence of the integral in (4.7). We claim that it converges if condition (4.2)

$$\beta\sum_{m=2}^l\frac{\gamma_m^2}{m!}-\frac{1}{2}<0$$

is fulfilled, it diverges, instead, if condition (4.3) holds.

The sufficient condition is clear. Indeed if (4.2) holds, then, for all $-1 \leq y \leq 1$, we have

$$\beta\frac{\gamma_2^2}{2} + \beta\sum_{m=3}^{l}\frac{\gamma_m^2}{m!}y^{m-2} \leq \beta\frac{\gamma_2^2}{2} + \beta\sum_{m=3}^{l}\frac{\gamma_m^2}{m!} < \frac{1}{2}.$$

Hence, setting $\frac{\eta}{2} = \frac{1}{2} - \beta\sum_{m=2}^{l}\frac{\gamma_m^2}{m!}$, we have

$$\exp\left\{\beta\gamma_1^2 y\sqrt{N} + Ny^2[\beta\sum_{m=2}^{l}\frac{\gamma_m^2}{m!}y^{m-2} - \frac{1}{2}]\right\} < \exp\left\{\beta\gamma_1^2 y\sqrt{N} - N\eta\frac{y^2}{2}\right\},$$

which implies

$$\begin{aligned}
&\frac{\sqrt{N}}{\sqrt{2\pi}}\int_{-1}^{1}\exp\left\{\sqrt{N}\beta\gamma_1^2 y + \beta N\sum_{m=2}^{l}\frac{\gamma_m^2}{m!}y^m - N\frac{y^2}{2}\right\}dy \\
\leq&\frac{\sqrt{N}}{\sqrt{2\pi}}\int_{-1}^{1}\exp\left\{\beta\gamma_1^2 y\sqrt{N} - N\eta\frac{y^2}{2}\right\}dy \\
=&\frac{1}{\sqrt{2\pi}}\int_{-\sqrt{N}}^{\sqrt{N}}\exp\left\{\beta\gamma_1^2 x - \eta\frac{x^2}{2}\right\}dx < \infty.
\end{aligned}$$

Viceversa let us assume that $\beta\sum_{m=2}^{l}\frac{\gamma_m^2}{m!} > \frac{1}{2}$. This might happen with either $\beta\gamma_2^2 > 1$ or $\beta\gamma_2^2 < 1$. The first case is trivial, indeed

$$\frac{\sqrt{N}}{\sqrt{2\pi}}\int_{-1}^{1}\exp\left\{\beta\gamma_1^2 y\sqrt{N} + N\frac{\beta\gamma_2^2 - 1}{2}y^2 + \beta\sum_{m=3}^{l}\frac{\gamma_m^2}{m!}y^m\right\}dy \longrightarrow +\infty \quad \text{as } N \to \infty.$$

In the second case, still there exists an $\epsilon > 0$ such that $2\beta\sum_{m=2}^{l}\frac{\gamma_m^2}{m!} - 1 = \eta > \epsilon$, so we split the interval $[0, 1]$ in k parts, with k large enough to have

$$\beta\gamma_2^2 - 1 + \beta(\frac{k-1}{k})^{l-2}\sum_{m=3}^{l}2\frac{\gamma_m^2}{m!} > \frac{\epsilon}{2}.$$

We have

$$
\begin{aligned}
&\sqrt{\frac{N}{2\pi}}\int_{-1}^{1}\exp\left\{\sqrt{N}\beta\gamma_1^2 y+N\frac{y^2}{2}(\beta\gamma_2^2-1+\beta\sum_{m=3}^{l}2\frac{\gamma_m^2}{m!}y^{m-2})\right\}dy\\
\geq&\sqrt{\frac{N}{2\pi}}\int_{0}^{1}\exp\left\{\sqrt{N}\beta\gamma_1^2 y+N\frac{y^2}{2}(\beta\gamma_2^2-1+\beta\sum_{m=3}^{l}2\frac{\gamma_m^2}{m!}y^{m-2})\right\}dy\\
=&\sqrt{\frac{N}{2\pi}}\sum_{j=1}^{k}\int_{\frac{j-1}{k}}^{\frac{j}{k}}\exp\left\{\sqrt{N}\beta\gamma_1^2 y+N\frac{y^2}{2}(\beta\gamma_2^2-1+\beta\sum_{m=3}^{l}2\frac{\gamma_m^2}{m!}y^{m-2})\right\}dy\\
\geq&\sqrt{\frac{N}{2\pi}}\int_{\frac{k-1}{k}}^{\frac{j}{k}}\exp\left\{\sqrt{N}\beta\gamma_1^2 y+N\frac{y^2}{2}(\beta\gamma_2^2-1+\beta\sum_{m=3}^{l}2\frac{\gamma_m^2}{m!}(\frac{k-1}{k})^{l-2})\right\}dy\\
\geq&\sqrt{\frac{N}{2\pi}}\left(1-\frac{k-1}{k}\right)\exp\left\{\frac{1}{2}(\frac{k-1}{k})^2\frac{\eta}{2}N+\sqrt{N}\beta\gamma_1^2(\frac{k-1}{k})\right\}\longrightarrow\infty
\end{aligned}
$$

as $N\to\infty$.

It remains to prove (4.6). In order to do so, let us notice that, if we denote by $F_N(x)$ and $\Phi(x)$ the distribution function of $\frac{S_N}{\sqrt{N}}$ and of a standard Normal respectively, then we may rewrite (4.6) as

$$
\left|\int_{\sqrt{N}}^{\sqrt{N}} f_N(x)d(F_N-\Phi)(x)\right|. \tag{4.8}
$$

For fixed N, the function f_N is continuous and both F_N and Φ are finite variation functions. The integral in (4.8) makes of course sense and, by the Lebesgue-Stiltjies integration by parts formula, one has

$$
\begin{aligned}
&\left|\int_{\sqrt{N}}^{\sqrt{N}} f_N(x)d(F_N-\Phi)(x)\right|\\
=&\left|[f_N(x)(F_N(x)-\Phi(x))]_{-\sqrt{N}}^{\sqrt{N}}-\int_{\sqrt{N}}^{\sqrt{N}}(F_N(x)-\Phi(x))df_N(x)\right|\\
\leq&\left|[f_N(x)(F_N(x)-\Phi(x))]_{-\sqrt{N}}^{\sqrt{N}}\right|+\int_{\sqrt{N}}^{\sqrt{N}}|F_N(x)-\Phi(x)|\,|f_N'(x)|dx\\
\leq&\left|[f_N(x)(F_N(x)-\Phi(x))]_{-\sqrt{N}}^{\sqrt{N}}\right|+2\int_{0}^{\sqrt{N}}|F_N(x)-\Phi(x)|\,|f_N'(x)|dx.
\end{aligned} \tag{4.9}
$$

We will evaluate the two terms separately. For the first one, let us remember that

$$P(\frac{S_N}{\sqrt{N}} \in [-\sqrt{N}, \sqrt{N}]) = 1, \quad \text{hence} \quad F_N(\sqrt{N}) = 1, F_N(-\sqrt{N}) = 0.$$

Moreover, by the symmetry of the Gaussian distribution, we have $(1-\Phi(\sqrt{N})) = \Phi(-\sqrt{N})$.

Therefore we get

$$\begin{aligned}
\left|[f_N(x)(F_N(x) - \Phi(x))]_{-\sqrt{N}}^{\sqrt{N}}\right| &= (1 - \Phi(\sqrt{N}))(f_N(\sqrt{N}) + f_N(-\sqrt{N})) \\
&\leq 2(1 - \Phi(\sqrt{N}))f_N(\sqrt{N}) \\
&= 2(1 - \Phi(\sqrt{N})) \exp\left\{\beta(\gamma_1^2\sqrt{N} + N\sum_{m=2}^{l}\frac{\gamma_m^2}{m!})\right\} \\
&\leq C\frac{1}{\sqrt{N}}\mathrm{e}^{-\frac{N}{2}} \exp\left\{\beta(\gamma_1^2\sqrt{N} + N\sum_{m=2}^{l}\frac{\gamma_m^2}{m!})\right\},
\end{aligned}$$

where we used Stirling's approximation of the Gaussian survival function

$$\frac{1}{\sqrt{2\pi}}\int_x^{\infty}\mathrm{e}^{-\frac{u^2}{2}}du \sim \frac{1}{\sqrt{2\pi}}\frac{1}{x}\mathrm{e}^{-\frac{x^2}{2}} \qquad \text{as } x \to \infty$$

(see [Bi]). Thus we may conclude that

$$\left|[f_N(x)(F_N(x) - \Phi(x))]_{-\sqrt{N}}^{\sqrt{N}}\right| \leq \frac{C}{\sqrt{N}}\exp\left\{-\frac{\eta}{2}N + \beta\gamma_1^2\sqrt{N}\right\},$$

which tends to zero, as $N \to \infty$.

As for the second term in (4.9), we recall that also $1 - F_N(x)$ has superexponential decay, namely

$$P(\frac{S_N}{N} \geq a_N) \leq \frac{1}{\sqrt{2\pi}}\frac{\mathrm{e}^{-\frac{a_N^2}{2}}}{a_N} \qquad \text{as } a_N \to \infty,$$

therefore the non uniform Berry-Eessen estimate of the Central Limit Theorem (see [S])

$$|F_N(x) - \Phi(x)| \leq \frac{C}{\sqrt{N}}\frac{1}{(1+|x|)^3} \tag{4.10}$$

can be obviously improved by

$$(4.11)\qquad |F_N(x)-\Phi(x)| \leq \min\left\{\frac{C}{\sqrt{N}}\frac{1}{(1+|x|)^3}, 2\max\{|F_N(x)|,|\Phi(x)|\}\right\}.$$

If x is large enough, for a properly chosen constant K, $\max\{F_N(x),\Phi(x)\} \leq K\frac{1}{x}\mathrm{e}^{-\frac{x^2}{2}}$ and it is easy to verify that for N large, for every $|x| \geq \sqrt{\ln N}$, it happens

$$\frac{C}{\sqrt{N}}\frac{1}{(1+|x|)^3} \geq K\mathrm{e}^{-\frac{x^2}{2}}.$$

In the light of the previous remarks, keeping in mind that

$$f_N'(x) = \beta\left(\gamma_1^2 + \sqrt{N}\sum_{m=2}^{l}\frac{\gamma_m^2}{(m-1)!}(\frac{x}{\sqrt{N}})^{m-1}\right) f_N(x),$$

we split the second term of (4.9) in two parts and, applying (4.11) to each part, we obtain

$$(4.12)\qquad \begin{aligned}&2\int_0^{\sqrt{N}} |F_N(x)-\Phi(x)|\,|f_N'(x)|dx\\ \leq& 2\frac{C}{\sqrt{N}}\int_0^{\sqrt{\ln N}}\frac{1}{(1+|x|)^3}|f_N'(x)|dx + 4K\int_{\sqrt{\ln N}}^{\sqrt{N}}\mathrm{e}^{-\frac{x^2}{2}}|f_N'(x)|dx.\end{aligned}$$

Dominating $f_N'(x)$ by its value at $\sqrt{\ln N}$, for the first part we may conclude

$$\begin{aligned}&\frac{C}{\sqrt{N}}\int_0^{\sqrt{\ln N}}\frac{1}{(1+|x|)^3}|f_N'(x)|dx\\ \leq&\frac{\beta C}{\sqrt{N}}[\gamma_1^2\sqrt{\ln N} + \sum_{m=3}^{l}\frac{\gamma_m^2}{(m-1)!}\frac{(\sqrt{\ln N})^{m-1}}{N^{m/2-1}}]\exp\left\{\beta\left(\gamma_1^2\sqrt{\ln N}+\sum_{m=2}^{l}\frac{\gamma_m^2}{m!}\frac{(\sqrt{\ln N})^m}{N^{m/2-1}}\right)\right\}\\ \sim&\frac{C'}{\sqrt{N}}\sqrt{\ln N}\mathrm{e}^{\frac{\beta\gamma_2^2}{2}\ln N}\end{aligned}$$

which again tends to zero, as $N\to\infty$, since $\dfrac{\beta\gamma_2^2}{2} < \dfrac{1}{2}$.

It remains to estimate the second part of (4.12). In $[\sqrt{\ln N},\sqrt{N}]$, we have

$$\begin{aligned}|f_N'(x)| &= |\beta(\gamma_1^2+\gamma_2^2 x + x\sum_{m=3}^{l}\frac{\gamma_m^2}{(m-1)!})|\exp\{\beta(\gamma_1^2 x + \frac{x^2}{2}\sum_{m=2}^{l}\frac{\gamma_m^2}{m!})\}\\ &\leq (C_1+C_2 x)\exp\{\beta(\gamma_1^2 x + \eta\frac{x^2}{2})\},\end{aligned}$$

for appropriate constants C_1 and C_2. The integral in (4.12) is therefore dominated by

$$K\int_{\sqrt{\ln N}}^{\sqrt{N}} \mathrm{e}^{-\frac{x^2}{2}}|f'_N(x)|dx \le K\int_{\sqrt{\ln N}}^{\sqrt{N}}(C_1+C_2x)\exp\left\{\beta\gamma_1^2 x-(1-\beta\eta)\frac{x^2}{2}\right\}dx$$
$$= K_1\int_{\sqrt{\ln N}}^{\sqrt{N}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}dx + K_2\int_{\sqrt{\ln N}}^{\sqrt{N}} x\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}dx,$$

where $\mu=\frac{\beta\gamma_1^2}{(1-\beta\eta)}$ and $\sigma=\sqrt{\frac{1}{(1-\beta\eta)}}$. By Stirling's approximation formula and integration by parts, we have

$$K\int_{\sqrt{\ln N}}^{\sqrt{\ln N}} \mathrm{e}^{-\frac{x^2}{2}}|f'_N(x)|dx \le K_3\frac{1}{\sqrt{N}}\mathrm{e}^{-\frac{(\sqrt{\ln N}-\mu)^2}{2\sigma^2}} + K_4(\mathrm{e}^{-\frac{(\sqrt{\ln N}-\mu)^2}{2\sigma^2}} - \mathrm{e}^{-\frac{(\sqrt{N}-\mu)^2}{2\sigma^2}}) \xrightarrow[N\to\infty]{} 0$$

which concludes the proof. $\square$

As it often happens in this type of problems, the convergence at the critical value $\beta\sum_{m=2}^{l}\frac{\gamma_m^2}{m!}=\frac{1}{2}$ turned out to be much harder to analyze and it might call for different techniques.

¿From the proof of theorem 4.1, it results clear that the result can be extended to any mean field spin glass model, provided it exhibits the same asymptotic behaviour as (4.1). Namely we have the following

Corollary 4.2: *Let $H_N(\sigma)$ be as in (2.1) and $\beta>0$ and l a fixed integer. Let $f_\alpha(N)$ be such that $f_\alpha(N)=0$ if $|\alpha|>l$ and*

$$\begin{cases} \lim\limits_{N\to+\infty}\dfrac{f_\alpha(N)}{N^{\frac{1}{4}}}=\gamma_1 & \text{if } |\alpha|=1\\ \lim\limits_{N\to+\infty}\dfrac{f_\alpha(N)}{N^{\frac{j-1}{2}}}=\gamma_j & \text{if } |\alpha|=j,\ 2\le j\le l, \end{cases}$$

where the constants $\gamma_j\ge 0$ for all j.

Then the corresponding partition function $Z_N(\beta)$ verifies theorem 2.2 if

$$\beta\sum_{m=2}^{l}\frac{\gamma_m^2}{m!}<\frac{1}{2}\qquad \text{and diverges if}\qquad \beta\sum_{m=2}^{l}\frac{\gamma_m^2}{m!}>\frac{1}{2}.$$

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