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Expectation Values of Observables in Time-Dependent Quantum Mechanics

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Abstract

Let $U(t)$ be the evolution operator of the Schrödinger equation generated by an hamiltonian of the form $H_0(t) + W(t)$ where $H_0(t)$ commutes for all t with a complete set of time independent projectors $\{P_j\}_{j=1}^{\infty}$. Consider the observable $A = \sum_j P_j \lambda_j$, where $\lambda_j \simeq j^\mu$, $\mu > 0$ for j large. Assuming that the "matrix elements" of $W(t)$ behave as $\|P_j W(t) P_k\| \simeq 1/|j-k|^p$, $j \neq k$ for $p > 0$ large enough, we prove estimates on the expectation value $\langle U(t)\varphi | A U(t)\varphi\rangle \equiv \langle A\rangle_\varphi(t)$ for large times of the type $\langle A\rangle_\varphi(t) \leq ct^\delta$, where $\delta > 0$ depends on p and μ. Typical applications concern the energy expectation $\langle H_0\rangle_\varphi(t)$ in case $H_0(t) \equiv H_0$ or the expectation of the position operator $\langle x^2\rangle_\varphi(t)$ on the lattice where $W(t)$ is the discrete laplacian or a variant of it and $H_0(t)$ is a time dependent multiplicative potential.

Key-Words: Time-dependent hamiltonians, Schrödinger operator, quantum stability, quantum dynamics.
May 1997
CPT-97/P.3479

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1 Introduction

This paper is concerned with estimates on the long time behaviour of expectation values of certain quantum observables. Such estimates pertain to the study of quantum stability and quantum diffusion as well. Indeed, although our main result is quite general, the two paradigms we have in mind are estimates on the energy expectation for general time-dependent driven systems and estimates on the moments of the position operator for time-dependent driven systems defined on the lattice. For more details on the origin of the problem, on quantum stability in general and on related results, the reader should consult [2], [14], [13], [5], [18], [17] and [15], for example. Let us start by describing our typical results for these cases.

The first type of systems are characterized by the self-adjoint hamiltonian $H(t) = H_0 + W(t)$ on some Hilbert space $\mathcal{H}$ which generates an evolution $U(t)$, $U(0) = \mathbb{1}$ and the energy expectation is defined by

$$\langle H_0\rangle_\varphi(t) = \langle U(t)\varphi | H_0 U(t)\varphi\rangle,$$

where $\varphi \in \mathcal{H}$. Let us assume here for simplicity that the spectrum of the time-independant part H_0 is discrete $\sigma(H_0) = \{\lambda_j\}_{j\in\mathbb{N}^*}$ and that the eigenvalues behave according to $\lambda_j \simeq j^\mu$, as $j \to \infty$, for some $\mu > 0$. We denote by P_j the corresponding spectral projectors. We further assume that the perturbation $W(t)$ is bounded with matrix elements characterized by

$$\|P_j W(t) P_k\| \leq w/|j-k|^p \text{ as } |j-k| \to \infty, \tag{1.1}$$

for some time-independant constant w and for some $p > 0$ large enough. We show that there exists a diffusion exponent $\delta > 0$ depending on μ and p such that

$$\langle H_0\rangle_\varphi(t) \leq ct^\delta \tag{1.2}$$

for some finite constant c and for any initial condition φ belonging to some dense domain in $\mathcal{H}$. More precisely, there exist real valued functions $P(\mu)$ and $\alpha(p)$ (see proposition 4.1) such that provided $p > P(\mu)$ and $\forall\varepsilon > 0$,

$$\delta = \mu/(1-\alpha(p))^2 + \varepsilon, \text{ where } \alpha(p) = \mathcal{O}(p^{-1}) \text{ as } p \to \infty, \tag{1.3}$$

and $P(\mu)$ is essentially linear in μ. In particular, when $\mu \leq 1$, $p > 2$ is enough. This estimate holds for any $\mu > 0$ if p large enough and regardless of the time dependence of $W(t)$.

Similar results were first obtained for general systems by Nenciu in [17]. He made use of recently developped tools in quantum adiabatic theory which require increasing gaps between successive eigenvalues, that is $\mu > 1$, and differentiability with respect to time of the perturbation $W(t)$. However, nothing is required on the behaviour of the matrix elements of the perturbation. The diffusion exponent obtained in [17] is of the form $\delta = \delta_0(\mu)/n$, where $n \in \mathbb{N}$, the order of differentiability of $W(t)$, is large enough. It was then showed by Joye in [15] that estimates of the type (1.2) can be obtained for any $\mu > 0$ and without differentiability of $W(t)$, provided the coupling induced by $W(t)$ with high energy levels is weak enough. Typically, if

$\|P_jW(t)\| \simeq j^{-\beta}$ with $2\beta > 1+\mu$, then $\delta = \mu/(\beta-1/2)$. In terms of matrix elements, this weak coupling condition means that $\|P_jW(t)P_k\| \to 0$ as $j^2+k^2 \to \infty$. The present results complete those of [15] to the physically important situation where the strength of the couplings between neighbouring levels is independent of the energy. This is caracterized by condition (1.1) requiring decay of the matrix elements $\|P_jW(t)P_k\|$ to zero as $|j-k| \to \infty$ only. As a result, the diffusion exponent (1.3) is bounded below by $\mu > 0$, uniformly in p, whereas the exponents obtained in [17] and [15] can be made arbitrarily small by adjusting the parameters of the perturbation $W(t)$. Note however that our bounds are optimal in some cases, as discussed below. Finally, in the specific case where $H_0 + W(t)$ is a time-dependent forced harmonic oscillator, the quantum dynamics can be solved explicitly, so that the asymptotic behavior of $\langle H_0\rangle_\varphi(t)$ can sometimes be derived. See for example [11], [3] and [4]. However, the perturbation $W(t)$ is unbounded in this situation.

Consider now the Hilbert space $l^2(\mathbb{Z}^d)$ whose vectors we denote by $u = \{u(n)\}_{n\in\mathbb{Z}^d}$. Let $H_0(t)$ be a time-dependent real-valued multiplicative operator, possibly unbounded, (potential) such that $(H_0(t)u)(n) = H_0(n,t)u(n)$, $\forall n \in \mathbb{Z}^d$. Let W be the discrete laplacian $(Wu)(n) = \sum_{j\in\mathbb{Z}^d,\ |j-n|=1} u(j)$, $\forall n \in \mathbb{N}$, and for fixed $m > 0$, let $|x|^m$ be the moment of order m defined by $|x|^m = \sum_{n\in\mathbb{Z}^d} |n|_e^m P_n$, where P_n is the projector on the site n and $|\ |_e$ is the euclidian norm. Under some regularity assumptions, the time-dependent Schrödinger operator $H_0(t) + W$ gives rise to an evolution $U(t)$, with $U(0) = \mathbb{1}$. We show that for any $m \geq 0$, the expectation value of $|x|^m$,

$$\langle |x|^m\rangle_\varphi(t) = \langle U(t)\varphi|\ |x|^m\ U(t)\varphi\rangle$$

satisfies the estimate

$$\langle |x|^m\rangle_\varphi(t) \leq ct^m, \tag{1.4}$$

for some finite constant c and for any φ in the dense domain $D(|x|^m) \in l^2(\mathbb{Z})$. Since this estimate holds in the free case as well, (1.4) is optimal. As a corollary, we note that it is impossible to accelerate a particle on the lattice. Such results are known in the time-independant case, see e.g. [19], [20]. Moreover, lower bounds are proven as well in case the potential is constant or periodic in time [9], [10], [1], [16]. However, such estimates are new for general time-dependent potentials. We also consider similar estimates when the discrete laplacian is replaced by a long range interaction W of the form

$$Wu(n) = \sum_{k\in\mathbb{Z}} \omega(n-k)u(k), \quad \forall n \in \mathbb{Z}, \tag{1.5}$$

where ω is real valued and belongs to $l^1(\mathbb{Z})$, we call long range laplacian. This allows comparison with a model of AC Stark effect studied recently by De Bievre and Forni [7] (see also Gallavotti [8], Bellissard [2]), for which they prove lower bounds on the expectation of $|x|^2$. Moreover, for some specific choices of potential $H_0(t)$, we are able to prove lower bounds on $\langle |x|^2\rangle_\varphi(t)$, some of which grow arbitrarily fast in time, although W defined in (1.5) is bounded.

We actually deal with both situations in a unified manner as follows. We consider a self-adjoint hamiltonian $H(t) = H_0(t) + W(t)$ such that $H_0(t)$ commutes with a complete set of time independent orthogonal projectors $\{P_j\}_{j\in\mathbb{N}^*}$. We estimate the expectation value of the positive observable A of the form $A = \sum_{j\in\mathbb{N}^*} \lambda_j P_j$, where $\{\lambda_j\}_{j\in\mathbb{N}*}$ is a positive stricly increasing sequence such that $\lambda_j \to \infty$ as $j \to \infty$. In order to estimate $\langle A\rangle_\varphi(t) = \langle U(t)\varphi|AU(t)\varphi\rangle$, the idea is to compare the evolution $U(t)$ generated by $H_0(t) + W(t)$ with the one generated by $H_0(t) + W_q^d(t)$, where $W_q^d(t)$ is obtained from $W(t)$ by replacing by zeros all matrix elements $\|P_jW(t)P_k\|$ such that $|j-k| > q(j+k)$, where $q : \mathbb{R}^+ \to \mathbb{R}^+$ is a real valued (typically increasing) function. $W_q^d(t)$ can be considered as a generalized band matrix, whose width increases as we move along the main diagonal. This new evolution, we call $V_q(t)$, differs from $U(t)$ by a term of order $W_q^o(t) = W(t) - W_q^d(t)$, by Duhamel's formula. Provided $\|A^\beta W_q^o(t)A^{-1/2}\|$ is uniformly bounded in time for some $\beta \geq 1/2$, $\langle A\rangle_\varphi(t)$ can be essentially estimated by $\langle V_q(t)\varphi|AV_q(t)\varphi\rangle$. The control of this expectation value with respect to the evolution $V_q(t)$, generated by the troncated perturbation $W_q(t)$ is the main technical point of the paper. We can get efficient estimates on this expectation value provided we can show that the quantity

$$S(N,t) = \sum_{j>N} \lambda_j \sup_{0\leq s\leq t} \|P_jV_q(t)V_q^{-1}(s)A^{-\beta}\|^2 \quad \beta \geq \frac{1}{2}\ , \tag{1.6}$$

decays to zero as $N \to \infty$, with a rate we control, as explained in the next section. By inserting the characteristics of the matrix elements of A and $W(t)$ roughly described above, and by making the choice $q(x) = x^\alpha$, $\alpha \in]0,1[$, we can achieve sufficient control on $S(N,t)$, by appropriate choice of the parameters α and β. Note that we have a conflict between the parameters β and α, i.e. between β and q. On the one hand, we wish to take q and β as large as possible to make V_q closer and closer to $U(t)$ and to take advantage of the decay induced by $A^{-\beta}$ in (1.6). On the other hand, $A^\beta W_q^o A^{-1/2}$ is more and more likely to be bounded when β is small and W_q^o is sparser and sparser, i.e. q is small. In case $W(t) = W_q(t)$ is a genuine band matrix, as is the case with the second application we deal with in section 5, this procedure is much easier. Actually, we found this example quite instructive and inspiring for the proof of the general case.

2 General Strategy

Let $H(t) = H_0(t) + W(t)$, $t \in \mathbb{R}^+$ be the generator of the Schrödinger equation defined on a separable Hilbert space $\mathcal{H}$.

Hypothesis H0:

$H_0(t)$ is self-adjoint on a dense domain D for any t and $W(t)$ is symmetric and relatively bounded with respect to $H_0(t)$, with relative bound $a < 1$.

There exists a set of t-independent complete orthogonal spectral projectors $\{P_j\}_{j=1}^\infty$ such that

$$[H_0(t), P_j] \equiv 0, \quad \forall j \in \mathbb{N}^*, t \in \mathbb{R}^+. \tag{2.7}$$

Let $q : \mathbb{R}^+ \to \mathbb{R}^+$. We define

$$W_q^d(t) = \sum_{|j-k|\leq q(j+k)} P_j W(t) P_k \quad \text{and} \quad W_q^o(t) = W(t) - W_q^d(t),$$

using a strong sum.

Note that with our notations, $W_0^d(t)$ corresponds to the diagonal operator $W_0^d(t) = \sum_j P_j W(t) P_j$.

Hypothesis H1:

$$\sup_{t\in\mathbb{R}^+} \|W_q^d(t) - W_0^d(t)\| < \infty.$$

We assume that the following evolution equations (where $' \equiv \frac{d}{dt}$*)*

$$\begin{aligned} iU'(t)\varphi &= (H_0(t) + W(t))U(t)\varphi, \quad U(0) = \mathbb{1} \\ iV_q'(t)\varphi &= (H_0(t) + W_q^d(t))V_q(t)\varphi, \quad V_q(0) = \mathbb{1} \\ iV_0'(t)\varphi &= (H_0(t) + W_0^d(t))V_0(t)\varphi, \quad V_0(0) = \mathbb{1}, \quad \varphi \in D, \end{aligned}$$

give rise to unitary operators which, together with there inverse, map D into D.

Let $\lambda_j > 0$, $\forall j \in \mathbb{N}^*$ and let

$$A = \sum_j \lambda_j P_j \text{ defined on } D(A) = \{\varphi \in \mathcal{H} | \sum_j \lambda_j^2 \|P_j\varphi\|^2 < \infty\}$$

be the positive observable the expectation value of which we will consider.

Hypothesis H2:

The eigenvalues of A are positive and form a strictly increasing sequence

$$\lambda_{j+1} > \lambda_j > 0, \quad \forall j \in \mathbb{N}^*, \tag{2.8}$$

and there exists two constants $0 < L_- \leq L_+ < \infty$ and $\mu > 0$ such that for any j large enough,

$$0 < L_- j^\mu \leq \lambda_j \leq L_+ j^\mu. \tag{2.9}$$

Let us describe the general strategy we adopt. Following [17, 15], we get from Duhamel's formula (in the strong sense),

$$\begin{aligned} U(t) &= V_q(t) - iV_q(t) \int_0^t V_q^{-1}(s) W_q^o(s) U(s) ds \\ &\equiv V_q(t) + R_q(t) \end{aligned}$$

where $\|R_q(t)\| \leq 2$. Let $\varphi \in D(A^\beta)$ with $\beta \geq 1/2$ and consider $\langle A\rangle_\varphi(t) \equiv \langle U(t)\varphi | AU(t)\varphi\rangle$. We have, due to the positivity of A

$$\langle A\rangle_\varphi(t) \leq 2\left(\langle V_q(t)\varphi|AV_q(t)\varphi\rangle + \langle R_q(t)\varphi|AR_q(t)\varphi\rangle\right). \tag{2.10}$$

Then, with the notation $V_q(t,s) = V_q(t)V_q^{-1}(s)$,

$$\begin{aligned}
&\|A^{1/2}R_q(t)\varphi\|^2 \qquad (2.11)\\
&\quad \le 4\lambda_N\|\varphi\|^2 + \sum_{j>N}\lambda_j \left\| P_j \int_0^t V_q(t,s)W_q^o(s)U(s)\varphi ds\right\|^2 \\
&\quad \le 4\lambda_N\|\varphi\|^2 + t^2\sum_{j>N}\lambda_j \left(\sup_{0\le s\le t}\|P_jV_q(t,s)A^{-\beta}\|\|A^\beta W_q^o(s)U(s)\varphi\|\right)^2 .
\end{aligned}$$

We work under the following hypothesis.

Hypothesis H3:

There exists $\beta \ge 1/2$ such that

$$\sup_{t\in\mathbb{R}^+}\|A^\beta W_q^o(t)A^{-1/2}\| \equiv \overline{\|A^\beta W_q^o A^{-1/2}\|} < \infty.$$

Under H3, we get from (2.10) and (2.11)

$$\begin{aligned}
\langle A\rangle_\varphi(t) \le\ & 2\|A^{1/2}V_q(t)\varphi\|^2 + 8\lambda_N\|\varphi\|^2 \\
& +2\overline{\|A^\beta W_q^o A^{-1/2}\|}^2 t^2 \sup_{0\le s\le t}\|A^{1/2}U(s)\varphi\|^2 \sum_{j>N}\lambda_j \sup_{0\le s\le t}\|P_jV_q(t,s)A^{-\beta}\|^2
\end{aligned}$$

Hence, noting that $\|A^{1/2}U(s)\varphi\|^2 = \langle A\rangle_\varphi(s)$ and writing $\overline{\langle A\rangle_\varphi}(t) = \sup_{0\le s\le t}\langle A\rangle_\varphi(s)$, we obtain

$$\begin{aligned}
\overline{\langle A\rangle_\varphi}(t) &\le 2\sup_{0\le s\le t}\|A^{1/2}V_q(t)\varphi\|^2 + 8\lambda_N\|\varphi\|^2 \qquad (2.12)\\
&+ 2\overline{\|A^\beta W_q^o A^{-1/2}\|}^2 t^2\overline{\langle A\rangle_\varphi}(t)\sum_{j>N}\lambda_j\sup_{0\le s\le t}\|P_jV_q(t,s)A^{-\beta}\|^2.
\end{aligned}$$

We can estimate the large t behaviour of $\langle A\rangle_\varphi(t)$ provided we control the large N behaviour of

$$S(N,t) = \sum_{j>N}\lambda_j\sup_{0\le s\le t}\|P_jV_q(t,s)A^{-\beta}\|^2 .$$

Indeed, $\|A^{1/2}V_q(t)\varphi\|^2$ is related to $S(N,t)$ by the estimates ($V_q(t)$ is unitary)

$$\begin{aligned}
\|A^{1/2}V_q(t)\varphi\|^2 &\le \lambda_N\|\varphi\|^2 + \sum_{j>N}\lambda_j\|P_jV_q(t)\varphi\|^2 \\
&\le \lambda_N\|\varphi\|^2 + S(N,t)\|A^\beta\varphi\|^2. \qquad (2.13)
\end{aligned}$$

Thus, provided we can show for some $\beta \ge 1/2$ that $S(N,t)\to 0$ as $N\to\infty$, we get from (2.12) and (2.13) under hypothesis H3

$$\overline{\langle A\rangle_\varphi}(t) \le 10\lambda_N\|\varphi\|^2 + 2S(N,t)\left(\|A^\beta\varphi\|^2 + \overline{\|A^\beta W_q^o A^{-1/2}\|}^2 t^2\overline{\langle A\rangle_\varphi}(t)\right) .$$

Hence we require $N(t)$ to be such that $2\overline{\|A^\beta W_q^o A^{-1/2}\|}^2 t^2 S(N(t),t) \le K < 1$ as $t\to\infty$, so that we can write by means of a bootstrap argument

$$\begin{aligned}
\langle A\rangle_\varphi(t) &\le \overline{\langle A\rangle_\varphi}(t) \\
&\le c(\lambda_{N(t)}\|\varphi\|^2 + S(N(t),t)\|A^\beta\varphi\|^2) \\
&\le c\lambda_{N(t)}(\|\varphi\|^2 + \|A^\beta\varphi\|^2),
\end{aligned}$$

for some finite constant c. Remark that the function q is not specified yet and that we don't make use of (2.9) at this point.

3 Main Result

In order to give explicit estimates on $\langle A\rangle_\varphi(t)$, we specify a little bit more our concern.

Proposition 3.1 *Assume H0, H1, H2 and H3 and let q be defined by*

$$\begin{aligned} q : \mathbb{R}^+ &\to \mathbb{R}^+ \\ x &\mapsto x^\alpha \end{aligned} \tag{3.14}$$

for some $\alpha \in]0,1[$. If $\mu(2\beta-1) > 1+\alpha$, there exists finite positive constants b (see (6.31)) and c_0 (see (6.34)) such that if

$$N(t) = [bt^{\frac{1}{(1-\alpha)^2}}] + 1, \quad (\ [\ .\ ]\ \text{ being the integer part})\ ,$$

we get for all t large enough,

$$S(N(t),t) = \sum_{j>N(t)} \lambda_j \sup_{0\le s\le t} \|P_j V_q(t,s) A^{-\beta}\|^2 \le c_0 t^{\frac{2-\mu(2\beta-1)}{(1-\alpha)^2}}. \tag{3.15}$$

Corollary 3.1 *Under the same hypotheses as above, and provided $\mu(2\beta-1) > 1+\alpha$, we get from (2.13)*

$$\sup_{0\le s\le t} \|A^{1/2} V_q(t,s) A^{-\beta}\|^2 \le c_1 (t^{\frac{2-\mu(2\beta-1)}{(1-\alpha)^2}} + t^{\frac{\mu}{(1-\alpha)^2}})$$

for some finite constant c_1.

Our results on $\langle A\rangle_\varphi(t)$ follow directly from the above considerations.

Theorem 3.1 *Assume the hypotheses of proposition*
If $\mu(2\beta-1) > 2 + 2(1-\alpha)^2$, we get for any $\varphi \in \mathcal{D}(A^\beta)$:

$$\langle A\rangle_\varphi(t) \le c_2 t^{\frac{\mu}{(1-\alpha)^2}} \left(\|A^\beta\varphi\|^2 + \|\varphi\|^2\right)\ ,$$

for some finite constant c_2.

The proofs of the above results are given in section 6

4 Applications

In order to apply the above results, we need to determine which perturbations $W(t)$ satisfy the hypotheses H3. Let us consider the typical situation where the perturbation $W(t)$ is characterized by its matrix elements in the large $P_jW(t)P_k$ in the fixed subspaces $P_j\mathcal{H}$, $j \in \mathbb{N}^*$.
Hypothesis H4:
There exist positive constants w_0 and p independent of t such that

$$\|P_jW(t)P_k\| \le \frac{w_0}{|k-j|^p} \tag{4.16}$$

for $|k-j|$ large enough.
The constraint on $W(t)$ expressed through H3 can be expressed in terms of p above and α using estimates on $\|A^\beta W_q^o A^{-1/2}\|$.

Lemma 4.1 *Under H2 and H4, for* $0 < \alpha < 1$, $\mu, p > 0$, $\beta \geq 1/2$, *one has*

$$\|A^{\beta} W_q^o A^{-1/2}\| < \infty \ \textit{if} \ \begin{cases} 2p - 2\beta\mu > 1 \\ \mu + \alpha(2p-1) - 2\beta\mu > 1 \end{cases} \tag{4.17}$$

and

$$\|A^{\beta} W_q^o A^{-1/2}\| < \infty \ \textit{if} \ \begin{cases} p - \beta\mu > 1 \\ \mu/2 + \alpha(p-1) \geq \beta\mu \end{cases} \tag{4.18}$$

Proof: Under assumption H4, one gets:

$$\|A^{\beta} W_q^0 A^{-\frac{1}{2}}\|^2 \leq \omega_0^2 \sum_{j \in \mathbb{N}^*} \sum_{|k-j| \geq (k+j)^{\alpha}} j^{2\beta\mu} \frac{1}{|j-k|^{2p} k^{\mu}} \equiv \omega_0^2 \sum_{j \in \mathbb{N}^*} j^{2\beta\mu} \sigma(j)$$

and

$$\sigma(j) \ \leq \ \sum_{k \geq j + 2^{\alpha} j^{\alpha}} \frac{1}{|j-k|^{2p} k^{\mu}} + \sum_{k \leq j - j^{\alpha}} \frac{1}{|j-k|^{2p} k^{\mu}}$$

By usual estimates (see e.g. [12], [15]), one obtains if $2p + \mu > 1$:

$$\sigma(j) \leq c\left(\frac{1}{j^{2p+\mu-1}} + \frac{1}{j^{2p}} + \frac{1}{j^{2p\alpha+\mu}} + \frac{\log j}{j^{2p}} + \frac{\log j}{j^{\alpha(2p-1)+\mu}}\right)$$

for some finite constant c, which implies (4.17). To prove (4.18), one estimates the Schur norm of $\|A^{\beta} W_q^0 A^{-\frac{1}{2}}\|$:

$$\|A^{\beta} W_q^0 A^{-\frac{1}{2}}\| \leq \omega_0 \max\Big\{ \sup_j j^{\beta\mu} \sigma'(j), \ \sup_j j^{-\frac{\mu}{2}} \sigma''(j) \Big\}$$

where

$$\sigma'(j) \ = \ \sum_{|k-j| \geq (k+j)^{\alpha}} \frac{1}{|k-j|^p k^{\frac{\mu}{2}}}$$

$$\sigma''(j) \ = \ \sum_{|k-j| \geq (k+j)^{\alpha}} \frac{k^{\beta\mu}}{|k-j|^p}$$

By using similar estimates as in [12], [15], one has, for some finite constant c, and if $p - \beta\mu - 1 > 0$:

$$\sigma'(j) \ \leq \ c\Big(\frac{1}{j^{p+\frac{\mu}{2}-1}} + \frac{1}{j^p} + \frac{1}{j^{\alpha p + \frac{\mu}{2}}} + \frac{\log j}{j^p} + \frac{1}{j^{\alpha(p-1)+\frac{\mu}{2}}}\Big) \ ,$$

and

$$\sigma''(j) \leq c\Big(\frac{1}{j^{\alpha p - \beta\mu}} + \frac{1}{j^{p-\beta\mu-1}} + \frac{j^{\beta\mu}}{j^{\alpha(p-1)}}\Big) \ .$$

which gives (4.18). $\square$

Gathering the conditions in the above lemma together with the bootstrap condition $\mu(2\beta - 1) > 2 + 2(1-\alpha)^2$ stated in Theorem 3.1, we get

Proposition 4.1 *Under the hypotheses H0, H1, H2 and H4, there exists $\beta > 1/2$ such that*

$$\langle A\rangle_\varphi(t) \le c_4\, t^{\frac{\mu}{(1-\alpha)^2}} \left(\|\varphi\|^2 + \|A^\beta\varphi\|^2\right),$$

for any parameter $\alpha \in]0,1[$ and $p < \infty$ such that

$$\begin{cases} p > 2 \text{ and } \alpha > \alpha(p) \equiv \frac{2p+3-\sqrt{(2p+3)^2-40}}{4}, & \text{if} \quad \mu \le 1 \\ p > 3 + \frac{\mu}{2} \text{ and } \alpha > \alpha(p) \equiv \frac{1+p-\sqrt{(1+p)^2-8}}{2}, & \text{if} \quad \mu > 1 \end{cases}$$

where

$$\begin{cases} \frac{\alpha(2p-1)+\mu-1}{2\mu} > \beta > \frac{1}{2} + \frac{1}{\mu} + \frac{(1-\alpha)^2}{\mu}, & \text{if} \quad \mu \le 1 \\ \beta = \frac{1}{2} + \frac{\alpha(p-1)}{\mu} & \text{if} \quad \mu > 1 \end{cases}$$

Remarks:
For $\mu > 1$, we have used (4.18). In this case, the values of p and α are not optimal and it is sometimes possible to improve them by doing more technical calculations using (4.17) and (4.18). If $\mu \le 1$, which is the situation where the analysis in [17] does not apply, the conditions on p and α are optimal according to (4.17). In particular, one can take p close to 2 provided α is close to 1 and we can take α close to zero, provided p is large enough: $\alpha(p) \to 1$ when $p \searrow 2$ and $\alpha(p) = \mathcal{O}(p^{-1})$ when $p \to \infty$.
Note also that in the limit $p \to \infty$, $W(t)$ becomes a band matrix. For such matrices, we derive the estimate $\langle A\rangle_\varphi \le c\, t^\mu$ in section 5 below, which is known to be optimal in certain cases (see e.g. [1], [6]).
It should be clear from the above computations that our results can be easily extended in order to accomodate more general asymptotic behaviours than these considered in (4.16)

5 Examples

The above result is quite general since it holds under very weak assumptions on the behaviour of the matrix elements of $W(t)$. We can get sharper estimates if we know more about the structure of the perturbation $W(t)$, as can be seen on the following examples. Actually, the first example also shows that there are cases where $\langle A\rangle_\varphi(t)$ can grow arbitrarily fast in time, eventhough $W(t)$ is uniformly bounded.

5.1 Long range laplacian

Let $\mathcal{H} = l^2(\mathbb{Z})$, $H_0(t)$ be the time-dependent multiplication operator

$$(H_0(t)u)(n) = a(t)nu(n), \quad \forall n \in \mathbb{Z},$$

where $a : \mathbb{R}^+ \to \mathbb{R}$ and $W(t)$ is a long range laplacian in $l^2(\mathbb{Z})$ defined as in (1.5).

$$(W(t)u)(n) = \sum_{k\in\mathbb{Z}} \omega(n-k,t)u(k), \quad \forall n \in \mathbb{Z},$$

where $\omega : \mathbb{Z} \times \mathbb{R}^+ \to \mathbb{R}$ is real valued, such that $\omega(m,t) = \omega(-m,t)$ for all $t \geq 0$ and belongs to $l^1(\mathbb{Z})$ uniformly in $t \in \mathbb{R}^+$. By the Schur condition, we deduce that $W(t)$ is uniformly bounded in t. We assume hypotheses H0 and H1 on the evolutions $U(t)$ and $V_0(t)$ and we consider $|x|^2 = \sum_{j\in\mathbb{Z}} j^2 |j\rangle\langle j|$. As is well known, see e.g. [8], [2], this system is explicitly soluble by Fourier series. Note that $|x|^2$ is not invertible, but it's expectation value behaves like the one of the operator $(|x|+1)^2$ which satisfies H2 with $P_j = |-j\rangle\langle -j| + |j\rangle\langle j|$, $j \in \mathbb{N}^*$ and $P_0 = |0\rangle\langle 0|$.

Proposition 5.1 *Let $H_0(t)$, $W(t)$ be as above. Let $g(n,t) = w(n,t)e^{in\int_0^t a(s)ds}$ and $\hat{g}(x,t) = \sum_{k\in\mathbb{Z}} e^{ikx} g(k,t) \in L^2[0,2\pi]$. Then*

$$(U(t)\varphi_0)(n) = \varphi(n,t) = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} e^{-i\int_0^t \hat{g}(x,s)ds} dx, \ \ \textit{where } \varphi_0(n) = \delta_{n,0} \ ,$$

and

$$\langle |x|^2 \rangle_{\varphi_0}(t) = \frac{1}{2\pi}\int_0^{2\pi} \left(\int_0^t \partial_x \hat{g}(x,s)ds\right)^2 dx = \sum_{n\in\mathbb{Z}} n^2 \left|\int_0^t g(n,s)ds\right|^2 . \tag{5.19}$$

It is readily seen from (5.19) that if $\omega(m,t) \leq \omega_0 |m|^{-p}$ with $p > 3/2$, uniformly in $t \in \mathbb{R}^+$, then $\langle |x|^2 \rangle_{\varphi_0}(t) \leq ct^2$, for some finite constant c, independently of the function $a(t)$. Moreover, this upper bound is optimal if $a(t) = E_0 + E_1 \cos(ft)$ ("alternating electric field"), for specific values of the parameters E_0, E_1 and f, as can be seen from [2], [8], [9] and [7]. Note that we recover this lower bound from our results in the limit $p \to \infty$.

However, if the decay of the matrix elements $\omega(j-k,t)$ of the uniformly bounded operator $W(t)$ is not fast enough as $|j-k| \to \infty$, the expectation value $\langle |x|^2 \rangle_{\varphi_0}(t)$ can grow arbitrarily fast, as shown in the example below.

Proposition 5.2 *Let $H_0(t)$ and $W(t)$ be as above with $a(t) = (1+t)^{-q}$, $q > 1$ and $\omega(m,t) = \omega(m) = |m|^{-p}$, where $1 < p < 3/2$. Then, there exists a constant $c > 0$ such that*

$$\langle x^2 \rangle_{\varphi_0}(t) \geq ct^{(3-2p)q+2p-1}, \ \ \textit{as } t \to \infty \ .$$

Remark: Since $p > 1$, the perturbation $W(t)$ is bounded. However, the domain of x is not invariant under W alone so the electric field helps stabilizing the system. The limit $a(t) \to 0$ as $t \to \infty$ thus explains the behaviour of $\langle x^2 \rangle_{\varphi_0}(t)$.

Proof: Consider

$$n^2 \left|\int_0^t g(n,s)ds\right|^2 = \left|\int_0^t ing(n,s)ds\right|^2 = \left|\int_0^t ine^{in\int_0^s a(u)du} ds \ \omega(n)\right|^2 .$$

Integrating by parts twice yields

$$\begin{aligned}&\int_0^t ine^{in\int_0^s a(u)du} ds = \mathrm{e}^{in\int_0^t a(u)du}/a(t) - 1/a(0) + \\ &\mathrm{e}^{in\int_0^t a(u)du} a'(t)/(ina(t)^3) - a'(0)/(ina(0)^3) - \\ &\int_0^t \mathrm{e}^{in\int_0^s a(u)du} \left(a'(s)/a(s)^3\right)' ds/in \ .\end{aligned}$$

Hence, with our choice of $a(t)$,

$$\left|\int_0^t ing(n,s)ds\right|^2 = |\omega(n)^2|(1+t)^{2q}\left\{1+\mathcal{O}((1+t)^{-q})+\mathcal{O}((1+t)^{q-1}/n)\right\} .$$

There exists a constant $K>0$ such that the last parentheses above is bigger than $1/2$, provided t is large enough and $|n|\geq K(1+t)^{q-1}$. Thus, for t large enough, we get

$$\begin{array}{l}\sum_{n\in\mathbb{Z}} n^2\left|\int_0^t g(n,s)ds\right|^2 \geq \sum_{|n|\geq K(1+t)^{q-1}} n^2\left|\int_0^t g(n,s)ds\right|^2 \geq \\ \frac{(1+t)^{2q}}{2}\sum_{|n|\geq K(1+t)^{q-1}}\frac{1}{|n|^{2p}} .\end{array}$$

It remains to use the estimate

$$\sum_{|n|\geq K(1+t)^{q-1}}\frac{1}{|n|^{2p}} \geq 2\int_{[K(1+t)^{q-1}]+1}^{\infty} x^{-2p}dx = \frac{2}{2p-1}([K(1+t)^{q-1}]+1)^{1-2p} ,$$

to end the proof. □

5.2 Time-dependent band matrix

The estimate we get for this second example is certainly not surprizing, but it has the merit of beeing optimal, which is useful for the sake of comparisons with general cases (see the remark below proposition 4.1). Moreover, the proof of it is relatively easy and contains some ideas we generalize to prove proposition 3.1 in section 6, so we give it here. $W(t)$ is a band matrix with time-dependent width, i.e. $W(t)$ is such that $\|W(t)-W_0^d(t)\|\leq\omega$, $\forall t\in\mathbb{R}^+$, and $P_jW(t)P_k=0$ if $|j-k|>q(t)$ where $q(t)$ is a positive real-valued function. We further assume that H0 and H1 holds for the evolutions $U(t)$ and $V_0(t)$ and we consider a generic observable A satisfying H2.

Proposition 5.3 *Let $H_0(t)$ and $W(t)$ be as above and let $q:\mathbb{R}^+\to\mathbb{R}^+$. Denoting $\overline{q}(t)\equiv\sup_{0\leq s\leq t}q(s)$, for all $\beta\geq 1$ there exist constants $c<\infty$ and $\gamma>0$ such that for any $\varphi\in D(A^\beta)$*

$$\langle A\rangle_\varphi(t)\leq c\left((t\overline{q}(t))^\mu\|\varphi\|^2+(\overline{q}(t)^{\mu+2}e^{-\gamma t}+(t\overline{q}(t))^{2-\mu(\beta-1)})\|A^\beta\varphi\|^2\right).$$

Furthermore, if there exists a $K>0$ such that $k>K$ implies $P_k\varphi=0$, then

$$\langle A\rangle_\varphi(t) \;\leq c\;\left((t\overline{q}(t))^\mu\|\varphi\|^2+\overline{q}(t)^{\mu+2}e^{-\gamma t}\|A\varphi\|^2\right).$$

In case H_0, W and q are independant of time, the second estimate can be found in [1].

Proof:

Several constants appear in the proof, which we all denote by the same symbol c. Due to hypothesis H1, the unitary operator $\Omega(t)\equiv V_0^{-1}(t)U(t)$ satisfies for any $\varphi\in D$

$$\begin{aligned} i\Omega'(t)\varphi &= V_0^{-1}(t)(W(t)-W_0^d(t))V_0(t)\Omega(t)\varphi \\ &= \widetilde{W}(t)\Omega(t)\varphi, \end{aligned}$$

where $\widetilde{W}(t)$ is bounded by ω. We can write, using Dyson's series

$$\Omega(t) = \sum_{n=0}^{\infty} \Omega_n(t) \text{ with } \Omega_n(t) = \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \widetilde{W}(s_1) \cdots \widetilde{W}(s_n) \ , \ \Omega_0(t) = \mathbb{1} \ ,$$

where, due to $[P_j, V_0(t)] \equiv 0$, $\forall j \in \mathbb{N}^*, t \in \mathbb{R}^+$,

$$\begin{aligned} &P_j \Omega_n(t) P_k = 0 \quad \text{if} \quad |j-k| > n\overline{q}(t) \ , \text{ and} \\ &\|P_j \Omega_n(t) P_k\| \leq \frac{(\omega t)^n}{n!} \leq \left(\frac{\omega t}{n}\right)^n . \end{aligned}$$

Moreover ($\Omega(t)$ is unitary and $\{\lambda_j\}_{j \in \mathbb{N}^*}$ is increasing),

$$\begin{aligned} \langle A \rangle_\varphi(t) &= \langle \Omega(t)\varphi | A \Omega(t) \varphi \rangle \\ &= \sum_{j \in \mathbb{N}^*} \|P_j A^{1/2} \Omega(t) \varphi\|^2 \\ &\leq \lambda_N \|\varphi\|^2 + \sum_{j > N} \lambda_j \|P_j \Omega(t) \varphi\|^2 \end{aligned}$$

for any $N \in \mathbb{N}^*$. Let $A(j,k) = \{n \in \mathbb{N} |\ |j-k| \leq \overline{q}(t) n\}$, $A_1(j) = \{k \in \mathbb{N}^* |\ |j-k| \geq at\overline{q}(t)\}$ and $A_2(j) = \mathbb{N}^* \backslash A_1(j)$, where a is determined below.

$$\begin{aligned} \lambda_j \|P_j \Omega(t) \varphi\|^2 &= \lambda_j \| \sum_{k \in A_1(j)} + \sum_{k \in A_2(j)} P_j \Omega(t) P_k \varphi \|^2 \\ &\leq c \left(\| \sum_{k \in A_1(j)} \sum_{n \in A(j,k)} j^{\mu/2} P_j \Omega_n(t) P_k \varphi \|^2 + \| \sum_{k \in A_2(j)} j^{\mu/2} P_j \Omega(t) P_k \varphi \|^2 \right) \\ &\leq c \left(\sum_{k \in A_1(j)} \sum_{n \in A(j,k)} j^{\mu/2} \|P_j \Omega_n(t) P_k\| \|P_k \varphi\| \right)^2 + \\ & \quad c \left(\sum_{k \in A_2(j)} \frac{j^{\beta\mu/2}}{j^{\mu(\beta-1)/2}} \|P_j \Omega(t)\| \|P_k \varphi\| \right)^2 \end{aligned} \tag{5.20}$$

By construction, if $k \in A_1(j)$ and $n \in A(j,k)$, one has:

$$j^{\beta\mu/2} \leq (k + \overline{q}(t) n)^{\beta\mu/2} \leq 2^{\beta\mu/2} k^{\beta\mu/2} (\overline{q}(t) n)^{\beta\mu/2} \ ,$$

and there exists a $\gamma(a) \to \infty$ as $a \to \infty$ such that the first term in (5.20) is bounded above by a constant times

$$\begin{aligned} & \frac{\overline{q}(t)^{\beta\mu}}{j^{\mu(\beta-1)}} \left(\sum_{k \in A_1(j)} k^{\beta\mu/2} \|P_k \varphi\| \sum_{n \in A(j,k)} n^{\beta\mu/2} \left(\frac{\omega t}{n}\right)^n \right)^2 \\ \leq\ & \frac{\overline{q}(t)^{\beta\mu}}{j^{\mu(\beta-1)}} \left(\sum_{k \in A_1(j)} k^{\beta\mu/2} \|P_k \varphi\| \sum_{n \in A(j,k)} n^{\beta\mu/2} \left(\frac{\omega}{a}\right)^n \right)^2 \\ \leq\ & \frac{\overline{q}(t)^{\beta\mu}}{j^{\mu(\beta-1)}} \left(\sum_{k \in A_1(j)} k^{\beta\mu/2} \|P_k \varphi\| \sum_{n \in A(j,k)} \mathrm{e}^{-\gamma(a) n} \right)^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\overline{q}(t)^{\beta\mu}}{j^{\mu(\beta-1)}}\left(\sum_{k\in A_1(j)} k^{\beta\mu/2}\|P_k\varphi\|\frac{\mathrm{e}^{-\gamma(a)|j-k|/\overline{q}(t)}}{1-\mathrm{e}^{-\gamma(a)}}\right)^2 \\
&\leq c\frac{\overline{q}(t)^{\beta\mu}}{N^{\mu(\beta-1)}}\mathrm{e}^{-\gamma(a)at}\left(\sum_{k\in A_1(j)} k^{\beta\mu/2}\|P_k\varphi\|\mathrm{e}^{-\gamma(a)|j-k|/(2\overline{q}(t))}\right)^2 .
\end{aligned}
$$

(where we used $j > N$). Introducing $f \in l^2(\mathbb{Z})$ and $g \in l^1(\mathbb{Z})$ by

$$
\begin{aligned}
f(k) &= \begin{cases} k^{\beta\mu/2}\|P_k\varphi\| \text{ if } k \geq 1 \\ 0 \text{ otherwise} \end{cases} \\
g(k) &= \mathrm{e}^{-\gamma(a)|k|/(2\overline{q}(t))} ,
\end{aligned}
$$

we can estimate the last parenthesis above by

$$
\|g\|^2_{l^1(\mathbb{Z})}\|f\|^2_{l^2(\mathbb{Z})} \leq c\overline{q}^2(t)\|A^\beta\varphi\|^2.
$$

The second term of (5.20) is bounded by a constant times

$$
\begin{aligned}
&\left(\sum_{k\in A_2(j)} \frac{j^{\beta\mu/2}}{j^{\mu(\beta-1)/2}}\|P_k\varphi\|\right)^2 \\
&\quad\leq \frac{1}{N^{\mu(\beta-1)}}\left(\sum_{j-at\overline{q}(t)<k<j+at\overline{q}(t)} k^{\beta\mu/2}\|P_k\varphi\|\,(j/k)^{\beta\mu/2}\right)^2 \\
&\quad\leq \frac{1}{N^{\mu(\beta-1)}}\left(\sum_{j-at\overline{q}(t)<k<j+at\overline{q}(t)} k^{\beta\mu}\|P_k\varphi\|\left(\frac{j}{j-at\overline{q}(t)}\right)^{\beta\mu/2}\right)^2
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_{j>N}\left(\sum_{k\in A_2(j)} j^{\mu/2}\|P_k\varphi\|\right)^2 \\
&\quad\leq \sum_{j>N}\frac{N^{\mu(1-\beta)}}{(1-at\overline{q}(t)/N)^{\mu\beta}}\left(\sum_{-at\overline{q}(t)\leq m\leq at\overline{q}(t)} (m+j)^{\beta\mu/2}\|P_{m+j}\varphi\|\right)^2 \\
&\quad\leq \frac{N^{\mu(1-\beta)}}{(1-at\overline{q}(t)/N)^{\mu\beta}}\sum_{-at\overline{q}(t)\leq m\leq at\overline{q}(t)}\left(\sum_{j>N}(m+j)^{\beta\mu}\|P_{m+j}\varphi\|^2\right)(2at\overline{q}(t)+1) \\
&\quad\leq c\frac{N^{\mu(1-\beta)}}{(1-at\overline{q}(t)/N)^{\mu\beta}}(t\overline{q}(t))^2\|A^\beta\varphi\|^2,
\end{aligned}
$$

where we need $at\overline{q}(t) < N$. Thus choosing $N(t) = [2at\overline{q}(t)] + 1$, we get by gathering these estimates

$$
\langle A\rangle_\varphi(t) \quad\leq\quad c\left((t\overline{q}(t))^\mu\|\varphi\|^2 + \left(\overline{q}(t)^{\mu+2}\mathrm{e}^{-\gamma(a)at} + (t\overline{q}(t))^{2+\mu(1-\beta)}\right)\|A^\beta\varphi\|^2\right).
$$

If there exists a $K > 0$ such that $k > K$ implies $P_k\varphi = 0$, the set $A_2(j)$ is empty if N is so large that $j > N(t) \geq at\overline{q}(t) + K$. Thus, with $N(t) = [2at\overline{q}(t)] + 1$ again, we get in this case for a large enough,

$$
\langle A\rangle_\varphi(t) \quad\leq\quad c\left((t\overline{q}(t))^\mu\|\varphi\|^2 + \overline{q}(t)^{\mu+2}\mathrm{e}^{-\gamma(a)at}\|A^\beta\varphi\|^2\right).
$$

□

5.3 Laplacian and time dependant potential on $l^2(\mathbb{Z}^d)$

Let W be the discrete laplacian on $l^2(\mathbb{Z}^d)$: $(Wu)(n) = \sum_{j\in\mathbb{Z}^d,\ |j-n|=1} u(j)$ and $H_0(t)$ be a time dependent multiplication operator defined on a dense domain D for any t, which satisfies hypothesis H1. Let

$$\Pi_j = |j\rangle\langle j| \ \ ,\ j \in \mathbb{Z}^d \ ,$$

and

$$P_k = \sum_{j\in\mathbb{Z}^d,\ |j|=k-1} \Pi_j \ \ ,\ k \in \mathbb{N}* \ ,$$

then for any $m > 0$, if

$$A = \sum_{k\in\mathbb{N}^*} k^m P_k \ , \quad \text{and} \quad |x|^m = \sum_{j\in\mathbb{Z}^d} |j|_e^m \Pi_j \ ,$$

where $|\ .\ |_e$ is the euclidian norm, one has, for all $\xi \in l^2(\mathbb{Z}^d)$

$$\langle |x|^m \xi \mid \xi\rangle \le \langle A\xi \mid \xi\rangle \le d^{m/2}\langle (|x|+1)^m \xi \mid \xi\rangle \ .$$

Now, for all $j, k \in \mathbb{N}^*$,

$$\begin{aligned} P_j W P_k &= \sum_{l\in\mathbb{Z}^d,\ |l|=j} \ \sum_{m\in\mathbb{Z}^d,\ |m|=k} \Pi_l W \Pi_m \\ &= 0 \quad \text{if} \quad |j-k| > 1 \ . \end{aligned}$$

Obviously, for all $j \in \mathbb{N}$, $[H_0(t), P_j] = 0$. Then, from the results of section 5.2 one obtains that for all $\varphi \in D(A^\beta)$, where $\beta = \max(2/m, 1)$, there exists $c < \infty$ such that for all $t \gg 1$,

$$\langle |x|^m\rangle_\varphi(t) \le c\, t^m(\|\varphi\|^2 + \|(|x|+1)^{\beta m}\varphi\|^2) \ .$$

6 Technicalities

Proof of Proposition 3.1:
From H1, one can easily prove that $(V_0^{-1}V_q)(t,s) \equiv V_0^{-1}(t)V_q(t,s)V_0(s)$ is the evolution operator associated to the bounded operator $\widetilde{W}(t) \equiv V_0^{-1}(t)(W_q^d - W_0^d)(t)V_0(t)$; thus one can write the norm convergent Dyson's series:

$$\begin{aligned} (V_0^{-1}V_q)(t,s) &= \sum_{j=0}^{\infty} \Omega_j(t,s) \ , \qquad \text{where} \\ \Omega_j(t,s) &= \int_s^t ds_1 \cdots \int_s^{s_{j-1}} ds_j \widetilde{W}(s_1)\cdots\widetilde{W}(s_j) \quad j \ge 1 \\ \Omega_0(t,s) &= \mathbb{1} \ . \end{aligned}$$

Moreover, if $w \equiv \sup_{t>0} \|(W_q^d - W_0^d)(t)\|$ then

$$\begin{aligned} \|P_j \Omega_n(t,s)\| &\le \frac{t^n w^n}{n!} \\ &\le e^{-\gamma n} \ , \quad \forall 0 \le s \le t \text{ and } \forall n \ge at \end{aligned} \tag{6.21}$$

for suitable non negative finite constants a and γ. For all $\zeta \in \mathcal{D}(A^\beta)$ one then obtains, using the unitarity of $V_0(s)$ and $[V_0(s), P_j] = 0$,

$$
\begin{aligned}
&\sum_{j>N(t)} \lambda_j \sup_{s\in[0,t]} \|P_j V_q(t,s)\zeta\|^2 \\
&\quad = \sum_{j>N(t)} \lambda_j \sup_{s\in[0,t]} \|P_j (V_0^{-1}V_q)(t,s) V_0^{-1}(s)\zeta\|^2 \\
&\quad \leq \sum_{j>N(t)} \lambda_j \sup_{s\in[0,t]} \|\sum_k \sum_n P_j \Omega_n(t,s) P_k V_0^{-1}(s)\zeta\|^2
\end{aligned} \tag{6.22}
$$

We now make use of the following lemma which is proven at the end of this section:

Lemma 6.1 *Under the assumptions H0, H1 and H2, there exists $K < \infty$ such that for all $s, t \in \mathbb{R}$,*

$$
P_k \Omega_n(t,s) P_j = 0 \quad \text{if} \quad |k-j| \geq K^{\frac{1}{1-\alpha}} \min(k,j)^\alpha n^{\frac{1}{1-\alpha}} \tag{6.23}
$$

¿From this, $\|P_k V_0^{-1}\zeta\| = \|P_k\zeta\|$, (6.21), and the definitions $B(j,k) \equiv \{n \in \mathbb{N} \text{ s.t. } |k-j| < K^{\frac{1}{1-\alpha}} \min(k,j)^\alpha n^{\frac{1}{1-\alpha}}\}$, $A_i \equiv \{k \in \mathbb{N}^* \mid \frac{(k-j)^{1-\alpha}}{j^{\alpha(1-\alpha)}} \geq Kat\}$, $A_{ii} \equiv \{k \in \mathbb{N}^* \mid \frac{(j-k)^{1-\alpha}}{k^{\alpha(1-\alpha)}} \geq Kat\}$ and $A_{iii} \equiv \mathbb{N}^* - \{A_i \cup A_{ii}\}$, one proves that the right hand side of (6.22) is bounded above by

$$
\begin{aligned}
& 3 \sum_{j>N(t)} \Big\{ (\lambda_j^{1/2} \sum_{k\in A_i} \sum_{n\in B(j,k)} \mathrm{e}^{-\gamma n} \|P_k\zeta\|)^2 + (\lambda_j^{1/2} \sum_{k\in A_{ii}} \sum_{n\in B(j,k)} \mathrm{e}^{-\gamma n} \|P_k\zeta\|)^2 \\
& \quad + (\lambda_j^{1/2} \sum_{k\in A_{iii}} \|P_k\zeta\|)^2 \Big\} \\
\leq \;& C_0 \sum_{j>N(t)} \Big\{ \mathrm{e}^{-\tilde{\gamma} at} (\sum_{k\in A_i} \lambda_k^{1/2} \|P_k\zeta\| \mathrm{e}^{-\frac{\tilde{\gamma}}{2}\frac{(k-j)^{1-\alpha}}{j^{\alpha(1-\alpha)}}})^2 \\
& \quad + \mathrm{e}^{-\tilde{\gamma} at} (\sum_{k\in A_{ii}} \lambda_k^{1/2} \|P_k\zeta\| \mathrm{e}^{-\frac{\tilde{\gamma}}{2}\frac{(j-k)^{1-\alpha}}{k^{\alpha(1-\alpha)}}})^2 + (\sum_{k\in A_{iii}} \lambda_j^{1/2} \|P_k\zeta\|)^2 \Big\}
\end{aligned} \tag{6.24}
$$

where $\tilde{\gamma} > 0$, and $C_0 < \infty$. Indeed, for fixed $j \in \mathbb{N}^*$, if $k \in A_i \cup A_{ii}$ then for all $n \in B(j,k)$, one has $n \geq at$ and if $k \in A_i$ and $n \in B(j,k)$

$$
j < k + K^{1/(1-\alpha)} k^\alpha n^{1/(1-\alpha)} \leq 2K^{1/(1-\alpha)} k n^{1/(1-\alpha)}
$$

so that from H2

$$
\lambda_j^{1/2} \leq c \lambda_k^{1/2} n^{\mu/2(1-\alpha)}
$$

where c is constant.

We deal with each term separately. Consider first

$$
\sigma_1(j) \equiv \sum_{k\in A_i} \lambda_k^{1/2} \mathrm{e}^{-\frac{\tilde{\gamma}}{2}\frac{(k-j)^{1-\alpha}}{j^{\alpha(1-\alpha)}}} \|P_k\zeta\|
$$

Since $\zeta = A^{-\beta}\varphi$ for some φ then $\|P_k\zeta\| = \frac{1}{\lambda_k^\beta}\|P_k\varphi\|$ and

$$
\begin{aligned}
\sigma_1(j) &\le C_1 \sum_{k\in A_i} \frac{\mathrm{e}^{-\frac{\tilde\gamma}{2}\frac{(k-j)^{1-\alpha}}{j^{\alpha}(1-\alpha)}}}{k^{\mu(\beta-1/2)}}\|P_k\varphi\| \\
&\le C_1 \sum_{k\ge j+(Kat)^{1/(1-\alpha)}j^\alpha} \frac{\mathrm{e}^{-\frac{\tilde\gamma}{2}\frac{(k-j)^{1-\alpha}}{k^{\alpha}(1-\alpha)}}}{k^{\mu(\beta-1/2)}}\|P_k\varphi\| , \qquad (6.25)
\end{aligned}
$$

since $k \ge j$, where C_1 is a finite constant. We then use the following lemma, the proof of which can be found at the end of this section

Lemma 6.2 *With the same notations as in proposition 3.1, if $\mu(2\beta-1) > 1+\alpha$ then there exists $c < \infty$ such that for all $t \gg 1$,*

$$
\sum_{j\in\mathbb{N}^*}\Big(\sum_{k\in\mathbb{N}^*} k^{\mu(1/2-\beta)}e^{-\frac{\tilde\gamma}{2}\frac{|k-j|^{1-\alpha}}{k^{\alpha}(1-\alpha)}}\|P_k\varphi\|\Big)^2 \le c\|\varphi\|^2 . \qquad (6.26)
$$

Thus, (6.25), (6.26) and $\mu(2\beta-1) > 1+\alpha$ then give

$$
\Sigma_1(t) \equiv \sum_{j>N(t)} \sigma_1(j)^2 \le c\|\varphi\|^2 \qquad (6.27)
$$

where c is a finite constant. Similarly, for

$$
\sigma_2(j) \equiv \sum_{k\in A_{ii}} \lambda_k^{1/2}\|P_k\zeta\|\mathrm{e}^{-\frac{\tilde\gamma}{2}\frac{(j-k)^{1-\alpha}}{k^{\alpha}(1-\alpha)}}
$$

one gets from the same lemma

$$
\begin{aligned}
\Sigma_2(t) \equiv \sum_{j>N(t)} \sigma_2(j)^2 &\le C_2 \sum_{j>N(t)}\Big(\sum_{1\le k\le l_t(j)} \frac{\mathrm{e}^{-\frac{\tilde\gamma}{2}\frac{(j-k)^{1-\alpha}}{k^{\alpha}(1-\alpha)}}}{k^{\mu(\beta-1/2)}}\|P_k\varphi\|\Big)^2 \\
&\le c\|\varphi\|^2 \qquad (6.28)
\end{aligned}
$$

where $l_t(j) = \sup\{x \in \mathbb{R} \text{ s.t. } j \ge x + (Kat)^{1/(1-\alpha)}x^\alpha\}$ and c is a finite constant. Finally,

$$
\sigma_3(j) \equiv \sum_{k\in A_{iii}} \lambda_j^{1/2}\|P_k\zeta\|
$$

is bounded above by

$$
\sum_{k\in A_{iii}} \frac{\lambda_j^{1/2}}{\lambda_k^\beta}\|P_k\varphi\| \le c \sum_{l_t(j)-j\le m\le(Kat)^{1/(1-\alpha)}j^\alpha} \frac{j^{\mu/2}}{(j+m)^{\mu\beta}}\|P_{j+m}\varphi\| \qquad (6.29)
$$

Since $m \ge l_t(j) - j$, one has

$$
\begin{aligned}
\frac{j^{\frac{\mu}{2}}}{(j+m)^{\mu\beta}} &= j^{\mu(\frac12-\beta)}\Big(\frac{1}{1+\frac{m}{j}}\Big)^{\mu\beta} \\
&\le j^{\mu(\frac12-\beta)}\Big(\frac{j}{l_t(j)}\Big)^{\mu\beta} \qquad (6.30)
\end{aligned}
$$

On the other hand, by the definition of $l_t(j)$, one has $j = l_t(j) + (Kat)^{\frac{1}{1-\alpha}} l_t(j)^\alpha$, which implies $\frac{l_t(j)}{j} \geq \frac{1}{2}$, if

$$N(t) = [2^{\frac{1}{1-\alpha}} (Kat)^{(\frac{1}{1-\alpha})^2}] + 1 \ . \tag{6.31}$$

This result together with (6.29) and (6.30) gives for some finite constant c_3:

$$\sigma_3(j) \ \leq \ c_3 \sum_{l_t(j)-j \leq m \leq (Kat)^{1/(1-\alpha)} j^\alpha} j^{\mu(\frac{1}{2}-\beta)} \|P_{j+m}\varphi\| \tag{6.32}$$

From (6.32) one obtains:

$$\begin{aligned}
\Sigma_3(t) \equiv \sum_{j>N(t)} \sigma_3(j)^2 \ &\leq \ c_3 \sum_{j>N(t)} j^{\mu(1-2\beta)} \Big(\sum_{|m| \leq (Kat)^{1/(1-\alpha)} j^\alpha} \|P_{j+m}\varphi\| \Big)^2 \\
&\leq \ c_3 \sum_{j>N} j^{\mu(1-2\beta)} \Big(\sum_{|m| \leq (Kat)^{1/(1-\alpha)} j^\alpha} 1 \Big) \Big(\sum_{|m| \leq (Kat)^{1/(1-\alpha)} j^\alpha} \|P_{j+m}\varphi\|^2 \Big) \\
&\leq \ c_4 \, \frac{(Kat)^{1/(1-\alpha)} \|\varphi\|^2}{N(t)^{\mu(2\beta-1)-\alpha-1}} \\
&\leq \ c_5 \, (Kat)^{\frac{\mu(1-2\beta)+2}{(1-\alpha)^2}} \|\varphi\|^2
\end{aligned} \tag{6.33}$$

where $c_4, \ c_5$ are a finite constants. Inequalities (6.27), (6.28) and (6.33) imply together with (6.24), for any $\zeta \in \mathcal{D}(A^{-\beta})$:

$$\begin{aligned}
\sum_{j>N(t)} \lambda_j \| \sum_{k>0} \sum_{n \in \mathbb{N}} P_j \Omega_n P_k \zeta\|^2 \ &\leq \ 3\, e^{-\tilde{\gamma} a t} \Big(\Sigma_1(t) + \Sigma_2(t) \Big) + 3\, \Sigma_3(t) \\
&\leq \ c_6 \left(\mathrm{e}^{-\tilde{\gamma} a t} + t^{\frac{\mu(1-2\beta)+2}{(1-\alpha)^2}} \right) \|A^\beta \zeta\|^2
\end{aligned} \tag{6.34}$$

for some finite constant c_6 independant of t and φ, which proves (3.15).

Proof of lemma 6.1:

For all $t > 0$ and for fixed n, we first calculate the values of j and k, $0 < j \leq k$ such that for all $p \in \{1, \cdots, n\}$ and for all $s_i \in [0, t]$, $(i \in \{1, \cdots, p\})$, one has $P_k \tilde{W}(s_1) \tilde{W}(s_2) \cdots \tilde{W}(s_p) P_j = 0$. For $n = 1$, consider the positive continuous concave function $f_1(x) = j + q(j + x)$, where $q(x) = x^\alpha$ as in proposition 3.1. Let $l_1^+(j)$ be it's unique positive fixed point; then for all $k > l_1^+(j)$, one has $k - j > q(k + j)$ which implies that $P_k \tilde{W}(s) P_j = 0$, for all $s \in [0, t]$, since $[V_0, P_j] = [V_0, P_k] = 0$; thus $P_k \Omega_1(t, s) P_j = 0$. Now define $f_2(x) = l_1^+(j) + q(l_1^+(j) + x)$ and let $l_2^+(j)$ be it's unique fixed point; then for all $s_1, s_2 \in [0, t]$:

$$P_k \tilde{W}(s_1) \tilde{W}(s_2) P_j = \sum_{l \in \mathbb{N}^*} P_k \tilde{W}(s_1) P_l \tilde{W}(s_2) P_j$$

If $l > l_1^+(j)$, $P_l \tilde{W}(s_2) P_j = 0$. If $l \leq l_1^+(j)$, then for all $k > l_2^+(j)$ one has

$$\begin{aligned}
k \ &> \ l_1^+(j) + q(k + l_1^+(j)) \\
&\geq \ l + q(k + l) \ ,
\end{aligned}$$

and thus $P_k \tilde{W}(s_1) P_l = 0$. Finally if $k > l_2^+(j)$, $P_k \Omega_2(t, s) P_j = 0$.

By using the same arguments, one can easily construct an increasing sequence $(l_n^+(j))_{n\in\mathbb{N}}$ $(l_0^+(j) \equiv j)$ of fixed points of $f_n(x) \equiv l_{n-1}^+(j) + q(l_{n-1}^+(j) + x)$ such that for any $m \in \mathbb{N}$, if $k > l_m^+(j)$, then $P_k\Omega_m(t,s)P_j = 0$. Note that $l_m^+(j)$ does not depend on s and t.
Similarly, there exists a decreasing sequence (possibly finite) of positive values $(l_n^-(j))$ solutions of the fixed point equations

$$l_{n-1}^-(j) - q(l_{n-1}^- + x) = x, \quad l_0^-(j) = j \ .$$

Those fixed points are such that $P_k\Omega_n(t,s)P_j = 0$ as soon as $k < l_n^-(j)$.

We now give an estimate of $l_n^\pm(j)$. One has:

$$\begin{aligned} l_n^+(j) - l_{n-1}^+(j) &= (l_n^+(j) + l_{n-1}^+(j))^\alpha \\ &< \frac{1+\alpha}{1-\alpha}(l_{n-1}^+(j))^\alpha \end{aligned} \tag{6.35}$$

Indeed, using the following estimate

$$\begin{aligned} l_n^+(j) - l_{n-1}^+(j) &< (l_{n-1}^+(j))^\alpha(1 + \alpha\frac{l_n^+(j)}{l_{n-1}^+(j)}) \\ &\leq (l_{n-1}^+(j))^\alpha\Big(1 + \alpha + \alpha\frac{(l_n^+(j) + l_{n-1}^+(j))^\alpha}{l_{n-1}^+(j)}\Big) \end{aligned}$$

which holds since $(1+x)^\alpha < 1 + \alpha x$ for $0 < \alpha \leq 1$ and $0 < x < 1$, we get

$$\begin{aligned} (l_n^+(j) + l_{n-1}^+(j))^\alpha &< (l_{n-1}^+(j))^\alpha\Big(\frac{1+\alpha}{1 - \alpha(l_{n-1}^+(j))^{\alpha-1}}\Big) \\ &\leq \frac{1+\alpha}{1-\alpha}(l_{n-1}^+(j))^\alpha \ . \end{aligned}$$

By induction, suppose that for some fixed m that for all $1 \leq i \leq m-1$,

$$l_i^+(j) - j < Kj^\alpha i^{\frac{1}{1-\alpha}} \ ,$$

then

$$\begin{aligned} (l_i^+(j))^\alpha &< (Kj^\alpha i^{\frac{1}{1-\alpha}} + j)^\alpha \\ &\leq 2^\alpha K^\alpha j^\alpha i^{\frac{\alpha}{1-\alpha}} \ , \end{aligned}$$

and

$$\begin{aligned} l_m^+(j) - j &= \Big(l_m^+(j) - l_{m-1}^+(j)\Big) + \Big(l_{m-1}^+(j) - l_{m-2}^+(j)\Big) + \cdots + \Big(l_1^+(j) - j\Big) \\ &< \frac{1+\alpha}{1-\alpha}\Big(l_{m-1}^+(j)^\alpha + l_{m-2}^+(j)^\alpha + \cdots l_1^+(j)^\alpha + j^\alpha\Big) \\ &\leq \frac{1+\alpha}{1-\alpha}2^\alpha K^\alpha j^\alpha \sum_{i=0}^{m-1} i^{\frac{\alpha}{1-\alpha}} \\ &\leq \frac{1+\alpha}{1-\alpha}2^\alpha K^\alpha j^\alpha\Big((1-\alpha)m^{\frac{1}{1-\alpha}} + \alpha\Big) \ . \end{aligned} \tag{6.36}$$

If $K \geq (2^\alpha \frac{1+\alpha}{1-\alpha})^{\frac{1}{1-\alpha}}$, the last inequality in (6.36) gives

$$l_m^+(j) - j \quad < \quad K j^\alpha m^{\frac{1}{1-\alpha}} \ . \tag{6.37}$$

Then from (6.37) and (6.35), with $n = 1$ and $K = \max\left\{(2^\alpha \frac{1+\alpha}{1-\alpha})^{\frac{1}{1-\alpha}}, \frac{1+\alpha}{1-\alpha}\right\}$, one obtains for all $n \in \mathbb{N}^*$:

$$l_n^+(j) - j < K j^\alpha n^{\frac{1}{1-\alpha}}$$

which proves (6.23) if $k > j$. With similar arguments on $l_n^-(j)$, one can prove for k and j such that $k < j$ that

$$P_k \Omega_n(t,s) P_j = 0 \quad \text{if} \quad j - k \geq K k^\alpha n^{\frac{1}{1-\alpha}} \ .$$

Proof of lemma 6.2: Since $\mu(2\beta - 1) > 1 + \alpha$, there exists $\kappa \in \mathbb{R}$ such that $2\kappa(1-\alpha) > 1$ and $\mu(1-2\beta) + 2\kappa\alpha(1-\alpha) < -1$; then for some finite constant C depending only on κ, and for all $t > 1$:

$$\begin{aligned}
&\sum_{j\in\mathbb{N}^*} \Big(\sum_{k\in\mathbb{N}^*} k^{\mu(1/2-\beta)} \mathrm{e}^{-\frac{\tilde{\gamma}}{2}\frac{|k-j|^{1-\alpha}}{k^{\alpha(1-\alpha)}}} \|P_k\varphi\| \Big)^2 \\
&\quad \leq C(\kappa) \sum_{j\in\mathbb{N}^*} \left(\left(\sum_{k\in\mathbb{N}^*, k\neq j} k^{\mu(1/2-\beta)} |k-j|^{-\kappa(1-\alpha)} k^{\kappa\alpha(1-\alpha)} \|P_k\varphi\| \right)^2 + \|P_j\varphi\|^2 \right) \\
&\quad \leq C(\kappa) \sum_{j\in\mathbb{N}^*} \left(\|\varphi\|^2 \sum_{k\in\mathbb{N}^*, k\neq j} k^{\mu(1-2\beta)} |k-j|^{-2\kappa(1-\alpha)} k^{2\kappa\alpha(1-\alpha)} \right) + \|\varphi\|^2 \\
&\quad \leq \tilde{C}(\kappa) \|\varphi\|^2 \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \frac{(|k|+1)^{\mu(1-2\beta)+2\kappa\alpha(1-\alpha)}}{(|k-j|+1)^{2\kappa(1-\alpha)}} \right) + 1 \right) ,
\end{aligned}$$

where $\tilde{C}(\kappa) = C(\kappa) 2^{\mu(2\beta-1)+2\kappa(1-\alpha)^2}$. By the choice of κ one has $f_1(k) \equiv (|k| + 1)^{\mu(1-2\beta)+2\kappa\alpha(1-\alpha)} \in l^1(\mathbb{Z})$ and $f_2(k) \equiv (|k|+1)^{-2\kappa(1-\alpha)} \in l^1(\mathbb{Z})$; then one gets

$$\begin{aligned}
\sum_{j\in\mathbb{N}} \Big(\sum_{k\in\mathbb{N}^*} k^{\mu(1/2-\beta)} \mathrm{e}^{-\frac{\tilde{\gamma}}{2}\frac{|k-j|^{1-\alpha}}{k^{\alpha(1-\alpha)}}} \|P_k\varphi\| \Big)^2 &\leq \tilde{C}(\kappa) \left(\|f_1 * f_2\|_{l^1(\mathbb{Z})} + 1 \right) \|\varphi\|^2 \\
&\leq \tilde{C}(\kappa) (\|f_1\|_{l^1(\mathbb{Z})} \|f_2\|_{l^1(\mathbb{Z})} + 1) \|\varphi\|^2 \ .
\end{aligned}$$

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