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Duality Symmetries and Noncommutative Geometry of String Spacetimes

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ABSTRACT

We examine the structure of spacetime symmetries of toroidally compactified string theory within the framework of noncommutative geometry. Following a proposal of Fröhlich and Gawędzki, we describe the noncommutative string spacetime using a detailed algebraic construction of the vertex operator algebra. We show that the spacetime duality and discrete worldsheet symmetries of the string theory are a consequence of the existence of two independent Dirac operators, arising from the chiral structure of the conformal field theory. We demonstrate that these Dirac operators are also responsible for the emergence of ordinary classical spacetime as a low-energy limit of the string spacetime, and from this we establish a relationship between T-duality and changes of spin structure of the target space manifold. We study the automorphism group of the vertex operator algebra and show that spacetime duality is naturally a gauge symmetry in this formalism. We show that classical general covariance also becomes a gauge symmetry of the string spacetime. We explore some larger symmetries of the algebra in the context of a universal gauge group for string theory, and connect these symmetry groups with some of the algebraic structures which arise in the mathematical theory of vertex operator algebras, such as the Monster group. We also briefly describe how the classical topology of spacetime is modified by the string theory, and calculate the cohomology groups of the noncommutative spacetime. A self-contained, pedagogical introduction to the techniques of noncommutative geometry is also included.

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1. Introduction

Duality has emerged as an important non-perturbative tool for the understanding of the spacetime structure of string theory and certain aspects of confinement in supersymmetric gauge theories (see [1, 2] for respective reviews). Recently, its principal applications have been in string theory within the unified framework of M theory [3], in which all five consistent superstring theories in ten-dimensions are related to one another by duality transformations. Target space duality, in its simplest toroidal version, i.e. T-duality, relates large and small compactification radius circles to one another. It therefore relates two different spacetimes in which the strings live and, implicitly, large and small distances. The quantum string theory is invariant under such a transformation of the target space. The symmetry between these inequivalent string backgrounds leads to the notion of a *stringy* or *quantum* spacetime which forms the moduli space of string vacua and describes the appropriate stringy modification of classical general relativity.

T-duality naturally leads to a fundamental length scale in string theory, which is customarily identified as the Planck length l_P. A common idea is that at distances smaller than l_P the conventional notion of a spacetime geometry is inadequate to describe its structure. As strings are extended objects, the notion of a 'point' in the spacetime may not make sense, just as the notion of a point in a quantum phase space is meaningless. In fact, it has been conjectured that the string configurations themselves obey an uncertainty principle, so that at small distances they become smeared out. A recent candidate theory for this picture is the effective matrix field theory for D-branes [4, 5] in which the spacetime coordinates are described by noncommuting matrices. In this paper we shall describe a natural algebraic framework for studying the geometry of spacetime implied by string theory.

Duality and string theory seem to point to a description of spacetime which goes beyond the one given by ordinary geometry with its concept of manifolds, points, dimensions etc. In this respect, a tool ideally suited to generalize the concept of ordinary differential geometry is *Noncommutative Geometry* [6]. Indeed, the foundations of noncommutative geometry were developed by von Neumann in an attempt to understand the geometry of a quantum phase space. The central idea of Noncommutative Geometry is that, since a generic (separable) topological space (for example a manifold) is completely characterized by the commutative C^*-algebra of continuous complex-valued functions defined on it, it may be useful to regard this algebra as the algebra generated by the coordinates of the space. Conversely, given a *commutative* C^*-algebra, it is possible to construct, with purely algebraic methods, a topological space. The terminology *noncommutative* geometry refers to the possibility of generalizing these concepts to the case where the algebra is noncommutative. This would be the case of some sort of space in which the coordinates do not commute. The structure of the theory is, however, even more powerful that this first generalization.

The commutative algebra gives not only the possibility to construct the points of a space, but also to give them a topology, again using only algebraic methods. Metric aspects (distances, etc.) can also be constructed by representing the algebra as bounded operators on a Hilbert space, and then defining on it a generalization of the Dirac operator. Differential forms are also represented as operators on the Hilbert space and gauge transformations act as conjugation by elements of the group of unitary operators of the algebra. This set of three objects, an algebra, a Hilbert space on which the algebra is represented, and the generalized Dirac operator, goes under the name of a *Spectral Triple*.

One is now led to ask if the generalization of spacetime hinted to by string theory can find a place in the framework of noncommutative geometry. Namely, if there is an algebra which can provide (very loosely speaking) the noncommutative coordinates of string theory. The structure should be such that in the low-energy limit, in which strings are effectively just point particles described by ordinary quantum field theory, one recovers the usual (commutative) spacetime manifolds. In the following we shall elaborate on a program initiated by Fröhlich and Gawędzki [7] (see also [8]) for understanding the geometry of string spacetimes using noncommutative geometry. This program has also been pursued recently by Fröhlich, Grandjean and Recknagel [9]. In [7] it was proposed that the algebra representing the stringy generalization of spacetime is the *vertex operator algebra* of the underlying conformal field theory for the strings. We shall analyse some of the properties of this spacetime through a detailed algebraic study of the properties of vertex operator algebras.

A conformal field theory has a natural chiral structure, from which we show how to naturally construct two Dirac operators. These operators are crucial to the construction of the low-energy limit of the noncommutative spacetime which gives the conventional spacetimes of classical general relativity at large distance scales. We will explicitly construct this limit, using the tools of noncommutative geometry, and show further how our Dirac operators are related to the more conventional ones that arise from $N = 1$ superconformal field theories. Chamseddine [10] has recently used these Dirac operators in the spectral action principle [11] of noncommutative geometry and shown that they lead to the desired effective superstring action. However, the Dirac operators that we incorporate into the spectral triple are more general and as such they illuminate the full structure of the duality symmetries of the string spacetime. All of the information concerning the target space dualities and discrete worldsheet symmetries of the string theory lies in the relationships between these two Dirac operators. They define isometric noncommutative spacetimes at the level of their spectral triples, and as such lead naturally to equivalences between their low-energy projective subspaces which imply the duality symmetries between classical spacetimes and the quantum string theory. From this we also deduce a non-trivial relation between T-duality transformations and changes of spin structure of the spacetime.

The main focus of this paper will be on the applications of noncommutative geometry to a systematic analysis of the symmetries of a string spacetime using the algebraic formulation of the theory of vertex operator algebras (see for example [12, 13, 14]). We will show (following [15]) that target space duality is, at the level of the string theory spectral triple, just a very simple inner automorphism of the vertex operator algebra, i.e. a gauge transformation (this was anticipated in part in [16]). This transformation leaves the algebra (which represents the noncommutative topology of the stringy spacetime) invariant, but changes the Dirac operator, and hence the metric properties. We shall also describe other automorphisms of the vertex operator algebra. For example, discrete worldsheet symmetries of the conformal field theory appear as outer automorphisms of the noncommutative geometry. For the commutative algebra of functions on a manifold there are no inner automorphisms (gauge symmetries), and the group of (outer) automorphisms coincides with the diffeomorphism group of the manifold. We will show that the outer automorphisms of the low-energy projective subalgebras of the vertex operator algebra, which define diffeomorphisms of the classical spacetime, are induced via the projections from *inner* automorphisms of the full string theory spectral triple. This implies that, in the framework of noncommutative geometry, general covariance appears naturally as a gauge symmetry of the quantum spacetime, and general relativity is therefore formulated as a gauge theory.

We shall also analyse briefly the problem of computing the full automorphism group of the noncommutative string spacetime. This ties in with the problem of finding a universal gauge group of string theory which overlies all of its dynamical symmetries. We are not aware of a full classification of the automorphims of a vertex operator algebra, nor will we attempt it here. Some very important mathematical aspects of this group are known, such as its relation to the Monster group [12], and we examine these properties within the context of the noncommutative geometry of the string spacetime. We also briefly discuss some properties of the noncommutative differential topology of the string spacetime and compare them with the classical topologies. From this we shall see the natural emergence of spacetime topology change.

For definitiveness and clarity we restrict ourselves in this paper to a detailed analysis of essentially the simplest string theory, the linear sigma-model, i.e. closed strings compactified on a flat n-dimensional torus. The advantage of studying this class of models is that everything can be represented in more or less explicit form. Already at this level we will find a very rich noncommutative geometrical structure, which illustrates how the geometry and topology relevant for general relativity must be embedded into a larger noncommutative structure in string theory. Our analysis indicates that noncommutative geometry holds one of the best promises of studying physics at the Planck scale, which at the same time incorporates the dynamical features governed by string theory. Our results also apply to $N = 1$ superconformal field theories (whose target spaces are effectively restricted to tori), and we also indicate along the way how the analysis generalizes to other conformal field theories. However, the most interesting generalization would be in the context of *open* strings, where the two chiral sectors of the theory merge and T-duality connects strings with D-branes, solitonic states of the theory that appear if one allows the endpoints of an open string to have Dirichlet rather than the customary Neumann boundary conditions. D-branes have been discussed in the context of Noncommutative Geometry recently in [17] (see also [9]). Duality now relates spectral triples of *different* string spacetimes, which could have remarkable implications in the context of M theory, the membrane theory which purports to contain (as different low energy regimes) all consistent superstring theories. A complete dynamical description of M theory is yet unknown, but in the context of this paper this could be achieved by some large vertex operator algebra. In this respect the conjecture of [5], which relates M-theory to a matrix model, could be interpreted as a particular truncation of a set of vertex operators to finite-dimensional $N \times N$ matrices. The large-N limit then recovers the aspects of the full theory.

Structure and Outline of the Paper

This paper has been written with the hope of being accessible to both string theorists with no prior knowledge of noncommutative geometry, and also to mathematicians and mathematical physicists with no detailed knowledge of string theory. It also merges two very modern branches of mathematics (both inspired in large part by physics), the algebraic theory of vertex operator algebras and noncommutative geometry. It therefore contains some review sections in the hope of keeping the presentation relatively self-contained. We start in section 2 with a brief, self-contained review of the basic techniques of noncommutative geometry in general, where the aspects relevant for our constructions are described, often in a heuristic way. In section 3 we then begin a systematic construction (following [7]) of the string spacetime. We begin this construction in a way that could be generalized to more general worldsheet geometries than the ones which we consider, and also to more complicated target spaces such as toroidal orbifolds. The Dirac operators of the theory are introduced in section 4, along with their alternative spin structures, and are related to similar objects which appear in supersymmetric sigma-models. The relevant algebra (the vertex operator algebra) is introduced in section 5. Here we aim at giving a more algebraic description of this space of operators than is usually presented, so that this section is somewhat foundational to the general study of noncommutative string spacetimes, in the sense that the full algebraic properties of it are not fully exploited in the remainder of the paper. A full analysis of the string spacetime that captures all of these rich algebraic aspects of the vertex operator algebra could lead to remarkable insights into the spacetime structure of string theory. Sections 6 and 7 then contain the main results of this paper. In section 6 the spectral triples are analysed and the low-energy sectors are constructed from the two Dirac operators. The duality symmetries are also presented as isomorphisms between the spectral triples. Section 7 discusses the symmetries and topology of the noncommutative string spacetime in a general setting, and some aspects of a universal string theory gauge group, Borcherds algebras and relations with the Monster group are presented. Some more formal algebraic aspects of vertex operator algebras, along with some features about how they construct the string spacetime, are included in an appendix at the end of the paper. This expands the descriptions already presented in section 5 with emphasis on how the analysis and results of this paper extend to more general string theories.

2. Elements of Noncommutative Geometry

In this section we will give, for the sake of completeness, a very brief introduction to the basic tools of noncommutative geometry. We shall work mostly with examples of commutative algebras, corresponding to ordinary (commutative) geometry. We will not be concerned with mathematical details, but aim mostly at giving an overview. The classical reference is Connes' book [6]. In the following we shall for the most part follow the introduction to noncommutative geometry given in the excellent forthcoming book by Landi [18], to which we refer the reader for details. Another introduction can be found in [19]. The reader already conversant in noncommutative geometry may wish to skip this section.

Topological Spaces from Algebras

The main idea behind noncommutative geometry is to study the topology (and geometrical properties) of a space by not seeing it as a set of points, but rather by investigating the set of fields defined on it. In this sense the tools of noncommutative geometry resemble the methods of modern theoretical physics. The main mathematical result which enables such a study lies in a series of theorems due to Gel'fand and Naimark (for a review see for example [20, 21]). They established a complete equivalence between *Hausdorff Topological Spaces* and *Abelian* C^**-algebras*.

It is worthwhile to recall these two concepts. A topological space is a set with a topology (a collection of open subsets obeying certain conditions) defined on it. A Hausdorff topology is one that makes the space separable, i.e. given two points it is always possible to find two disjoint open sets each containing one of the two points. The common topological spaces encountered in physics (for example manifolds) are separable*. A C^*-algebra $\mathcal{A}$ is, first of all, an associative algebra over the complex numbers $\mathbb{C}$, i.e. a set on which two operations, sum (commutative) and product (not necessarily commutative), are defined with the following properties:

*For an example of a *non-Hausdorff* space in the context of noncommutative geometry see [22].

1) $\mathcal{A}$ is a vector space over $\mathbb{C}$, i.e. $\alpha a + \beta b \in \mathcal{A}$ for $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$.

2) It is distributive over addition with respect to left and right multiplication, i.e. $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$, $\forall\ a, b, c \in \mathcal{A}$.

$\mathcal{A}$ is further required to be a Banach algebra:

3) It is complete with respect to a norm $\|\cdot\| : \mathcal{A} \to \mathbb{R}$ with the usual properties

   a) $\|a\| \geq 0\ , \quad \|a\| = 0 \iff a = 0$

   b) $\|\alpha a\| = |\alpha|\,\|a\|$

   c) $\|a + b\| \leq \|a\| + \|b\|$

   d) $\|ab\| \leq \|a\|\,\|b\|$

The Banach algebra $\mathcal{A}$ is called a C^*-algebra if, in addition to the properties above, it also has defined on it a conjugation operation $*$ (analogous to the one usually defined for complex numbers) with the properties

4) $a^{**} = a$

5) $(ab)^* = b^* a^*$

6) $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$

7) $\|a^*\| = \|a\|$

8) $\|a^* a\| = \|a\|^2$

for any $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$, where $\bar{\alpha}$ denotes the usual complex conjugate of $\alpha \in \mathbb{C}$.

Note that the existence of an identity $\mathbb{I}$ is not an axiom. If an identity with the property $a\,\mathbb{I} = \mathbb{I}\,a = a \quad \forall a \in \mathcal{A}$ exists then the algebra is called *unital*. The simplest example of a C^*-algebra is the algebra of complex numbers $\mathbb{C}$ itself. Matrix algebras, as well as algebras of bounded (or compact) operators on a separable Hilbert space, are also examples of C^*-algebras, in general noncommutative. The $*$ conjugation in these cases is the usual Hermitian conjugation $\dagger$, while the norm is the Hilbert-Schmidt operator norm, i.e. $\|T\|^2$ is the largest eigenvalue of $T^\dagger T$. Another example of an abelian C^*-algebra is the algebra $C(M)$ of *continuous* complex-valued functions on a topological set M [†]. The norm in this case is the L^∞-norm, i.e. the maximum value attained by the function:

$$\|f\|_\infty = \sup_{x \in M} |f(x)|\ . \tag{2.1}$$

[†] In the case where the space is non-compact one has to further restrict to functions which vanish on the frontier of the space.

The Gel'fand-Naimark theorem states that *there is a one-to one correspondence between* Hausdorff *topological spaces and* commutative *C^*-algebras*. The correspondence in one direction is rather simple. Given a topological space M it is possible to naturally construct the algebra $C(M)$ of continuous complex-valued functions on it. This is a commutative C^*-algebra, and moreover it depends (through the continuity criterion) on the topology of the original space. The correspondence in the other direction is much more sophisticated and can be seen in a constructive way. Given an abelian C^*-algebra $\mathcal{A}$, i.e. a set of elements with two commutative operations, a norm and a conjugation, one can *reconstruct* the topological space M for which $\mathcal{A} \cong C(M)$ is the algebra of continuous complex-valued functions on M.

Topological Space as a Set of Ideals

Let us now briefly describe this reconstruction process. An *ideal* $I \subset \mathcal{A}$ of an algebra $\mathcal{A}$ is a subalgebra with the property:

$$ab \in I \text{ and } ba \in I \qquad \forall a \in I, b \in \mathcal{A} \tag{2.2}$$

That is, not only is the ideal closed under summation and multiplication (as it is a subalgebra), but the product of any element of the ideal with any element of the whole algebra is also in the ideal. A *maximal ideal* is an ideal which is not contained in any other ideal (apart from the trivial ideal which is the whole algebra $\mathcal{A}$).

The relevant example of an ideal is the collection of continuous functions which vanish on some subset of a topological space M. It is easy to check that they form an ideal, and moreover that the sets I_x of functions which vanish at a single point $x \in M$ are maximal ideals. Note that if $I_{x,y}$ denotes the ideal of functions which vanish at two points $x, y \in M$, then $I_{x,y} = I_x \cap I_y \subseteq I_x, I_y$. In this sense a reduction in the number of points is equivalent to an increase in the number of functions in the ideal.

We will therefore identify the points of the space under reconstruction with the maximal ideals of the given commutative algebra $\mathcal{A}$ [‡]. We must at this point reconstruct the topology of the space, i.e. the relations between the points. This can be done using the *Hull kernel* or *Jacobson* topology. As is well-known the topology of a space can be given in various equivalent ways, for example in terms of open sets, or their complement closed sets. Another equivalent way is to give the closure of every set. Closed subsets are then defined as the ones which coincide with their closure. The closure operation must satisfy some axioms due to Kuratowski [18]:

1. $\overline{\emptyset} = \emptyset \;$;
2. $W \subseteq \overline{W} \,, \quad \forall\, W \;$;

[‡]If the algebra has no unit then a further requirement (modularity) must be imposed [20].

3. $\overline{\overline{W}} = \overline{W}$, $\quad \forall\, W$;

4. $\overline{W_1 \cup W_2} = \overline{W}_1 \cup \overline{W}_2$, $\quad \forall\, W_1, W_2$.

In an abelian C^*-algebra we can define the closure of any subset W of the set of maximal ideals. Being the union of ideals, W is itself an ideal, although not a maximal one in general. We define the closure $\overline{W}$ of W as the set of maximal ideals I_m with

$$\overline{W} \equiv \left\{ I_m \subseteq \mathcal{A} : \bigcap_{I \in W} I \subseteq I_m \right\} \tag{2.3}$$

That is, the point corresponding to the ideal I_m belongs to the closure of W if I_m contains the intersection of all ideals I of W. All maximal ideals which comprise W belong to $\overline{W}$. It is possible to prove that the above definition of closure satisfies the Kuratowski axioms [18].

As an example let us find the closure of the open interval. In terms of the above construction, W is the ideal of all functions which vanish at some point in the open interval. The intersection of all of these ideals is the ideal of functions vanishing on the open interval. An ideal I_m therefore belongs to the closure of W if it contains all functions which vanish in the open interval. Recall now that the functions we are considering are *continuous* (this carries the information about the topology), and, therefore, if they vanish in the open interval, they vanish at the endpoints of the interval as well. Thus the functions which vanish at any point of the *closed* interval belong to the closure of W.

Topological Space as a Set of Irreducible Representations

There is an alternative construction of the space and its topology. In this construction the points are the irreducibile representations of the algebra (or rather their equivalence classes modulo algebra-isomorphism). This space is called the *structure space* of $\mathcal{A}$ and is denoted $\hat{\mathcal{A}}$. In the case of commutative algebras the irreducible representations are all one-dimensional. They are thus the (non-zero) $*$-linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ which are multiplicative, i.e. $\phi(ab) = \phi(a)\phi(b)$ for any $a, b \in \mathcal{A}$. It follows that for unital algebras $\phi(\mathbb{I}) = 1$, $\forall \phi \in \hat{\mathcal{A}}$. For commutative algebras the set of irreducible representations and the set of maximal ideals coincide [20]. We can consider a representation ϕ_x as the evaluation map which gives the values of the elements of the algebra at a point x:

$$\phi_x(f) \equiv f(x) \quad , \quad f \in \mathcal{A} \tag{2.4}$$

The topology in the structure space is given using the concept of pointwise convergence. A sequence ϕ_{x_n} of representations is said to converge to ϕ_x if and only if for any $a \in \mathcal{A}$ the sequence $\phi_{x_n}(a) \to \phi_x(a)$ in the usual topology of $\mathbb{C}$. It is not at all trivial to show that the two topologies we described above are equivalent [18, 20], and in the proof all of the ingredients that make up a C^*-algebra play a role.

Yet another way to reconstruct the topological space, which is very similar to this latter method, is to consider the space of *characters* of the algebra. A character is a multiplicative $*$-linear functional

$$\phi : \mathcal{A} \to \mathbb{C} \quad ; \quad \phi(ab) = \phi(a)\phi(b) \quad \forall a, b \in \mathcal{A} \tag{2.5}$$

For abelian algebras the space of characters is the same as the space of irreducible representations. The space of characters of the algebra is therefore also called the structure space of the algebra. The connection with the space of maximal ideals is easily done by considering the space of *primitive ideals*. An ideal I is primitive if given an irreducible representation π,

$$I = \ker \pi \tag{2.6}$$

For commutative algebras a primitive ideal is also maximal, and vice-versa, so that the two spaces coincide. For a noncommutative algebra, for which not all irreducible representations are one-dimensional, this is no longer true.

The important aspect that we want to stress is that the investigation of the topology of a space M in terms of the relations among its points is completely equivalent to the analysis of the algebra of functions defined on it. Therefore the study of topological spaces (manifolds etc.) can be substituted by a study of C^*-algebras. In the case where M is a manifold, in addition to the topology the differentiable structure is determined by the algebra $C^\infty(M)$ of smooth functions on M. The algebra $C^\infty(M)$ is not a C^*-algebra (but only a $*$-algebra) because it is not complete with respect to the L^∞-norm. The fact that with it we reconstruct the same space as $C(M)$ is therefore not a violation of the Gel'fand-Naimark theorem, since in fact the completion of $C^\infty(M)$ is $C(M)$. Manifolds are characterized by the presence of tangent bundles. In this respect an important theorem due to Serre and Swan [23] provides a one-to-one correspondence between smooth sections of a vector bundle over a manifold M and finitely-generated projective modules over the algebra $C^\infty(M)$, i.e. vector spaces on which the algebra acts and which can be generated using a projector acting on finitely-many tensor products of the algebra. Thus the study of the differentiable structures of manifolds is equivalent to the study of finitely-generated projective modules.

Metric Spaces from Dirac Operators

A physical space has much more structure to it than just topology, which is in fact the most basic aspect of it, having to do mostly with global properties. Connes [6] has shown that a metric and other local aspects can be encoded at the level of algebras, and how fermionic and bosonic actions can be written. The key property is another important result due to Gel'fand, the fact that any C^*-algebra $\mathcal{A}$ (commutative or otherwise) can be represented faithfully as a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on an infinite-dimensional separable Hilbert space $\mathcal{H}$. In the following we will not distinguish between the algebra and this representation, and we take $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$. The norm of $\mathcal{A}$ is represented by the Hilbert-Schmidt operator norm on $\mathcal{B}(\mathcal{H})$,

$$\|a\| = \sup_{|\psi\rangle, |\phi\rangle \in \mathcal{H}} |\langle \psi | a | \phi \rangle| \tag{2.7}$$

The metric structure, as well as the noncommutative generalization of differential and integral calculus, is obtained via an operator which is the generalization of the usual Dirac operator. We shall call it a Dirac operator even in this generalized context.

From the point of view of noncommutative geometry the Dirac operator is a (not necessarily bounded) operator D on $\mathcal{H}$ with the following properties:

1. D is self-adjoint, i.e. $D = D^\dagger$ on the common domain of the two operators.

2. The commutator $[D, a]$ $(a \in \mathcal{A})$ is bounded on a dense subalgebra of $\mathcal{A}$.

3. D has compact resolvent, i.e. $(D - \lambda)^{-1}$ for $\lambda \notin \mathbb{R}$ is a compact operator on $\mathcal{H}$.

If the algebra is commutative, and therefore it has a structure space which determines a corresponding algebra of continuous functions on a topological space, then the Dirac operator enables one to give the topological space a metric structure via the definition of the *distance*. We define the distance between two points of the structure space as

$$d(x, y) = \sup_{a \in \mathcal{A}} \left\{ |a(x) - a(y)| \ : \ \|[D, a]\| \leq 1 \right\} \quad . \tag{2.8}$$

This notion of distance coincides with the usual definition of geodesic distance for a Riemannian manifold with Euclidean-signature metric $g_{\mu\nu}$:

$$\bar{d}(x, y) = \inf_{\gamma} \ \ell_\gamma(x, y) \tag{2.9}$$

where $\ell_\gamma(x, y)$ is the length of the path γ from x to y with respect to $g_{\mu\nu}$.

It is easy to convince oneself of the equality of these two definitions in a simple one-dimensional example. In this case the algebra is the set of functions on a line (or interval or circle) and the Dirac operator is the derivative $D = i\frac{d}{dx}$. The distance between x and y is simply $|x - y|$ if we use the usual metric. The definition (2.8) says that we must take the supremum over those functions whose derivatives have norm ≤ 1. This means that $\frac{da}{dx}$ must be nowhere greater than 1. It follows that then $|a(x) - a(y)| \leq |x - y|$, the inequality being saturated by any function with the property $a(t) = t$ for $t \in [x, y]$.

To see the relation between the two definitions in a higher-dimensional example, consider the usual Dirac operator $D = i\gamma^\mu \partial_\mu$ acting on some dense domain of the Hilbert space $\mathcal{H} = L^2(\mathrm{spin}(\mathbb{R}^d))$ of square-integrable spinors $\psi(x)$ defined on $\mathbb{R}^d$. The real-valued gamma-matrices generate the Euclidean Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$. We take $\mathcal{A}$ to be the algebra of continuous complex-valued functions on $\mathbb{R}^d$ which vanish at infinity, and consider them as operators on $\mathcal{H}$ acting by pointwise multiplication. The commutator $[D, a]$, acting on the spinor field ψ, is

$$[D, a]\psi = (i\gamma^\mu \partial_\mu a)\psi + ia\gamma^\mu \partial_\mu \psi - ia\gamma^\mu \partial_\mu \psi = i(\gamma^\mu \partial_\mu a)\psi \tag{2.10}$$

The L^∞-norm of the commutator is thus the maximum value of the $\mathbb{C}^d$ norm $\sqrt{\partial^\mu a^* \partial_\mu a}$, which is also equal [24] to the Lipschitz norm of a:

$$\|[D, a]\|_\infty = \sup_{x \in \mathbb{R}^d} \sqrt{\partial^\mu a^*(x) \partial_\mu a(x)} = \|a\|_{\mathrm{Lip}} \equiv \sup_{x \neq y} \frac{|a(x) - a(y)|}{\bar{d}(x, y)} \tag{2.11}$$

To check that the definitions of distance in (2.8) and (2.9) agree, note that the condition on the norm of $[D, a]$ in (2.8) along with (2.11) imply

$$d(x, y) \leq \bar{d}(x, y) \quad . \tag{2.12}$$

But $\bar{a}(t) \equiv \bar{d}(x, t)$ as a function of t (suitably regularized) satisfies

$$\|[D, \bar{a}(t)]\|_\infty = \|\bar{d}(x, t)\|_{\mathrm{Lip}} = 1 \tag{2.13}$$

which gives

$$d(x, y) = \bar{d}(x, y) \quad . \tag{2.14}$$

Given the notion of distance between pairs of points, the other properties of a metric space can be obtained by the traditional techniques. The set of data $(\mathcal{A}, \mathcal{H}, D)$, i.e. a C^*-algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$ and a Dirac operator D on $\mathcal{H}$, is called a *spectral triple*, and it encodes the geometry and topology of a space under consideration. The pair $(\mathcal{H}, D)$ is called a *Dirac K-cycle* for $\mathcal{A}$ and it can be used to describe the cohomology of a space using K-theory [6]. A different choice of Dirac operator will alter the metric properties of the space. Let us go back to the simple one-dimensional example above and suppose that we choose as Dirac operator

$$D' = i\frac{d}{dF(x)} = i\left(\frac{dF(x)}{dx}\right)^{-1} \frac{d}{dx} \tag{2.15}$$

for a monotonic real-valued function F. It is then easy to see that

$$[D', a] \leq 1 \quad \Rightarrow \quad \frac{da(x)}{dx} \leq \frac{dF(x)}{dx} \quad \forall x \tag{2.16}$$

and that therefore

$$|a(x) - a(y)| \leq |F(x) - F(y)| \quad , \tag{2.17}$$

again the equality being attained for $a = F$.

The change in Dirac operator in this simple example corresponds of course to the use of the measure $dF(x)$ instead of dx. But we want to stress the point that a change of measure is, in this context, a change of Dirac operator. Later on we will see how different choices of the Dirac operator for the case of string theory corresponds to spacetimes whose metric structures are related by T-duality and other geometric transformations.

Distance between States of an Algebra

The notion of distance between points does not generalize to the noncommutative case, where it is in general impossible to speak of points in a topological space, let alone of distances between them. It is, however, possible to speak of distances between the *states* of the algebra $\mathcal{A}$. A state is a positive definite unit norm map,

$$\Psi : \mathcal{A} \to \mathbb{C} \quad ; \quad \Psi(a^*a) \geq 0 \ , \forall a \in \mathcal{A}, \quad \|\Psi\|_\infty = 1 \ . \tag{2.18}$$

If the algebra is represented on a Hilbert space $\mathcal{H}$, then vectors $|\psi\rangle \in \mathcal{H}$ define states via the expectation values[§]:

$$\Psi(a) = \langle\psi|a|\psi\rangle \ . \tag{2.19}$$

States are, however, a more general concept, as the 'delta-functions' defined in (2.4) are states in the sense of (2.18) as well although, as is well-known, they do not correspond to the expectation value of any vector of the Hilbert space. We denote the space of states by $\mathcal{S}(\mathcal{A})$. Since

$$\lambda\Psi + (1-\lambda)\Phi \in \mathcal{S}(\mathcal{A}) \ , \quad \forall \Psi, \Phi \in \mathcal{S}(\mathcal{A}), \ \lambda \in [0,1] \tag{2.20}$$

the set of all states of an algebra $\mathcal{A}$ is a convex space. Being a convex space $\mathcal{S}(\mathcal{A})$ has a boundary whose elements are called *pure states*. The delta-functions (2.4) are examples of pure states. Namely, a state is called pure if it cannot be written as the convex combination of (two) other states. Another characterization of the structure space is as the space of pure states, which in the commutative case coincides with the set of irreducible representations, and hence the space of characters, and moreover coincides with the space of maximal ideals of $\mathcal{A}$.

The distance between states (pure or mixed) is then defined as

$$d(\Psi, \Phi) = \sup_{a \in \mathcal{A}} \left\{ |\Psi(a) - \Phi(a)| \ : \ \|[D,a]\| \leq 1 \right\} \ . \tag{2.21}$$

This distance is much more general than the distance between the points (pure states) as it could be applied to calculate the distance between any two quantum states of a physical system.

Differential Forms on Noncommutative Spaces

Another important role played by the Dirac operator D is in the construction of the algebra of differential forms in the context of noncommutative geometry. The key idea is

[§]Even if one starts with an abstract algebra $\mathcal{A}$ it is always possible to associate a representation of $\mathcal{A}$ to a state via a construction due to Gel'fand, Naimark and Segal. The space of states of an algebra (after quotienting out the states for which $\Psi(a^*a) = 0$) can be made into a Hilbert space. In the cases under consideration in this paper, however, a Hilbert space is provided from the onset by the quantum theory of the given physical problem.

to also represent differential forms as operators on $\mathcal{H}$, on a par with D and $\mathcal{A}$. We first define the (abstract) *universal differential algebra of forms* as the $\mathbb{Z}$-graded algebra

$$\Omega^*\mathcal{A} = \bigoplus_{p\geq 0} \Omega^p\mathcal{A} \tag{2.22}$$

which is generated as follows:

$$\Omega^0\mathcal{A} = \mathcal{A} \tag{2.23}$$

and $\Omega^1\mathcal{A}$ is generated by a set of abstract symbols da which satisfy:

$$d(ab) = (da)b + adb \ , \quad \forall\ a, b \in \mathcal{A} \quad \text{(Leibnitz Rule)} \tag{2.24}$$

$$d(\alpha a + \beta b) = \alpha da + \beta db \ , \quad \forall\ a, b \in \mathcal{A} \ , \quad \alpha, \beta \in \mathbb{C} \quad \text{(Linearity)} \tag{2.25}$$

Elements of $\Omega^p\mathcal{A}$ are linear combinations of elements of the form

$$\omega = a_0 da_1 \cdots da_p \tag{2.26}$$

Because of (2.24) a generic p-form can be written as a linear combination of forms of the kind (2.26), with a_0 possibly a multiple of $\mathbb{I}$ in the unital case. This makes $\Omega^p\mathcal{A}$ a $\mathbb{Z}_2$-graded $\mathcal{A}$-module. The graded exterior derivative operator is the nilpotent linear map $d : \Omega^p\mathcal{A} \to \Omega^{p+1}\mathcal{A}$ defined by

$$d(a_0 da_1 \cdots da_p) = da_0 da_1 \cdots da_p \tag{2.27}$$

We define a linear representation $\pi_D : \Omega^*\mathcal{A} \to \mathcal{B}(\mathcal{H})$ of the universal algebra of abstract forms by

$$\pi_D(a_0 da_1 \cdots da_p) = a_0 [D, a_1] \cdots [D, a_p] \tag{2.28}$$

Notice, however, that $\pi_D(\omega) = 0$ does not necessarily imply $\pi_D(d\omega) = 0$. Forms ω for which this happens are called *junk forms*. They generate a $\mathbb{Z}$-graded ideal in $\Omega^*\mathcal{A}$ and have to be quotiented out [6, 18]. Then the noncommutative differential algebra is represented by the quotient space

$$\Omega^*_D\mathcal{A} = \pi_D\left[\Omega^*\mathcal{A}/(\ker \pi_D \oplus d \ker \pi_D)\right] \tag{2.29}$$

which we note depends explicitly on the particular choice of Dirac operator D on the Hilbert space $\mathcal{H}$.

The algebra $\Omega^*_D\mathcal{A}$ determines a DeRham complex whose cohomology groups can be computed using the conventional methods. With this machinery, it is also possible to naturally define (formally at this level) a vector bundle E over $\mathcal{A}$ as a finitely-generated projective left $\mathcal{A}$-module, and along with it the usual definitions of connection, curvature, and so on [25]. However, in what follows we shall for the most part use only the trivial bundle over a unital C^*-algebra $\mathcal{A}$. For this we define a gauge group $\mathcal{U}(\mathcal{A})$ as the group of unitary elements of $\mathcal{A}$,

$$\mathcal{U}(\mathcal{A}) = \left\{ u \in \mathcal{A} \mid u^\dagger u = u u^\dagger = \mathbb{I} \right\} \tag{2.30}$$

Alternatively, the gauge group can be defined as the subgroup of unimodular elements $u \in \mathcal{U}(\mathcal{A})$, i.e. the unitary operators of unit determinant $\det u = 1$. The presence of one-forms is then tantamount to the possibility of defining a connection, which is a generic Hermitian one-form $\rho = \sum_i a_i[D, b_i]$, and with it a covariant Dirac operator $D_\rho = D + \rho$. The curvature of a connection ρ is defined to be

$$\theta = [D, \rho] + \rho^2 \tag{2.31}$$

Integration in Noncommutative Geometry

The final ingredient of differential geometry we discuss is the integral. Since we are representing all of the objects as operators on $\mathcal{H}$, it is natural to define the integral as a trace. It is in fact [6] a regularized trace called the *Dixmier trace* which is defined as follows. Consider a generic bounded operator L on $\mathcal{H}$ of discrete spectrum with eigenvalues λ_n ordered according to modulus and counted with the appropriate multiplicities. We then define its Dixmier trace $\mathrm{tr}_\omega L$ to be

$$\mathrm{tr}_\omega L = \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \lambda_n \tag{2.32}$$

For the algebra of continuous functions on a p-dimensional compact manifold M, this definition then yields [6]

$$\int_M f(x)\, d^p x = \mathrm{tr}_\omega\, f|D|^{-p} \tag{2.33}$$

and we see that in a loose sense the Dirac operator D is the "inverse" of the infinitesimal dx. The right-hand side of (2.33) can be calculated using heat-kernel methods.

Example of a Manifold

Let us work out the example¶ of a p-dimensional spin-manifold M with metric $g_{\mu\nu}$ and the usual Dirac operator $D = i\gamma^\mu \nabla_\mu$. The real-valued gamma-matrices generate the corresponding Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ of the spin bundle $\mathrm{spin}(M)$ of M, and ∇_μ is the usual covariant derivative constructed from the Levi-Civita spin-connection of M. Notice how these latter two dependences of D characterize the Riemannian geometry of M, and that, in this example, D acts densely on the Hilbert space, i.e. it maps a dense subspace of $\mathcal{H} = L^2(\mathrm{spin}(M))$ into itself. Zero-forms are just complex-valued functions, while we can represent an exact one-form as

$$\pi_D(\partial_\mu f\, dx^\mu) = i\partial_\mu f \gamma^\mu \tag{2.34}$$

so that a generic one-form is represented as $f_\mu \gamma^\mu$, i.e. $\Omega^1_D\mathcal{A}$ is a free $\mathcal{A}$-module with basis $\{\gamma^\mu\}$. As we build two-forms we run into the problem of junk forms. Consider the form

¶The beginning of this subsection is a very simplified version of section 6.2.1 of [18].

$\alpha = f df - (df)f$. As a form in the universal algebra it is non-zero. Its representation does, however, vanish:

$$\pi_D(\alpha) = \pi_D(f df - (df)f) = i\gamma^\mu(f\partial_\mu f - (\partial_\mu f)f) = 0 \tag{2.35}$$

while the representation of its differential does not vanish:

$$\pi_D(d\alpha) = -\gamma^\mu\gamma^\nu\partial_\mu f\partial_\nu f = -2g^{\mu\nu}\partial_\mu f\partial_\nu f \tag{2.36}$$

We can therefore identify junk forms as the symmetric part of the product of two one-forms. This has to be quotiented out leaving only the antisymmetric part, so that a generic two-form is represented as $f_{\mu\nu}\gamma^{\mu\nu}$ where $\gamma^{\mu\nu} = (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)/2$, i.e. $\Omega^2_D\mathcal{A}$ is a free $\mathcal{A}$-module with basis $\{\gamma^{\mu\nu}\}$. Analogously one constructs higher-degree forms by antisymmetrizations of the γ's. Forms of degree $p+1$ or higher are all junk forms.

A connection is a generic one-form $\rho = A_\mu\gamma^\mu$ and the curvature defined in (2.31) is the familiar Maxwell tensor:

$$\theta = \tfrac{1}{2}F_{\mu\nu}\gamma^{\mu\nu} \tag{2.37}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Dixmier trace reduces to the usual spacetime integral along with a trace over spinor indices.

Note that in this case arbitrary iterated commutators of the form $[D,[D,[\cdots,[D,a]\cdots]$ are bounded only on the dense subalgebra $C^\infty(M) \subset C(M)$. Thus to ensure boundedness of all operators under consideration, one should restrict attention to the $*$-algebra $C^\infty(M)$, rather than the C^*-algebra $C(M)$. As mentioned before, nothing is lost in such a restriction, and therefore in the following we shall for the most part not be concerned with the completeness of the algebra in the spectral triple. Notice also that $C^\infty(M)$ acts densely on $\mathcal{H}$.

Fermionic and Bosonic Actions; The Standard Model

It is also possible to define the action of a 'noncommutative gauge theory'. The fermionic action is defined as

$$S_F = \mathrm{tr}_\omega\ \psi^\dagger D_\rho\psi = \langle\psi|D_\rho|\psi\rangle \tag{2.38}$$

where the last equality gives the usual inner product on the Hilbert space of square-integrable spinors, while the bosonic action is

$$S_B = \mathrm{tr}_\omega\ \theta^2 \tag{2.39}$$

Other forms of the bosonic action have been introduced more recently in [11, 26]. There is also another version of gauge theories in noncommutative geometry which employs Lie superalgebras [27].

The fermionic and bosonic actions for the abelian case described in the example above combine into the Dirac-Maxwell action for quantum electrodynamics. But the most interesting perspective is of course the generalization to the case of noncommutative algebras. Connes and Lott [28] have shown how the application of this machinery gives a simple generalization of spacetime to a two-sheeted spacetime, described by the algebra of functions with values in the space of diagonal 2×2 matrices, that has a bosonic action which naturally gives not only a Yang-Mills theory, but also the Higgs potential with its biquadratic form. Connes [29] (for a review including more recent developments see [30]) has generalized the model to the full standard model with a noncommutative algebra

$$\mathcal{A}_{SM} = C^{\infty}(M) \otimes [\mathbb{C} \oplus \mathbb{H} \oplus M(3, \mathbb{C})] \tag{2.40}$$

where $\mathbb{H}$ is the algebra of quaternions and $M(3, \mathbb{C})$ is the algebra of 3×3 complex-valued matrices. The unimodular group of this algebra is the familiar gauge group $U(1) \times SU(2) \times SU(3)$ of the standard model. The action contains the Yang-Mills action and the Higgs term. The input parameters are the masses of all fermions and the coupling constants of the gauge group, while the (classical) mass of the Higgs boson is a prediction of the model. Other predictions for masses at current energies can also be made [31]. The actions of [11, 26] also enable the introduction of gravity into the model.

3. Quantum Spacetimes and the Fröhlich-Gawędzki Construction

In this section we will now try to generalize the material developed in the previous section to the generalizations of spaces suggested by noncommutative geometry. We first discuss some generalities on noncommutative spaces, leading up to the spacetime of string theory. We then begin the construction of the spectral triples pertinent to string theory, as suggested by the work of Fröhlich and Gawędzki [7]. In the latter half of this section we will present a systematic construction of the string Hilbert space, following [7] for the most part, by analysing in detail the basic quantum fields which will compose the quantum spacetime.

Noncommutative Spaces

There are various examples of noncommutative spaces which are close in character to the context which will be described in the following. Some of them are the *Fuzzy Sphere* [32], *noncommutative lattices* [22], and the possibility of having a spacetime with noncommuting coordinates as described in [4, 5, 33]. Quantum groups [34] are noncommutative spaces, in the sense that the algebra of functions defined on them is noncommutative, while another example is the quantum plane [35].

What is common to this variety of noncommutative spaces is that they are described by the noncommutative algebra defined upon them. It might in general be a misnomer to call them "spaces", in the sense that the concept of point is not appropriate. The problem in dealing with noncommutative C^*-algebras is that the various sets described in the previous section, of maximal ideals, primitive ideals, irreducible representations, characters, and pure states, and the topologies that these concepts induce, are no longer equivalent. The identification of the points of the space therefore becomes ambiguous, and has to be abandoned. However, another important feature, which is common to these examples, is that there exists some regime, usually when a scale goes to zero, in which it is possible to recognize a topological space. Such a "low energy" regime is of course necessary if one wants to identify at some level the space in which we live and do experiments.

In string theory a low energy regime is one in which no excited, vibrational states of the string are present and, in the case of closed strings, no string modes wind around the spacetime. In this regime the theory is well described by an ordinary (point particle) quantum field theory. In this respect we look for a noncommutative algebra which describes the "space" of interacting strings. *Vertex operators* were originally introduced in string theory to describe the interactions of strings. They operate on the Hilbert space of strings as insertions on the worldsheet corresponding to the emission or absorption of string states. The "coordinates" of a string are the Fubini-Veneziano fields X (which we will construct formally in the next subsection), and the basic vertex operators are objects of the form e^{ipX}. These vertex operators act as a basis of a "smeared" set of vertex operators which form a noncommutative algebra. For toroidally compactified strings restrictions are imposed on the momenta p, which have to lie on an even self-dual lattice. We will discuss this algebra in more detail in section 5, and we have described some of the more formal algebraic properties of vertex operator algebras in the appendix. This is the spacetime that was proposed by Fröhlich and Gawędzki in [7]. In their construction the algebra is the vertex operator algebra of the string theory under consideration. The spacetime is thus described by the operator algebra which describes the relations among the quantum fields of the conformal field theory.

String theories come with a scale (the string tension), which is usually set to be of the order of the Planck mass. Usually the dimensions of compactified directions are taken to be of a similar scale, so that "low energy" in this context means to neglect higher (vibrational) excited states of the strings, as well as the non-local states which correspond to the strings winding around a compactified direction. In this limit the theory becomes a theory of point particles, and we should thus expect a commutative spacetime. In fact, a quantum theory of point particles on a spin-manifold M naturally supplies the Hilbert space $\mathcal{H} = L^2(\mathrm{spin}(M))$ of physical states and the commutative C^*-algebra $\mathcal{A} = C^\infty(M)$ of observables. Thus the low-energy limit of the noncommutative string spacetime should be represented by a spectral triple $(C^\infty(M), L^2(\mathrm{spin}(M)), ig^{\mu\nu}\gamma_\mu\nabla_\nu)$ corresponding to an ordinary spacetime manifold M at large distance scales.

Let us briefly remark that there is another approach to the noncommutative spacetime which is inspired by D-brane field theory [9, 17]. An effective low-energy description of open superstrings is provided by supersymmetric $U(N)$ Yang-Mills theory in ten dimensions. Dimensional reduction of this gauge theory down to a $(p+1)$-dimensional manifold M_{p+1} describes the low-energy dynamics of N Dp-branes, with $U(N)$ gauge symmetry. In [9, 17] the spacetime is thus described in analogy with the Connes-Lott formulation of the standard model by choosing as algebra

$$\mathcal{A}_D = C^\infty(M_{p+1}) \otimes M(N, \mathbb{C}) \tag{3.1}$$

This construction is rather different in spirit from the one which we shall present in the following, in that it utilizes a different low-energy regime of the string theory.

Fubini-Veneziano Fields

We now begin the construction of the spectral triple describing the noncommutative spacetime of string theory. From the point of view of the corresponding conformal field theory, it is more natural to start with the Hilbert space of physical states rather than the algebra since the former, with its oscillatory Fock space component, is one of the immediate characterizations of the string theory. We shall take this to be the space of states on which the quantum string configurations act. We consider a linear sigma model with target space the flat n-torus

$$T^n = (S^1)^n \cong \mathbb{R}^n / 2\pi\Gamma \tag{3.2}$$

where Γ is a lattice of rank n with inner product $g_{\mu\nu}$ of Euclidean signature. The action of the model is (in units where $l_P = 1$)

$$S[X] = \frac{1}{2\pi} \int_\Sigma \left[g_{\mu\nu} dX^\mu \wedge \star dX^\nu + 2X^*(\beta) \right] \tag{3.3}$$

where $X^*(\beta) = \frac{1}{2}\beta_{\mu\nu} dX^\mu \wedge dX^\nu$ is the pull-back of the constant two-form β to the world-sheet Σ by the embedding fields $X : \Sigma \to T^n$, and $\star$ denotes the Hodge dual. The kinetic term in (3.3) is defined by the constant Riemannian metric $g \equiv \frac{1}{2} g_{\mu\nu} dX^\mu \otimes dX^\nu$ on T^n, and it leads to local propagating degrees of freedom in the field theory. The second term in (3.3) is the topological instanton term and it depends only on the cohomology class of β. In fact, as we shall show, at the quantum level the two-form β takes values in the torus $\beta \in H^2(T^n; \mathbb{R}) / H^2(T^n; \mathbb{Z})$, where the real cohomology represents the local gauge transformations $\beta \to \beta + d\lambda$ while the integer cohomology represents the large gauge transformations $\beta \to \beta + 4\pi^2 C$ with C a closed two-form with integer periods. We shall find it convenient to introduce the non-singular 'background' matrices

$$[d^\pm_{\mu\nu}] = [g_{\mu\nu} \pm \beta_{\mu\nu}] \tag{3.4}$$

which, when n is even, determine both the complex and Kähler structures of T^n.

To highlight the relevant features of the construction, in this paper we shall consider only the simplest non-trivial case where the Riemann surface Σ is taken to be an infinite cylinder with local coordinates $(\tau, \sigma) \in \mathbb{R} \times S^1$. On a cylinder the action (3.3) can be written in terms of the local worldsheet coordinates as

$$S[X] = \frac{1}{4\pi} \int d\tau \oint d\sigma \; [g_{\mu\nu} \left(\partial_\tau X^\mu \partial_\tau X^\nu - \partial_\sigma X^\mu \partial_\sigma X^\nu\right) - 2\beta_{\mu\nu} \partial_\sigma X^\mu \partial_\tau X^\nu] \tag{3.5}$$

and we assume periodic boundary conditions for the embedding fields $\{X^\mu(\tau, \sigma)\} \in (S^1)^n$ (corresponding to closed strings). Canonical quantization identifies the momentum conjugate to the field $X^\mu(\tau, \sigma)$ as

$$\Pi_\mu = \tfrac{1}{2\pi} \left(g_{\mu\nu} \partial_\tau X^\nu + \beta_{\mu\nu} \partial_\sigma X^\nu\right) \tag{3.6}$$

which leads to the canonical equal-time quantum commutator

$$[X^\mu(\tau, \sigma), \Pi_\nu(\tau, \sigma')] = -i \; \delta^\mu_\nu \; \delta(\sigma, \sigma') \tag{3.7}$$

The one-forms dX^μ have winding numbers w^μ which represent the number of times that the worldsheet circle S^1 wraps around the $\mu^{\rm th}$ circle of the torus $(S^1)^n$, i.e.

$$\frac{1}{2\pi} \oint_{S^1} dX^\mu = w^\mu \qquad \text{with} \qquad \{w^\mu\} \in \Gamma \tag{3.8}$$

These winding numbers define a homotopy invariant of the target space which, as we will see below, label the connected components of the Hilbert space. In addition to the periodicity condition (3.8), we also require that

$$\frac{1}{2\pi} \int_\tau dX^\mu = g^{\mu\nu} p_\nu \qquad \text{with} \qquad \{p_\mu\} \in \Gamma^* \tag{3.9}$$

where Γ^* is the lattice dual to Γ (obtained by joining the centers of all plaquettes of Γ). Equations (3.9) and (3.6) define the momenta p_μ of the winding modes in the target space. The periodicity condition follows from the fact that the translation generator $e^{ip_\mu X^\mu}$ on the target space should be invariant under the windings of $X^\mu \in S^1$. In addition to these momentum windings, from (3.6) and (3.8) it follows that there are also the translations generated by the instanton windings which are given by $\beta_{\mu\nu} w^\nu$.

The closed one-form dX can be written as

$$dX = dY + 2\pi h \tag{3.10}$$

where $Y : \Sigma \to \mathbb{R}^n$ is a single-valued function and $h \in H^1(\Sigma; \Gamma)$ is a harmonic vector-valued one-form, $dh = d \star h = 0$. In the case of the cylinder, we have from (3.8) that $h^\mu = w^\mu d\sigma$. The Euler-Lagrange equations determine the local configurations Y as the solutions of the two-dimensional wave equation

$$\Box Y^\mu(\tau, \sigma) = 0 \tag{3.11}$$

whose solutions are given by meromorphic expansions of the chiral spin-1 currents

$$-i\partial_\pm Y^\mu_\pm(\tau\pm\sigma)=\sum_{k=-\infty}^{\infty}\alpha_k^{(\pm)\mu}\ \mathrm{e}^{i(k+1)(\tau\pm\sigma)} \tag{3.12}$$

with $(\alpha_k^{(\pm)\mu})^\dagger=\alpha_{-k}^{(\pm)\mu}$ and the light-cone derivatives are $\partial_\pm=\partial_\tau\pm\partial_\sigma$.

Combining the above configurations, and splitting the fields up into chiral sectors, we find that the configurations of the sigma-model (3.5) are given by the Fubini-Veneziano fields

$$X^\mu_\pm(\tau\pm\sigma)=x^\mu_\pm+g^{\mu\nu}p^\pm_\nu(\tau\pm\sigma)+\sum_{k\neq 0}\frac{1}{ik}\ \alpha_k^{(\pm)\mu}\ \mathrm{e}^{ik(\tau\pm\sigma)} \tag{3.13}$$

where, from the canonical quantum commutator (3.7), the zero-modes $x^\mu_\pm$ (the center of mass coordinates of the string) and the (center of mass) momenta $p^\pm_\mu$ are canonically conjugate variables,

$$[x^\mu_\pm,p^\pm_\nu]=-i\delta^\mu_\nu \tag{3.14}$$

with all other commutators vanishing. The left-right momenta are

$$p^\pm_\mu=\tfrac{1}{\sqrt{2}}\left(p_\mu\pm d^\pm_{\mu\nu}w^\nu\right) \tag{3.15}$$

The set of momenta $\{(p^+_\mu,p^-_\mu)\}$ along with the integer-valued quadratic form

$$\langle p,q\rangle_\Lambda\equiv p^+_\mu g^{\mu\nu}q^+_\nu-p^-_\mu g^{\mu\nu}q^-_\nu=p_\mu v^\mu+q_\mu w^\mu \quad , \tag{3.16}$$

where $q^\pm_\mu=\frac{1}{\sqrt{2}}(q_\mu\pm d^\pm_{\mu\nu}v^\nu)$, form an even self-dual Lorentzian lattice

$$\Lambda=\Gamma^*\oplus\Gamma \tag{3.17}$$

of rank $2n$ and signature (n,n) which is called the Narain lattice [36]. The even and self-duality properties, $\langle p,p\rangle_\Lambda=2p_\mu w^\mu\in 2\mathbb{Z}$ and $\Lambda^*=\Lambda$, guarantee modular invariance of the worldsheet theory [37]. The functions (3.13) define chiral multi-valued quantum fields of the sigma-model. The oscillatory modes $\alpha_k^{(\pm)\mu}$ in (3.13) yield bosonic creation and annihilation operators (acting on some vacuum states $|0\rangle_\pm$) with the non-vanishing commutation relations (see (3.7))

$$\left[\alpha_k^{(\pm)\mu},\alpha_m^{(\pm)\nu}\right]=k\ g^{\mu\nu}\ \delta_{k+m,0} \tag{3.18}$$

where $g^{\mu\lambda}g_{\lambda\nu}=\delta^\mu_\nu$.

The Hilbert space of states of this quantum field theory is thus

$$\mathcal{H}_X=L^2((S^1)^n,\textstyle\prod_{\mu=1}^n\frac{dx^\mu}{\sqrt{2\pi}})^\Gamma\otimes\mathcal{F}^+\otimes\mathcal{F}^- \tag{3.19}$$

where

$$L^2((S^1)^n,\textstyle\prod_{\mu=1}^n\frac{dx^\mu}{\sqrt{2\pi}})^\Gamma=\displaystyle\bigoplus_{\{w^\mu\}\in\Gamma}L^2((S^1)^n,\textstyle\prod_{\mu=1}^n\frac{dx^\mu}{\sqrt{2\pi}}) \tag{3.20}$$

with $L^2((S^1)^n,\prod_{\mu=1}^n \frac{dx^\mu}{\sqrt{2\pi}})$ the space of square integrable functions on T^n with its Riemannian volume form. Periodic functions of $x^\mu=\frac{1}{\sqrt{2}}(x_+^\mu+x_-^\mu)$ act in (3.20) by multiplication and p_μ as the derivative operator $i\frac{\partial}{\partial x^\mu}$ in each L^2-component labelled by the winding numbers w^μ. The Hilbert space (3.20) is spanned by the eigenvectors $|p^+;p^-\rangle=\mathrm{e}^{-ip_\mu x^\mu}$ of $i\frac{\partial}{\partial x^\mu}$ in each component of the direct sum. The spaces $\mathcal{F}^\pm$ are two commuting copies of the bosonic Fock space

$$\mathcal{F}^\pm=\bigoplus_{k>0}<\ \alpha_{n_1}^{(\pm)\mu_1}\cdots\alpha_{n_k}^{(\pm)\mu_k}|0\rangle_\pm\ |\ n_j<0\quad> \tag{3.21}$$

built on the vacuum states $|0\rangle_\pm$ with $\alpha_k^{(\pm)\mu}|0\rangle_\pm=0$ for $k>0$. The unique vacuum state of $\mathcal{H}_X$ is

$$|\mathrm{vac}\rangle\equiv|0;0\rangle\otimes|0\rangle_+\otimes|0\rangle_- \tag{3.22}$$

with $p_\mu^\pm|\mathrm{vac}\rangle=\alpha_k^{(\pm)\mu}|\mathrm{vac}\rangle=0\ \ \forall k>0$.

4. Spacetime Symmetries and Dirac Operator

In this section we shall introduce a Dirac operator that will yield the Riemannian geometry of the eventual string spacetime. The Dirac operator that we construct is related to the two fundamental, infinite-dimensional continuous symmetries of the conformal field theory (3.5). We shall also briefly show how this Dirac operator is related to some previous approaches to describing the geometrical and topological properties of a spacetime using supersymmetric sigma-models. Superconformal field theories have been emphasized recently as the correct field theoretical structure for the description of stringy spacetimes in the framework of noncommutative geometry [7]–[10],[38].

Gauge Symmetry and the Dirac-Ramond Operator

The first fundamental symmetry of the string theory is the target space reparametrization symmetry $X_\pm^\mu(\tau\pm\sigma)\to X_\pm^\mu(\tau\pm\sigma)+\delta X_\pm^\mu(\tau\pm\sigma)$, where $\delta X_\pm^\mu(\tau\pm\sigma)$ are arbitrary periodic functions. Varying the action (3.5) shows that this symmetry is generated on the Hilbert space (3.19) by the $u(1)_+^n\oplus u(1)_-^n$ Kac-Moody algebra at level 2 with conserved currents

$$J_\pm^\mu(\tau\pm\sigma)=\partial_\pm X_\pm^\mu(\tau\pm\sigma)=\sum_{k=-\infty}^{\infty}\alpha_k^{(\pm)\mu}\ \mathrm{e}^{ik(\tau\pm\sigma)} \tag{4.1}$$

obeying the commutation relations (3.18), where we have defined $\alpha_0^{(\pm)\mu}\equiv g^{\mu\nu}p_\nu^\pm$. Then the Hilbert space (3.19) is a direct sum of the irreducible highest-weight representations of the current algebra acting in $|p^+;p^-\rangle\otimes\mathcal{F}^+\otimes\mathcal{F}^-$ and labelled by the $U(1)_\pm^n$ charges $p_\mu^\pm$.

The currents (4.1) can be used to define the Dirac operator which describes the DeRham cohomology and Riemannian geometry of the effective spacetime of the sigma-model (3.5). For this, we endow the toroidal spacetime T^n with a spin structure. Note that there are 2^n possibilities corresponding to a choice of Neveu-Schwarz or Ramond fermionic boundary conditions around each of the n circles of T^n. We then introduce two anti-commuting copies of the spin(n) Clifford algebra $\mathcal{C}(T^n)^\pm$ whose corresponding Dirac generators $\gamma_\mu^\pm = (\gamma_\mu^\pm)^*$ obey the non-vanishing anti-commutation relations

$$\left\{\gamma_\mu^\pm, \gamma_\nu^\pm\right\} = 2g_{\mu\nu} \tag{4.2}$$

To define the appropriate Dirac operator we must first enlarge the Hilbert space (3.19) to include the spin structure. Thus replace $\mathcal{H}_X$ by

$$\mathcal{H} = \bigoplus_{S[\mathcal{C}(T^n)]} L^2(\mathrm{spin}(T^n))^\Gamma \otimes \mathcal{F}^+ \otimes \mathcal{F}^- \tag{4.3}$$

where

$$L^2(\mathrm{spin}(T^n))^\Gamma = \bigoplus_{\{w^\mu\}\in\Gamma} S_{\{w^\mu\}}[\mathcal{C}(T^n)] \otimes L^2((S^1)^n, \textstyle\prod_{\mu=1}^n \frac{dx^\mu}{\sqrt{2\pi}}) \tag{4.4}$$

is (a local trivialization of) the space of square integrable spinors, i.e. L^2-sections of the spin bundle spin(T^n) of the n-torus, with $S_{\{w^\mu\}}[\mathcal{C}(T^n)]$ unitary irreducible representations of the double Clifford algebra $\mathcal{C}(T^n) = \mathcal{C}(T^n)^+ \oplus \mathcal{C}(T^n)^-$. These modules have the form

$$S_{\{w^\mu\}}[\mathcal{C}(T^n)] = \begin{cases} S_{\{w^\mu\}}[\mathcal{C}(T^n)]^+ \otimes S_{\{w^\mu\}}[\mathcal{C}(T^n)]^- & \text{for } n \text{ even} \\ S_{\{w^\mu\}}[\mathcal{C}(T^n)]^+ \otimes S_{\{w^\mu\}}[\mathcal{C}(T^n)]^- \otimes \mathbb{C}^2 & \text{for } n \text{ odd} \end{cases} \tag{4.5}$$

where the bundle of the spinors contains both chiralities of fermion fields in the even-dimensional case. These various representations of the Clifford algebra need not be the same, but, to simplify notation in the following, we shall typically omit the explicit representation labels for the spinor parts of the Hilbert space (4.3).

We now define two anti-commuting Dirac operators acting on the Hilbert space (4.3) by

$$\not{D}^\pm(\tau \pm \sigma) = \sqrt{2}\,\gamma_\mu^\pm \otimes J_\pm^\mu(\tau \pm \sigma) = \sum_{k=-\infty}^{\infty} \not{D}_k^\pm\ \mathrm{e}^{ik(\tau\pm\sigma)} \tag{4.6}$$

where

$$\not{D}_k^\pm = \sqrt{2}\,\gamma_\mu^\pm \otimes \alpha_k^{(\pm)\mu} \tag{4.7}$$

That these Dirac operators are appropriate for the target space geometry can be seen by noting that $J_\pm^\mu \sim \frac{\delta}{\delta X_\pm^\mu}$ when acting on spacetime functions of $X_\pm^\mu$. Furthermore, as we shall see, their squares determine the appropriate Laplace-Beltrami operator for the target space geometry which can also be found from conventional conformal field theory.

The Dirac operator introduced above is a low-energy limit of Witten's Dirac-Ramond operator for the full superstring theory corresponding to the two-dimensional $N = 1$

supersymmetric sigma-model [39, 40]. For the simple linear sigma-model under consideration here, this field theory is obtained by adding to the action (3.5) a free fermion term, so that the total action is

$$S[X,\psi]=S[X]+\frac{i}{2\pi}\int d\tau\ \oint d\sigma\ g_{\mu\nu}\left(\psi_+^\mu\partial_-\psi_+^\nu+\psi_-^\mu\partial_+\psi_-^\nu\right) \tag{4.8}$$

where $\psi_\pm^\mu(\tau,\sigma)$ are Majorana spinor fields. Varying (4.8) we find that the worldsheet supersymmetry generators are the $N=1$ supercharges

$$Q^\pm=\tfrac{1}{\sqrt{2}}\ g_{\mu\nu}\ \psi_\pm^\mu\partial_\pm X^\nu \tag{4.9}$$

and canonical quantization of the action (4.8) leads to the non-vanishing equal-time canonical anti-commutators

$$\left\{\psi_\pm^\mu(\tau,\sigma),\psi_\pm^\nu(\tau,\sigma')\right\}=g^{\mu\nu}\ \delta(\sigma,\sigma') \tag{4.10}$$

The equations of motion $\partial_-\psi_+^\mu=\partial_+\psi_-^\mu=0$ for the fermion fields imply that they have the Weyl mode decompositions

$$\psi_\pm^{(\epsilon)\mu}(\tau\pm\sigma)=\sum_{k\in\mathbb{Z}+\epsilon}\psi_k^{(\pm)\mu}\ \mathrm{e}^{i(k+1/2)(\tau\pm\sigma)} \tag{4.11}$$

so that the supercharges (4.9) are also Majorana-Weyl spinor fields, i.e. $\partial_\mp Q^\pm=0$. Here $(\psi_k^{(\pm)\mu})^\dagger=\psi_{-k}^{(\pm)\mu}$, and $\epsilon=\frac{1}{2}$ for Neveu-Schwarz boundary conditions while $\epsilon=0$ for Ramond boundary conditions corresponding to the two possible choices of spin structure on the worldsheet circle S^1. The canonical anti-commutator (4.10) implies that the fermionic modes obey the non-vanishing anti-commutation relations

$$\left\{\psi_k^{(\pm)\mu},\psi_m^{(\pm)\nu}\right\}=g^{\mu\nu}\ \delta_{k+m,0} \tag{4.12}$$

In the Ramond sector the fermionic zero modes $\psi_0^{(\pm)\mu}$ generate the Clifford algebras (4.2) and thus coincide with the gamma-matrices introduced above in specific irreducible representations $S_F[\mathcal{C}(T^n)]^\pm$, i.e.

$$\gamma_\mu^\pm=\tfrac{1}{\sqrt{2}}\ g_{\mu\nu}\,\psi_0^{(\pm)\nu} \tag{4.13}$$

The Ramond Fock space for the fermion fields is thus built

$$\mathcal{F}_{\mathrm{R}}^\pm=\bigoplus_{k>0}<\ \psi_{n_1}^{(\pm)\mu_1}\cdots\psi_{n_k}^{(\pm)\mu_k}|\Psi_\pm\rangle\ |\ n_j\in\mathbb{Z}^-\ ,\ \Psi_\pm\in S_F[\mathcal{C}(T^n)]^\pm\ > \tag{4.14}$$

on the Clifford vacua $|\Psi_\pm\rangle$ with $\psi_k^{(\pm)\mu}|\Psi_\pm\rangle=0$ for $k>0$. The Neveu-Schwarz Fock space is built

$$\mathcal{F}_{\mathrm{NS}}^\pm=\bigoplus_{k>0}<\ \psi_{n_1}^{(\pm)\mu_1}\cdots\psi_{n_k}^{(\pm)\mu_k}|0\rangle_\pm^F\ |\ n_j\in\mathbb{Z}^-+\tfrac{1}{2}\ > \tag{4.15}$$

on the vacuum states $|0\rangle_\pm^F$ with $\psi_k^{(\pm)\mu}|0\rangle_\pm^F=0$ for $k>0$. The total fermionic Hilbert space of the model is the Fock space

$$\mathcal{F}_F=\mathcal{F}_F^+\otimes\mathcal{F}_F^-\qquad\text{with}\quad\mathcal{F}_F^\pm=\mathcal{F}_{\mathrm{R}}^\pm\oplus\mathcal{F}_{\mathrm{NS}}^\pm \tag{4.16}$$

and the total Hilbert space of the supersymmetric sigma-model (4.8) is

$$\mathcal{H}_{X,\psi} = \mathcal{F}_F \otimes \mathcal{H}_X \tag{4.17}$$

where the bosonic Hilbert space $\mathcal{H}_X$ is defined in (3.19).

In the Ramond sector the quantized fermionic zero-modes of the worldsheet supercharges (4.9) coincide with the generalized Dirac operators (4.6) acting in the representation sector of the Hilbert space (4.3) determined by the fermion fields. More precisely, we define a mode expansion $Q^{\pm}_{\epsilon}(\tau\pm\sigma) = \sum_{k\in\mathbb{Z}+\epsilon} Q^{\pm}_{k,\epsilon}\ \mathrm{e}^{i(k+1/2)(\tau\pm\sigma)}$ of the worldsheet supercharges with the operators

$$Q^{\pm}_{k,\epsilon} = \sum_{m\in\mathbb{Z}+\epsilon} g_{\mu\nu}\ \psi^{(\pm)\mu}_{m}\alpha^{(\pm)\nu}_{k-m} \tag{4.18}$$

acting on the full Hilbert space (4.17) of the supersymmetric sigma-model. The orthogonal projection of the Hilbert space $\mathcal{H}_{X,\psi}$, with projector $\mathcal{P}^{(0)}_{\mathrm{R}}$, onto the fermionic zero-modes is the subspace

$$\mathcal{H}^{(0)}_{\mathrm{R}} \equiv \mathcal{P}^{(0)}_{\mathrm{R}}\mathcal{H}_{X,\psi} = \mathcal{F}^{(0)+}_{\mathrm{R}} \otimes \mathcal{F}^{(0)-}_{\mathrm{R}} \otimes \mathcal{H}_X \tag{4.19}$$

where

$$\mathcal{F}^{(0)\pm}_{\mathrm{R}} = \Big\{ |\Psi_{\pm}\rangle \ \big|\ \Psi_{\pm} \in S_F[\mathcal{C}(T^n)]^{\pm} \Big\} \cong S_F[\mathcal{C}(T^n)]^{\pm} \tag{4.20}$$

From (4.13) it follows that under this orthogonal projection the Hilbert space (4.17) reduces to (4.3) and the supersymmetry charges (4.9) coincide with the Dirac operators (4.6) in the representation $S_F[\mathcal{C}(T^n)]$ of the Clifford algebra determined by the fermion fields,

$$\mathcal{H}\ \big|_{S_F[\mathcal{C}(T^n)]} \cong \mathcal{H}^{(0)}_{\mathrm{R}} \qquad , \qquad \not{D}^{\pm}_{k} = \mathcal{P}^{(0)}_{\mathrm{R}}\, Q^{\pm}_{k,0}\, \mathcal{P}^{(0)}_{\mathrm{R}} \tag{4.21}$$

Note that the analogous projection onto the Neveu-Schwarz sector of (4.17),

$$\mathcal{H}^{(0)}_{\mathrm{NS}} = |0\rangle^{F}_{+} \otimes |0\rangle^{F}_{-} \otimes \mathcal{H}_X \cong \mathcal{H}_X \tag{4.22}$$

projects out the original bosonic Hilbert space (3.19). This supersymmetric construction thus exhibits an algebraic, field-theoretical origin for the Dirac operators introduced above.

Conformal Symmetry and the Witten Complex

The other basic symmetry that the field theory (3.5) possesses is worldsheet conformal invariance under transformations which act by reparametrizations of the light-cone coordinates $\tau\pm\sigma$. At the quantum level, this symmetry is represented on the spin-extended Hilbert space (4.3) by a commuting pair of Virasoro algebras with the conserved stress-energy tensors

$$\begin{aligned} T^{\pm}(\tau\pm\sigma) &= -\tfrac{1}{2}\ :\not{D}^{\pm}(\tau\pm\sigma)^2: = -\tfrac{1}{2}\ \mathbb{I}\otimes g_{\mu\nu}\ :\partial_{\pm}X^{\mu}_{\pm}(\tau\pm\sigma)\partial_{\pm}X^{\nu}_{\pm}(\tau\pm\sigma): \\ &= \sum_{k=-\infty}^{\infty} \mathbb{I}\otimes L^{\pm}_{k}\ \mathrm{e}^{i(k+2)(\tau\pm\sigma)} \end{aligned} \tag{4.23}$$

where $\mathbb{I}$ is the identity operator and the Sugawara-Virasoro generators

$$L_k^\pm = \frac{1}{2} \sum_{m=-\infty}^{\infty} g_{\mu\nu} \ :\alpha_m^{(\pm)\mu} \alpha_{k-m}^{(\pm)\nu}: \tag{4.24}$$

act on the bosonic Hilbert space (3.19). The Wick normal ordering is defined by

$$\begin{aligned} :\alpha_k^{(\pm)\mu} \alpha_m^{(\pm)\nu}: &= \alpha_k^{(\pm)\mu} \alpha_m^{(\pm)\nu} \quad \text{for} \quad k < m \\ &= \alpha_m^{(\pm)\nu} \alpha_k^{(\pm)\mu} \quad \text{for} \quad k > m \end{aligned} \tag{4.25}$$

and also by putting the operators $x_\pm^\mu$ to the left of $p_\mu^\pm$. The operators (4.24) generate the Virasoro algebra

$$\left[L_k^\pm, L_m^\pm\right] = (k-m) L_{k+m}^\pm + \tfrac{c}{12}\left(k^3 - k\right) \delta_{k+m,0} \tag{4.26}$$

with central charge $c = n$, the dimension of the toroidal spacetime.

Using these Virasoro operators we can construct the global spacetime symmetry generators of the Poincaré algebra. The momentum operator generating space translations is $P = L_0^+ - L_0^-$. The Hamiltonian operator is $H = L_0^+ + L_0^- - \frac{c}{12}$, which can be written explicitly as

$$H = \frac{1}{2} g^{\mu\nu} p_\mu^+ p_\nu^+ + \frac{1}{2} g^{\mu\nu} p_\mu^- p_\nu^- + \sum_{k>0} g_{\mu\nu} \ \alpha_{-k}^{(+)\mu} \alpha_k^{(+)\nu} + \sum_{k>0} g_{\mu\nu} \ \alpha_{-k}^{(-)\mu} \alpha_k^{(-)\nu} - \frac{n}{12} \tag{4.27}$$

The Hamiltonian (4.27) has unique vacuum eigenstate (3.22) (with eigenvalue $-\frac{n}{12}$) at the bottom of its spectrum. It determines the Laplace-Beltrami operator of the Riemannian geometry of the effective string spacetime [7], and it is the square of the Dirac operator (4.6) in the sense of (4.23). This geometrical property can be made somewhat more precise by turning to the supersymmetric sigma-model (4.8). The fermion fields also generate two commuting Virasoro algebras of central charge $c = n/2$ defined by the fermionic stress-energy tensors

$$t_\epsilon^\pm(\tau \pm \sigma) = -\frac{1}{2} \ g_{\mu\nu} \ :\psi_\pm^{(\epsilon)\mu}(\tau \pm \sigma) \partial_\mp \psi_\pm^{(\epsilon)\nu}(\tau \pm \sigma): = \sum_{k \in \mathbb{Z}+\epsilon} \ell_{k,\epsilon}^\pm \ \mathrm{e}^{i(k+2)(\tau \pm \sigma)} \tag{4.28}$$

where the Virasoro operators are

$$\ell_{k,\epsilon}^\pm = -\frac{1}{2} \sum_{m \in \mathbb{Z}+\epsilon} m \ g_{\mu\nu} \ :\psi_m^{(\pm)\mu} \psi_{k-m}^{(\pm)\nu}: + \frac{3n}{48} \ \delta_{\epsilon,0} \delta_{k,0} \tag{4.29}$$

The supersymmetric modes (4.18) together with the Virasoro generators

$$\mathcal{L}_{k,\epsilon}^\pm = L_k^\pm + \ell_{k,\epsilon}^\pm \tag{4.30}$$

generate the $N = 1$ supersymmetric extension of the $c = 3n/2$ Virasoro algebra (4.26),

$$\begin{aligned} \left[\mathcal{L}_{k,\epsilon}^\pm, Q_{m,\epsilon}^\pm\right] &= \left(\tfrac{k}{2} - m\right) Q_{k+m,\epsilon}^\pm \\ \left\{Q_{k,\epsilon}^\pm, Q_{m,\epsilon}^\pm\right\} &= 2\mathcal{L}_{k+m,\epsilon}^\pm + \tfrac{c}{3}\left(k^2 - \tfrac{1}{4}\right) \delta_{k+m,0} \\ \left[Q_{k,\epsilon}^\pm, \mathcal{L}_{m,\epsilon}^\mp\right] &= \left\{Q_{k,\epsilon}^\pm, Q_{m,\epsilon}^\mp\right\} = 0 \end{aligned} \tag{4.31}$$

In particular, in the Ramond sector we have $(Q^\pm_{0,0})^2=\mathcal{L}^\pm_{0,0}-\frac{c}{24}$ and $\{Q^+_{0,0},Q^-_{0,0}\}=0$ which defines the global supersymmetry algebra associated with the spacetime Poincaré group. Thus the spacetime Poincaré generators of the bosonic sigma-model (3.5) are given by the projections

$$\mathbb{I}\otimes(P\pm H)=\tfrac{1}{2}\mathcal{P}^{(0)}_{\rm R}\,(Q^\pm_{0,0})^2\,\mathcal{P}^{(0)}_{\rm R} \tag{4.32}$$

onto the spin-extended Hilbert space (4.3). The operators $(Q^\pm_{0,0})^2$ also annihilate all states of the form $|\Psi_\pm\rangle\otimes|{\rm vac}\rangle$, so that

$$\ker Q^\pm_{0,0}\cong S_F[\mathcal{C}(T^n)]^\pm \tag{4.33}$$

Thus the global supersymmetry is unbroken and there exists a whole family of supersymmetric ground states of the field theory (4.8). When restricted to Hilbert space states of spacetime momentum $P=0$, the fields of (4.8) generate the DeRham complex of the target space T^n [39]. The $P=0$ projected supercharges $\mathcal{P}_{P=0}Q_0\mathcal{P}_{P=0}$ and $\mathcal{P}_{P=0}\bar{Q}_0\mathcal{P}_{P=0}$, with

$$Q_0=\tfrac{1}{\sqrt2}\left(Q^+_{0,0}+Q^-_{0,0}\right)\qquad,\qquad \bar{Q}_0=\tfrac{1}{\sqrt2}\left(Q^+_{0,0}-Q^-_{0,0}\right) \tag{4.34}$$

realize the exterior derivative d and the co-derivative $\star{\rm d}\star$, respectively, when acting on the Ramond sector of the Hilbert space of the supersymmetric sigma-model. Moreover, the projected fermionic zero-modes $\mathcal{P}_{P=0}\psi^\mu\mathcal{P}_{P=0}$ and $\mathcal{P}_{P=0}\bar\psi^\mu\mathcal{P}_{P=0}$, with

$$\psi^\mu=\psi^{(+)\mu}_{0,0}+\psi^{(-)\mu}_{0,0}\qquad,\qquad \bar\psi^\mu=\psi^{(+)\mu}_{0,0}-\psi^{(-)\mu}_{0,0} \tag{4.35}$$

correspond, respectively, to basis differential one-forms and basis vector fields, and the Poincaré-Hodge duality is realized by (Hermitian) conjugation in the sense of mappings $\psi^\pm\to\pm\psi^\pm$ of left and right chiral sectors in the light-cone parametrizations above.

5. Vertex Operator Algebra

We finally introduce an appropriate operator algebra acting on the Hilbert space (3.19) which will give the necessary topology and differentiable structure to the string spacetime. We want to use an algebra that acts on (3.19) densely, i.e. it maps a dense subspace of $\mathcal{H}_X$ into itself, so as to capture the full structure of the string spacetime as determined by the Fubini-Veneziano fields (3.13). For this, we now introduce the basic (single-valued) quantum fields of the sigma-model. We define the holomorphic coordinates $z_\pm={\rm e}^{-i(\tau\pm\sigma)}$, which after a Wick rotation of the worldsheet temporal coordinate maps the cylinder onto the complex plane. We then define the mutually local holomorphic and anti-holomorphic vertex operators

$$V_{q^\pm}(z_\pm)=\,:{\rm e}^{-iq^\pm_\mu X^\mu_\pm(\tau\pm\sigma)}: \tag{5.1}$$

where single-valuedness restricts their momenta to

$$q^\pm_\mu=\tfrac{1}{\sqrt2}\left(q_\mu\pm d^\pm_{\mu\nu}v^\nu\right)\qquad\text{with}\quad \{q_\mu\}\in\Gamma^*,\{v^\nu\}\in\Gamma \tag{5.2}$$

so that $(q^+, q^-) \in \Lambda$. The complete, left-right symmetric local vertex operators of the conformal field theory are

$$V_{q^+q^-}(z_+, z_-) = c_{q^+q^-}(p^+, p^-) V_{q^+}(z_+) V_{q^-}(z_-) = c_{q^+q^-}(p^+, p^-) : \mathrm{e}^{-iq_\mu^+ X_+^\mu(\tau+\sigma) - iq_\mu^- X_-^\mu(\tau-\sigma)} : \tag{5.3}$$

The operator-valued phases

$$c_{q^+q^-}(p^+, p^-) = (-1)^{[(d^+)^{\lambda\rho} q_\rho^+ + (d^-)^{\lambda\rho} q_\rho^-] d_{\lambda\mu} g^{\mu\nu} [p_\nu^+ - p_\nu^-]} = (-1)^{q_\mu w^\mu} \tag{5.4}$$

where $d = [(d^+)^{-1} + (d^-)^{-1}]^{-1}$, are 2-cocycles of the lattice algebra generated by the complexification $\Lambda^c = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, and they are inserted to correct the algebraic transformation properties (both gauge and conformal) of the vertex operators. They also enable the vertex operator construction of affine Kac-Moody algebras [41].

The local vertex operators generate an important algebraic structure which we now describe in some detail. We consider the n-fold Heisenberg-Weyl operator algebra spanned by the oscillator modes,

$$\hat{h}_\pm = \left\{ q_\mu^\pm \alpha_m^{(\pm)\mu} \;\Big|\; (q^+, q^-) \in \Lambda \;,\; m \in \mathbb{Z} \right\} \tag{5.5}$$

and the algebra of polynomials on the bosonic creation operators which is the symmetric vector space

$$S(\hat{h}_\pm^{(-)}) = \bigoplus_{k>0} \left\{ \prod_{i=1}^k q_\mu^{(i)\pm} \alpha_{-m_i}^{(\pm)\mu} \;\Big|\; (q^{(i)+}, q^{(i)-}) \in \Lambda \;,\; m_i > 0 \right\} \tag{5.6}$$

Next we consider the group algebra $\mathbb{C}[\Lambda] = \mathbb{C}[\Lambda]^+ \times \mathbb{C}[\Lambda]^-$ of the complexified lattice Λ^c, which is the abelian group generated by the translation operators $\mathrm{e}^{iq_\mu^\pm x_\pm^\mu}$, $(q^+, q^-) \in \Lambda$. Because of the inclusion of the lattice cocycles in the definition of the complete vertex operators (5.3), we "twist" these group algebra generators by defining the operators

$$\varepsilon_{q^+q^-} \equiv \mathrm{e}^{iq_\mu^+ x_+^\mu + iq_\mu^- x_-^\mu} \; c_{q^+q^-}(p^+, p^-) \tag{5.7}$$

Using the Baker-Campbell-Hausdorff formula it is straightforward to show that

$$\varepsilon_{q^+q^-} \; \varepsilon_{r^+r^-} = c_{q^+q^-}(r^+, r^-) \; \varepsilon_{(q^++r^+)(q^-+r^-)} \tag{5.8}$$

and hence that the operators (5.7) generate the clock algebra

$$\varepsilon_{q^+q^-} \; \varepsilon_{r^+r^-} = (-1)^{\langle q,r \rangle_\Lambda} \; \varepsilon_{r^+r^-} \; \varepsilon_{q^+q^-} \tag{5.9}$$

with the 2-cocycles $(-1)^{\langle q,r \rangle_\Lambda}$ in the lattice algebra generated by Λ^c.

Notice that the 2-cocycles (5.4) are maps $c : \Lambda^c \oplus \Lambda^c \to \mathbb{Z}_2$. This means that the twisted operators (5.7) generate the group algebra of the double cover $\hat{\Lambda}^c = \mathbb{Z}_2 \times \Lambda^c$ of the complexified lattice Λ^c. The multiplication in the lattice algebra of $\hat{\Lambda}^c$ is $(\rho; q^+, q^-) \cdot (\sigma; r^+, r^-) = (c_{q^+q^-}(r^+, r^-)\rho\sigma; q^+ + r^+, q^- + r^-)$ for $\rho, \sigma \in \mathbb{Z}_2$ and $(q^+, q^-), (r^+, r^-) \in \Lambda^c$.

Thus taking the twisted group algebra $\mathbb{C}\{\Lambda\}$ generated by the operators $\varepsilon_{q^+q^-}$, the vertex operators (5.3) are formally endomorphisms of the $\mathbb{Z}$-graded Fock space of operators

$$\widehat{\mathcal{H}}_X(\Lambda) = \mathbb{C}\{\Lambda\} \otimes S(\hat{h}_+^{(-)}) \otimes S(\hat{h}_-^{(-)}) \tag{5.10}$$

which defines the space of twisted endomorphisms of the Hilbert space (3.19). The oscillator modes $\alpha_k^{(\pm)\mu}$ for $k<0$ act in the latter tensor products of (5.10) by multiplication, and for $k>0$ their (adjoint) action is defined by (3.18). The zero mode operators $\alpha_0^{(\pm)\mu}$ act on $\mathbb{C}\{\Lambda\}$ by

$$\alpha_0^{(\pm)\mu}\ \varepsilon_{q^+q^-} = g^{\mu\nu} q_\nu^\pm\ \varepsilon_{q^+q^-} \tag{5.11}$$

while the action of $\varepsilon_{q^+q^-}$ on the twisted group algebra is given by (5.8).

Explicitly, we can expand the exponentials in (5.3) in terms of the Schur polynomials $P_m[t_1,\dots,t_m]$ defined by

$$\exp\left(\sum_{k>0}\frac{t_k}{k}\ z^k\right) = \sum_{m=0}^\infty P_m[t_1,\dots,t_m]\ z^m \tag{5.12}$$

to get

$$\begin{aligned} V_{q^+q^-}(z_+,z_-) &= \varepsilon_{q^+q^-}\sum_{k,m=0}^\infty P_k[q_{\mu_1}^+\alpha_{-1}^{(+)\mu_1},\dots,q_{\mu_k}^+\alpha_{-k}^{(+)\mu_k}] \\ &\qquad \times P_m[q_{\nu_1}^-\alpha_{-1}^{(-)\nu_1},\dots,q_{\nu_m}^-\alpha_{-m}^{(-)\nu_m}]\ z_+^k z_-^m \quad \in \widehat{\mathcal{H}}_X(\Lambda)[z_+,z_-] \end{aligned} \tag{5.13}$$

More generally, to a typical homogeneous element

$$\Psi = \varepsilon_{q^+q^-}\otimes\prod_j r_\mu^{(j)+}\alpha_{-n_j}^{(+)\mu}\otimes\prod_k r_\nu^{(k)-}\alpha_{-m_k}^{(-)\nu} \tag{5.14}$$

of $\widehat{\mathcal{H}}_X(\Lambda)$ with $(q^+,q^-),(r^{(i)+},r^{(i)-})\in\Lambda$, we associate the higher-spin vertex operator

$$\begin{aligned} \mathcal{V}(\Psi;z_+,z_-) &\equiv \mathcal{V}_{q^+q^-}^\Omega(z_+,z_-) \\ &= \ :\ i\,V_{q^+q^-}(z_+,z_-)\ \prod_j\frac{r_\mu^{(+)j}}{(n_j-1)!}\ \partial_{z_+}^{n_j}X_+^\mu\ \prod_k\frac{r_\nu^{(-)k}}{(m_k-1)!}\ \partial_{z_-}^{m_k}X_-^\nu\ : \end{aligned} \tag{5.15}$$

where $\Omega=\{(r^{(i)+},r^{(i)-});n_j,m_k\}$ labels the fields and $\partial_{z_\pm}X_\pm^\mu(z_\pm)=\sum_{k\in\mathbb{Z}} i\alpha_k^{(\pm)\mu}z_\pm^{-k-1}$. Extending this by linearity we obtain a well-defined one-to-one mapping from $\widehat{\mathcal{H}}_X(\Lambda)$ into the algebra of endomorphism-valued Laurent power series in the variables $z_\pm$,

$$\widehat{\mathcal{H}}_X(\Lambda)\to(\mathrm{End}\ \widehat{\mathcal{H}}_X(\Lambda))[z_+^{\pm1},z_-^{\pm1}] \tag{5.16}$$

which defines the space of quantum conformal fields of the sigma-model. For example, the stress-energy tensors (4.23) are given by

$$T^\pm(z_\pm)=\mathbb{I}\otimes\mathcal{V}(\omega^\pm;z_\pm,1) \qquad \text{with}\ \ \omega^\pm=\tfrac12\,\mathbb{I}\otimes g_{\mu\nu}(\vec{e}^{\,\mu})_\lambda(\vec{e}^{\,\nu})_\rho\ \alpha_{-1}^{(\pm)\lambda}\alpha_{-1}^{(\pm)\rho} \tag{5.17}$$

where $\{\vec{e}^\mu\}$ is an arbitrary basis of the lattice Γ.

The fundamental (tachyonic) vertex operators (5.3) have some noteworthy algebraic properties which define the algebra of the primary fields. Since the sigma-model under consideration is essentially a free field theory, these features follow from the normal-ordering property

$$\begin{aligned}&V_{q^+q^-}(z_+,z_-)V_{r^+r^-}(w_+,w_-)\\&=\,:V_{q^+q^-}(z_+,z_-)V_{r^+r^-}(w_+,w_-):\;\exp\left\{\langle\mathrm{vac}|q^+_\mu X^\mu_+(z_+)+r^+_\mu X^\mu_+(w_+)|\mathrm{vac}\rangle\right\}\\&\qquad\times\exp\left\{\langle\mathrm{vac}|q^-_\mu X^\mu_-(z_-)+r^-_\mu X^\mu_-(w_-)|\mathrm{vac}\rangle\right\}\end{aligned}\tag{5.18}$$

Evaluating the vacuum expectation values explicitly yields the operator product algebra

$$\begin{aligned}V_{q^+q^-}(z_+,z_-)V_{r^+r^-}(w_+,w_-) &= (z_+-w_+)^{g^{\mu\nu}q^+_\mu r^+_\nu}(z_--w_-)^{g^{\mu\nu}q^-_\mu r^-_\nu}\\&\qquad\times\;:V_{q^+q^-}(z_+,z_-)V_{r^+r^-}(w_+,w_-):\end{aligned}\tag{5.19}$$

which represents the product of two vertex operators in terms of another one (in general with singular local coefficient). Interchanging the order of the two vertex operators in (5.19) and using (5.9) we then find the local cocycle relation

$$\begin{aligned}&V_{q^+q^-}(z_+,z_-)V_{r^+r^-}(w_+,w_-)\\&=V_{r^+r^-}(w_+,w_-)V_{q^+q^-}(z_+,z_-)\;\exp\left\{-i\pi g^{\mu\nu}q^+_\mu r^+_\nu\;\mathrm{sgn}(\arg z_+-\arg w_+)\right\}\\&\qquad\times\exp\left\{-i\pi g^{\mu\nu}q^-_\mu r^-_\nu\;\mathrm{sgn}(\arg z_--\arg w_-)\right\}\end{aligned}\tag{5.20}$$

where we have chosen the branch of $\log(-1)$ on the worldsheet for which the imaginary part of the logarithm lies in the interval $[-i\pi,+i\pi]$.

The algebraic relations (5.19) and (5.20) determine, by differentiation of the appropriate vertex operators, corresponding relations among the general higher-spin vertex operators (5.15). For instance, from the identities

$$\begin{aligned}\partial_{z_\pm}V_{q^+q^-}(z_+,z_-) &= \sum_{k=-\infty}^{\infty}\;:\;q^\pm_\mu\alpha_k^{(\pm)\mu}z_\pm^{-k-1}\;V_{q^+q^-}(z_+,z_-)\;:\\\partial_{z_\pm}\mathcal{V}(\alpha_{-m}^{(\pm)\mu};z_+,z_-) &= \mathcal{V}(m\alpha_{-m-1}^{(\pm)\mu};z_+,z_-)\end{aligned}\tag{5.21}$$

we have

$$\partial_{z_\pm}\mathcal{V}(\Psi;z_+,z_-)=\mathcal{V}(L^\pm_{-1}\Psi;z_+,z_-)\qquad\forall\Psi\in\widehat{\mathcal{H}}_X(\Lambda)\tag{5.22}$$

In fact, the coefficients of the monomials in the variables $z_\pm$, that arise in an expansion of products of the basis vertex operators

$$\mathcal{V}^{(R)}_{[q^+q^-]}[z_+,z_-]=\,:\prod_{i=1}^{R}V_{q^{(i)+}q^{(i)-}}(z_+^{(i)},z_-^{(i)})\;:\tag{5.23}$$

in terms of Schur polynomials, span the linear space (5.10) as R and the momenta $(q^{(i)+},q^{(i)-})\in\Lambda$ are varied. The complete set of relations of the conformal field algebra is therefore determined by the operator product expansion formula for the operators (5.23), which is

$$\begin{aligned}\mathcal{V}^{(R)}_{[q^+q^-]}[z_+,z_-]\mathcal{V}^{(S)}_{[r^+r^-]}[w_+,w_-]&=\mathcal{V}^{(S)}_{[r^+r^-]}[w_+,w_-]\mathcal{V}^{(R)}_{[q^+q^-]}[z_+,z_-]\\ \times\prod_{1\leq(i,j)\leq(R,S)}&\exp\left\{-i\pi g^{\mu\nu}q^{(i)+}_\mu r^{(j)+}_\nu\ \mathrm{sgn}(\arg z^{(i)}_+-\arg w^{(j)}_+)\right\}\\ &\times\exp\left\{-i\pi g^{\mu\nu}q^{(i)-}_\mu r^{(j)-}_\nu\ \mathrm{sgn}(\arg z^{(i)}_--\arg w^{(j)}_-)\right\}\end{aligned}\tag{5.24}$$

The fundamental vertex operators (5.3) are primary fields of both the Kac-Moody and Virasoro algebras of the sigma-model. Specifically, $V_{q^+q^-}(z_+,z_-)$ transform as local fields of $U(1)^n_\pm$ charges $q^\pm_\mu$ under the local gauge transformations generated by the Kac-Moody currents,

$$\left[\alpha^{(\pm)\mu}_k,V_{q^+q^-}(z_+,z_-)\right]=g^{\mu\nu}q^\pm_\nu\ z^k_\pm\ V_{q^+q^-}(z_+,z_-)\tag{5.25}$$

Moreover,

$$\left[L^\pm_k,V_{q^+q^-}(z_+,z_-)\right]=\left(z^{k+1}_\pm\partial_{z_\pm}+(k+1)\Delta_{q^\pm}z^k_\pm\right)V_{q^+q^-}(z_+,z_-)\tag{5.26}$$

so that the local vertex operators (5.3) also transform under conformal transformations as tensors of weight

$$\Delta_{q^\pm}=\tfrac12 g^{\mu\nu}q^\pm_\mu q^\pm_\nu\tag{5.27}$$

However, (5.25) holds only at $k=0$ and (5.26) only for $k=0$ and $k=-1$ in general for the higher-spin vertex operators $\mathcal{V}^\Omega_{q^+q^-}(z_+,z_-)$. Thus the general operators (5.15) also have $U(1)^n_\pm$ charges $q^\pm_\mu$ and scaling dimensions

$$\Delta^\Omega_{q^+}=\tfrac12 g^{\mu\nu}q^+_\mu q^+_\nu+\textstyle\sum_j n_j\qquad,\qquad\Delta^\Omega_{q^-}=\tfrac12 g^{\mu\nu}q^-_\mu q^-_\nu+\textstyle\sum_k m_k\tag{5.28}$$

The general transformation property (5.26) holds for the vertex operators (5.15) which are primary fields of the sigma-model (of weights $\Delta^\Omega_{q^\pm}$), i.e. the endomorphisms acting on the subspaces

$$\widehat{\mathcal{P}}_{\Delta^\Omega_q}(\Lambda)\equiv\left\{\Psi\in\widehat{\mathcal{H}}_X(\Lambda)\mid L^\pm_0\Psi=\Delta^\Omega_{q^\pm}\Psi\ \ ,\ \ L^\pm_k\Psi=0\ \ \forall k>0\right\}\tag{5.29}$$

of highest-weight operators in $\widehat{\mathcal{H}}_X(\Lambda)$. The space (5.10) is then a direct sum of the irreducible highest-weight representations of the Virasoro algebra acting in the subspaces (5.29) and labelled by the conformal weights $\Delta^\Omega_{q^\pm}$. Similarly, one can decompose (5.10) into irreducible highest-weight representations of the current algebra acting in subspaces of definite $U(1)^n_\pm$ charges $q^\pm_\mu$.

In fact, in addition to defining the grading of the space $\widehat{\mathcal{H}}_X(\Lambda)$, the conformal dimension and spacetime momentum eigenvalues grade the Hilbert space (3.19). This follows from the operator-state correspondence which relates the states

$$|\varphi^\Omega_{q^+q^-}\rangle=|q^+;q^-\rangle\otimes\prod_j r^{(j)++}_\mu\alpha^{(+)\mu}_{-n_j}|0\rangle_+\otimes\prod_k r^{(k)-}_\nu\alpha^{(-)\nu}_{-m_k}|0\rangle_-\in\mathcal{H}_X\tag{5.30}$$

to the higher-spin vertex operators (5.15), where Ω labels the quantum numbers of the states. For example, the spin-0 tachyon state corresponds to the basis vertex operators themselves. In this case, the basis primary fields correspond to the Hilbert space states

$$|q^+;q^-\rangle\otimes|0\rangle_+\otimes|0\rangle_- = \lim_{z_\pm\to 0} V_{q^+q^-}(z_+,z_-)|\mathrm{vac}\rangle \in L^2((S^1)^n,\textstyle\prod_{\mu=1}^n\frac{dx^\mu}{\sqrt{2\pi}}) \tag{5.31}$$

The (primary) tachyon vectors (5.31) are eigenstates of $\alpha_0^{(\pm)\mu}$ with $U(1)^n_\pm$ charge eigenvalues $q^\pm_\mu$ and of $L_0^\pm$ with conformal weight eigenvalues $\Delta_{q^\pm}$.

Likewise, the general states (5.30) have spacetime momentum eigenvalues $q^\pm_\mu$ and conformal dimension eigenvalues (5.28). An important state is the graviton state which corresponds to the minimal spin-2 operator

$$\mathcal{V}^{\mu\nu}_{q^+q^-}(z_+,z_-) = :\ i\,V_{q^+q^-}(z_+,z_-)\ \partial_{z_+}X^\mu_+\ \partial_{z_-}X^\nu_-\ : \tag{5.32}$$

which represents the basic conformal structure embedded into the content of the algebra under construction here (see (5.17)). The operator (5.32) creates a graviton of polarization $(\mu\nu)$ and represents the Fourier modes of the background matrices $d^\pm_{\mu\nu}$, i.e. of the metric $g_{\mu\nu}$ and instanton form $\beta_{\mu\nu}$. It corresponds to the string state $|q^+;q^-\rangle\otimes\alpha_{-1}^{(+)\mu}|0\rangle_+\otimes\alpha_{-1}^{(-)\nu}|0\rangle_-\in\mathcal{H}_X$, which is the lowest-lying vector with non-trivial string oscillations and will play an important role in section 7 (see also the appendix). As another example, the massless dilaton state corresponds to the minimal spin-0 operator $:iV_{q^+q^-}(z_+,z_-)\,\frac{1}{2}g_{\mu\nu}\partial_{z_+}X^\mu_+\partial_{z_-}X^\nu_-:$ which is orthogonal to the tachyon field. In the general case, the operator-state correspondence is achieved by the action of the smeared vertex operators

$$V_\Omega(q^+,q^-) = \int\frac{dz_+\ dz_-}{4\pi z_+z_-}\ \mathcal{V}^\Omega_{q^+q^-}(z_+,z_-)f_S(z_+,z_-)\in\widehat{\mathcal{H}}_X(\Lambda) \tag{5.33}$$

on the vacuum state of the Hilbert space, so that

$$|\varphi^\Omega_{q^+q^-}\rangle = V_\Omega(q^+,q^-)|\mathrm{vac}\rangle \tag{5.34}$$

Here f_S is an appropriate Schwartz space test function which smears out the operator-valued distributions $\mathcal{V}^\Omega_{q^+q^-}(z_+,z_-)$ *. The operator (5.33) is said to create a string state of type Ω and momentum $(q^+,q^-)\in\Lambda$. The smeared vertex operators represent spacetime Fourier transformations of the string state operators inserted on the worldsheet. They span the space of primary fields of the conformal sigma model and generate the set of all functionals on the vector space (5.10), i.e. they span the twisted dual of the Hilbert space (3.19). The quantities (5.33) are naturally (non-local) elements of the operator algebra (5.10) and are well-defined operators on (3.19). In fact, they map a dense subspace of $\mathcal{H}_X$ into itself, as follows from (5.34) and the completeness of the primary states (5.30)

*For ease of notation in the following, we suppress the explicit dependence on the test function f_S in (5.33).

in $\mathcal{H}_X$. The operator-state correspondence is thus the Fock space mapping between the Hilbert space (3.19) and its twisted endomorphisms (5.10)

$$|\varphi^{\Omega}_{q^+q^-}\rangle \leftrightarrow V_\Omega(q^+,q^-) : \mathcal{H}_X \leftrightarrow \widehat{\mathcal{H}}_X(\Lambda) \tag{5.35}$$

Note that this mapping is not one-to-one as there are many smeared vertex operators which can be associated to a given conformal state of the Hilbert space (3.19).

The algebraic make-up of the vertex operators described above has the formal mathematical structure of a vertex operator algebra [12] (see also [13, 14] for concise introductions). A vertex operator algebra is the formal mathematical definition of a chiral algebra in conformal field theory [42], which is defined to be the operator product algebra of (primary) holomorphic fields in the conformal field theory (see (5.19) in the present case). In this context the Klein transformations (5.4), which in the above were introduced to correct signs in the Kac-Moody and Virasoro commutation relations with the vertex operators, are needed to adjust some signs in the so-called 'Jacobi identities' of the vertex operator algebra. In conformal field theory, these Jacobi identities are the precise statement of the Ward identities associated with the conformal invariance on the 3-punctured Riemann sphere. The general properties of vertex operator algebras as they pertain to this paper are briefly discussed in the appendix.

The smeared vertex operators $V_\Omega(q^+,q^-)$ generate a noncommutative unital $*$-algebra $\mathcal{A}_X$ which contains two Virasoro and two Kac-Moody subalgebras. The operator-state correspondence (5.35) then implies that

$$\mathcal{A}_X\mathcal{H}_X = \mathcal{H}_X \tag{5.36}$$

The noncommutativity of $\mathcal{A}_X$ is expressed in terms of the cocycle relation (5.24) which determines the complete set of relations between the smeared vertex operators. As a vertex operator algebra, the identity (5.24) in fact leads immediately to the Jacobi identity, which in turn encodes many other non-trivial algebraic relations among the elements of $\mathcal{A}_X$, including the noncommutativity, associativity, locality of matrix elements, and the duality (or crossing symmetry) of 4-point (and m-point) correlation functions on the Riemann sphere. The complicated nature of the 3-term Jacobi relation among the elements of the vertex operator algebra $\mathcal{A}_X$ is what distinguishes the string spacetime from not only a classical (commutative) spacetime, but also from the conventional examples of noncommutative spaces. The full set of relations of the vertex operator algebra $\mathcal{A}_X$ are presented in a more general context in the appendix, where it is also discussed how the algebraic structure of $\mathcal{A}_X$ leads to a construction of the corresponding string spacetime along the lines described in section 2. In particular, they illustrate the complicated nature of the string spacetime determined by the algebra $\mathcal{A}_X$, and how more general string spacetimes constructed from other conformal field theories arise from the general structure of vertex operator algebras.

Note that the chiral and anti-chiral vertex operators (5.1) generate local chiral algebras $\mathcal{E}^{\pm}$ whose products combine into the full algebra $\mathcal{E} = \mathcal{E}^{+} \otimes \mathcal{E}^{-}$ corresponding to the usual decomposition of the operator product algebra of primary holomorphic fields in conformal field theory. As mentioned before, the twisting of this chirally-symmetric algebraic structure by the cocyle factors (5.4) is required to compensate signs in the Jacobi identity for $\mathcal{A}_X$. But we shall see that the non-trivial duality structure of the effective string spacetime is represented essentially as a left-right chirality symmetry between the Dirac operators introduced in section 4. The twistings then yield more complicated duality transformations, and, as we will discuss in section 7, at certain special points in the quantum moduli space of the sigma-model the chiral symmetry is restored and the algebra $\mathcal{E}$ controls the structure of the quantum spacetime.

6. Quantum Geometry of Toroidal Spacetimes

We shall now begin applying the formalism developed thus far to a systematic analysis of the geometrical properties of the string spacetime. With the above constructions, we obtain the spectral triple

$$\mathcal{T} \equiv (\mathcal{A}\ ,\ \mathcal{H}\ ,\ D) \tag{6.1}$$

associated to the sigma-model with target space the n-torus $(S^1)^n$ with metric $g_{\mu\nu}$ and torsion form $\beta_{\mu\nu}$. The triple (6.1) encodes the effective target space geometry of the linear sigma-model, i.e. the moduli space of this class of conformal field theories. Here D is an appropriately defined Dirac operator on the spin-extended Hilbert space $\mathcal{H}$ which encodes the effective geometry and topology of the string spacetime described by (6.1). It will be constructed below from the two Dirac operators presented in section 4. The algebra $\mathcal{A} \equiv \mathbb{I} \otimes \mathcal{A}_X$ acts trivially on the spinor part of $\mathcal{H}$ *. As we have pointed out in section 2, different selections of D lead to different geometries for the string spacetime. Below we shall examine the properties of the string spacetime with appropriate selections of this operator.

The topology and differentiable structure of the spacetime is encoded via the complicated, noncommutative vertex operator algebra $\mathcal{A}_X$ that we described in the previous section. The effective spacetime of the strings is in this sense horribly complicated. The

*An algebraic, field-theoretical construction of (6.1) can be naturally given as a low-energy limit of the canonical spectral triple associated with the $N = 1$ superconformal field theory (4.8). There we take the Hilbert space (4.17), the algebra generated by the superconformal primary fields corresponding to states of $\mathcal{H}_{X,\psi}$, and a Dirac operator Q constructed from the worldsheet supersymmetry generators (4.9). The resulting unital $*$-algebra is described as a vertex operator superalgebra (see the appendix). The projection onto fermionic zero-modes gives $\mathcal{H}_{\rm R}^{(0)}$, the Dirac operator $\mathcal{P}_{\rm R}^{(0)} Q \mathcal{P}_{\rm R}^{(0)}$, and the algebra $\mathcal{A}\mathcal{P}_{\rm R}^{(0)}$ where $\mathcal{A} = \mathbb{I} \otimes \mathcal{A}_X$ is generated by the primary conformal fields corresponding to states of $\mathcal{H}_{\rm NS}^{(0)}$. This is the construction that has been elucidated on in [7, 8, 9, 38].

smearing of the vertex operators effectively achieves the non-locality property of noncommuting spacetime coordinate fields. We can get some insight into the various symmetries of the stringy geometries by viewing the emergence of ordinary spacetime (i.e. one with a commutative geometry) as a low-energy limit of the quantum spacetime of the strings. The low-energy limit of the sigma-model is the limit wherein the oscillatory modes $\alpha_k^{(\pm)\mu}$ of the string vanish, leaving only the particle-like, center of mass degrees of freedom $x_\pm^\mu, p_\mu^\pm$. Thus in the low-energy limit of the string theory, the spectral triple (6.1) should contain a subspace

$$\mathcal{T}_0 = \Big(C^\infty(T^n) \ , \ L^2(\mathrm{spin}(T^n)) \ , \ i g^{\mu\nu} \gamma_\mu \partial_\nu \Big) \tag{6.2}$$

which represents the ordinary spacetime manifold T^n at large distance scales. We shall see that these commutative spacetimes are indeed small subspaces of the larger string spacetime given by the spectral triple $\mathcal{T}$. The various string theoretic symmetries that determine the quantum moduli space of the toroidal sigma-models will then appear naturally via the possibility of introducing more than one Dirac operator D corresponding to isometries of $\mathcal{T}$.

The moduli space of linear sigma-models is determined by the symmetries of the lattice Γ and its dual Γ^* which define the toroidal spacetime. The symmetry group of the lattice Λ with inner product (3.16) is the non-compact orthogonal group $O(n,n)$ in $(n+n)$-dimensions. However, the Hamiltonian (4.27) and hence the spectrum of the toroidal sigma-model change under an $O(n,n)$ rotation. Since the chiral momenta $p_\mu^\pm$ transform as vectors under $O(n,n)$, the quantum sigma-model is invariant under rotations by the maximal compact subgroup $O(n) \times O(n) \subset O(n,n)$ representing the rotations of the lattice Γ defining the torus T^n itself (i.e. the set of allowed winding modes) and the rotations of the dual lattice Γ^* (i.e. the set of allowed momentum modes). Thus the "classical" moduli space is the right coset manifold

$$\mathcal{M}_{\rm cl} = O(n,n)/(O(n) \times O(n)) \tag{6.3}$$

The homogeneous space (6.3) is isomorphic to the Grassmannian $\mathrm{Gr}(n,n)$ of maximal positive subspaces of $\mathbb{R}^{n,n}$ which is parametrized by the constant $n\times n$ matrix $d^+_{\mu\nu}$ (equivalently $d^-_{\mu\nu}$). In the following we shall see how the appropriate quantum string modification of the moduli space (6.3) appears naturally in the framework of the noncommutative geometry of the sigma-model. We will see that the discrete group of automorphisms of the quantum sigma-model, the so-called "duality group", can be readily identified by viewing the string spacetime as the spectral triple (6.1). By construction, this point of view automatically establishes the equivalence between the original quantum field theory and its dual (i.e. that the correlation functions of the models are also equivalent to each other), this being encoded in the quantum field algebra $\mathcal{A}$.

The spacetime duality maps are, by definition, those which lead to isomorphisms between inequivalent low-energy spectral triples (6.2). As we will show, they emerge from the possibility of defining the two independent (smeared) Dirac operators

$$\begin{aligned}
\not{D} &= \int \frac{dz_+ \, dz_-}{4\pi z_+ z_-} \, \tfrac{1}{\sqrt{2}} \left(\not{D}^+(z_+) + \not{D}^-(z_-) \right) f_S(z_+, z_-) \\
\bar{\not{D}} &= \int \frac{dz_+ \, dz_-}{4\pi z_+ z_-} \, \tfrac{1}{\sqrt{2}} \left(\not{D}^+(z_+) - \not{D}^-(z_-) \right) f_S(z_+, z_-)
\end{aligned} \tag{6.4}$$

in the spectral triple (6.1). The main point is that there exists several unitary transformations $T : \mathcal{H} \to \mathcal{H}$ with

$$\bar{\not{D}} = T \not{D} T^{-1} \tag{6.5}$$

which define automorphisms of the vertex operator algebra, i.e. $T\mathcal{A}T^{-1} = \mathcal{A}$. This then immediately leads to the isomorphism of spectral triples

$$\mathcal{T}_{\not{D}} \equiv (\mathcal{A} \ , \ \mathcal{H} \ , \ \not{D}) \cong (\mathcal{A} \ , \ \mathcal{H} \ , \ \bar{\not{D}}) \equiv \mathcal{T}_{\bar{\not{D}}} \tag{6.6}$$

The isomorphism (6.6) is a special case of the spectral action principle [11] which describes Riemannian manifolds which are isospectral (i.e. their Dirac K-cycles have the same spectrum) but not necessarily isometric. It states that the noncommutative string spacetime determined by the spectral triple $\mathcal{T}$ with $D = \not{D}$ is identical to that defined with $D = \bar{\not{D}}$. As such a change of Dirac operator in noncommutative geometry simply represents a change of metric structure on the spacetime, the isomorphism (6.6) is simply the statement of general covariance of the noncommutative string spacetime represented as an isometry of $\mathcal{T}$.

From this point of view, target space duality can be represented symbolically by the commutative diagram

$$\begin{array}{ccc}
\mathcal{T}_{\not{D}} & \xrightarrow{T} & \mathcal{T}_{\bar{\not{D}}} \cong \mathcal{T}_{\not{D}} \\
{\scriptstyle \mathcal{P}_0} \downarrow & & \downarrow {\scriptstyle \bar{\mathcal{P}}_0} \\
\mathcal{T}_0 & \xrightarrow{T_0} & \bar{\mathcal{T}}_0
\end{array} \tag{6.7}$$

The operators $\mathcal{P}_0$ and $\bar{\mathcal{P}}_0$ project the full spectral triples onto their respective low-energy subspaces $\mathcal{T}_0$ and $\bar{\mathcal{T}}_0$ (these projections will be defined formally below). The triple $\mathcal{T}_0$ is the commutative spacetime (6.2) while $\bar{\mathcal{T}}_0$ represents a duality transformed commutative spacetime. From the point of view of classical general relativity, these two spacetimes are inequivalent. However, the isomorphism T in the top line of (6.7) makes the diagram commutative, so that $T_0\mathcal{P}_0 = \bar{\mathcal{P}}_0 T$ and T_0 is an isomorphism of subspaces of the noncommutative spacetime. Thus as subspaces of the full quantum spacetime (6.1), the commutative spacetimes $\mathcal{T}_0$ and $\bar{\mathcal{T}}_0$ are equivalent. This is the essence of duality and the stringy modification of classical general relativity.

T-duality, Low-energy Projections and Spin Structures

The first symmetry of the string spacetime that we explore is T-duality which relates large and small radius tori on equal footing and is responsible for the fundamental length scale in string theory. In terms of the isomorphism (6.6), it is the unitary mapping $T\equiv T_S\otimes T_X:\mathcal{H}\to\mathcal{H}$ in (6.5) which is defined as follows†. T_X acts trivially on the spinor part of $\mathcal{H}$ and on the bosonic Hilbert space $\mathcal{H}_X$ itself it is defined by

$$\begin{aligned}T_X\,|p^+;p^-\rangle\otimes|0\rangle_+\otimes|0\rangle_- &= c_{p^+p^-}(p^+,p^-)|(d^+)^{-1}p^+;-(d^-)^{-1}p^-\rangle\otimes|0\rangle_+\otimes|0\rangle_-\\ T_X\,\alpha_k^{(\pm)\mu}\,T_X^{-1} &= \pm g_{\nu\lambda}(d^\mp)^{\mu\nu}\alpha_k^{(\pm)\lambda}\end{aligned}\tag{6.8}$$

T_S acts trivially on $\mathcal{H}_X$ and on the generators of the spin bundle as

$$T_S\,\gamma_\mu^\pm\,T_S^{-1}=g^{\nu\lambda}d_{\mu\nu}^\mp\gamma_\lambda^\pm\tag{6.9}$$

Thus the operator T simply redefines the bosonic basis of the Hilbert space, and it also changes the choice of spin structure on the target space by defining a different representation of the double Clifford algebra (4.2). This latter property implies that the target space metric $g_{\mu\nu}$ changes to its dual

$$\tilde{g}^{\mu\nu}=(d^+)^{\mu\lambda}g_{\lambda\rho}(d^-)^{\rho\nu}\tag{6.10}$$

which defines an inner product on the dual lattice Γ^*. Note that in the case where the spacetime has torsion ($\beta\neq 0$), the duality (and other properties) are determined by the mutually inverse background matrices $(d^\pm)^{-1}$, rather than just $g^{\mu\nu}$ itself which controls the dual torsion-free theory. The quadratic form (3.16) of the lattice Λ is invariant under these transformations.

The Hilbert space (4.3) is thus invariant under this mapping between the two Dirac operators (6.4), as is the Hamiltonian (4.27) (and hence the spectrum of the theory). Furthermore, the action of T on the smeared vertex operator algebra is given by

$$T_X\,V_\Omega(q^+,q^-)\,T_X^{-1}=c_{q^+q^-}(p^+,p^-)V_\Omega(q^+(d^+)^{-1},-q^-(d^-)^{-1})\tag{6.11}$$

which amounts to simply producing the same type of vertex operator with a redefined twisted momentum in $\hat{\Lambda}^c$. The algebra $\mathcal{A}=T\mathcal{A}T^{-1}$ is thus invariant under the unitary mapping T. The transformation T defined by (6.8)–(6.11) which yields an explicit realization of the isomorphism (6.6) is the noncommutative geometry version of the celebrated 'T-duality' transformation of string theory which exchanges the torus T^n with its dual $(T^n)^*$, and at the same time interchanges momenta and winding numbers in the spectrum of the compactified string theory. It corresponds to an inversion of the background matrices $d^\pm\to(d^\pm)^{-1}$ and is the n-dimensional analog of the $R\to 1/R$ circle duality [1]. From the point of view of the noncommutative geometry formalism, T-duality thus appears quite naturally as a very simple geometric invariance property of the noncommutative spacetime. As we will see, the choice and independence of spin structure in the string theory is also intimately related to the T-duality symmetry.

†In the next section we shall present explicit operator expressions for the duality transformations T.

We now describe the low-energy projections in detail and explore the consequences of the above duality map in this sector of the string spacetime. For this, we consider the subspace

$$\bar{\mathcal{H}}_0 \equiv \ker \not{D} \cong \left\{ |\psi; p^+, p^-\rangle \in \bigoplus_{S[\mathcal{C}(T^n)]} L^2(\mathrm{spin}(T^n))^\Gamma \;\middle|\; \not{D}_0 |\psi; p^+, p^-\rangle = 0 \right\} \subset \mathcal{H} \tag{6.12}$$

where

$$\not{D}_0 = \tfrac{1}{\sqrt{2}}\, g^{\mu\nu} \left[\left(\gamma_\mu^+ + \gamma_\mu^-\right) \otimes p_\nu + \left(d_{\nu\lambda}^+ \gamma_\mu^+ - d_{\nu\lambda}^- \gamma_\mu^-\right) \otimes w^\lambda \right] \tag{6.13}$$

The equivalence in (6.12) shows that all oscillator modes are suppressed, leaving only the center of mass zero modes of the strings. The subspace $\bar{\mathcal{H}}_0$ can be decomposed into a direct sum of 2^n smaller subspaces via

$$\bar{\mathcal{H}}_0 = \bigotimes_{\mu=1}^{n} \left(\bar{\mathcal{H}}_0^{(+)\mu} \oplus \bar{\mathcal{H}}_0^{(-)\mu} \right) \tag{6.14}$$

where

$$\begin{aligned} \bar{\mathcal{H}}_0^{(+)\mu} &= \left\{ |\psi\rangle \otimes |p^+; p^-\rangle \in \bar{\mathcal{H}}_0 \mid g^{\nu\lambda} d_{\lambda\mu}^+ \gamma_\nu^+ |\psi\rangle = g^{\nu\lambda} d_{\lambda\mu}^- \gamma_\nu^- |\psi\rangle \;\;,\;\; p_\mu = 0 \right\} \\ \bar{\mathcal{H}}_0^{(-)\mu} &= \left\{ |\psi\rangle \otimes |p^+; p^-\rangle \in \bar{\mathcal{H}}_0 \mid \gamma_\mu^+ |\psi\rangle = -\gamma_\mu^- |\psi\rangle \;\;,\;\; w^\mu = 0 \right\} \end{aligned} \tag{6.15}$$

Each subspace (6.15) represents a particular choice of chiral or anti-chiral representation of the double Clifford algebra $\mathcal{C}(T^n) = \mathcal{C}(T^n)^+ \oplus \mathcal{C}(T^n)^-$, respectively (see (4.5)). Each such representation in turn encodes a particular choice of one of the 2^n possible spin structures of the torus.

The Hilbert space $\bar{\mathcal{H}}_0$ contains the subspace of highest-weight vectors which belong to complex-conjugate pairs of left-right representations of the $u(1)_+^n \oplus u(1)_-^n$ current algebra. The 2^n subspaces in (6.14) are all naturally isomorphic to the canonical anti-chiral subspace

$$\bar{\mathcal{H}}_0^{(-)} = \bar{\mathcal{H}}_0^{(-)1} \otimes \bar{\mathcal{H}}_0^{(-)2} \otimes \cdots \otimes \bar{\mathcal{H}}_0^{(-)n} \tag{6.16}$$

The explicit isomorphism maps $\bar{\mathcal{H}}_0^{(+)\mu} \leftrightarrow \bar{\mathcal{H}}_0^{(-)\mu}$ by $g^{\nu\lambda} d_{\lambda\mu}^\pm \gamma_\nu^\pm \leftrightarrow \pm\gamma_\mu^\pm$ and $g^{\mu\nu} p_\nu \leftrightarrow w^\mu$ (or equivalently $g^{\mu\nu} p_\nu^\pm \leftrightarrow \pm (d^\pm)^{\mu\nu} p_\nu^\pm$). These isomorphisms themselves determine a sort of partial T-duality transformation that exchanges $m \leq n$ momenta with winding numbers. We will see below that they are related to another type of duality called 'factorized duality'. Thus the various chiral and anti-chiral subspaces (6.15) are all naturally isomorphic to one another under such "partial" T-duality transformations. Here we shall make the canonical choice of anti-chiral low-energy subspace (6.16) corresponding to the representation of the double Clifford algebra for which $\gamma_\mu^+ = -\gamma_\mu^- \equiv \gamma_\mu \;\; \forall \mu$. The isomorphisms above demonstrate explicitly the independence of quantities on the choice of spin structure for the spacetime. It is intriguing that a change of spin structure is manifested as a T-duality symmetry of the string theory.

In the subspace $\bar{\mathcal{H}}_0^{(-)}$, we have $w^\mu = 0 \ \ \forall\mu$, as required since in the low-energy sector there should be no global winding modes of the string around the compactified directions of the target space. It follows that $\sqrt{2}p_\mu^+ = \sqrt{2}p_\mu^- = p_\mu \in \Gamma^*$, and thus the subspace (6.16) is naturally isomorphic to the Hilbert space

$$\bar{\mathcal{H}}_0^{(-)} \cong \varrho^-[\mathcal{C}(T^n)] \otimes L^2((S^1)^n, \textstyle\prod_{\mu=1}^n \frac{dx^\mu}{\sqrt{2\pi}}) \tag{6.17}$$

which is (a local trivialization of) the bundle of anti-chiral square-integrable spinors on the torus. Here ϱ^- denotes the anti-chiral spinor representation and the explicit L^2-isomorphism in (6.17) maps $|p;p\rangle \leftrightarrow \mathrm{e}^{-ip_\mu x^\mu}$ in the restriction to the L^2-component of winding number 0. The anti-holomorphic Dirac operator $\bar{\not{D}}$ acting in (6.17) is

$$\bar{\not{D}}\,\bar{\mathcal{P}}_0^{(-)} = ig^{\mu\nu}\gamma_\mu \otimes \tfrac{\partial}{\partial x^\nu} \tag{6.18}$$

where $\bar{\mathcal{P}}_0^{(-)}$ is the operator that projects $\mathcal{H}$ orthogonally onto $\bar{\mathcal{H}}_0^{(-)}$, and we have defined $x^\mu = \frac{1}{\sqrt{2}}(x_+^\mu + x_-^\mu)$. In particular, $\bar{\mathcal{H}}_0^{(-)}$ is an invariant subspace of the Dirac operator, $\bar{\not{D}}\bar{\mathcal{H}}_0^{(-)} = \bar{\mathcal{H}}_0^{(-)}$, and the full low-energy Hilbert space (6.14) is a maximal subspace of $\mathcal{H}$ with this invariance property.

Next we need a corresponding low-energy projection $\bar{\mathcal{A}}_0$ of the vertex operator algebra $\mathcal{A}$. We define $\bar{\mathcal{A}}_0$ to be the commutant of the Dirac operator restricted to the Hilbert space $\bar{\mathcal{H}}_0$,

$$\bar{\mathcal{A}}_0 = \bar{\mathcal{P}}_0\,(\mathrm{comm}\ \bar{\not{D}})\,\bar{\mathcal{P}}_0 \equiv \{V \in \mathcal{A} \mid [\bar{\not{D}}, V]\,\bar{\mathcal{P}}_0 = 0\} \tag{6.19}$$

It is the largest subalgebra of $\mathcal{A}$ with the property

$$\bar{\mathcal{A}}_0\bar{\mathcal{H}}_0 = \bar{\mathcal{H}}_0 \tag{6.20}$$

To describe the restriction of $\bar{\mathcal{A}}_0$ to $\bar{\mathcal{H}}_0^{(-)}$, consider a typical homogenous smeared vertex operator $V_\Omega(q^+,q^-) \in \mathcal{A}_X$ of type Ω and momentum $(q^+,q^-) \in \Lambda$. Since

$$\bar{\mathcal{P}}_0^{(-)}\left[\bar{\not{D}}, \mathbb{I}\otimes V_\Omega(q^+,q^-)\right]\bar{\mathcal{P}}_0^{(-)} = g^{\mu\nu}\gamma_\mu \otimes \left(q_\nu^+ - q_\nu^-\right)\bar{\mathcal{P}}_0^{(-)}\,V_\Omega(q^+,q^-)\,\bar{\mathcal{P}}_0^{(-)} \tag{6.21}$$

it follows that the subalgebra $\bar{\mathcal{P}}_0^{(-)}\bar{\mathcal{A}}_0\bar{\mathcal{P}}_0^{(-)}$ consists of those vertex operators which create string states of identical left and right chiral momentum, i.e. $\sqrt{2}q_\mu^+ = \sqrt{2}q_\mu^- = q_\mu \in \Gamma^*$, which again agrees heuristically with the zero winding number restriction of the low-energy sector. For the basis (tachyon) vertex operators of $\mathcal{A}$ we have in general that

$$\bar{\mathcal{P}}_0\,(\mathbb{I}\otimes V_{q^+q^-}(1,1))|\psi;p^+,p^-\rangle = |\psi;q^++p^+,q^-+p^-\rangle \tag{6.22}$$

It follows that $V(q,q)$ generate $\bar{\mathcal{P}}_0^{(-)}\bar{\mathcal{A}}_0\bar{\mathcal{P}}_0^{(-)}$ and, in particular, we have

$$(\mathbb{I}\otimes V_{qq}(z_+,z_-))\,\bar{\mathcal{P}}_0^{(-)} = \mathrm{e}^{-iq_\mu x^\mu}\,(z_+z_-)^{q_\mu g^{\mu\nu}p_\nu/2} \tag{6.23}$$

Thus the smeared tachyon generators of $\bar{\mathcal{P}}_0^{(-)}\bar{\mathcal{A}}_0\bar{\mathcal{P}}_0^{(-)}$ coincide with the spacetime functions $\mathrm{e}^{-iq_\mu x^\mu}$ which constitute a basis for the algebra $C^\infty(T^n)$ of smooth (single-valued) functions on the toroidal target space. Thus the low-energy algebra $\bar{\mathcal{A}}_0$ yields a natural isomorphism with the abelian algebra

$$\bar{\mathcal{P}}_0^{(-)}\,\bar{\mathcal{A}}_0\,\bar{\mathcal{P}}_0^{(-)} \cong C^\infty(T^n) \tag{6.24}$$

To summarize then, we have proven that there is a natural isomorphism between the spectral triples:

$$\bar{\mathcal{P}}_0^{(-)}\mathcal{T}_{\bar{D\!\!\!/}} \equiv \left(\bar{\mathcal{P}}_0^{(-)}\,\bar{\mathcal{A}}_0\,\bar{\mathcal{P}}_0^{(-)}\ ,\ \bar{\mathcal{H}}_0^{(-)}\ ,\ \bar{D\!\!\!/}\,\bar{\mathcal{P}}_0^{(-)}\right) \cong \left(C^\infty(T^n)\ ,\ L^2(\mathrm{spin}^-(T^n))\ ,\ ig^{\mu\nu}\gamma_\mu\partial_\nu\right) \tag{6.25}$$

This says that the low-energy projection of the spectral triple (6.1) determined by the kernel of the Dirac operator $\bar{D\!\!\!/}$ coincides with the spectral triple that describes the ordinary (commutative) spacetime geometry of the n-torus $T^n \cong \mathbb{R}^n/2\pi\Gamma$ with metric $g_{\mu\nu}$. Thus the full noncommutative spacetime (6.1) of the string theory can be projected onto an ordinary, commutative spacetime via an explicit choice of Dirac operator. A key feature of these low-energy projections is that the corresponding algebras $\bar{\mathcal{A}}_0$ consist only of the zero-mode components of the tachyon vertex operators. The full noncommutative spacetime is then built from the highest-weight states of the current algebra which are eigenstates of the Dirac operator $D\!\!\!/$,

$$\left[D\!\!\!/,\mathbb{I}\otimes V_\Omega(q^+,q^-)\right] = g^{\mu\nu}\left(\gamma_\mu^+\otimes q_\nu^+ + \gamma_\mu^-\otimes q_\nu^-\right)V_\Omega(q^+,q^-) \tag{6.26}$$

which is just another feature of the spectral action principle [11].

Now let us treat the second Dirac operator $\bar{D\!\!\!/}$ in (6.4) in an analogous way. It yields another low-energy subspace of the Hilbert space $\mathcal{H}$,

$$\mathcal{H}_0 \equiv \ker\,\bar{D\!\!\!/} \cong \bigotimes_{\mu=1}^n\left(\mathcal{H}_0^{(+)\mu}\oplus\mathcal{H}_0^{(-)\mu}\right) \tag{6.27}$$

where

$$\begin{aligned}\mathcal{H}_0^{(+)\mu} &= \left\{|\psi\rangle\otimes|p^+;p^-\rangle\in\mathcal{H}_0\mid\gamma_\mu^+|\psi\rangle=\gamma_\mu^-|\psi\rangle\ \ ,\ \ w^\mu=0\right\}\\ \mathcal{H}_0^{(-)\mu} &= \left\{|\psi\rangle\otimes|p^+;p^-\rangle\in\mathcal{H}_0\mid g^{\nu\lambda}d_{\lambda\mu}^+\gamma_\nu^+|\psi\rangle=-g^{\nu\lambda}d_{\lambda\mu}^-\gamma_\nu^-|\psi\rangle\ \ ,\ \ p_\mu=0\right\}\end{aligned} \tag{6.28}$$

define the chiral and anti-chiral subspaces of the kernel of $\bar{D\!\!\!/}_0$ which is obtained from (6.13) by the replacements $\gamma_\mu^\pm\leftrightarrow\pm\gamma_\mu^\pm$. Again the 2^n spin structure subspaces in (6.27) are all naturally isomorphic under partial T-duality, and we therefore take the canonical anti-chiral subspace $\mathcal{H}_0^{(-)}=\mathcal{H}_0^{(-)1}\otimes\mathcal{H}_0^{(-)2}\otimes\cdots\otimes\mathcal{H}_0^{(-)n}$ with the representation of the double Clifford algebra for which $g^{\nu\lambda}d_{\lambda\mu}^+\gamma_\nu^+=-g^{\nu\lambda}d_{\lambda\mu}^-\gamma_\nu^-\equiv\tilde{\gamma}_\mu\ \ \forall\mu$ (which is dual to the anti-chirality condition of the subspace (6.16)). In this subspace we have $p_\mu=0\ \ \forall\mu$ so that $\sqrt{2}(d^+)^{\mu\nu}p_\nu^+=-\sqrt{2}(d^-)^{\mu\nu}p_\nu^-=w^\mu\in\Gamma$. $\mathcal{H}_0^{(-)}$ is also naturally isomorphic to the low-energy particle Hilbert space (6.17), with the dual anti-chiral spinor representation $(\varrho^-)^*$, under the identification $|d^+w;-d^-w\rangle\leftrightarrow \mathrm{e}^{-id_{\mu\lambda}^-g^{\lambda\rho}d_{\rho\nu}^+w^\nu x^\mu}$, and the holomorphic Dirac operator action is given by

$$D\!\!\!/\,\mathcal{P}_0^{(-)} = i\tilde{g}^{\mu\nu}\tilde{\gamma}_\mu\otimes\frac{\partial}{\partial x^\nu} \tag{6.29}$$

where the dual metric $\tilde{g}_{\mu\nu}$ is defined by (6.10). Again the full low-energy Hilbert space $\mathcal{H}_0$ is a maximal invariant subspace for the Dirac operator $\not{\bar{D}}$.

Finally, we define the maximal subalgebra

$$\mathcal{A}_0 = \mathcal{P}_0\,(\mathrm{comm}\ \bar{\not{D}})\,\mathcal{P}_0 \tag{6.30}$$

of the smeared vertex operator algebra $\mathcal{A}$ with the property $\mathcal{A}_0\mathcal{H}_0 = \mathcal{H}_0$. Since

$$\mathcal{P}_0^{(-)}\left[\bar{\not{D}},\mathbb{I}\otimes V_\Omega(q^+,q^-)\right]\mathcal{P}_0^{(-)} = \tilde{\gamma}_\mu\otimes\left((d^+)^{\nu\mu}q_\nu^+ + (d^-)^{\nu\mu}q_\nu^-\right)\mathcal{P}_0^{(-)}\,V_\Omega(q^+,q^-)\,\mathcal{P}_0^{(-)} \tag{6.31}$$

it follows that the subalgebra $\mathcal{P}_0^{(-)}\mathcal{A}_0\mathcal{P}_0^{(-)}$ consists of smeared vertex operators $V_\Omega(q^+,q^-)$ with $\sqrt{2}(d^+)^{\nu\mu}q_\nu^+ = -\sqrt{2}(d^-)^{\nu\mu}q_\nu^- = v^\mu \in \Gamma$. $\mathcal{P}_0^{(-)}\mathcal{A}_0\mathcal{P}_0^{(-)}$ is generated by the smeared tachyon vertex operators $V(d^+v,-d^-v)$ which coincide with the basis spacetime functions $\mathrm{e}^{-i\tilde{g}_{\mu\nu}v^\nu x^\mu}$ of $C^\infty(T^n)$. Thus there is also the natural isomorphism of spectral triples,

$$\mathcal{P}_0^{(-)}\mathcal{T}_{\not{D}} \equiv \left(\mathcal{P}_0^{(-)}\,\mathcal{A}_0\,\mathcal{P}_0^{(-)}\ ,\ \mathcal{H}_0^{(-)}\ ,\ \not{D}\,\mathcal{P}_0^{(-)}\right) \cong \left(C^\infty(T^n)\ ,\ L^2(\mathrm{spin}^-(T^n)^*)\ ,\ i\tilde{g}^{\mu\nu}\tilde{\gamma}_\mu\partial_\nu\right) \tag{6.32}$$

which identifies the low-energy projection of $\mathcal{T}$ determined by the kernel of the Dirac operator $\bar{\not{D}}$ with the commutative spacetime geometry of the dual n-torus $(T^n)^* \cong \mathbb{R}^n/2\pi\Gamma^*$ with metric $\tilde{g}^{\mu\nu}$.

According to the T-duality symmetry (6.6) of the effective string spacetime, as subspaces of the quantum spacetime we have the isomorphism

$$\left(C^\infty(T^n)\ ,\ L^2(\mathrm{spin}^-(T^n))\ ,\ ig^{\mu\nu}\gamma_\mu\partial_\nu\right) \cong \left(C^\infty(T^n)\ ,\ L^2(\mathrm{spin}^-(T^n)^*)\ ,\ i\tilde{g}^{\mu\nu}\tilde{\gamma}_\mu\partial_\nu\right) \tag{6.33}$$

which is the usual statement of the T-duality $T^n \leftrightarrow (T^n)^*$ of string theory compactified on an n-torus. Notice how the statement that this duality symmetry corresponds to the target space symmetry $g_{\mu\nu} \leftrightarrow \tilde{g}^{\mu\nu}$ and the symmetry under interchange of momentum and winding numbers in the compactified string spectrum arise very naturally from the point of view of noncommutative geometry. It appears as a discrete $\mathbb{Z}_2$-symmetry of the noncommutative geometry.

Worldsheet Parity

T-duality was shown above to be the linear isomorphism of the string spacetime which relates the anti-chiral low-energy subspaces defined by the pair of Dirac operators (6.4). There are several other symmetries of the string spacetime which also arise from the above construction, represented by the isomorphisms between other pairs of subspaces in (6.14) and (6.27). For example, suppose we consider the chiral subspace of (6.27),

$$\mathcal{H}_0^{(+)} = \mathcal{H}_0^{(+)1}\otimes\mathcal{H}_0^{(+)2}\otimes\cdots\otimes\mathcal{H}_0^{(+)n} \tag{6.34}$$

In this subspace we have $\gamma_\mu^+ = \gamma_\mu^- \equiv \gamma_\mu$ and $w^\mu = 0$ for all μ. The Hilbert space $\mathcal{H}_0^{(+)}$ is also isomorphic to (6.17), with the chiral spinor representation ϱ^+, and the holomorphic Dirac operator action is now

$$\not{D}\,\mathcal{P}_0^{(+)} = -ig^{\mu\nu}\gamma_\mu \otimes \tfrac{\partial}{\partial x^\nu} = -\bar{\not{D}}\,\bar{\mathcal{P}}_0^{(-)} \tag{6.35}$$

Moreover, one finds that again $\mathcal{P}_0^{(+)}\mathcal{A}_0\mathcal{P}_0^{(+)}$ is generated by the smeared tachyon vertex operators $V(q,q) \sim \mathrm{e}^{-iq_\mu x^\mu}$. According to the equivalences discussed above, we then have the isomorphism of low-energy spectral triples

$$\Big(C^\infty(T^n)\ ,\ L^2(\mathrm{spin}^-(T^n))\ ,\ ig^{\mu\nu}\gamma_\mu\partial_\nu\Big) \cong \Big(C^\infty(T^n)\ ,\ L^2(\mathrm{spin}^+(T^n))\ ,\ -ig^{\mu\nu}\gamma_\mu\partial_\nu\Big) \tag{6.36}$$

This quantum symmetry of the string spacetime is the worldsheet parity symmetry of string theory. It is the left-right chirality symmetry which reflects the worldsheet spatial coordinate $\sigma \to -\sigma$ and thus acts on the string background as $\beta \to -\beta$, i.e. $d_{\mu\nu}^\pm \to d_{\mu\nu}^\mp$. It acts on the Hilbert space $\mathcal{H}$ as $\alpha_k^{(\pm)\mu} \to \alpha_k^{(\mp)\mu}$, $p_\mu^\pm \to p_\mu^\mp$ and $\gamma_\mu^\pm \to \pm\gamma_\mu^\mp$. Because it interchanges the left and right chirality sectors of the string theory, it flips the sign of the Lorentzian quadratic form (3.16) and is thus not an automorphism of the lattice Λ. In terms of the full spectral triple describing the quantum spacetime, it is the $\mathbb{Z}_2$-transformation $T \equiv W_S \otimes W_X : \mathcal{H} \to \mathcal{H}$ that achieves (6.5) and (6.6) by mapping

$$\begin{aligned} W_X\,\alpha_k^{(\pm)\mu}\,W_X^{-1} &= \alpha_k^{(\mp)\mu} \\ W_S\,\gamma_\mu^\pm\,W_S^{-1} &= \pm\gamma_\mu^\mp \\ W_X\,V_\Omega(q^+,q^-)\,W_X^{-1} &= V_\Omega(q^-,q^+) \end{aligned} \tag{6.37}$$

which amounts to the interchange $\pm \leftrightarrow \mp$ on both $\mathcal{H}$ and $\mathcal{A}$. This worldsheet quantum symmetry of the sigma-model thus also arises as a change of Dirac operator for the noncommutative geometry, and so the isometries of the spectral triple (6.1) account for both target space *and* discrete worldsheet symmetries of the quantum geometry.

Factorized Duality and Spacetime Topology Change

The next generalization of the T-duality isomorphism of low-energy sectors is to compare the anti-chiral subspace (6.16) with the subspaces

$$\mathcal{H}_0^{[+;\mu]} = \mathcal{H}_0^{(+)1} \otimes \cdots \otimes \mathcal{H}_0^{(+)\mu-1} \otimes \mathcal{H}_0^{(-)\mu} \otimes \mathcal{H}_0^{(+)\mu+1} \otimes \cdots \otimes \mathcal{H}_0^{(+)n} \tag{6.38}$$

which are defined for each $\mu = 1,\dots,n$. In $\mathcal{H}_0^{[+;\mu]}$ we have $\gamma_\nu^+ = \gamma_\nu^- \equiv \gamma_\nu$ and $w^\nu = 0$ for all $\nu \neq \mu$, while $g^{\lambda\rho}d_{\lambda\mu}^+\gamma_\rho^+ = -g^{\lambda\rho}d_{\lambda\mu}^-\gamma_\rho^- \equiv \tilde{\gamma}_\mu$ and $p_\mu = 0$. This Hilbert space is isomorphic to (6.17) with the mixed chirality spinor representation $\varrho^{(\mu)}$ determined by the spinor conditions of (6.38), where the explicit L^2-isomorphism maps the bosonic states of (6.38) to the functions

$$f^{(\mu)}(x) = \exp\Big(-i\textstyle\sum_{\nu\neq\mu} p_\nu x^\nu - i\sum_\lambda \tilde{g}_{\mu\lambda}w^\mu x^\lambda\Big) \tag{6.39}$$

The restriction of the holomorphic Dirac operator to (6.38) is

$$\not{D}\,\mathcal{P}_0^{[+;\mu]} = -i\sum_{\nu\neq\mu} g^{\nu\lambda}\gamma_\nu\otimes\frac{\partial}{\partial x^\lambda} + i\sum_\lambda \tilde{g}^{\mu\lambda}\tilde{\gamma}_\mu\otimes\frac{\partial}{\partial x^\lambda} \tag{6.40}$$

The algebra $\mathcal{P}_0^{[+;\mu]}\mathcal{A}_0\mathcal{P}_0^{[+;\mu]}$ is generated by the smeared tachyon vertex operators which are given by the basis spacetime functions (6.39) of $C^\infty(T^n)$, and we arrive at the spectral triple isomorphism

$$\begin{aligned}&\Big(C^\infty(T^n)\ ,\ L^2(\mathrm{spin}^-(T^n))\ ,\ ig^{\mu\nu}\gamma_\mu\partial_\nu\Big)\\ &\qquad\cong\Big(C^\infty(T^n)\ ,\ L^2(\mathrm{spin}^{(\mu)}(T^n))\ ,\ -i\sum_{\nu\neq\mu}\sum_\lambda g^{\nu\lambda}\gamma_\nu\partial_\lambda + i\sum_\lambda\tilde{g}^{\mu\lambda}\tilde{\gamma}_\mu\partial_\lambda\Big)\end{aligned} \tag{6.41}$$

The symmetry (6.41) of the string spacetime is called 'factorized duality'. For each $\mu = 1,\dots,n$ it is a generalization of the $R\to 1/R$ circle duality in the X^μ direction of T^n. This becomes more transparent if we choose a particular basis of the lattice Γ that splits the n-torus into a product of a circle S^1 of radius R_μ and an $(n-1)$-dimensional background T^{n-1}. The factorized duality map then takes $R_\mu\to 1/R_\mu$ leaving T^{n-1} unchanged, and at the same time interchanges the $\mu^{\rm th}$ momentum and winding mode $g^{\mu\nu}p_\nu\leftrightarrow w^\mu$ leaving all others invariant. Acting on the full spectral triple of the noncommutative spacetime it is described formally as follows. Let $(E_\mu)^{\nu\lambda} = \delta^\nu_\mu\delta^\lambda_\mu$ be the n-dimensional step operators, and consider the unitary transformation $T\equiv\mathcal{D}_S^{(\mu)}\otimes\mathcal{D}_X^{(\mu)}:\mathcal{H}\to\mathcal{H}$ in (6.5) defined by

$$\begin{aligned}\mathcal{D}_X^{(\mu)}\,|p^+,p^-\rangle\otimes|0\rangle_+\otimes|0\rangle_- &= (-1)^{p_\mu w^\mu}\,|\tilde{p}^+;\tilde{p}^-\rangle\otimes|0\rangle_+\otimes|0\rangle_-\\ \mathcal{D}_X^{(\mu)}\,\alpha_k^{(\pm)\nu}\,\mathcal{D}_X^{(\mu)-1} &= \Big[(\delta^\nu_\lambda - g_{\lambda\rho}(E_\mu)^{\nu\rho})\pm(E_\mu)^{\nu\rho}d^\mp_{\rho\lambda}\Big]\,\alpha_k^{(\pm)\lambda}\\ \mathcal{D}_S^{(\mu)}\,\gamma^\pm_\nu\,\mathcal{D}_S^{(\mu)-1} &= \Big[\Big(\delta^\lambda_\nu - g_{\nu\rho}(E_\mu)^{\rho\lambda}\Big) + g_{\nu\sigma}g^{\alpha\lambda}(E_\mu)^{\sigma\rho}d^\mp_{\rho\alpha}\Big]\,\gamma^\pm_\lambda\\ \mathcal{D}_X^{(\mu)}\,V_\Omega(q^+,q^-)\,\mathcal{D}_X^{(\mu)-1} &= (-1)^{q_\mu w^\mu}\,V_\Omega(\tilde{q}^+,\tilde{q}^-)\end{aligned} \tag{6.42}$$

where $\tilde{p}^\pm_\nu = p^\pm_\nu\ \ \forall\nu\neq\mu$ and $\tilde{p}^\pm_\mu = \pm g_{\mu\lambda}(d^\pm)^{\lambda\rho}p^\pm_\rho$ (or equivalently $g^{\mu\nu}p_\nu\leftrightarrow w^\mu$). The mapping (6.42) acts on the background matrices and the metric tensor as

$$\begin{aligned}d^\pm_{\nu\rho}&\to\Big[\Big(\delta^\lambda_\nu - g_{\nu\alpha}(E_\mu)^{\alpha\lambda}\Big)\,d^\pm_{\lambda\gamma} + g_{\nu\alpha}g_{\gamma\beta}(E_\mu)^{\alpha\beta}\Big]\Big([E_\mu\cdot d^\pm + (I_n - g\cdot E_\mu)]^{-1}\Big)^\gamma_{\ \rho}\\ g_{\nu\rho}&\to\Big[\Big(\delta^\lambda_\nu - g_{\nu\alpha}(E_\mu)^{\alpha\lambda}\Big) + g_{\nu\alpha}g^{\lambda\beta}(E_\mu)^{\alpha\gamma}d^-_{\gamma\beta}\Big]\,g_{\lambda\sigma}\,\Big[\Big(\delta^\sigma_\rho - g_{\rho\alpha}(E_\mu)^{\alpha\sigma}\Big) + g_{\rho\alpha}g^{\beta\sigma}(E_\mu)^{\alpha\gamma}d^+_{\gamma\beta}\Big]\end{aligned} \tag{6.43}$$

where I_n is the $n\times n$ identity matrix. As before, the transformation (6.42) leads to the isomorphism (6.6) of the full noncommutative geometry.

When n is even, the factorized duality map yields the famous 'mirror symmetry' of string theory which expresses the equivalence of string spacetimes under the interchange of the complex and Kähler structures of T^n. It is equivalent to the interchange of the Dolbeault cohomology groups $H^{p,q}(T^n)$ and $H^{\frac{n}{2}-p,q}(T^n)$ which gives a mirror reflection along the diagonals of the corresponding Hodge diamonds. The change of Dirac operator in (6.5) therefore also yields the stringy phenomenon of *spacetime topology change*. It arises in the present point of view from the non-trivial chirality structure which is present in the spin bundle of T^n when n is even (see (4.5)) ‡. Furthermore, the above analysis shows that generally a factorized duality transformation in the $\mu^{\rm th}$ direction must be accompanied by a worldsheet parity transformation in all of the other $n-1$ directions. This is somewhat anticipated from our earlier remarks about the relationship between the spin structures of T^n and T-duality, and it agrees with some basic statements concerning mirror symmetry when n is even. Thus the noncommutative geometry formulation of the quantum geometry shows that worldsheet parity is in fact a crucial part of the duality symmetries. The remaining isomorphisms between pairs of subspaces in (6.14) and (6.27) are then combinations of the discrete duality transforms exhibited above.

Lattice Isomorphism

The duality symmetries which were essentially deduced above from the various isomorphisms that exist between the low-energy spectral triples exhaust the transformations (6.5) which lead to non-trivial equivalences between spacetimes of distinct geometry and topology. There are, however, two other discrete "internal" spacetime symmetries that leave each of the Dirac operators in (6.4) invariant and trivially leave the corresponding spectral triples unaffected. These transformations do not affect the classical spacetimes, but they do lead to non-trivial effects in the quantum field theory and are therefore associated with symmetries of the quantum geometry.

The first one is a change of basis of the compactification lattice Γ, which is described by an invertible, integer-valued matrix $[A^\mu_\nu]\in GL(n,\mathbb{Z})$ that acts on the spacetime metric as

$$g_{\mu\nu}\to(A^{-1})^\lambda_\mu\,g_{\lambda\rho}\,(A^{-1})^\rho_\nu \tag{6.44}$$

In general, all covariant tensors from which the spectral triples are built transform under A^{-1} while all contravariant tensors transform under A. This leads to a simple reparametrization of all quantities composing the spectral triples, thus leaving them unaltered. This change of basis is therefore also trivially a symmetry of the low-energy commutative string spacetime (6.2). For instance, some of these $GL(n,\mathbb{Z})$ transformations simply permute the spacetime dimensions, while others reflect the configurations $X^\mu\to -X^\mu$. T-duality can be shown to be the composition of a succession of factorized dualities and dimension permutations in all of the directions of T^n. This is naturally evident from the isomorphisms between (6.14) and (6.27), where appropriate permutations of the spin structures map factorized duality onto T-duality.

‡More complicated duality symmetries, such as mirror symmetry between distinct, curved Calabi-Yau manifolds [1], can be obtained by introducing a larger set of Dirac operators which are related to, for instance, an $N=2$ supersymmetric sigma-model. The resulting spectral triple contains a larger symmetry than just the chiral-antichiral one used in this paper. Some of these points are addressed in [7, 9].

Torsion Cohomology

The final quantum symmetry is the shift

$$\beta_{\mu\nu} \to \beta_{\mu\nu} + C_{\mu\nu} \tag{6.45}$$

of the spacetime torsion form by an antisymmetric, integer-valued matrix $C_{\mu\nu}$. This shift corresponds to a change of integer cohomology class of the instanton form, and thus only affects the winding numbers in the target space T^n. It can be absorbed by a shift $p_\mu \to p_\mu - C_{\mu\nu}w^\nu$ which simply yields a reparametrization of the momenta $\{p_\mu\} \in \Gamma^*$. All other quantities are left invariant by this shift, and the action of the Dirac operators on $(\mathcal{A}, \mathcal{H})$ is unaffected by this discrete transformation. Again we trivially have the equivalence between the corresponding string spacetimes.

Quantum Moduli Space

It can be shown that the discrete transformations exhibited in this section generate the duality group of the string theory, which is the semi-direct product

$$G_d = O(n,n;\mathbb{Z}) \otimes_{\rm S} \mathbb{Z}_2 \tag{6.46}$$

of the lattice automorphism group $O(n,n;\mathbb{Z})$ (i.e. the group of transformations of Λ that preserve the quadratic form (3.16)) by the action of the reflection group $\mathbb{Z}_2$ corresponding to worldsheet parity. The group $O(n,n;\mathbb{Z})$ is the arithmetic subgroup of $O(n,n)$ and it acts on the background matrices $d^{\pm}_{\mu\nu}$ by linear fractional transformations. Note that inside the duality group G_d lies the discrete geometrical subgroup $SL(n,\mathbb{Z}) \subset O(n,n;\mathbb{Z})$ which represents the group of large diffeomorphisms of T^n. The *quantum* modification of (6.3) is therefore given by the Narain moduli space [36]

$$\mathcal{M}_{\rm qu} = O(n,n;\mathbb{Z}) \setminus O(n,n)/(O(n) \times O(n)) \otimes_{\rm S} \mathbb{Z}_2 \tag{6.47}$$

where the quotient by the infinite discrete group $O(n,n;\mathbb{Z})$ acts on $O(n,n)$ from the left. As we have discussed, the duality group (6.46) is a discrete subgroup of the group $\mathrm{Aut}(\mathcal{A})$ of automorphisms of the vertex operator algebra $\mathcal{A}$. In the next section we shall discuss the structure of $\mathrm{Aut}(\mathcal{A})$ in a somewhat more general setting.

7. Symmetries of the Noncommutative String Spacetime

In the previous section we examined the symmetries of the quantum spacetime by determining the discrete automorphisms of the spectral triples which led to isomorphisms between their low-energy projection subspaces. The full duality group (6.46) was thus determined as the set of all isomorphisms between subspaces of (6.14) and (6.27). Along the way we showed that this way of viewing the target space duality in noncommutative geometry led to new insights into the relationships between choices of spin structure, worldsheet parity, and T-duality. However, there are many more possible automorphisms of the vertex operator algebra $\mathcal{A}$ than just those which preserve the commutative subspaces. In fact, the elements of the duality group G_d arise from the equivalences between the zero-mode eigenspaces of the Dirac operators (6.4), while the general isospectral automorphisms of the spectral triple (6.1) are determined by unitary transformations (such as (6.5)) between different Dirac operators that have the same spectrum [11]. Indeed, the structure of the full string spacetime is determined by the spectrum of the Dirac K-cycle $(\mathcal{H}, \not{D})$ or $(\mathcal{H}, \bar{\not{D}})$ (see (6.26))), which in turn incorporates the non-zero oscillatory modes of the strings. In this final section we shall examine the geometrical symmetries of the string spacetime in a more general setting by viewing them as automorphisms of the vertex operator algebra. This point of view will, among other things, naturally establish the framework for viewing duality as a gauge symmetry [1]. We shall also comment on some non-metric aspects of the noncommutative geometry.

Automorphisms of the Vertex Operator Algebra

The basic symmetry group of the noncommutative string spacetime (6.1) is $\mathrm{Aut}(\mathcal{A})$. An *automorphism* of the vertex operator algebra is a unitary transformation $g : \mathcal{H} \to \mathcal{H}$ which preserves both the vacuum state and conformal vectors $\omega^\pm$ defined in (5.17), i.e. $g|\mathrm{vac}\rangle = |\mathrm{vac}\rangle$ and $g\omega^\pm = \omega^\pm$, such that the actions of g and $\mathcal{V}(\Psi; z_+, z_-)$ on $\widehat{\mathcal{H}}_X(\Lambda)$ are compatible in the sense that

$$g\, \mathcal{V}(\Psi; z_+, z_-)\, g^{-1} = \mathcal{V}(g\Psi; z_+, z_-) \quad , \quad \forall \Psi \in \widehat{\mathcal{H}}_X(\Lambda) \tag{7.1}$$

Thus g preserves the stress-energy tensors, and hence the representations of the Virasoro subalgebras, and the mapping $\mathcal{V}$ on (5.16) is equivariant with respect to the natural adjoint actions of g. The automorphism g also preserves the grading of $\widehat{\mathcal{H}}_X(\Lambda)$ (i.e. the subspaces (5.29)) which can be decomposed into a direct sum of the eigenspaces of g,

$$\widehat{\mathcal{H}}_X(\Lambda) = \bigoplus_{j \in \mathbb{Z}_r} \widehat{\mathcal{H}}_X^{(j)}(\Lambda) \tag{7.2}$$

where $r > 0$ is the order of g and $\widehat{\mathcal{H}}_X^{(j)}(\Lambda) = \{\Psi \in \widehat{\mathcal{H}}_X(\Lambda) \mid g\Psi = \eta^j \Psi\}$ with η the generator of $\mathbb{Z}_r$. Note that, from (4.23), the invariance of the stress-energy tensors automatically implies invariances among the Dirac operators $\not{D}^\pm$. Two immediate examples with $r = \infty$ are provided by the Kac-Moody and Virasoro transformation groups of the sigma-model. The former group decomposes $\mathcal{A}$ in terms of the spectrum of the Dirac operators. Generally, given a subgroup $G \subset \mathrm{Aut}(\mathcal{A})$, the Fock space $\widehat{\mathcal{H}}_X(\Lambda)$ decomposes into a direct sum of irreducible G-modules $\widehat{\mathcal{H}}_X^{[R(G)]}(\lambda)$, $\widehat{\mathcal{H}}_X(\Lambda) = \bigoplus_{R(G)} \widehat{\mathcal{H}}_X^{[R(G)]}(\Lambda)$.

In the ordinary commutative case of a manifold M, the group $\mathrm{Diff}(M)$ of diffeomorphisms of M is naturally isomorphic to the group of automorphisms of the abelian algebra

$\mathcal{A} = C^\infty(M)$. To each $\varphi \in \mathrm{Diff}(M)$ one associates the algebra-preserving map $g_\varphi : \mathcal{A} \to \mathcal{A}$ by $g_\varphi(f) = f \circ \varphi^{-1}$ $\forall f \in \mathcal{A}$. In the general noncommutative case, the group $\mathrm{Aut}(\mathcal{A})$ has a natural normal subgroup $\mathrm{Inn}(\mathcal{A}) \subset \mathrm{Aut}(\mathcal{A})$ consisting of *inner* automorphisms of $\mathcal{A}$, i.e. the algebra-preserving maps $g_u : \mathcal{A} \to \mathcal{A}$ that act on the algebra as conjugation by elements of the group (2.30) of unitary operators in $\mathcal{A}$,

$$g_u(a) = uau^\dagger \quad , \quad \forall a \in \mathcal{A} \tag{7.3}$$

The exact sequence of groups

$$\mathbb{I} \to \mathrm{Inn}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to \mathbb{I} \tag{7.4}$$

defines the remaining *outer* automorphisms in $\mathrm{Aut}(\mathcal{A})$ such that the automorphism group is the semi-direct product

$$\mathrm{Aut}(\mathcal{A}) = \mathrm{Inn}(\mathcal{A}) \otimes_{\mathrm{S}} \mathrm{Out}(\mathcal{A}) \tag{7.5}$$

of $\mathrm{Inn}(\mathcal{A})$ by the natural action of $\mathrm{Out}(\mathcal{A})$.

Note that for an *abelian* algebra $\mathcal{A}$ the group of inner automorphisms $\mathrm{Inn}(\mathcal{A}) = \{\mathbb{I}\}$ is trivial, so that in the case of a commutative space M the diffeomorphisms of the manifold correspond to outer automorphisms. Recall from section 2 that the group (2.30) of unitary elements of an algebra $\mathcal{A}$ defines a natural gauge group of the space. In this context inner automorphisms then correspond to gauge transformations. For example, the automorphism group (7.5) of the noncommutative algebra (2.40) of the standard model is the semi-direct product

$$\mathrm{Aut}(\mathcal{A}_{SM}) = C^\infty(M, U(1) \times SU(2) \times U(3)) \otimes_{\mathrm{S}} \mathrm{Diff}(M) \tag{7.6}$$

of the group of local gauge transformations by the natural action of the diffeomorphism group of M. The inner automorphisms in this case are therefore associated with the local internal gauge invariance of the model while the outer automorphisms represent the spacetime general covariance dictated by general relativity. In fact, (7.6) is the canonical invariance group of the standard model coupled to Einstein gravity, modulo an overall $U(1)$ phase group which can be eliminated by restricting to the unimodular group of $\mathcal{A}_{SM}$. In the general case then, one can identify the outer automorphisms of the noncommutative string spacetime as general coordinate transformations and the inner automorphisms as internal gauge symmetry transformations [43], i.e. internal fluctuations of the noncommutative geometry corresponding to the rotations (7.3) of the elements of $\mathcal{A}$. We shall see that these symmetry structures of the string spacetime have some remarkable features.

Duality Transformations as Inner Automorphisms and Gauge Symmetries

We shall now begin to explore the structure of the symmetry group $\mathrm{Aut}(\mathcal{A})$. For illustration, we start by giving the explicit duality maps T that were exhibited in the previous section and show that they correspond to inner automorphisms of the vertex operator algebra. A general formalism for viewing symmetries of string theory as inner automorphisms of the vertex operator algebra has been given in [44] and applied to duality transformations in [15]. The basic idea behind this approach is that such an inner automorphism represents a deformation of the conformal field theory by a marginal operator, and as such it represents the same point in the corresponding moduli space. Here we wish to stress the fact that such automorphisms arise quite naturally from the point of view of the noncommutative geometry formalism and lead immediately to the well-known interpretation of duality as a gauge symmetry [1].

Recall that the Dirac operators were constructed from the generators (4.1) of the fundamental $U(1)_+^n\times U(1)_-^n$ gauge symmetry of the theory. It turns out that this symmetry group is augmented at the fixed point of the T-duality transformation of the string spacetime. T-duality is tantamount to the inversion $d^\pm\to(d^\pm)^{-1}$ of the background matrices. This transformation has a unique fixed point $(d^\pm)^2=I_n$ given by $g_{\mu\nu}=\delta_{\mu\nu}$ and $\beta_{\mu\nu}=0$. At this single fixed point the generic $U(1)_+^n\times U(1)_-^n$ gauge symmetry is 'enhanced' to a level 1 representation of the affine Lie group $\widehat{SU(2)}_+^n\times\widehat{SU(2)}_-^n$ [45]. Thus the fixed point $\Lambda_0\in\mathcal{M}_{\rm qu}$ in the Narain moduli space of toroidal compactifications coincides with the occurrence of 'enhanced gauge symmetries'. It is due to the appearance of extra dimension $(\Delta^+,\Delta^-)=(1,0)$ and $(0,1)$ operators in the theory. To describe this structure, let $k_\mu^{(i)}$, $i=1,\dots,n$, be a suitable basis of (constant) Killing forms on T^n which are the generators of isometries of the spacetime metric $g_{\mu\nu}$. Then the operators

$$J_\pm^{\alpha(i)}(z_\alpha)=\,:\,{\rm e}^{\pm ik_\mu^{(i)}X_\alpha^\mu(z_\alpha)}\,:\qquad,\qquad J_3^{\alpha(i)}(z_\alpha)=ik_\mu^{(i)}J_\alpha^\mu(z_\alpha)\tag{7.7}$$

where $\alpha=\pm$, generate a level 1 $su(2)_+^n\oplus su(2)_-^n$ Kac-Moody algebra,

$$\left[J_{3,k}^{\alpha(i)}\,,\,J_{\pm,m}^{\alpha(i)}\right]=\pm J_{\pm,k+m}^{\alpha(i)}\qquad,\qquad\left[J_{+,k}^{\alpha(i)}\,,\,J_{-,m}^{\alpha(i)}\right]=2J_{3,k+m}^{\alpha(i)}+2k\delta_{k+m,0}\tag{7.8}$$

with all other commutators vanishing, and where we have defined the mode expansions $J_a^{\alpha(i)}(z_\alpha)=\sum_{k\in\mathbb{Z}}J_{a,k}^{\alpha(i)}z_\alpha^{-k-1}$.

Associated with the $SU(2)_+^n\times SU(2)_-^n$ gauge symmetry of the theory is the corresponding gauge group element

$$g={\rm e}^{i\mathcal{G}_\chi}\tag{7.9}$$

where the generator $\mathcal{G}_\chi$ is defined as the smeared operator

$$\mathcal{G}_\chi=\int\frac{dz_+\,dz_-}{4\pi z_+z_-}\,\left(\chi_{+,\mu}^a[X]\,J_a^{+(\mu)}(z_+)+\chi_{-,\mu}^a[X]\,J_a^{-(\mu)}(z_-)\right)f_S(z_+,z_-)\tag{7.10}$$

and the gauge parameter functions $\chi_{\pm,\mu}^a[X]$, $a=1,2,3$, $\mu=1,\dots,n$, are sections of the spin bundle of T^n. Here and in the following we define $X=\frac{1}{\sqrt2}(X_++X_-)$. The unitary operators (7.9) locally decompose as $g=g_S\otimes g_X$, as in the previous section, where the operators g_X act as inner automorphisms of $\mathcal{A}_X$. The automorphisms g_S are defined by their corresponding actions on the generators $\gamma_\mu^\pm$ and lead to reparametrizations of the spin structure of the target space.

The operators (7.9) that implement spacetime duality transformations of the string theory have been constructed in [15]. The $\mu^{\rm th}$ factorized duality map corresponds to the action of the inner automorphism (7.9) with $\mathcal{G}_\chi = \mathcal{G}^{(\mu)}$ where

$$\mathcal{G}^{(\mu)} = \frac{\pi}{2i} \int \frac{dz_+ \, dz_-}{4\pi z_+ z_-} \, \left(J_+^{+(\mu)} J_+^{-(\mu)} - J_-^{+(\mu)} J_-^{-(\mu)} \right) f_S \tag{7.11}$$

Another duality map which comes from the enhanced gauge symmetry follows from choosing $k_\mu^{(i)} = \delta_\mu^i$ and $\mathcal{G}_\chi = \mathcal{G}_+^{(\mu)} + \mathcal{G}_-^{(\mu)}$ where

$$\mathcal{G}_\pm^{(\mu)} = \frac{\pi}{2i} \int \frac{dz_+ \, dz_-}{4\pi z_+ z_-} \, \left(J_+^{\pm(\mu)} - J_-^{\pm(\mu)} \right) f_S \tag{7.12}$$

The inner automorphism generated by (7.12) corresponds to a reflection $X^\mu \to -X^\mu$ of the coordinates of T^n and is part of the lattice isomorphism group $GL(n, \mathbb{Z})$. Thus factorized dualities and spacetime reflections are enhanced gauge symmetries of the noncommutative geometry, and as such they are intrinsic properties of the string spacetime.

The remaining $O(n, n; \mathbb{Z})$ transformations are abelian gauge symmetries. By the definition of the currents (4.1), a general spacetime coordinate transformation $X \to \xi(X)$, with $\xi(X)$ a local section of spin(T^n), is generated by $\mathcal{G}_\chi = \mathcal{G}_\xi$ with

$$\mathcal{G}_\xi = \int \frac{dz_+ \, dz_-}{4\pi z_+ z_-} \, \xi_\mu(X) \left(J_+^\mu(z_+) + J_-^\mu(z_-) \right) f_S(z_+, z_-) \tag{7.13}$$

Taking the large diffeomorphism $\xi_\mu(X) = \xi_\mu^{(\pi)}(X) = \frac{\pi}{2} {\rm sgn}(\pi) g_{\pi(\mu),\nu} X^\nu$ then yields a permutation $\pi \in S_n$ of the coordinates of T^n (corresponding to another lattice isomorphism). Combining this with the factorized duality transformations above yields T-duality in the form of an inner automorphism. As such, T-duality corresponds to the global gauge transformation in the Weyl subgroup $\mathbb{Z}_2$ of $SU(2)$. Next, the local gauge transformations $\beta \to \beta + d\lambda$ of the torsion two-form are generated by $\mathcal{G}_\chi = \mathcal{G}_\lambda$ with

$$\mathcal{G}_\lambda = \int \frac{dz_+ \, dz_-}{4\pi z_+ z_-} \, \lambda_\mu(X) \left(J_+^\mu(z_+) - J_-^\mu(z_-) \right) f_S(z_+, z_-) \tag{7.14}$$

Taking the gauge transformation $\lambda_\mu(X) = C_{\mu\nu} X^\nu$, with $C_{\mu\nu}$ a constant antisymmetric matrix, gives effectively the torsion shift (6.45). Singlevaluedness of the corresponding group element (7.9) then forces $C_{\mu\nu} \in \mathbb{Z} \;\; \forall \mu, \nu$ yielding a large gauge transformation. The final $O(n, n; \mathbb{Z})$ transformations correspond to large diffeomorphisms $\xi_\mu(X) = T_{\mu\nu} X^\nu$ of the n-torus. Again singlevaluedness of the corresponding gauge group element puts $[T_{\mu\nu}] \in SL(n, \mathbb{Z})$.

As anticipated, the $\mathbb{Z}_2$ part of the duality group G_d representing worldsheet parity corresponds to an *outer* automorphism of the vertex operator algebra. This is because it corresponds to the automorphism $W_X \in \mathrm{Aut}(\mathcal{A})$ that interchanges the left and right chiral algebras $\mathcal{E} = \mathcal{E}^+ \otimes \mathcal{E}^- \xrightarrow{W_X} \mathcal{E}^- \otimes \mathcal{E}^+$. Clearly no inner automorphism of $\mathcal{A}$ can achieve this transformation, and indeed worldsheet parity is the outer automorphism of $\mathcal{A}$ represented by the $\mathbb{Z}_2$ generator

$$W_X = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{7.15}$$

acting in the two-dimensional space labelled by the chiral components $\mathcal{E} = \mathcal{E}^+ \otimes \mathcal{E}^-$. This $\mathbb{Z}_2$-symmetry is actually a discrete subgroup of the $U(1)$ worldsheet symmetry group that acts by rotating the chiral sectors among each other. Associated to the spin structure of the string worldsheet there is a representation of $\mathrm{spin}(2) \cong \mathbb{R}$ on the Hilbert space $\mathcal{H}_X$ that implements the group $SO(2) \cong U(1)$ with generator

$$W_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad , \quad \theta \in [0, 2\pi) \tag{7.16}$$

The discrete vertex operator automorphisms, corresponding to large gauge transformations of the internal string spacetime, also represent inner automorphisms of the finite-dimensional Lie group $O(n,n)$. In this context the target space duality group $O(n,n;\mathbb{Z})$ is generated by the groups $G_{\mathrm{Weyl}}(n)$ and $\mathbb{Z}_2$, where the Weyl group $G_{\mathrm{Weyl}}(n)$ of order n is generated by the spacetime coordinate permutations $\xi_\mu^{(\pi)}$, the $SL(n,\mathbb{Z})$ generators $T_{\mu\nu}$, and the torsion cohomology shifts (6.45), while the reflection group $\mathbb{Z}_2$ is the inner automorphism of $O(n,n)$ that permutes two coordinates at the point Λ_0 in the Narain moduli space with identity background matrices. This is the usual description of the duality group $O(n,n;\mathbb{Z})$ as a spontaneously broken gauge symmetry of the toroidal sigma-model. On the other hand, worldsheet parity $\mathbb{Z}_2$ cannot be interpreted in terms of any gauge symmetry and represents a large diffeomorphism of the noncommutative string spacetime.

Thus, as a start, we can identify the infinite-dimensional symmetry algebras which contain the target space duality group $O(n,n;\mathbb{Z})$ as inner automorphisms at the fixed point $\Lambda_0 \in \mathcal{M}_{\mathrm{qu}}$,

$$\mathrm{Inn}(\mathcal{A}_{\Lambda_0}) \supset \left(\widehat{SU(2)}_+^n \times \widehat{SU(2)}_-^n\right) \otimes_{\mathrm{S}} \left(\mathrm{Vir}^+ \times \mathrm{Vir}^-\right) \supset \left(\widehat{U(1)}_+^n \times \widehat{U(1)}_-^n\right) \otimes_{\mathrm{S}} \left(\mathrm{Vir}^+ \times \mathrm{Vir}^-\right) \tag{7.17}$$

where $\mathrm{Vir}^\pm$ denotes the chiral Virasoro groups. The abelian gauge symmetry group in (7.17) (the maximal torus of the nonabelian one) is present at a generic point $\Lambda \neq \Lambda_0$. Furthermore, worldsheet parity is a finite subgroup of the outer automorphism group of $\mathcal{A}$ which contains a finite-dimensional worldsheet rotation group,

$$\mathrm{Out}(\mathcal{A}) \supset O(2) \tag{7.18}$$

It is interesting to note what happens to these automorphisms when projected onto the low-energy sector $\bar{\mathcal{P}}_0^{(-)} \bar{\mathcal{A}}_0 \bar{\mathcal{P}}_0^{(-)}$ representing the ordinary spacetime T^n. Only the inner automorphisms (7.13) act non-trivially on this subalgebra of $\mathcal{A}$ and represent the generators of $\mathrm{Diff}(T^n)$ in terms of the canonically conjugate center of mass variables x^μ, p_μ. The other transformations when restricted to this subalgebra act as the identity $\mathbb{I}$, i.e. as inner automorphisms. Thus, the subgroup (7.17) of the *inner* automorphism group of $\mathcal{A}$, representing internal gauge symmetries of the string spacetime, is projected onto the full group of *outer* automorphisms of the low-energy target space T^n, corresponding to diffeomorphisms of the manifold. This approach therefore naturally identifies the usual invariance principles of general relativity as a *gauge symmetry* of the stringy modification. The diffeomorphisms of the full string spacetime are completely unobservable in the low-energy sector (for instance in the anti-chiral projection onto $\bar{\mathcal{H}}_0^{(-)}$ the operator (7.15) acts as I_n), as are the gauge symmetries corresponding to the duality transformations.

This is indeed another essence of the target space duality in string theory. It is only observable as a symmetry of the huge noncommutative spacetime represented by the full spectral triple (6.1) and acts trivially on the corresponding low-energy projections representing the conventional spacetimes. In fact, the duality automorphisms naturally partition the full vertex operator algebra into sectors, each of which project onto the various low-energy spacetimes we described in the last section. Each such low-energy sector is distinct at the classical level but related to the other ones by the duality maps. At the level of the full spectral triple, there are two sectors corresponding to the two eigenspaces of the duality maps as dictated by the decomposition (7.2). For example, in the case of worldsheet parity, the two eigenspaces consist of holomorphic and anti-holomorphic combinations, respectively, of the chirality sectors of the vertex operator algebra. In a low-energy projection, where the notion of chirality is absent, the effects of duality are unobservable. Similar decompositions can also be made for the larger symmetry groups in (7.17) and (7.18) in terms of their irreducible representations. Note that, given a subgroup G of automorphisms, the G-invariant subspace $\widehat{\mathcal{H}}_X^{(0)}(\Lambda)$ in (7.2) (corresponding to the one-dimensional trivial representation of G) defines a vertex operator subalgebra and hence leads to a subspace of the string spacetime which is invariant under the G-transformations. In particular, the corresponding low-energy subspace from the decomposition with respect to a duality map is then completely unaffected by the duality transformation. Thus, the above presentation of duality (and other symmetries of the string spacetime) naturally leads to a systematic construction of the low-energy projective subspaces that we presented in the previous section.

Universal Gauge Groups and Monster Symmetry

At present, the general structure of the unitary group of the vertex operator algebra $\mathcal{A}$ is not known, nor are its general automorphisms. The group $\mathcal{U}(\mathcal{A})$ represents the complete internal (gauge) symmetry group of the noncommutative spacetime and appears to be quite non-trivial. Even at the commutative level where $\mathcal{A} = C^\infty(M)$, the unitary group is the complicated, infinite-dimensional loop group $C^\infty(M,S^1)$ of the manifold M. The inner automorphism group of the noncommutative string spacetime includes spacetime diffeomorphisms, two copies of the Virasoro group, and the Kac-Moody symmetry groups in (7.17) which contain the spacetime duality symmetries. There are a number of additional infinite-dimensional subalgebras of $\mathcal{A}$ that have been identified as subspaces of the inner automorphism algebra $\mathrm{inn}(\mathcal{A})$, such as the algebras of area-preserving (W_∞) and volume-preserving diffeomorphisms in $n=2$ dimensions [46] and also the weighted tensor algebras described in [47]. In all of these instances the inner automorphisms define appropriate mixings among the chiral Dirac operators $D\!\!\!\!/^{\pm}$ which preserve the conformal invariance of the theory. Indeed, the chiral and conformal properties of the worldsheet theory are, as we have extensively shown in this paper, crucial aspects of the string spacetime.

A classification of $\mathcal{U}(\mathcal{A})$ would ultimately lead to a 'universal symmetry group' of string theory that would contain all unbroken gauge groups and represent the true stringy symmetries of the quantum spacetime. The problem with such a classification scheme though is that the "size" of $\mathcal{U}(\mathcal{A})$ appears to be very sensitive to the lattice $\Lambda\in\mathcal{M}_{\rm qu}$ from which the vertex operator algebra is built. A natural Lie group G_Λ of automorphisms arises from exponentiating the Lie algebra $\mathcal{L}_\Lambda$ associated with the lattice Λ [12] (see the appendix). Then G_Λ acts continuously and faithfully on the vertex operator algebra $\mathcal{A}$. The construction of $\mathcal{U}(\mathcal{A})$ has been discussed by Moore in [48] who considered the appearance of enhanced symmetry points in the Narain moduli space, i.e. points $\Lambda\in\mathcal{M}_{\rm qu}$ at which extra dimension (1,0) and (0,1) operators appear and generate new symmetries of the conformal field theory. For this, one analytically continues in the spacetime momenta and extends the lattice Λ to a module over the Gaussian integers,

$$\Lambda^{(\rm G)}=\Lambda\otimes_{\mathbb{Z}}\mathbb{Z}[i]\subset\Lambda^c \tag{7.19}$$

We then form the corresponding operator Fock space (5.10) based on $\Lambda^{(\rm G)}$ by

$$\widehat{\mathcal{H}}_X(\Lambda^{(\rm G)})=\mathbb{C}\{\Lambda^{(\rm G)}\}\otimes S(\hat h_+^{(-)})\otimes S(\hat h_-^{(-)}) \tag{7.20}$$

so that the corresponding Lie algebra of dimension 1 primary fields (5.29) is (see the appendix)

$$\mathcal{L}_U\equiv\widehat{\mathcal{P}}_1(\Lambda^{(\rm G)})/\bigcup_{k\geq1}\widehat{\mathcal{P}}_1(\Lambda^{(\rm G)})\cap\left(L_{-k}^+\otimes L_{-k}^-\right)\widehat{\mathcal{H}}_X(\Lambda^{(\rm G)}) \tag{7.21}$$

Moore proved that, since the action of $O(n,n;\mathbb{Z})$ on $\mathcal{M}_{\rm qu}$ is transitive, the Lie algebra (7.21) generates a *universal* symmetry group of the string theory, i.e. if $\mathcal{L}_\Lambda$ is the (affine) Lie algebra that appears at an enhanced symmetry point Λ, then there is a natural Lie subalgebra embedding $\mathcal{L}_\Lambda\hookrightarrow\mathcal{L}_U$. We refer to [48] for the details.

We would like to stress that, from the point of view of the noncommutative geometry formalism that we have discussed, not only is the interpretation of duality symmetries as being part of some mysterious gauge group [1] now clarified, but the Lie group generated by (7.21) now has a natural geometrical description in terms of the theory of vertex operator algebras and the noncommutative geometry of $\mathcal{A}$. The Lie algebra $\mathcal{L}_U$ naturally overlies all symmetries of the string spacetime obtained from marginal deformations of the conformal field theory, and geometrically it contains many of the internal rotational symmetries of the noncommutative geometry. This by no means exhausts all of the inner automorphisms of $\mathcal{A}$, but it provides a geometric, universal way of identifying gauge symmetries.

To get an idea of how large the gauge group $\mathcal{U}(\mathcal{A})$ can be, it is instructive to consider a specific example. We consider an $n = (25 + 1)$ dimensional toroidal spacetime defined by Wick rotating the target space coordinate X^{26}. The change from Euclidean to hyperbolic compactification lattices Γ is well-known to have dramatic effects on the structures of the corresponding vertex operator algebra [12, 14] and on the Narain moduli space [48]. Consider the unique 26-dimensional even unimodular Lorentzian lattice $\Gamma = \mathit{\Pi}_{25,1}$. It can be shown [48] that $\Lambda_* = \mathit{\Pi}_{25,1} \oplus \mathit{\Pi}_{25,1} \in \mathcal{M}_{\rm qu}$ is the unique point in the Narain moduli space at which the vertex operator algebra $\mathcal{A}$ completely factorizes between its left and right chiral sectors,

$$\widehat{\mathcal{H}}_X(\Lambda_*) = \mathcal{C}^+ \otimes \mathcal{C}^- \tag{7.22}$$

where

$$\mathcal{C}^{\pm} = \mathbb{C}\{\mathit{\Pi}_{25,1}\} \otimes S(\hat{h}_{\pm}^{*(-)}) \tag{7.23}$$

and $\hat{h}^*_{\pm}$ is the Heisenberg-Weyl algebra (5.5) built on Λ_*. The distinguished point $\Lambda_* \in \mathcal{M}_{\rm qu}$ is an enhanced symmetry point and the corresponding Lie algebra $\mathcal{L}_{\Lambda_*}$ is a *maximal* symmetry algebra, in the sense that it contains all unbroken gauge symmetry algebras. $\mathcal{L}_{\Lambda_*}$ is not, however, universal since the gauge symmetries are not necessarily embedded into it as Lie subalgebras. Again the framework of noncommutative geometry naturally constructs $\mathcal{L}_{\Lambda_*}$ as a symmetry algebra of the string theory.

The Lie algebra $\mathcal{L}_{\Lambda_*} = \mathcal{B} \oplus \mathcal{B}$ is an example of a mathematical entity known as a *Borcherds* or *generalized Kac-Moody* algebra [49], where

$$\mathcal{B} = \widehat{\mathcal{P}}_1(\mathit{\Pi}_{25,1}) / \ker\langle\cdot,\cdot\rangle \tag{7.24}$$

and $\langle\cdot,\cdot\rangle$ is the bilinear form on the Lie algebra of primary fields of weight one defined in the appendix. The root lattice of $\mathcal{B}$ is $\mathit{\Pi}_{25,1}$ along with the set of positive integer multiples of the Weyl vector $\vec{\rho} = (1, 2, \ldots, 25; 70) \in \mathit{\Pi}_{25,1}$, each of multiplicity 24. It is generated by $\varepsilon_{q^{\pm}}$, $\varepsilon_{-q^{\pm}}$, $q^{\pm}_{\mu}\alpha^{(\pm)\mu}_{-1}$ (in each chiral sector) and $e^{m\vec{\rho}}$ where $q^{\pm} \in \mathit{\Pi}_{25,1}$ and $m \in \mathbb{Z}$. The first three of these generators span an infinite-dimensional Kac-Moody algebra of infinite rank. The simple roots of $\mathcal{B}$ are the simple roots of this Kac-Moody algebra, and the positive-norm simple roots of the lattice $\mathit{\Pi}_{25,1}$ lie in the Leech lattice $\Gamma_{\rm Leech}$, which is the unique 24-dimensional even unimodular Euclidean lattice with no vectors of square length two. The symmetries of its Dynkin diagram can be classified according to the automorphism group of the Leech lattice. The Lie algebra $\mathcal{B}$ is called the *fake Monster Lie algebra* [12, 14, 49].

Thus the fake Monster Lie algebra (7.24) is a maximal symmetry algebra of the string theory, so that Borcherds algebras, when interpreted as generalized symmetry algebras of the noncommutative geometry, seem to be relevant for the construction of a universal symmetry of string theory. These algebras, being a natural generalization of affine Lie algebras, may emerge as new symmetry algebras for string spacetimes within the unified framework of vertex operator algebras and noncommutative geometry. The fake Monster Lie algebra can also be used to construct the Monster Lie algebra [12]. This latter algebra is constructed from the Monster vertex operator algebra which is built from the 24-dimensional Leech lattice $\Gamma_{\rm Leech}$ and an order 2 automorphism g induced from the -1 isometry of the lattice. The resulting vertex operator algebra is then 'twisted' by the action of g, i.e. one restricts to the g-invariant vertex operator subalgebra generated by $\widehat{\mathcal{H}}_X^{(0)}(\Gamma_{\rm Leech})$ in (7.2). The underlying operator vector space $\widehat{\mathcal{H}}_X^{(0)}(\Gamma_{\rm Leech})$ is called the *moonshine module* and it generates the *Monster vertex operator algebra*. It can be naturally obtained [14] from the fake Monster vertex operator algebra since $\mathit{\Pi}_{25,1}=\Gamma_{\rm Leech}\oplus\mathit{\Pi}_{1,1}$, where $\mathit{\Pi}_{1,1}$ is the unique two-dimensional even unimodular Lorentzian lattice, and it also arises from the vertex operator algebra associated to a twisted heterotic string theory [50]. Mathematically, the most interesting aspect of this construction is that a subgroup of the automorphism group of the Monster vertex operator algebra is the celebrated Monster group, which is the full automorphism group of the 196884-dimensional Griess algebra that is constructed from the Monster Lie algebra and the moonshine module along the lines described above and in the appendix. The Monster group is the largest finitely-generated simple sporadic group.

The appearance of this Monster symmetry as a gauge symmetry of the noncommutative spacetime emphasizes the point that these exotic mathematical structures, such as those contained in the content of Borcherds algebras, might play a role as a sort of dynamical Lie algebra which changes the Dirac operators and the Fock space gradings. But the underlying noncommutative geometrical structure of the string spacetime remains unchanged. We know of no complete classification of such vertex operator algebra automorphisms, and, in the context of this paper, this remains an important problem to be carried out in order to understand the full set of geometrical symmetries that underlie the stringy modification of classical general relativity.

Differential Topology of the Quantum Spacetime

As a final example of the formalism developed in this paper, we look briefly at the problem of computing the cohomology groups of the noncommutative string spacetime and compare them with the known (DeRham) cohomology groups of the ordinary n-torus

$$H^k(T^n;\mathbb{R})=\begin{cases}\mathbb{R}^{\binom{n}{k}} & \text{for } \ 0\leq k\leq n\\ \{0\} & \text{otherwise}\end{cases}\tag{7.25}$$

The rigorous way to describe the non-metric aspects of noncommutative geometry is through noncommutative K-theory [6], but we shall not enter into this formalism here. Here we shall simply compute the cohomology groups in analogy with the example of a manifold that was presented in section 2. This approach is based on a natural generalization of the Witten complex of section 4 which describes the cohomology (7.25).

We assume, for simplicity, that n is even and that a basis for the compactification lattice Γ has been chosen so that $g_{\mu\nu} = \delta_{\mu\nu}$. In light of our analysis above, no loss of generality occurs with this choice of point in the Narain moduli space. In that case the spin(n) Clifford algebras $\mathcal{C}(T^n)^\pm$ each possess a chirality matrix

$$\gamma_c^\pm = \gamma_1^\pm \gamma_2^\pm \cdots \gamma_n^\pm \tag{7.26}$$

whose actions on the generators of the spin bundle are

$$\left\{\gamma_c^\pm, \gamma_\mu^\pm\right\} = 0 \qquad , \qquad \left[\gamma_c^\pm, \gamma_\mu^\mp\right] = 0 \tag{7.27}$$

The chirality matrices are of order 2, $(\gamma_c^\pm)^2 = \mathbb{I}$.

Our first observation is that the two Dirac operators in (6.4) are related by

$$\bar{\not{D}} = \gamma_c^- \not{D} \gamma_c^- \tag{7.28}$$

As we shall see below, the chirality operators (7.26) define a Klein operator $\tilde{\gamma}$, which provides a natural $\mathbb{Z}_2$-grading, and a Hodge duality operator $\star$ by*

$$\tilde{\gamma} = \gamma_c^+ \gamma_c^- \qquad , \qquad \star = \gamma_c^- \tag{7.29}$$

Recalling the description of section 2 and the discussion in section 3 concerning the Witten complex, we can identify the holomorphic Dirac operator $\not{D}$ as an exterior derivative operator d. According to (7.28) and (7.29), the anti-holomorphic Dirac operator $\bar{\not{D}}$ can then be identified with the co-derivative $\mathrm{d}^\dagger = \star \mathrm{d} \star$. The duality isomorphisms of section 6 then state that the string spacetime is invariant under the exchange between the exterior derivative and its dual $\mathrm{d} \leftrightarrow \mathrm{d}^\dagger = \star\mathrm{d}\star$, which is another well-known characterization of target space duality in string theory.

We can now proceed to construct the complex of differential forms of the noncommutative string spacetime as described in section 2. As always our starting point is

$$\Omega^0_{\not{D}} \mathcal{A} = \mathcal{A} \tag{7.30}$$

Next, we can compute, for $V = \mathbb{I} \otimes V_\Omega \in \mathcal{A}$, the exact one-form

$$\pi_{\not{D}}(dV) = [\not{D}, V] \tag{7.31}$$

*For a definition of the Hodge duality operator in a more general setting which can be applied to the cases where n is odd, see [38].

Since the vertex operators of definite spacetime momentum $(q^+,q^-)\in\Lambda$ span the vertex operator algebra $\mathcal{A}_X$, it follows that all commutators of the form $[J^\mu_\pm,V_\Omega]$ sweep out the space $\mathcal{A}_X$ as V_Ω is varied, i.e. the Dirac operator $\not{D}$ acts densely on $\mathcal{H}$, just as in the commutative case of section 2. Using the explicit form of the Dirac operator we can thus identify the linear space of differential one-forms as

$$\Omega^1_{\not{D}}\mathcal{A}=\mathcal{A}\otimes_{\mathbb{R}}\Big(\mathcal{C}(T^n)^+\oplus\mathcal{C}(T^n)^-\Big)\qquad,\qquad \dim_{\mathcal{A}}\Omega^1_{\not{D}}\mathcal{A}=2n \tag{7.32}$$

with basis $\{\gamma^+_\mu,\gamma^-_\mu\}_{\mu=1}^n$. Similarly, one can proceed to calculate the higher-degree spaces $\Omega^k_{\not{D}}\mathcal{A}$ just as in the example of a spin-manifold described in section 2. As occurred there, we will encounter junk forms for $k\geq 2$, which can be eliminated by antisymmetrizations of the gamma-matrices in each chiral sector. Since the left and right chiral sector gamma-matrices already anticommute, this need not be done for mixed chirality products of the γ's. We therefore arrive at

$$\Omega^k_{\not{D}}\mathcal{A}=\mathcal{A}\otimes_{\mathbb{R}}\left(\bigoplus_{i=0}^k\mathcal{C}(T^n)^{+[i]}\otimes\mathcal{C}(T^n)^{-[k-i]}\right)\qquad,\qquad \dim_{\mathcal{A}}\Omega^k_{\not{D}}\mathcal{A}=\sum_{i=0}^k\binom{n}{i}\binom{n}{k-i} \tag{7.33}$$

where $\mathcal{C}(T^n)^{\pm[j]}$ is the linear space spanned by the antisymmetrized products $\gamma^\pm_{[\mu_1}\cdots\gamma^\pm_{\mu_j]}=\frac{1}{j!}\sum_{\pi\in S_j}\mathrm{sgn}\;\pi\prod_{l=1}^j\gamma^\pm_{\mu_{\pi(l)}}$.

The linear space (7.33) is defined for all $0\leq k\leq n$. What is interesting about the noncommutative differential complex is that, unlike that of the torus T^n, forms of degree higher than n exist. To construct $\Omega^l_{\not{D}}\mathcal{A}$ for $l>n$, we exploit the interpretation of the chirality matrices above as Hodge duality operators. However, on the differential complex, we use a slightly different representation than that given in (7.29) to avoid forms of negative degree. This is simply a matter of convenience, and the entire differential topology can be instead given using the Hodge dual in (7.29). Thus we take

$$\pi_{\not{D}}(\star)=m_{\mathcal{C}}\circ\tilde{\gamma} \tag{7.34}$$

where $m_{\mathcal{C}}$ is the multiplication operator on the double Clifford algebra $\mathcal{C}(T^n)$. If one now proceeds to construct differential l-forms by antisymmetrizations of products of l gamma-matrices $\gamma^\pm_\mu$ for $l>n$, it is straightforward to see that

$$\Omega^l_{\not{D}}\mathcal{A}\cong\tilde{\gamma}\cdot\Omega^{2n-l}_{\not{D}}\mathcal{A}\cong\Omega^{2n-l}_{\not{D}}\mathcal{A}\qquad\text{for}\quad l>n \tag{7.35}$$

This process will terminate at $l=2n$, so that the algebra of differential forms of the noncommutative spacetime is

$$\Omega^*_{\not{D}}\mathcal{A}=\bigoplus_{k=0}^{2n}\Omega^k_{\not{D}}\mathcal{A} \tag{7.36}$$

The action of the Dirac operator $\not{D}$ as defined in (7.31) gives a nilpotent linear map $d:\Omega^k_{\not{D}}\mathcal{A}\to\Omega^{k+1}_{\not{D}}\mathcal{A}$ with $d(\Omega^{2n}_{\not{D}}\mathcal{A})=\{0\}$, while that of $\bar{\not{D}}$ defined by (7.31) and (7.34) gives the adjoint nilpotent map $d^\dagger=\star d\star:\Omega^k_{\not{D}}\mathcal{A}\to\Omega^{k-1}_{\not{D}}\mathcal{A}$ with $d^\dagger(\Omega^0_{\not{D}}\mathcal{A})=\{0\}$.

Thus the chirality structure of the worldsheet theory leads to a "doubling" in the differential complex of the string spacetime. Since the $\mathcal{A}$-module $\Omega^1_{\not{D}}\mathcal{A}$ is finitely-generated and projective, it can be viewed as a cotangent bundle, and one can proceed to equip it with connections, although at the level of a toroidal target space there is really no modification from the classical (commutative) case. Since $\Omega^1_{\not{D}}\mathcal{A}$ is free with basis given by $\{\gamma_\mu^\pm\}_{\mu=1}^n$, we can define a connection $\nabla_{\not{D}}:\Omega^1_{\not{D}}\mathcal{A}\to\Omega^1_{\not{D}}\mathcal{A}\otimes_{\mathcal{A}}\Omega^1_{\not{D}}\mathcal{A}$ by [7, 25]

$$\nabla_{\not{D}}\,\omega=[\not{D},\omega] \tag{7.37}$$

Thus $\{\gamma_\mu^\pm\}_{\mu=1}^n$ constitutes a parallel basis for $\Omega^1_{\not{D}}\mathcal{A}$, i.e. $\nabla_{\not{D}}(\gamma_\mu^\pm\otimes\mathbb{I})=0$, and $\nabla_{\not{D}}$ has vanishing curvature. Thus the string spacetime is a noncommutative space with flat connections of zero torsion, and the curvature properties of the toroidal general relativity are unchanged by stringy effects.

To compute the cohomology ring of the noncommutative spacetime, we define the cohomology group $H^k_{\not{D}}(\mathcal{A})$ to be the linear space spanned by the harmonic differential k-forms, i.e. the k-forms annihilated by both $\not{D}$ and $\bar{\not{D}}$ in the representation $\pi_{\not{D}}$ defined above. For instance, the harmonic zero-forms are the vertex operators $V\in\mathcal{A}$ with

$$[\not{D},V]=0 \tag{7.38}$$

so that

$$H^0_{\not{D}}(\mathcal{A})\cong\mathrm{comm}\ \not{D} \tag{7.39}$$

The situation for the higher degree cohomology groups is similar, except that now one obtains higher-dimensional spaces corresponding to the global string oscillations around the cycles of T^n. After some calculation, we find

$$H^k_{\not{D}}(\mathcal{A})\cong\begin{cases}\mathrm{comm}\ \not{D} & \text{for}\quad k=0\\ \mathcal{A}_{\not{D},\bar{\not{D}}}\otimes_{\mathbb{R}}\mathbb{R}^{\dim_{\mathcal{A}}\Omega^k_{\not{D}}\mathcal{A}} & \text{for}\quad 0<k<2n\\ \mathrm{comm}\ \bar{\not{D}} & \text{for}\quad k=2n\\ \{0\} & \text{otherwise}\end{cases} \tag{7.40}$$

where

$$\mathcal{A}_{\not{D},\bar{\not{D}}}\equiv\mathrm{comm}\ \not{D}\cap\mathrm{comm}\ \bar{\not{D}} \tag{7.41}$$

and the dimension of $\Omega^k_{\not{D}}\mathcal{A}$ is given in (7.33) and by (7.35).

The intermediate cohomology groups for $0<k<2n$ are characterized by vertex operators $V\in\mathcal{A}$ with

$$[\not{D},V]=[\bar{\not{D}},V]=0 \tag{7.42}$$

These harmonic k-forms are the vertex operators which constitute 'isometries' of the string spacetime. From section 6, it follows that $\mathcal{A}_{\not{D},\bar{\not{D}}}$ contains smeared tachyon vertex operators $\mathbb{I}\otimes V(q^+,q^-)$ with $q^+=q^-=0$. In the low-energy projection onto $\bar{\mathcal{H}}_0^{(-)}$, one finds that

$\bar{\mathcal{P}}_0^{(-)}\mathcal{A}_{\not{D},\bar{\not{D}}}\bar{\mathcal{P}}_0^{(-)} \cong \mathbb{C}$. But there are still higher-spin vertex operators of zero charge that survive in (7.41) in the general case. This is wherein most of the stringy modification of the topology of the classical spacetime lies, in that higher-spin oscillatory modes of the strings "excite" the cohomology groups (7.25) leading to generalized, infinitely-many connected components in the string spacetime. The spaces (7.40) essentially represent the vertex operators which are invariant under the global $U(1)_+^n \times U(1)_-^n$ Kac-Moody gauge symmetry of the string theory, and as such they represent the globally diffeomorphism-invariant spacetime observables of the noncommutative geometry. Explicit calculations can eliminate potential vertex operators from belonging to (7.41), for instance the graviton field (5.32). This space consists of those states which belong to the simultaneous zero-mode eigenspaces of the two Dirac operators $\not{D}$ and $\bar{\not{D}}$. At this stage though we have not found any elegant way of characterizing the cohomology (7.40) and it would be interesting to explore these spaces further.

The cohomology for $k = 0$ is determined by the vertex operators of zero winding number but non-zero spacetime momentum, and vice-versa for $k = 2n$. The number of independent "k-cycles" is larger in general than $\binom{n}{k}$ because in the generic high-energy sector the string spacetime distinguishes between chirality combinations and accounts for string oscillations and windings about the circles of T^n. Using the $\mathbb{Z}_2$-grading $\tilde{\gamma}$ (Klein operator) and the Hodge dual $\star$ defined in (7.29), it is also possible to compute topological invariants, such as the Euler characteristic and the Hirzebruch signature, of the noncommutative geometry, in analogy to the Witten complex [9, 38, 39].

Notice that in a low-energy projection the cohomology groups (7.40) do not coincide with (7.25). One needs to first project the complex (7.36) and Dirac operators and *then* compute the cohomology groups along the lines described in section 2. The key feature to this is that then the chirality sectors of the spaces (7.33) become equivalent, and for $l > n$ the spaces (7.35) "fold" back onto the n linear spaces in (7.33). In the general case the cohomology (7.40) leads immediately to the mirror symmetry of the string spacetime. We can naturally define "Dolbeault" cohomology groups $H_{\not{D}}^{k,l}(\mathcal{A})$ by using the chiral and anti-chiral Clifford algebra decompositions in (7.33) to split the spaces (7.40) into holomorphic and anti-holomorphic combinations. Using the chirality matrix we then have $\gamma_c^- \cdot H_{\not{D}}^{k,l}(\mathcal{A}) \cong H_{\not{D}}^{k,n-l}(\mathcal{A})$. Comparing the respective projections onto $\bar{\mathcal{H}}_0^{(-)}$ and $\mathcal{H}_0^{[+;\mu]}$ as described in section 6 leads immediately to the usual statement of mirror symmetry between the "Dolbeault" cohomology groups arising from "foldings" in (7.40).

Of course this analysis is only meant to be somewhat heuristic since, as mentioned before, a complete analysis of the topological properties of a noncommutative spacetime entails more sophisticated techniques. But the above results show how the algebraic structures inherent in the vertex operator algebra modify the geometry and topology, as well as the general symmetry principles of general relativity. More non-trivial structures could be displayed by the Wess-Zumino-Witten (WZW) models studied in [7, 9], and even in the generalizations of the conformal field theory (3.3) to arbitrary compact Riemann surfaces Σ or to embedding fields X^μ which live in toroidal orbifold target spaces. The methods emphasized in this paper can be more or less straightforwardly extended to the analysis of such string theories.

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Appendix. Properties of Vertex Operator Algebras and Construction of Quantum Spacetimes

In string theory, vertex operators generate the algebra of observables of the underlying two-dimensional conformal quantum field theory whose action on the vacuum state $|\mathrm{vac}\rangle$ forms a dense subspace of vectors of the corresponding Hilbert space $\mathcal{H}$ of physical states. The (anti-)chiral algebra $\mathcal{E}^+$ ($\mathcal{E}^-$) is defined to be the operator product algebra of the (anti-)holomorphic fields in the conformal field theory. The chiral algebras contain, in particular, copies of the Virasoro algebras characterizing the conformal invariance of the string theory. A rational conformal field theory is completely characterized by its chiral algebra, and thus the classification problem of rational conformal field theories can be reduced to that of vertex operator algebras.

Generally, in a two-dimensional conformal field theory we can always choose a basis of primary operators ϕ_i of fixed conformal weights $\Delta_i^\pm$ and normalize their 2-point functions as

$$\langle\mathrm{vac}|\phi_i(z_+,z_-)\phi_j(w_+,w_-)|\mathrm{vac}\rangle = \delta_{ij}\ (z_+-w_+)^{-2\Delta_i^+}(z_--w_-)^{-2\Delta_i^-} \tag{A.1}$$

The principles of conformal invariance suggest that the usual Wilson operator product expansion of primary fields should converge, rather than just representing a formal asymptotic expansion. For operators of fixed scaling dimensions, one can define the (constant) operator product expansion coefficients C_{ijk} by

$$\phi_i(z_+,z_-)\phi_j(w_+,w_-) = \sum_k C_{ijk}\ (z_+-w_+)^{\Delta_k^+-\Delta_i^+-\Delta_j^+}(z_--w_-)^{\Delta_k^--\Delta_i^--\Delta_j^-}\,\phi_k(w_+,w_-) \tag{A.2}$$

for $z_\pm \to w_\pm$, where the sum runs over a complete set of primary fields. The coefficients C_{ijk} are then symmetric in i,j,k.

However, it is well-known that the operator product expansion is a consequence of some more elementary relations among the vertex operators. A primary field can be decomposed

$$\phi_i(z_+,z_-) \equiv \sum_{k,l} D_i^{kl}~\varphi_k^+(z_+)\otimes\varphi_l^-(z_-) \tag{A.3}$$

in terms of chiral and anti-chiral vertex operators $\varphi_k^\pm(z_\pm)$ which span $\mathcal{E}^\pm$. If $\mathbf{h}^\pm$ denotes the Hilbert spaces on which $\mathcal{E}^\pm$ act densely, then the local fields $\phi_i(z_+,z_-)$ act as operator-valued distributions from the Hilbert space $\mathcal{H}=\mathbb{C}^{\{D\}}\otimes\mathbf{h}^+\otimes\mathbf{h}^-$ onto itself, where $\mathbb{C}^{\{D\}}$ is the multiplicity space which labels the different left-right sewing determined by the complex-valued sewing coefficients D_i^{kl}. After smearing they become well-defined and densely-defined operators on $\mathcal{H}$. The chiral and anti-chiral vertex operators obey the *braiding relations*

$$\varphi_i^\pm(z_\pm)\varphi_j^\pm(w_\pm) = \sum_{k,l}(R^\pm)_{ij}^{kl}~\varphi_k^\pm(w_\pm)\varphi_l^\pm(z_\pm) \tag{A.4}$$

when the points $z_\pm$ and $w_\pm$ are interchanged along some paths on the worldsheet Σ, where $(R^\pm)_{ij}^{kl}$ are braiding matrices. An example of the braiding commutation relations are the local cocycle relations between the vertex operators discussed in section 5. Furthermore, the vertex operators obey the *fusion equations*

$$\varphi_i^\pm(z_\pm)\varphi_j^\pm(w_\pm) = \sum_{k,l}(F^\pm)_{ij}^{kl}~\varphi_k^\pm(w_\pm)\circ\varphi_l^\pm(z_\pm-w_\pm) \tag{A.5}$$

where the composition of operators on the right-hand side of (A.5) refers to their action on the Hilbert spaces $\mathbf{h}^\pm$, and $(F^\pm)_{ij}^{kl}$ are fusion matrices. These relations then immediately lead to the operator product expansion (A.2) in which the coefficients C_{ijk} can be determined in terms of the sewing, braiding and fusion coefficients introduced above.

The above relations can be further shown to lead to the property of *locality*, i.e. that the quantum fields $\phi_i(\tau,\sigma)$ and $\phi_j(\tau',\sigma')$ commute whenever their arguments are space-like separated, and also the property of *duality*, i.e. crossing-symmetry of the 4-point functions,

$$\sum_p C_{ijp}C_{pkl} = \sum_p C_{ipl}C_{jkp} \tag{A.6}$$

This identity immediately implies the associativity of the operator product algebra (A.2). The identities that we have presented here can be expressed in terms of a single, compact relation by turning to the formal notion of a Vertex Operator Algebra. This single identity is known as the 'Jacobi identity' and it encodes the full non-triviality of the structure of the Vertex Operator Algebra (as the above relations do for the local conformal field theory). The standard discussion above can be cast into such formal form that is useful in the more algebraic applications of conformal field theory, such as that required in the noncommutative geometry of string spacetimes. In the remainder of this appendix we will define formally a Vertex Operator Algebra and describe some of its algebraic properties in the context of the construction of quantum spacetimes. We will be very sketchy, and for more details the reader is invited to consult, for example, the introductory reviews [13, 14], the book [12], and references therein.

Axiomatics

A *vertex operator algebra* consists of a $\mathbb{Z}$-graded complex vector space

$$\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}_n \tag{A.7}$$

and a linear map $\mathcal{V}$ which associates to each element $\Psi \in \mathcal{F}$ an endomorphism of $\mathcal{F}$ that can be expressed as a formal sum in a variable z:

$$\mathcal{V}(\Psi, z) = \sum_{n\in\mathbb{Z}} \Psi_n \; z^{-n-1} \tag{A.8}$$

with $\Psi_n \in \mathcal{F}_n$. The element $\Psi \in \mathcal{F}$ is called a *state* and the endomorphism $\mathcal{V}(\Psi, z)$ is called a *vertex operator*. The vertex operators $\mathcal{V}(\Psi, z)$ are required to satisfy the following axioms:

1. Given any $\Phi \in \mathcal{F}$ we have:

$$\Psi_n \Phi = 0 \qquad \text{for } n \text{ sufficiently large.} \tag{A.9}$$

2. There is a preferred vector $\mathbf{1}$ called the *vacuum* such that

$$\mathbf{1}_n = \delta_{n+1,0} \tag{A.10}$$

   and therefore $\mathcal{V}(\mathbf{1}, z)\Phi = \Phi, \ \forall \Phi \in \mathcal{F}$.

3. $\Psi_n = 0 \ \forall n \in \mathbb{Z} \Longleftrightarrow \Psi = 0$.

4. There exists a *conformal vector* $T_{n+1} = L_n$, with the L_n satisfying the Virasoro Algebra (4.26) for some central charge $c \in \mathbb{C}$. This vector provides as well a translation generator:

$$\mathcal{V}(L_{-1}\Psi, z) = \frac{d}{dz}\mathcal{V}(\Psi, z) \tag{A.11}$$

   and the grading of $\mathcal{F}$:

$$L_0 \Psi_n = n\Psi_n \qquad \forall \Psi_n \in \mathcal{F}_n \ . \tag{A.12}$$

5. The spectrum of L_0 is bounded from below.

6. The eigenspaces $\mathcal{F}_n$ of L_0 are finite-dimensional.

7. The vertex operators must also satisfy a *Jacobi identity*:

$$\sum_{i\geq 0} (-1)^i \binom{l}{i} \Big(\Psi_{l+m-i}(\Phi_{n+i}\Xi) - (-1)^l \Phi_{l+n-i}(\Psi_{m+i}\Xi)\Big) = \sum_{i\geq 0} \binom{m}{i} (\Psi_{l+i}\Phi)_{m+n-i}\Xi \tag{A.13}$$

   for all $\Psi, \Phi, \Xi \in \mathcal{F}, \ l, m, n \in \mathbb{Z}$.

The axioms 1.–7. define a Vertex Operator Algebra, which contains the Virasoro algebra (Axiom 4.). The Jacobi identity (A.13) contains the most information about the algebra. Three special cases of it are particularly interesting. They represent associativity:

$$(\Psi_m\Phi)_n = \sum_{i\geq 0}(-1)^i\binom{m}{i}\left(\Psi_{m-i}\Phi_{n+i} - (-1)^m\Phi_{m+n-i}\Psi_i\right) \quad , \tag{A.14}$$

the commutator formula:

$$[\Psi_m, \Phi_n] = \sum_{i\geq 0}\binom{m}{i}(\Psi_i\Phi)_{m+n-i} \quad , \tag{A.15}$$

and skew-symmetry:

$$\Psi_n\Phi = (-1)^{n+1}\Phi_n\Psi + \sum_{i\geq 1}\frac{1}{i!}\,(-1)^{i+n+1}(L_{-1})^i(\Phi_{n+i}\Psi) \tag{A.16}$$

for all $\Psi, \Phi \in \mathcal{F},\ m, n \in \mathbb{Z}$. The Jacobi identity therefore encodes the complete noncommutativity (as well as other nontrivial properties) of the vertex operator algebra.

Adjoint

The adjoint of a vertex operator is defined as:

$$\mathcal{V}^\dagger(\Psi, z) = \sum_{n\in\mathbb{Z}}\Psi_n^\dagger\ z^{-n-1} \equiv \mathcal{V}(\mathrm{e}^{zL_1}(-z^2)^{L_0}\Psi, z^{-1}) \tag{A.17}$$

It can be shown [14] that with this definition $L_n^\dagger = L_{-n}$. This yields the $*$-conjugation on the vertex operator algebra.

Translation and Scaling

The Virasoro generators L_{-1} and L_0 generate translations and scale transformations, respectively:

$$\begin{aligned} \mathrm{e}^{wL_{-1}}\,\mathcal{V}(\Psi, z)\ \mathrm{e}^{-wL_{-1}} &= \mathcal{V}(\Psi, w+z) \\ \mathrm{e}^{wL_0}\,\mathcal{V}(\Psi, z)\ \mathrm{e}^{-wL_0} &= \mathrm{e}^{w\Delta_\Psi}\,\mathcal{V}(\Psi, \mathrm{e}^w z) \end{aligned} \tag{A.18}$$

where Δ_Ψ is the conformal weight of the vector Ψ (defined by (A.12)).

Conformal Highest Weight Vectors

Vertex operators generate states when applied to the vacuum (this is the original motivation for their name):

$$\mathcal{V}(\Psi, z)\mathbf{1} = \mathrm{e}^{zL_{-1}}\Psi \tag{A.19}$$

In general, the conformal highest weight vectors (or primary fields) are states which satisfy:

$$L_0\Psi = \Delta_\Psi\Psi \qquad , \qquad L_n\Psi = 0\ \ \forall n > 0 \tag{A.20}$$

and in particular the vacuum is a primary state of weight zero.

Algebra of Primary Fields of Weight One

The primary fields of weight one form a Lie algebra

$$\mathcal{L} \equiv \mathcal{F}_1/(\mathcal{F}_1 \cap L_{-1}\mathcal{F}) = \mathcal{F}_1/L_{-1}\mathcal{F}_0 \tag{A.21}$$

with antisymmetric bracket:

$$[\Psi, \Phi] \equiv \Psi_0 \Phi \tag{A.22}$$

and $\mathcal{L}$-invariant bilinear form:

$$\langle \Psi, \Phi \rangle \equiv \Psi_1 \Phi \tag{A.23}$$

provided that the spectrum of L_0 is non-negative and the weight zero subspace $\mathcal{F}_0$ is one-dimensional. The former constraint is usually imposed out of physical considerations, as often L_0 is identified with the Hamiltonian of the system. The quotient of $\mathcal{F}_1$ in (A.21) is by the set of *spurious* states. The classical Jacobi identity for the Lie bracket (A.22) follows from the Jacobi identity for the vertex operator algebra, while the symmetry of the inner product (A.23) follows directly from the skew-symmetry property.

Setting $\Psi = \Phi = \mathbf{1}$ in (A.13), using the vacuum axiom 2. and the definition (A.8), we obtain the usual Cauchy theorem of classical complex analysis. Thus the Jacobi identity for vertex operator algebras is a combination of the classical Jacobi identity for Lie algebras and the Cauchy residue formula for meromorphic functions. It is also possible to define, in certain instances, more exotic subalgebras using the Jacobi identity, such as a commutative non-associative algebraic structure $\Psi \times \Phi \equiv \Psi_1 \Phi$ on the space of fields of weight two, and a commutative associative product $\Psi \cdot \Phi \equiv \Psi_{-1}\Phi$ as well as the Lie bracket (A.22) on the quotient space $\mathcal{F}/\mathcal{F}_{-2}\mathcal{F}$.

Vertex Operators and Even Lattices

One of the most important results in the theory of vertex operator algebras (and the one of great relevance to the present work) is the theorem which states that [51]:

Associated with any even positive-definite lattice Γ *there is a vertex operator algebra.*

The proof of the theorem is a constructive procedure. The construction is in fact the one which associates a bosonic string theory compactified on a torus, as we did in section 5. In this paper we had a Fock space as our starting point. In general this is not necessary, and the Fock space can actually be constructed starting from the lattice Γ, so that the lattice is the only ingredient necessary for the construction. The proof that the algebra one constructs is indeed a vertex operator algebra, as well as the details of the formal construction, can be found in [12, 14].

In the case of the vertex operator algebra of section 5, the algebra one constructs is actually the chiral algebra $\mathcal{E}^{\pm}$, the endomorphisms on $\mathbb{C}\{\Gamma\}^{\pm} \otimes S(\hat{h}^{(-)})$ (respectively on Γ^*). The full vertex operator algebra is then constructed using the sewing transformation

(5.3). For the chiral algebras, the Jacobi identity follows from the analogous braiding relations (5.24) which lead to the fusion relations [12]

$$\begin{aligned}\mathcal{V}\left(\mathcal{V}(\mathcal{V}^{(R)}_{[q^\pm]}[z_\pm],z_\pm)\mathcal{V}^{(S)}_{[r^\pm]}[w_\pm],w_\pm\right) &= \prod_{1\leq(i,j)\leq(R,S)}\left(z_\pm+(z^{(i)}_\pm-w^{(j)}_\pm)\right)^{g^{\mu\nu}q^{(i)\pm}_\mu r^{(j)\pm}_\nu}\\ &\qquad\times\ :\ \mathcal{V}(\mathcal{V}^{(R)}_{[q^\pm]}[z_\pm],z_\pm+w_\pm)\mathcal{V}(\mathcal{V}^{(S)}_{[r^\pm]}[w_\pm],w_\pm)\ :\end{aligned} \tag{A.24}$$

By the completeness of the chiral vertex operators $\mathcal{V}^{(R)}_{[q^\pm]}[z_\pm]$ on the respective Hilbert spaces, the relation (A.24) leads immediately to the Jacobi identity for the chiral vertex operator algebras $\mathcal{E}^\pm$.

Note that self-duality $\Gamma=\Gamma^*$ is not a requirement. In the general case, we have the coset decomposition

$$\Gamma^*=\bigcup_{x\in\Gamma^*/\Gamma}(\Gamma+\lambda_x) \tag{A.25}$$

of the dual lattice with $\lambda_0=0$. It can be shown [12, 51] that $\{R_x\mid x\in\Gamma^*/\Gamma\}$ is the complete set of inequivalent irreducible representations of the vertex operator algebra, where

$$R_x=\mathbb{C}\{\Gamma+\lambda_x\}\otimes S(\hat{h}^{(-)}) \tag{A.26}$$

Then $\widehat{\mathcal{H}}_X(\Gamma^*)=\bigoplus_{x\in\Gamma^*/\Gamma}R_x$, and the representations (A.26) can be identified with the "points" of a noncommutative spacetime. Furthermore, the corresponding linear space of characters $\phi_{R_x}\equiv\mathrm{tr}_{R_x}q^{L_0-c/24}$, where $q=\mathrm{e}^{\pi i\tau}$ with $\mathrm{Im}\,\tau>0$, is modular invariant with respect to $SL(2,\mathbb{Z})$ [52]. If the lattice Γ is self-dual then the vertex operator algebra is *holomorphic*, i.e. it is its only irreducible representation. This shows how the symmetries of Γ control the structure of the associated spacetime.

In the cases studied throughout this paper, we can now see how the structure of the quantum spacetime is intimately tied to the properties of the compactification lattice. Those lattices associated to large symmetries of the spacetime, such as that associated with the Monster group, give smaller quantum spacetimes that those associated to non-symmetrical lattices, i.e. large gauge symmetries essentially exhaust the full structure of the spacetime. This increase in symmetry of the spacetime from a decrease in the number of its "points" is similar to the effect of increasing the number of elements of an algebra to gain an decrease in the number of points of a topological space (see section 2). Furthermore, given a compact automorphism group G of the vertex operator algebra as in section 7, each G-module $\widehat{\mathcal{H}}^{[R(G)]}_X(\Lambda)$ is an irreducible representation of the G-invariant subalgebra $\widehat{\mathcal{H}}^{(0)}_X(\Lambda)$ [53]. This exemplifies, in particular, how the construction of the commutative low-energy projective subspaces carries through from the structure of the vertex operator algebra. Moreover, the above results show that more general theories than the ones we have presented in this paper will also lead to vertex operator algebras, and thus similar noncommutative spacetimes. For example, the allowed momenta and winding modes of heterotic string theory live on an $(n+16,n)$-dimensional even self-dual

Lorentzian lattice [36], and the construction of section 6 can be used to show that the target space duality group in this case is isomorphic to $O(n+16, n; \mathbb{Z})$ [1].

In this general class of vertex operator algebras, $\mathcal{F}_1$ is a Lie algebra with generators ε_q and $q_\mu \alpha^\mu_{-1}$, where $q \in \Gamma$. Its root lattice is precisely the lattice Γ, and the affinization of $\mathcal{F}_1$ then yields the usual Frenkel-Kac construction of affine Lie algebras [41]. Note that, as a subspace of the noncommutative spacetime, the subalgebra $\mathcal{F}_1$ contains the lowest non-trivial oscillatory modes of the strings, so that the universal gauge symmetry of the string spacetime coincides with smallest excitations of the commutative subspaces. It would be interesting to further give a spacetime interpretation to the commutative structure of the quotient space $\mathcal{F}/\mathcal{F}_{-2}\mathcal{F}$. Thus, more general spacetime gauge symmetric structures can also be inputed into the constructions presented in this paper (leading to analogs of the results of section 7), so that our results extend naturally to a larger class of models than just the linear sigma-models described here. The key underlying feature is always a vertex operator algebra which leads to a horribly complicated quantum spacetime.

Vertex Operators in other Conformal Field Theories

At present the only other known class of examples of vertex operator algebras arise from a restricted subalgebra of observables of WZW conformal field theories (equivalently conformal field theory in a general group manifold target space). The proof is due to Frenkel and Zhu [54] who constructed vertex operator algebras corresponding to the highest weight representations of Kac-Moody and Virasoro algebras where the highest weights are multiples of the highest weight of the fundamental representation. For a representation-theoretic description of the corresponding local conformal algebra, see [7, 9]. In [9] a description of the quantum spacetime associated with the $SU(2)$ WZW model is presented and related to the Fuzzy 3-sphere. Generally, the low-energy (semi-classical) target space manifolds of these conformal field theories are quantum deformations of the group manifold target space. In fact, in the general case, there is an intimate connection between the modules of a rational vertex operator algebra and the modules of a Hopf algebra associated with the vertex operator algebra. This makes more precise the relationship between conformal field theory and the theory of quantum groups within a framework suited to the techniques of noncommutative geometry.

The complete list of concrete vertex operator algebras thus includes the moonshine vertex operator algebra that we discussed in section 7, vertex operator algebras based on Heisenberg-Weyl algebras (equivalently Fock spaces) and even positive-definite lattices, and the vertex operator algebras associated with the integrable representations of affine Lie algebras, Virasoro algebras, and W-algebras. There are also various generalizations via twistings and orbifold theories which could be used to describe string geometries associated with toroidal orbifold sigma-models, for example, and also the notion of a vertex operator superalgebra [55] leading to effective target space geometries associated with superconformal field theories [9]. A more abstract approach is due to Zhu [52] who established a one-to-one correspondence between the set of inequivalent irreducible (twisted) modules for a given vertex operator algebra, and the set of inequivalent irreducible modules for an associative algebra associated with the vertex operator algebra and an automorphism of it. Thus a vertex operator algebra can be represented as an *associative* algebra, a process which we implicitly carried out in section 5 when we represented the algebra $\mathcal{A}_X$ as operators on the Hilbert space $\mathcal{H}_X$. There is even a geometric procedure for constructing rational conformal field theories from two given vertex operator algebras (representing the left and right chiral algebras) which have only finitely many irreducible modules [13].

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