# On semibounded restrictions of self-adjoint operators

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## Abstract

After the von Neumann's remark [10] about "pathologies" of unbounded symmetric operators and an abstract theorem about stability domain [9], we develope here a general theory allowing to construct semibounded restrictions of self-adjoint operators in explicit form. We apply this theory to quantum-mechanical momentum (position) operator to describe corresponding stability domains. Generalization to the case of measurable functions of these operators is considered. In conclusion we discuss spectral properties of self-adjoint extensions of constructed self-adjoint restrictions.

## Mathematics Subject Classification

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## Keywords

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## Contents

# 1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space and let $H$ be a (closed or unclosed) densely defined symmetric operator on $\mathcal{H}$. Assume that $H$ is not semibounded from below. Naturally, the question arises whether for a given real number $b \in \mathbf{R}^1$ there is a linear dense subset $\mathcal{D}_b \subseteq \mathrm{dom}(H)$ such that $H|\mathcal{D}_b$ is bounded from below by $b$, i.e.

$$1.1(Hf, f) \geq b(f, f), \qquad f \in \mathcal{D}_b. \tag{1.1}$$

In the following we call such a domain $\mathcal{D}_b$ a $b$-stability domain of $H$. It turns out that for each $b \in \mathbf{R}^1$ a $b$-stability domain always exists provided the symmetric operator is not semibounded from above. The result was proven in [10] and, independently, in [9]. Moreover, in [9] it was found that there is always a maximal $b$-stability domain. A $b$-stability domain $\mathcal{D}_b$ is called maximal if for each $b$-stability domain $\mathcal{D}'_b$ obeying $\mathcal{D}'_b \supseteq \mathcal{D}_b$ one has $\mathcal{D}'_b = \mathcal{D}_b$.

In [9, 10] the proofs are constructive but not practicable in order to find explicitly a stability domain for a given symmetric operator. Nevertheless, for the differential operator $H_{\frac{1}{\kappa}}$, $0 < \kappa < 1$, on $\mathcal{H} = L^2(\mathbf{R}^1)$ given by

$$1.2\mathrm{dom}(H) = C_0^\infty(\mathbf{R}^1 \setminus \{0\}) \tag{1.2}$$

and

$$1.3(H_{\frac{1}{\kappa}} f)(x) = -\frac{d^2}{dx^2} f(x) - \frac{1}{4\kappa} \frac{1}{x^2} f(x), \qquad f \in \mathrm{dom}(H), \tag{1.3}$$

maximal stability domains for each $b \in \mathbf{R}^1$ are constructed in [9] in explicit form. Moreover, these domains have a physical interpretation: they correspond to an infinite sequence of repulsive $\delta$-barriers screening the attractive centre.

The results of [9, 10] are applicable to the momentum operator $D$ on $\mathcal{H} = L^2(\mathbf{R}^1)$ given by

$$1.4(Df)(x) = -i\frac{d}{dx} f(x), \qquad f \in \mathrm{dom}(D) = W^{1,2}(\mathbf{R}^1). \tag{1.4}$$

Therefore for each $b \in \mathbf{R}^1$ there is a maximal $b$-stability domain for $D$, in particular, there is 0-stability domain $\mathcal{D}_0$, i.e.

$$1.5(Df, f) \geq 0, \qquad f \in \mathcal{D}_0 \subseteq \mathrm{dom}(D). \tag{1.5}$$

In contrast to the operator $H_{\frac{1}{\kappa}}$ there is no clear physical intuition to visualize stability domains for $D$.

Since there is a certain interest among physicists to have an idea about structure of such stability domains for $D$, in this paper we make an attempt to describe more or less explicitly those domains. The chosen approach is applicable to any self-adjoint operator $H$. Our method clearly shows general structure of stability domains for self-adjoint operators and, furthermore, is practicable for the momentum operator. So we are able to construct stability domains for $D$ explicitly.

Technically, our method presupposes some knowledge of domain of the square root of non-negative part $H_+$ of the self-adjoint operator $H$. This comes from the necessity to find a contraction $\Pi$ acting from the non-negative subspace $\mathcal{H}_+$ of $H$ into the negative subspace $\mathcal{H}_-$ of $H$ such that

$$1.6\mathrm{ima}(\Pi^*) \cap \mathrm{dom}(\sqrt{H_+}) = \{0\}, \tag{1.6}$$

cf. Corollary II.6. For the momentum operator $D$ this means that using the fractional Sobolev space $W^{1/2,2}(\mathbf{R}^1)$, see [1], we choose a contraction $\Pi : H^2_+(\mathbf{R}^1) \longrightarrow H^2_-(\mathbf{R}^1)$ such that

$$1.7\mathrm{ima}(\Pi^*) \cap W^{1/2,2}(\mathbf{R}^1) = \{0\}, \tag{1.7}$$

where $H^2_\pm(\mathbf{R}^1)$ denote Hardy spaces [8]. For the operator $D$ this problem is complicated by the fact that in general elements of $W^{1/2,2}(\mathbf{R}^1)$ are not continuous.

In fact, the method is applicable to functions $\phi(D)$ of the momentum operator. We consider measurable functions $\phi(\cdot) : \mathbf{R}^1 \longrightarrow \mathbf{R}^1$ which satisfy the conditions

$$1.8\phi(t) > 0 \quad \text{a.e. in} \quad \mathbf{R}^1_+ \qquad \text{and} \qquad \phi(t) < 0 \quad \text{a.e. in} \quad \mathbf{R}^1_-, \tag{1.8}$$

where $\mathbf{R}^1_- = (-\infty, 0)$ and $\mathbf{R}^1_+ = (0, +\infty)$, and

$$1.9\phi(t) \geq Ct^\alpha, \quad \text{a.e. in} \quad \mathbf{R}^1_+, \quad \alpha > 1, \quad \text{and} \quad C > 0. \tag{1.9}$$

Now, instead of the fractional Sobolev space $W^{1/2,2}(\mathbf{R}^1)$ one has to use here the fractional Sobolev space $W^{\alpha/2,2}(\mathbf{R}^1)$ for $\alpha > 1$ which simplifies the method due to the fact that $W^{\alpha/2,2}(\mathbf{R}^1)$ can be embeded into continuous function space.

The paper is organized as follows. In the next section we present an general method allowing to construct stability domains for self-adjoint operators. In section three we apply it to the momentum operator $D$ and describe explicitly its 0-stability domains. Finally, in section four we apply the method to functions of the momentum operator. The last section is reserved for some conclusions and comments concerning previous results.

## 2 General Theory

Let $H$ be a self-adjoint operator on the separable $\mathcal{H}$ and let $E_H(\cdot)$ be its spectral measure. We set

$$2.0 - 1\mathcal{H}_- = E_H(\mathbf{R}^1_-)\mathcal{H} \qquad \text{and} \qquad \mathcal{H}_+ = E_H(\overline{\mathbf{R}^1_+})\mathcal{H} \tag{2.1}$$

as well as

$$2.0 - 2H_{-} = -H|\mathrm{dom}(H) \cap \mathcal{H}_{-} \qquad \text{and} \qquad H_{+} = H|\mathrm{dom}(H) \cap \mathcal{H}_{+}. \tag{2.2}$$

Note that $H_{\pm} \geq 0$ and that one obviously has

$$2.1\mathcal{H} = \mathcal{H}_{-} \oplus \mathcal{H}_{+}, \tag{2.3}$$

and

$$2.2H = -H_{-} \oplus H_{+}, \tag{2.4}$$

Furthermore, we note that

$$2.3\,\ker(H_{-}) = \{0\}. \tag{2.5}$$

Assume that $H$ is not semibounded neither from below and nor from above. Then it makes sense to look for 0-stability domains $\mathcal{D}_0$ of $H$. Let $\mathcal{D}_0$ be such a 0-stability domain. For each $f \in \mathcal{D}_0$ one has decomposition

$$2.4f = f_{-} \oplus f_{+}, \qquad P_{\pm}f = f_{\pm}, \tag{2.6}$$

where $P_{\pm}$ are the orthogonal projections from $\mathcal{H}$ onto $\mathcal{H}_{\pm}$. We define operator $T : \mathcal{H}_{+} \longrightarrow \mathcal{H}_{-}$ by:

$$2.5Tf_{+} \stackrel{\mathrm{def}}{=} f_{-}, \qquad f_{+} \in \mathrm{dom}(T) = P_{+}\mathcal{D}_0. \tag{2.7}$$

First one has to justify correctness of this definition. It would be correct if $f_{+} = 0$ implies $f_{-} = 0$ for $f \in \mathcal{D}_0$. Since $\mathcal{D}_0$ is a 0-stability domain, one has

$$2.6(Hf, f) = (H_{+}f_{+}, f_{+}) - (H_{-}f_{-}, f_{-}) \geq 0, \qquad f \in \mathcal{D}_0. \tag{2.8}$$

If $f_{+} = 0$, then (2.6) gives

$$2.7(H_{-}f_{-}, f_{-}) \leq 0. \tag{2.9}$$

Since $H_{-} \geq 0$ and $\ker(H_{-}) = \{0\}$, (2.7) implies $f_{-} = 0$. Hence, definition (2.5) is correct.

Using $T$ we define the operator $K : \mathcal{H}_{+} \longrightarrow \mathcal{H}$,

$$2.8Kf_{+} \stackrel{\mathrm{def}}{=} Tf_{+} \oplus f_{+}, \qquad f_{+} \in \mathrm{dom}(K) = \mathrm{dom}(T). \tag{2.10}$$

Note that

$$2.9\mathrm{ima}(K) = \mathcal{D}_0. \tag{2.11}$$

**Theorem 2.1** *II.1 Operator $H$ has a 0-stability domain $\mathcal{D}_0$ if and only if there is a densely defined linear operator $T : \mathcal{H}_{+} \longrightarrow \mathcal{H}_{-}$ obeying* *(i)* $\mathrm{dom}(T) \subseteq \mathrm{dom}(H_+)$,

*(ii)* $\mathrm{ima}(T) \subseteq \mathrm{dom}(H_-)$,

*(iii)* $\mathrm{dom}(T^*) = \{0\}$,

*(iv)* $(H_-Tf_+, Tf_+) \leq (H_+f_+, f_+), \qquad f_+ \in \mathrm{dom}(T)$.

*Proof:* First we show necessity of conditions (i) - (iv). To this end let $\mathcal{D}_0$ be a 0-stability domain of $H$. Since $\mathcal{D}_0$ is dense in $\mathcal{H}$, the domain $\mathrm{dom}(T) = P_+\mathcal{D}_0$ is dense in $\mathcal{H}_+$. Then we get:

(i) The subspace $\mathcal{H}_+$ reduces the self-adjoint operator $H$. Hence for each $f \in \mathrm{dom}(H)$ one has $P_+f \in \mathrm{dom}(H_+)$. Therefore, $P_+\mathcal{D}_0 \subseteq \mathrm{dom}(H_+)$.

(ii) Similarly one gets $P_-\mathcal{D}_0 \subseteq \mathrm{dom}(H_-)$ which proves that $\mathrm{ima}(T) \subseteq \mathrm{dom}(H_-)$.

(iii) Let $g_- \in \mathrm{dom}(T^*)$. We set $g_+ \stackrel{\text{def}}{=} -T^*g_-$. Then one gets

$$2.10(Kf_+, g_- \oplus g_+) = \\ (Tf_+ \oplus f_+, g_- \oplus -T^*g_-) = (Tf_+, g_-) - (f_+, T^*g_-) = 0, \qquad f_+ \in \mathrm{dom}(T). \tag{2.12}$$

Hence $g_- \oplus -T^*g_- \perp \mathcal{D}_0$. However, the domain $\mathcal{D}_0 = \mathrm{ima}(K)$ is dense in $\mathcal{H}$. This yields $g_- \oplus -T^*g_- = 0$, in particular, $g_- = 0$. Hence $\mathrm{dom}(T^*) = \{0\}$.

(iv) Since $\mathcal{D}_0$ is a 0-stability domain, we obtain

$$2.110 \leq (Hf, f) = -(H_-Tf_+, Tf_+) + (H_+f_+, f_+), \qquad f \in \mathcal{D}_0, \tag{2.13}$$

which verifies (iv).

To prove the converse, we start from $K$ and $\mathcal{D}_0$ defined by (2.8) and (2.9), respectively. First, we prove that $\mathcal{D}_0$ is dense in $\mathcal{H}$. If $g \perp \mathcal{D}_0$, then

$$2.13(g, Kf_+) = (g_-, Tf_+) + (g_+, f_+) = 0, \qquad f_+ \in \mathrm{dom}(T), \tag{2.14}$$

which implies that $g_- \in \mathrm{dom}(T^*)$ and $g_+ = -T^*g_-$. By (iii) we get $g_- = 0$ which yields $g_+ = 0$. Hence $g = 0$. Next we verify that $\mathcal{D}_0 \subseteq \mathrm{dom}(H)$. An element $f \in \mathcal{H}$ belongs to $\mathrm{dom}(H)$ if and only if $f_+ = P_+f$ and $f_- = P_-f$ belong to $\mathrm{dom}(H_+)$ and $\mathrm{dom}(H_-)$, respectively. Hence $Kf_+ \in \mathrm{dom}(H)$ if and only if $f_+ \in \mathrm{dom}(H_+)$ and $Tf_+ \in \mathrm{dom}(H_-)$. Taking into account (i) and (ii) we immediately obtain $\mathcal{D}_0 = K\mathrm{dom}(T) \subseteq \mathrm{dom}(H)$. It remains to show that $\mathcal{D}_0$ is a 0-stability domain. By virtue of (iv) we have

$$2.14(HKf_+, Kf_+) = (H_+f_+, f_+) - (H_-Tf_+, Tf_+) \geq 0, \tag{2.15}$$

which completes the proof. $\square$

**Corollary 2.2** *II.2 If linear operator $T$ obeys (i) - (iv), then $T$ has a dense range.*

*Proof:* Let $g_- \perp \text{ima}(T)$. Then for any $f_+ \in \text{dom}(T)$ one obtains

$$2.15(g_-, Tf_+) = 0, \tag{2.16}$$

which implies $g_- \in \text{dom}(T^*)$. By (iii) we get $g_- = 0$. □

It is natural to ask whether one can characterize maximality of stability domains [9] in terms of the operator $T$.

**Corollary 2.3** *II.2-1 Let $\mathcal{D}_0$ be a 0-stability domain of $H$.*

*(v) Domain $\mathcal{D}_0$ of $H$ can be extended to a larger 0-stability domain if and only if there is an extension $T' \supseteq T$ obeying conditions (i), (ii) and (iv).*

*(vi) Domain $\mathcal{D}_0$ is maximal if and only if $T_0$ does not admit any (proper) extension $T'$ which satisfies conditions (i), (ii) and (iv).*

*(vii) If $\text{dom}(T) = \text{dom}(H_+)$, then $\mathcal{D}_0$ is maximal.*

*Proof:* (v) If $\mathcal{D}'_0 \supseteq \mathcal{D}_0$, then $T' \supseteq T$ and $T'$ satisfies (i), (ii) and (iv) of Theorem II.1. Conversely, let $T' \supseteq T$ and let conditions (i), (ii) and (iv) be satisfied. Since $\text{dom}(T'^*) \subseteq \text{dom}(T^*)$ by Theorem II.1 (iii) we get $\text{dom}(T'^*) = \{0\}$. Hence $\mathcal{D}'_0 = \text{ima}(K')$ is a 0-stability domain of $H$ and $\mathcal{D}_0 = \text{ima}(K) \subseteq \text{ima}(K') = \mathcal{D}'_0$.

(vi) By (v) this assertion is obvious.

(vii) Suppose that $\mathcal{D}'_0 \supseteq \mathcal{D}_0$ is also a 0-stability domain of $H$. Since $\text{dom}(H_+) \supseteq P_+\mathcal{D}'_0 = \text{dom}(T') \supseteq \text{dom}(T) = \text{dom}(H_+)$ one has $\text{dom}(T') = \text{dom}(T) = \text{dom}(H_+)$. Hence $T' = T$. Then by (vi) the domain $\mathcal{D}_0$ is maximal. □

Theorem II.1 reduces the problem of 0-stability domains to existence of densely defined linear operators with certain properties. However, the condition (iv) is difficult to handle. That is why our next aim will be to reduce (iv) to a simpler condition.

**Lemma 2.4** *II.3 Let $A : \mathcal{H} \longrightarrow \mathcal{H}$ be a self-adjoint operator and let $B : \mathcal{H} \longrightarrow \mathcal{K}$ be a linear operator ($\mathcal{K}$ is a Hilbert space) such that:*

$$2.16\text{dom}(B) \subseteq \text{dom}(A) \tag{2.17}$$

*and* $\mathrm{dom}(B)$ *is core of* $A$*. If*

$$2.17\|Bf\|_{\mathcal{K}} \le \|Af\|_{\mathcal{H}}, \qquad f \in \mathrm{dom}(B), \tag{2.18}$$

*then there is a uniquely determined contraction* $\Gamma : \overline{\mathrm{ima}(A)} \longrightarrow \mathcal{K}$ *such that we have representation*

$$2.18Bf = \Gamma A f, \qquad f \in \mathrm{dom}(B). \tag{2.19}$$

*Proof:* Since $\mathrm{dom}(B)$ is a core of $A$, by (2.17) the operator $B$ admits a unique extension $\hat{B}$ such that $\mathrm{dom}(\hat{B}) = \mathrm{dom}(A)$ and

$$2.19\|\hat{B}f\|_{\mathcal{K}} \le \|Af\|_{\mathcal{H}}, \qquad f \in \mathrm{dom}(A). \tag{2.20}$$

Let $\mathcal{H}_0 = \ker(A)$ and $\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0$. Then with respect to decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ the operator $A$ gets the form

$$2.20A = 0 \oplus A_1, \tag{2.21}$$

where $\ker(A_1) = \{0\}$ and $\overline{\mathrm{ima}(A_1)} = \mathcal{H}_1$. Let

$$2.21\Gamma_1 f \stackrel{\mathrm{def}}{=} \hat{B}A_1^{-1}f, \qquad f \in \mathrm{dom}(\Gamma_1) = \mathrm{ima}(A_1). \tag{2.22}$$

By $\mathrm{ima}(A_1) = \mathrm{ima}(A)$ (see (2.20)) we have

$$2.22\Gamma_1 Af = \hat{B}A_1^{-1}Af = \hat{B}P_1 f, \qquad f \in \mathrm{dom}(A), \tag{2.23}$$

where $P_1$ denotes orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_1$. By (2.19) we obtain $\ker(\hat{B}) \supseteq \ker(A)$. Hence

$$2.23\hat{B}f = \hat{B}P_1 f = \Gamma_1 A f, \qquad f \in \mathrm{dom}(A). \tag{2.24}$$

Using again (2.19) we find

$$2.24\|\Gamma_1 Af\|_{\mathcal{K}} \le \|Af\|_{\mathcal{H}}, \qquad f \in \mathrm{dom}(A), \tag{2.25}$$

which implies

$$2.25\|\Gamma_1 g\|_{\mathcal{K}} \le \|g\|_{\mathcal{H}}, \qquad g \in \mathrm{ima}(A). \tag{2.26}$$

Hence $\Gamma_1$ can be uniquely extended to a contraction $\Gamma : \mathcal{H}_1 \longrightarrow \mathcal{K}$. By (2.23) we finally get

$$2.26\hat{B}f = \Gamma A f, \qquad f \in \mathrm{dom}(A). \tag{2.27}$$

If $f \in \mathrm{dom}(B) \subseteq \mathrm{dom}(\hat{B})$, then (2.26) implies (2.18). □

Let $\mathcal{D}_+ \stackrel{\mathrm{def}}{=} \mathrm{dom}(T) = P_+\mathcal{D}_0$, see (2.5), and

$$2.27\dot{H}_+ \stackrel{\mathrm{def}}{=} H_+|\mathcal{D}_+. \tag{2.28}$$

The operator $\dot{H}_+$ is symmetric, non-negative but in general non-selfadjoint. Let $H_+^F$ be its Friedrichs extension. Using $H_+^F$ the condition (iv) can be rewritten as

$$2.28\|\sqrt{H_-}Tf_+\| \le \|\sqrt{H_+^F}f_+\|, \qquad f_+ \in \mathcal{D}_+. \tag{2.29}$$

We note that $\mathcal{D}_+$ is a core of $\sqrt{H_+^F}$. Then by Lemma II.3 there is a unique contraction $\Gamma_F : \mathcal{R}_+^F \longrightarrow \mathcal{H}_-$, $\mathcal{R}_+^F \stackrel{\text{def}}{=} \overline{\sqrt{H_+^F}\mathcal{D}_+}$, such that

$$2.29\sqrt{H_-}Tf_+ = \Gamma_F\sqrt{H_+^F}f_+, \qquad f_+ \in \mathcal{D}_+. \tag{2.30}$$

Since

$$2.30\|\sqrt{H_+}f_+\| = \|\sqrt{H_+^F}f_+\|, \qquad f_+ \in \mathcal{D}_+, \tag{2.31}$$

again by Lemma II.3 there is a contraction $V : \mathcal{R}_+^F \longrightarrow \mathcal{H}_+$, such that

$$2.31\sqrt{H_+}f_+ = V\sqrt{H_+^F}f_+, \qquad f_+ \in \mathcal{D}_+. \tag{2.32}$$

The operator $V$ is a partial isometry acting from the subspace $\mathcal{R}_+^F$ onto the subspace

$$2.31 - 1\mathcal{R}_+ \stackrel{\text{def}}{=} \overline{\sqrt{H_+}\mathcal{D}_+}. \tag{2.33}$$

Hence the representation

$$2.32\sqrt{H_+^F}f_+ = V^*\sqrt{H_+}f_+, \qquad f_+ \in \mathcal{D}_+, \tag{2.34}$$

holds. Setting

$$2.33\Gamma = \Gamma_F V^* : \mathcal{R}_+ \longrightarrow \mathcal{H}_- \tag{2.35}$$

we obtain a contraction $\Gamma$ such that

$$2.34\sqrt{H_-}Tf_+ = \Gamma\sqrt{H_+}f_+, \qquad f_+ \in \mathcal{D}_+. \tag{2.36}$$

Since $\mathrm{ima}(T) \subseteq \mathrm{dom}(H_-)$ one has $\Gamma\sqrt{H_+}f_+ \subseteq \mathrm{ima}(\sqrt{H_-})$, $f_+ \in \mathcal{D}_+$. Using that and $\ker(H_-) = \{0\}$ one gets the representation

$$2.35Tf_+ = H_-^{-1/2}\Gamma\sqrt{H_+}f_+, \qquad f_+ \in \mathcal{D}_+. \tag{2.37}$$

Now we can characterize all densely defined linear operators $T$ obeying conditions (i) - (iv) in terms of contractions $\Gamma$. For simplicity we assume below that $H_-$ is boundedly invertible. As we shall see this condition gives no restrict on possibilities to construct stability domains.

**Theorem 2.5** *II.4 Assume that $H_-$ is boundedly invertible. A corresponding operator $H$ has a 0-stability domain $\mathcal{D}_0$ if and only if there is a dense subset $\mathcal{D}_+ \subseteq \mathrm{dom}(H_+)$ and contraction $\Gamma : \mathcal{R}_+ \longrightarrow \mathcal{H}_-$, such that* *(a)* $\overline{\mathrm{ima}(H_-^{-1/2}\Gamma)} = \mathcal{H}_-$,

*(b)* $\Gamma\sqrt{H_+}\mathcal{D}_+ \subseteq \mathrm{dom}(\sqrt{H_-})$,

*(c)* $\Gamma^*\mathrm{dom}(\sqrt{H_-}) \cap \mathrm{dom}(L_+^*) = \{0\}, \qquad L_+ \stackrel{\mathrm{def}}{=} \sqrt{H_+}|\mathcal{D}_+.$

*Proof:* Let $\mathcal{D}_0$ be a 0-stability domain. Then existence of contraction $\Gamma$ is shown by reasoning (2.27) - (2.35) above. Therefore, it rests to verify conditions (a) - (c). To this end we set $\mathcal{D}_+ \stackrel{\mathrm{def}}{=} P_+\mathcal{D}_0$. By (2.35) and Corollary II.2 one gets that

$$2.36\overline{H_-^{-1/2}\Gamma H_+^{1/2}\mathcal{D}_+} = \mathcal{H}_-. \tag{2.38}$$

Since $\sqrt{H_+}\mathcal{D}_+$ is dense in $\mathcal{R}_+ = \overline{\sqrt{H_+}\mathcal{D}_+}$ and $\Gamma : \mathcal{R}_+ \longrightarrow \mathcal{H}_-$ is a bounded operator, we obtain condition (a). Moreover, by Theorem II.1 (ii) we have

$$2.37H_-^{-1/2}\Gamma H_+^{1/2}\mathcal{D}_+ \subseteq \mathrm{dom}(H_-) \tag{2.39}$$

which implies condition (b). Let $g_- \in \mathrm{dom}(\sqrt{H_-})$ and

$$2.38\Gamma^* g_- \in \mathrm{dom}(L_+^*). \tag{2.40}$$

Setting $f_- = \sqrt{H_-}g_-$ and $L_+ = \sqrt{H_+}|\mathcal{D}_+$ and using (2.35) we get

$$2.39(Tf_+, f_-) = (H_-^{-1/2}\Gamma\sqrt{H_+}f_+, \sqrt{H_-}g_-) = (L_+f_+, \Gamma^*g_-), \qquad f_+ \in \mathcal{D}_+. \tag{2.41}$$

Then (2.38) implies

$$2.40(Tf_+, f_-) = (f_+, L_+^*\Gamma^*H_-^{-1/2}f_-), \qquad f_+ \in \mathcal{D}_+, \tag{2.42}$$

which shows that $f_- \in \mathrm{dom}(T^*)$. Since $\mathrm{dom}(T^*) = \{0\}$, cf. Theorem II.1 (iii), one gets $f_- = 0$. Since $\ker(H_-) = \{0\}$, one gets $g_- = 0$. Therefore, $\Gamma^*g_- = 0$ and, consequently, $\Gamma^*\mathrm{dom}(\sqrt{H_-}) \cap \mathrm{dom}(L_+^*) = \{0\}$ which proves condition (c).

To prove the converse we define by (2.35) the operator $T$. Since $\mathrm{dom}(T) = \mathcal{D}_+$, the operator $T$ is densely defined in $\mathcal{H}_+$. Then the property (i) (see Theroem II.1) is satisfied by construction. Taking into account condition (b) we verify (ii). To prove (iii) let $f_- \in \mathrm{dom}(T^*)$. Then we get

$$2.41(Tf_+, f_-) = (L_+f_+, \Gamma^*H_-^{-1/2}f_-), \qquad f_+ \in \mathrm{dom}(T) = \mathcal{D}_+. \tag{2.43}$$

Since $f_- \in \mathrm{dom}(T^*)$ one has

$$2.42(L_+f_+, \Gamma^*H_-^{-1/2}f_-) = (f_+, T^*f_-), \tag{2.44}$$

which yields $\Gamma^* H^{-1/2} f_- \in \mathrm{dom}(L_+^*)$. By condition (c) this yields $\Gamma^* H_-^{-1/2} f_- = 0$. From condition (a) we obtain $\ker(\Gamma^* H_-^{-1/2}) = \{0\}$ which implies $f_- = 0$. Hence $\mathrm{dom}(T^*) = \{0\}$ which verifies (iii) of Theroem II.1. Finally, (iv) follows from relation

$$2.43(H_- T f_+, T f_+) = \qquad (2.45)$$
$$\|\Gamma\sqrt{H_+} f_+\|^2 \le \|\sqrt{H_+} f_+\|^2 = (H_+ f_+, f_+), \qquad f_+ \in \mathrm{dom}(T) = \mathcal{D}_+,$$

which completes the proof. □

Then Corollary II.2-1 should to be modified as follows.

**Corollary 2.6** *II.4-1 Let $H_-$ be boundedly invertible.*

*(d) The 0-stability domain $\mathcal{D}_0$ of $H$ can be extended to a larger 0-stability domain if and only if there is a domain $\mathcal{D}'_+$ obeying*

$$2.43 - 1 P_+ \mathcal{D}_0 = \mathcal{D}_+ \subseteq \mathcal{D}'_+ \subseteq \mathrm{dom}(H_+) \qquad (2.46)$$

*and a contraction $\Gamma' : \mathcal{R}'_+ \longrightarrow \mathcal{H}_-$, $\mathcal{R}'_+ = \overline{\sqrt{H_+}\mathcal{D}'_+} \supseteq \mathcal{R}_+$, such that*

$$2.43 - 2\Gamma' |\mathcal{R}_+ = \Gamma \qquad (2.47)$$

*and*

$$2.43 - 3\Gamma' \sqrt{H_+}\mathcal{D}'_+ \subseteq \mathrm{ima}(\sqrt{H_-}). \qquad (2.48)$$

*(e) The 0-stability domain $\mathcal{D}_0$ is maximal if and only if there is no a (properly) larger domain $\mathcal{D}'_+ \supseteq \mathcal{D}_+$, $\mathcal{D}'_+ \subseteq \mathrm{dom}(H_+)$, and a contraction $\Gamma' : \mathcal{R}'_+ \longrightarrow \mathcal{H}_-$ such that conditions (2.43-2) and(2.43-3) are satisfied.*

*(f) There is a maximal 0-stability domain $\mathcal{D}_0$ of $H$ obeying $P_+\mathcal{D}_0 = \mathrm{dom}(H_+)$ if and only if there is a contraction $\Gamma : \mathcal{R}_+ \longrightarrow \mathcal{H}_-$, $\mathcal{R}_+ = \overline{\sqrt{H_+}\mathrm{dom}(H_+)}$ such that conditions (a), (b) and*

$$2.43 - 4\Gamma^* \mathrm{dom}(\sqrt{H_-}) \cap \mathrm{dom}(\sqrt{H_+}) = \{0\} \qquad (2.49)$$

*are satisfied. The domain $\mathcal{D}_0$ is given by $\mathcal{D}_0 = K\mathrm{dom}(H_+)$, where $K$ is defined by (2.8) and (2.35).*

*Proof:* (d) Let $\mathcal{D}'_0 \supseteq \mathcal{D}_0$ be a larger 0-stability domain of $H$. As above we associate with $\mathcal{D}'_0$ the contraction $\Gamma' : \mathcal{R}'_+ \longrightarrow \mathcal{H}_-$, $\mathcal{R}'_+ = \overline{\sqrt{H_+}\mathcal{D}'_+}$, $\mathcal{D}'_+ = P_+\mathcal{D}'_0$. Since $\mathcal{D}'_0 \supseteq \mathcal{D}_0$ one has $\mathcal{D}'_+ \supseteq \mathcal{D}_+$ which yields $\mathcal{R}'_+ \supseteq \mathcal{R}_+$. Applying Corollary II.2-1 (v) and using (2.34) we find $\Gamma'|\mathcal{R}_+ = \Gamma$.

Since $\Gamma'$ satisfies (b), we obtain (2.43-3). Conversely, if the conditions (2.43-1) - (2.43-3) hold, then $\Gamma'$ obeys (a) and (b). To verify that $\Gamma'$ satisfies (c) we note that

$$2.43 - 5\Gamma'^{*}\mathrm{dom}(\sqrt{H_-}) \cap \mathrm{dom}(L'^{*}_{+}) \subseteq \Gamma'^{*}\mathrm{dom}(\sqrt{H_-}) \cap \mathrm{dom}(L^{*}_{+}) \tag{2.50}$$

because $L'_+ \supseteq L_+$ yields $\mathrm{dom}(L'^{*}_{+}) \subseteq \mathrm{dom}(L^{*}_{+})$. Furthermore, one has

$$2.43 - 6\mathrm{ima}(\Gamma'^{*}) \subseteq \mathrm{ima}(\Gamma^{*}) \oplus \ker(L^{*}_{+}). \tag{2.51}$$

Let $g_- \in \mathrm{dom}(\sqrt{H_-})$. Then

$$2.43 - 7\Gamma'^{*}g_- = \Gamma^{*}g_- + h, \qquad h \in \ker(L^{*}_{+}). \tag{2.52}$$

If $\Gamma'^{*}g_- \in \mathrm{dom}(L^{*}_{+})$,

$$2.43 - 8\Gamma^{*}g_- \in \mathrm{dom}(L^{*}_{+}). \tag{2.53}$$

By property (c) for $\Gamma$ we obtain $\Gamma^{*}g_- = 0$. Let $f_- = \sqrt{H_-}g_-$, then $\Gamma^{*}H_-^{-1/2}f_- = 0$. By (a) we have $\ker(\Gamma^{*}H_-^{-1/2}) = \{0\}$. Therefore, $f_- = \sqrt{H_-}g_- = 0$. Since $\ker(H_-) = \{0\}$, we finally obtain $g_- = 0$. Hence $\Gamma'^{*}g_- = 0$ which is equivalent to

$$2.43 - 9\Gamma'^{*}\mathrm{dom}(\sqrt{H_-}) \cap \mathrm{dom}(L^{*}_{+}) = \{0\} \tag{2.54}$$

and which proves property (c) for $\Gamma'$. Defining $T'$ by (2.35) we find $T' \supseteq T$. Applying Corollary II.2-1 (v) we complete the proof of this part.

(e) Due to (d) the assertion is obvious.

(f) By Theorem II.4 there is a contraction $\Gamma$ such that (a) - (c) are satisfied. Since $P_+\mathcal{D}_0 = \mathrm{dom}(H_+) = \mathcal{D}_+$ and $L^{*}_{+} = \sqrt{H_+}$, $L_+ = \sqrt{H_+}|\mathcal{D}_+$, the property (c) reduces to (2.43-4). Conversely, (a), (b) and (2.43-4) imply (a) - (c). By Theorem II.4 there is 0-stability domain $\mathcal{D}_0$ which is given by $\mathcal{D}_0 = K\mathrm{dom}(H_+)$ where $K$ and $T$ are defined by (2.8) and (2.35), respectively. Obviously, we have $P_+\mathcal{D}_0 = \mathrm{dom}(H_+)$. By Corollary II.2-1 (vii) the domain $\mathcal{D}_0$ is maximal. □

Next we study problem whether a given 0-stability domain $\mathcal{D}_0$ can be extended to a maximal stability domain $\mathcal{D}'_0$ which satisfies the condition $P_+\mathcal{D}'_0 = \mathrm{dom}(H_+)$

**Corollary 2.7** *II.5 Let $\mathcal{D}_0$ be a 0-stability domain of $H$ and let $H_-$ be boundedly invertible. If*

$$2.44\mathrm{ima}(\Gamma) \subseteq \mathrm{dom}(\sqrt{H_-}), \tag{2.55}$$

*then there is maximal stability domain $\mathcal{D}'_0 \supseteq \mathcal{D}_0$ obeying $P_+\mathcal{D}'_0 = \mathrm{dom}(H_+)$.*

*Proof:* We set

$$\mathcal{D}'_+ = \mathrm{dom}(H_+)2.45 \tag{2.56}$$

$$\mathcal{R}'_+ = \overline{\sqrt{H_+}\mathcal{D}'_+} \supseteq \mathcal{R}_+ 2.46 \tag{2.57}$$

$$\Gamma' = \Gamma P^{\mathcal{R}'_+}_{\mathcal{R}_+} : \mathcal{R}'_+ \longrightarrow \mathcal{H}_-, 2.47 \tag{2.58}$$

where $P^{\mathcal{R}'_+}_{\mathcal{R}_+}$ is the orthogonal projection from $\mathcal{R}'_+$ onto $\mathcal{R}_+$. The operator $\Gamma'$ is obviously a contraction such that $\Gamma'|\mathcal{R}_+ = \Gamma$, cf. (2.43-2). Since $\mathrm{ima}(\Gamma) = \mathrm{ima}(\Gamma')$, the condition (2.43-3) is satisfied. Moreover, by

$$2.48\Gamma'^*\mathrm{dom}(\sqrt{H_-}) = \Gamma^*\mathrm{dom}(\sqrt{H_-}) \tag{2.59}$$

one gets

$$\begin{aligned} &2.49\Gamma'^*\mathrm{dom}(\sqrt{H_-}) \cap \mathrm{dom}(\sqrt{H_+}) = \\ &\qquad \Gamma^*\mathrm{dom}(\sqrt{H_-}) \cap \mathrm{dom}(\sqrt{H_+}) \subseteq \Gamma^*\mathrm{dom}(\sqrt{H_-}) \cap \mathrm{dom}(L^*_+) = \{0\}, \end{aligned} \tag{2.60}$$

which verifies (2.43-4). Using relation (2.35) we define operator $T'$ with $\mathrm{dom}(T') = \mathcal{D}'_+ = \mathrm{dom}(H_+)$. Finally, by Corollary II.4-1 (f) the corresponding 0-stability domain $\mathcal{D}'_0$ is maximal. □

**Corollary 2.8** *II.6 Let $H_- \geq I_{\mathcal{H}_-}$, where $I_{\mathcal{H}_-}$ is the identity operator on $\mathcal{H}_-$. If there is a contraction $\Pi : \mathcal{H}_+ \longrightarrow \mathcal{H}_-$ with dense range such that*

$$2.51\mathrm{ima}(\Pi^*) \cap \mathrm{dom}(\sqrt{H_+}) = \{0\}, \tag{2.61}$$

*then $H$ has a maximal 0-stability domain $\mathcal{D}_0$ obeying $P_+\mathcal{D}_0 = \mathrm{dom}(H_+)$. The domain $\mathcal{D}_0$ is given by $\mathcal{D}_0 = K\mathrm{dom}(H_+)$, where*

$$2.52 - 1Kf_+ = Tf_+ \oplus f_+, \qquad f_+ \in \mathrm{dom}(K) = \mathrm{dom}(T) = \mathrm{dom}(H_+) \tag{2.62}$$

*and*

$$2.52Tf_+ = H_-^{-1}\Pi\sqrt{H_+}f_+ \qquad f_+ \in \mathrm{dom}(T) = \mathrm{dom}(H_+). \tag{2.63}$$

*Proof:* We set $\mathcal{D}_+ = P_+\mathrm{dom}(H_+)$ and

$$2.53\Gamma = H_-^{-1/2}\Pi : \mathcal{H}_+ \longrightarrow \mathcal{H}_-. \tag{2.64}$$

Since $H_- \geq I_{\mathcal{H}_-}$, the operator $\Gamma$ is a contraction. Furthermore, since $\Pi$ has a dense range, condition (a) is satisfied. Condition (b) holds by construction. Finally, one has

$$2.54\Gamma^*\mathrm{dom}(\sqrt{H_-}) \cap \mathrm{dom}(\sqrt{H_+}) \subseteq \mathrm{ima}(\Pi^*) \cap \mathrm{dom}(\sqrt{H_+}) = \{0\} \tag{2.65}$$

which proves (2.43-4). Then applying Corollary II.4-1 (f) we finish the proof. □

# 3 Momentum Operator

In order to apply the previous theory to the momentum operator we use so-called Hardy spaces $H^2_\pm(\mathbf{R}^1)$. An element $f_\pm \in L^2(\mathbf{R}^1)$ belongs to $H^2_\pm(\mathbf{R}^1)$ if and only if $f_\pm$ admits an analytic continuation to $\mathbf{C}_\pm = \{z \in \mathbf{C} : \pm\Im\mathrm{m}(z) > 0\}$, respectively, such that

$$4.1 \sup_{y>0} \int_{-\infty}^{+\infty} |f_\pm(x \pm iy)|^2 dx < +\infty. \tag{3.1}$$

Hardy spaces are closed subspaces of $L^2(\mathbf{R}^1)$ such that

$$4.2 L^2(\mathbf{R}^1) = H^2_-(\mathbf{R}^1) \oplus H^2_+(\mathbf{R}^1). \tag{3.2}$$

Using the Fourier transform $F : L^2(\mathbf{R}^1) \longrightarrow L^2(\mathbf{R}^1)$,

$$4.3 \hat{f}(t) \stackrel{\mathrm{def}}{=} (Ff)(t) \stackrel{\mathrm{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx e^{-itx} f(x), \qquad f \in L^2(\mathbf{R}^1), \tag{3.3}$$

one gets

$$4.4 F H^2_\pm(\mathbf{R}^1) = L^2(\mathbf{R}^1_\pm). \tag{3.4}$$

Consider momentum operator $D$,

$$4.5 (Df)(x) = -i\frac{d}{dx} f(x), \qquad f \in \mathrm{dom}(D) = W^{1,2}(\mathbf{R}^1), \tag{3.5}$$

Via the Fourier transform the momentum operator is unitarily equivalent to multiplication operator $\hat{M}$,

$$4.6 (\hat{M}\hat{f})(t) = tf(t), \qquad \hat{f} \in \mathrm{dom}(\hat{M}) = \{f \in L^2(\mathbf{R}^1) : t\hat{f}(t) \in L^2(\mathbf{R}^1)\}, \tag{3.6}$$

i.e. $FDF^{-1} = \hat{M}$.

Since $FDF^{-1} = \hat{M}$ and $F^{-1}L^2(\mathbf{R}^1_\pm) = H^2_\pm(\mathbf{R}^1)$, subspaces $\mathcal{H}^2_\pm(\mathbf{R}^1)$ reduces the momentum operator. We set

$$4.7 D_- = -D|\mathrm{dom}(D) \cap H^2_-(\mathbf{R}^1) \qquad \text{and} \qquad D_+ = D|\mathrm{dom}(D) \cap H^2_+(\mathbf{R}^1). \tag{3.7}$$

Obviously we have

$$4.8 FD_\pm F^{-1} = \hat{M}_\pm, \tag{3.8}$$

where

$$4.9 \hat{M}_- = -\hat{M}|\mathrm{dom}(\hat{M}) \cap L^2(\mathbf{R}^1_-) \qquad \text{and} \qquad \hat{M}_+ = \hat{M}|\mathrm{dom}(\hat{M}) \cap L^2(\mathbf{R}^1_+). \tag{3.9}$$

Since $\hat{M}_\pm \geq 0$, one gets $D_\pm \geq 0$. To apply the general theory of Section 2 we set $\mathcal{H}_\pm = H^2_\pm(\mathbf{R}^1)$,

$$4.10 H_- = D_- + I_{\mathcal{H}_-} \qquad \text{and} \qquad H_+ = D_+ \tag{3.10}$$

and consider operator $H = -H_- \oplus H_+$. Our intention is to give a simple description of 0-stability domains of $H$.

To this end we introduce operator $J : \mathcal{H} \longrightarrow \mathcal{H}$ given by

$$4.11 (Jf)(x) = f(-x), \qquad f \in L^2(\mathbf{R}^1), \quad x \in \mathbf{R}^1. \tag{3.11}$$

The operator $J$ is isometrical and satisfies $J\mathcal{H}_\pm = \mathcal{H}_\mp$. Note that

$$4.11 - 1 D_- J f_+ = J D_+ f_+, \qquad f_+ \in \mathrm{dom}(D_+). \tag{3.12}$$

Next we choose a measurable function $\Theta(\cdot) : \mathbf{R}^1 \longrightarrow \mathbf{R}^1_+$ such that

$$4.12 0 < \Theta(x) \leq \frac{1}{\sqrt{1+x^2}}, \quad \text{a.e. in} \quad \mathbf{R}^1, \tag{3.13}$$

and

$$4.13 \int_{-\infty}^{+\infty} dx \, \frac{|\log(\Theta(x))|}{1+x^2} < +\infty. \tag{3.14}$$

Moreover, let $\Theta(\cdot)$ be smooth except at points $\{x_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} x_n = 0$. At that points it is assumed that the boundary values $\Theta(x_n \pm 0) = \lim_{\epsilon\to 0} \Theta(x_n \pm \epsilon)$ exist but

$$4.14 \Theta(x_n - 0) \neq \Theta(x_n + 0), \qquad n = 1, 2, \ldots. \tag{3.15}$$

To give an example we choose

$$4.15 x_n = \frac{1}{n}, \qquad n = 1, 2, \ldots. \tag{3.16}$$

Then we introduce intervals

$$4.16 \Delta_n = (\tfrac{1}{n+1}, \tfrac{1}{n}], \qquad n = 1, 2, \ldots, \tag{3.17}$$

and set

$$4.17 \Theta_0(x) = \begin{cases} 1 & : \quad x \in (-\infty, 0] \\ \frac{1}{n} & : \quad x \in \Delta_n, \qquad n = 1, 2, \ldots \\ 1 & : \quad x \in (1, +\infty) \end{cases}. \tag{3.18}$$

Now we can define function

$$4.18 \Theta(x) = \Theta_0(x) e^{-\sqrt{|x|}}, \qquad x \in \mathbf{R}^1. \tag{3.19}$$

One can verify that both conditions (4.12) and (4.13) are satisfied. Moreover, we have

$$4.19 \Theta(\tfrac{1}{n} - 0) = \tfrac{1}{n+1} e^{-n^{-1/2}} \neq \tfrac{1}{n} e^{-n^{-1/2}} = \Theta(\tfrac{1}{n} + 0), \qquad n = 1, 2, \ldots. \tag{3.20}$$

We set

$$4.20\Omega(z) = \exp\left\{\frac{i}{\pi}\int_{-\infty}^{+\infty}\left[\frac{1}{z-t}+\frac{t}{1+t^2}\right]\log(\Theta(t))\,dt\right\}, \qquad z \in \mathbf{C}_+. \tag{3.21}$$

The function $\Omega(z)$, $z \in \mathbf{C}_+$ is an outer function, see [8]. Boundary values $\Omega(x) = \lim_{\epsilon\to 0}\Omega(x+i\epsilon)$ exist a.e. in $\mathbf{R}^1$ and

$$4.21|\Omega(x)| = \Theta(x), \qquad \text{a.e. in} \quad \mathbf{R}^1. \tag{3.22}$$

Hence $|\Omega(x)| \le 1$ a.e in $\mathbf{R}^1$ which yields $\Omega \in L^\infty(\mathbf{R}^1)$. Moreover, one has

$$4.22|\Omega(x)| \le e^{-\sqrt{|x|}}, \qquad \text{a.e. in} \quad \mathbf{R}^1. \tag{3.23}$$

By means of $\Omega$ we can introduce a bounded multiplication operator $M(\Omega)$,

$$4.23(M(\Omega)f)(x) = \Omega(x)f(x), \qquad f \in \mathcal{H}. \tag{3.24}$$

Since $\Omega$ is an outer function, one has

$$4.24M(\Omega)\mathcal{H}_+ \subseteq \mathcal{H}_+. \tag{3.25}$$

Therefore, we can define operator $M_+(\Omega) = M(\Omega)|\mathcal{H}_+ : \mathcal{H}_+ \longrightarrow \mathcal{H}_+$. The adjoint operator $M_+(\Omega)^* : \mathcal{H}_+ \longrightarrow \mathcal{H}_+$ is given by

$$4.25M_+(\Omega)^* f_+ = P_+ M(\overline{\Omega}) f_+, \qquad f_+ \in \mathcal{H}_+, \tag{3.26}$$

where $P_\pm$ denote orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}_\pm$. If $M_+(\Omega)f_+ = 0$, then

$$4.26\Omega(x)f_+(x) = 0 \qquad \text{a.e. in} \quad \mathbf{R}^1. \tag{3.27}$$

By (4.12) and (4.21) one gets $f_+(x) = 0$ a.e. in $\mathbf{R}^1$. Hence $\ker(M_+(\Omega)) = \{0\}$ or, equivalently,

$$4.27\overline{\mathrm{ima}(M_+(\Omega)^*)} = \mathcal{H}_+. \tag{3.28}$$

Now we define contraction $\Pi : \mathcal{H}_+ \longrightarrow \mathcal{H}_-$

$$4.28\Pi \stackrel{\text{def}}{=} Je^{-D_+}M_+(\Omega)^*. \tag{3.29}$$

Since $\overline{\mathrm{ima}(e^{-D_+})} = \mathcal{H}_+$, the operator $\Pi$ has a dense range.

Let us show that

$$4.29\Pi^* f \in \mathrm{dom}(\sqrt{D_+}) \tag{3.30}$$

yields $\Pi^* f = 0$. To this end we note that

$$4.30\mathrm{dom}(\sqrt{D_+}) = W^{1/2,2}(\mathbf{R}^1) \cap H^2_+(\mathbf{R}^1). \tag{3.31}$$

By Theorem 7.48 of [1] one has that $u \in W^{1/2,2}(\mathbf{R}^1)$ if and only if

$$4.31 \int_{\mathbf{R}^1} dx\, |u(x)|^2 + \int_{\mathbf{R}^1}\int_{\mathbf{R}^1} dx\, dy \frac{|u(x)-u(y)|^2}{|x-y|^2} < +\infty \tag{3.32}$$

Since

$$4.32 (e^{-D_+} f_+)(x) = f_+(x+i), \qquad f_+ \in H^2_+(\mathbf{R}^1), \tag{3.33}$$

one gets

$$4.33 (\Pi^* f_-)(x) = \Omega(x) f_+(x+i), \qquad f_+ \stackrel{\mathrm{def}}{=} J^* f_-, \qquad f_- \in \mathcal{H}_-. \tag{3.34}$$

Therefore, if $\Pi^* f \in \mathrm{dom}(\sqrt{D_+})$, then $\Omega(x) f_+(x+i) \in W^{1/2,2}(\mathbf{R}^1)$ and hence

$$4.34 \int_{\mathbf{R}^1}\int_{\mathbf{R}^1} dx\, dy \frac{|\Omega(x) f_+(x+i) - \Omega(y) f_+(y+i)|^2}{|x-y|^2} < +\infty. \tag{3.35}$$

Using representation

$$\begin{aligned} 4.35 \Omega(x) f_+(x+i) - \Omega(y) f_+(y+i) = \\ (f_+(x+i) - f_+(y+i))\Omega(x) + f_+(y+i)(\Omega(x) - \Omega(y)) \end{aligned} \tag{3.36}$$

we get

$$\begin{aligned} 4.36 \int_{\mathbf{R}^1}\int_{\mathbf{R}^1} dx\, dy \frac{|\Omega(x) f_+(x+i) - \Omega(y) f_+(y+i)|^2}{|x-y|^2} \geq \\ \Bigg| \left\{ \int_{\mathbf{R}^1}\int_{\mathbf{R}^1} dx\, dy \frac{|f_+(x+i) - f_+(y+i)|^2 |\Omega(x)|^2}{|x-y|^2} \right\}^{1/2} - \\ \left\{ \int_{\mathbf{R}^1}\int_{\mathbf{R}^1} dx\, dy |f_+(y+i)|^2 \frac{|\Omega(x) - \Omega(y)|^2}{|x-y|^2} \right\}^{1/2} \Bigg|^2. \end{aligned} \tag{3.37}$$

Since $f_+(x+i) \in W^{1/2,2}(\mathbf{R}^1)$ and $|\Omega(x)| \leq 1$ a.e. in $\mathbf{R}^1$, we find

$$4.36-1 \int_{\mathbf{R}^1}\int_{\mathbf{R}^1} dx\, dy \frac{|f_+(x+i) - f_+(y+i)|^2 |\Omega(x)|^2}{|x-y|^2} < +\infty. \tag{3.38}$$

Hence (4.34), (4.36) and (4.36-1) imply

$$4.37 \int_{\mathbf{R}^1}\int_{\mathbf{R}^1} dx\, dy |f_+(y+i)|^2 \frac{|\Omega(x) - \Omega(y)|^2}{|x-y|^2} < +\infty. \tag{3.39}$$

Assume that

$$4.38 f_+(x_n + i) \neq 0. \tag{3.40}$$

for some $n = 1, 2, \ldots$, cf. (4.15). Since $f_+(x+i)$ is continuous, there is a neighbourhood $\delta_n$ of $x_n$ such that

$$4.39 m_n = \inf_{x \in \delta_n} |f_+(x+i)| > 0. \tag{3.41}$$

By (4.37) one gets estimate

$$4.40 m_n \int_{\delta_n} \int_{\delta_n} dx\, dy \frac{|\Omega(x) - \Omega(y)|^2}{|x-y|^2} \le \int_{\delta_n} \int_{\delta_n} dx\, dy |f_+(y+i)|^2 \frac{|\Omega(x) - \Omega(y)|^2}{|x-y|^2} < +\infty. \tag{3.42}$$

Since

$$4.41 |\, |\Omega(x)| - |\Omega(y)|\, | \le |\Omega(x) - \Omega(y)|, \tag{3.43}$$

we finally obtain

$$4.42 m_n \int_{\delta_n} \int_{\delta_n} dx\, dy \frac{|\, |\Omega(x)| - |\Omega(y)|\, |^2}{|x-y|^2} < +\infty. \tag{3.44}$$

By (4.21) this is equivalent to

$$4.43 m_n \int_{\delta_n} \int_{\delta_n} dx\, dy \frac{|\Theta(x) - \Theta(y)|^2}{|x-y|^2} < +\infty. \tag{3.45}$$

Since $\Theta(x)$ is smooth except at points $x_n$ and $\Theta(x_n - 0) \neq \Theta(x_n + 0)$ we can get that

$$4.44 d_n \stackrel{\text{def}}{=} \inf_{\{x,y \in \tilde{\delta}_n : x < x_n < y\}} |\Theta(x) - \Theta(y)| > 0 \tag{3.46}$$

provided $\tilde{\delta}_n$ is small enough. Hence

$$4.45 m_n d_n \int \int_{\{(x,y) \in \tilde{\delta}_n \times \tilde{\delta}_n : x < x_n < y\}} dx\, dy \frac{1}{|x-y|^2} < +\infty. \tag{3.47}$$

Since

$$4.46 \int \int_{\{(x,y) \in \tilde{\delta}_n \times \tilde{\delta}_n : x < x_n < y\}} dx\, dy \frac{1}{|x-y|^2} = +\infty \tag{3.48}$$

one obtains $m_n = 0$. Hence the assumption $f_+(x_n + i) \neq 0$ is false. Therefore, $f_+(x_n + i) = 0$ for each $n = 1, 2, \ldots$. Since the sequence $\{x_n\}_{n=1}^{\infty}$ (4.15) has limit point and $f_+(z)$ is analytic in $\mathbf{C}_+$, one gets $f_+ \equiv 0$ which proves

$$4.47 \mathrm{dom}(\sqrt{D_+}) \cap \mathrm{ima}(\Pi^*) = \{0\}. \tag{3.49}$$

Now, following our general construction, see (2.53) and (4.28), we define contraction operator

$$4.48 \Gamma = H_-^{-1/2} J e^{-D_+} M_+(\Omega)^* \tag{3.50}$$

which gives representation

$$4.50 g = K f_+ = H_-^{-1} J e^{-D_+} M_+(\Omega)^* \sqrt{D_+} f_+ \oplus f_+, \qquad f_+ \in \mathrm{dom}(D_+), \tag{3.51}$$

for elements $g \in \mathcal{D}_0$, see Corollary II.6, where the operator $T$ is given by

$$4.49 T f_+ = H_-^{-1} J e^{-D_+} M_+(\Omega)^* \sqrt{D_+} f_+, \qquad f_+ \in \mathrm{dom}(T) = \mathrm{dom}(D_+). \tag{3.52}$$

By (4.11-1) we obtain

$$4.51 H_-^{-1} J P_+ = J P_+ (H_+ + I_{\mathcal{H}_+})^{-1} = J(D_+ + I_{\mathcal{H}_+})^{-1}, \tag{3.53}$$

which yields

$$4.51 - 1 g = J(D_+ + I_{\mathcal{H}_+})^{-1} e^{-D_+} M_+(\Omega)^* \sqrt{D_+} f_+ \oplus f_+, \qquad f_+ \in \mathrm{dom}(D_+). \tag{3.54}$$

Here $(D_+ + I_{\mathcal{H}_+})^{-1} e^{-D_+} M_+(\Omega)^*$ is an integral operator given by

$$4.52 (D_+ + I_{\mathcal{H}_+})^{-1} e^{-D_+} M_+(\Omega)^* f_+)(x) = \int_{-\infty}^{+\infty} dy\, E(y - x) \overline{\Omega(y)} f_+(y), \qquad f_+ \in H_+^2(\mathbf{R}^1), \tag{3.55}$$

where $E(\cdot)$ is defined by

$$4.53 E(\xi) = \frac{1}{2\pi} \int_0^{+\infty} dt\, e^{-i\xi t} e^{-t} (1+t)^{-1}, \qquad \xi \ \in \mathbf{R}^1. \tag{3.56}$$

We note that that $E(\cdot)$ can be expressed by

$$4.55 E(\xi) = \frac{1}{2\pi} e^{1+i\xi} E_1(1 + i\xi). \tag{3.57}$$

where $E_1(\cdot)$ denotes incomplete $\Gamma$-function given by

$$4.54 E_1(x) = \int_x^{+\infty} dt\, e^{-t} t^{-1}, \tag{3.58}$$

cf. definition 9.7 (1) and relation 9.7 (14) of [3]. From (4.51-1) we get representation

$$4.54 - 1 g(x) = (K f_+)(x) = \int_{-\infty}^{+\infty} dy\, E(y + x) \overline{\Omega(y)} (\sqrt{D_+} f_+)(y) + f_+(x), \quad f_+ \in \mathrm{dom}(D_+). \tag{3.59}$$

Setting $h_+ = \sqrt{D_+} f_+$ for $g \in \mathcal{D}_0$, one obtains that

$$4.55 - 2 g = H_-^{-1} J e^{-D_+} M_+(\Omega)^* h_+ \oplus D^{-1/2} h_+, \qquad h_+ \in \mathcal{L} \stackrel{\text{def}}{=} \mathrm{dom}(D_+^{1/2}) \cap \mathrm{dom}(D_+^{-1/2}), \tag{3.60}$$

which gives representation

$$4.56 g(x) = \int_{-\infty}^{+\infty} dy\, E(y + x) \overline{\Omega(y)} h_+(y) + (D_+^{-1/2} h_+)(x), \qquad h_+ \in \mathcal{L}. \tag{3.61}$$

Since

$$4.57 \mathrm{dom}(D_+^{-1/2}) = \{h_+ \in H_+^2(\mathbf{R}^1) : \int_{-\infty}^{+\infty} \int_0^{+\infty} dx\, dt\, |h(x + it)|^2 < +\infty\}, \tag{3.62}$$

the set $\mathcal{L}$ can be characterized by

$$4.58 \mathcal{L} = \{h_+ \in W^{1/2,2}(\mathbf{R}^1) \cap H_+^2(\mathbf{R}^1) : \int_{-\infty}^{+\infty} \int_0^{+\infty} dx\, dt\, |h(x + it)|^2 < +\infty\}. \tag{3.63}$$

Note that $D_+^{-1/2}$ admits representation

$$4.59(D_+^{-1/2}h_+)(x) = \frac{1}{\Gamma(\frac{3}{2})}\int_0^{+\infty} dt\, \frac{1}{\sqrt{t}}h_+(x+it) \tag{3.64}$$

a.e. in $\mathbf{R}^1$. Since $E(\cdot) \in C^\infty(\mathbf{R}^1) \cap L^2(\mathbf{R}^1)$ and

$$4.60g_-(x) = \int_{-\infty}^{+\infty} dy\, E(y+x)\overline{\Omega(y)}h_+(y), \qquad h_+ \in \mathcal{L}, \tag{3.65}$$

one always has $g_- \in C^\infty(\mathbf{R}^1)$ and

$$4.61 \lim_{x\to\pm\infty} \frac{d^m}{dx^m}g_-(x) = 0, \qquad m = 0,1,2,\ldots. \tag{3.66}$$

Furthermore, taking into account (4.55) and relation 9.7 (7) of [3] we obtain:

$$4.61-1 \lim_{x\to\pm\infty} xg_-(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dy\, \overline{\Omega(y)}h_+(y) = 0. \tag{3.67}$$

The last equality is due to the fact that $\overline{\Omega} \in H^2_-(\mathbf{R}^1)$ and $h_+ \in H^2_+(\mathbf{R}^1)$.

The domain $\mathcal{D}_0$, which is given by elements $g$ of the form (4.54-1), is a maximal 0-stability domain for $H = -H_- \oplus H_+$, where $H_- = D_- + I_{\mathcal{H}_-}$ and $H_+ = D_+$. We show that $\mathcal{D}_0$ is also a maximal 0-stability domain for the momentum operator $D$. Since $\mathrm{dom}(H) = \mathrm{dom}(D)$, one has $\mathcal{D}_0 \subseteq \mathrm{dom}(D)$. By

$$\begin{aligned} 4.61-20 \leq (Hg,g) = -(H_-g_-,g_-) + (H_+g_+,g_+) \leq \\ -(D_-g_-,g_-) + (D_+g_+,g_+) = (Dg,g), \qquad g \in \mathcal{D}_0 \end{aligned} \tag{3.68}$$

the domain $\mathcal{D}_0$ is a 0-stability domain for $D$. The operator $T$ is the same for $H$ and $D$. Since $\mathrm{dom}(T) = \mathrm{dom}(H_+) = \mathrm{dom}(D_+)$, by Corollary II.2-1 (vii) the 0-stability domain is maximal for $D$.

Finally, since the momentum operator $D$ is unitarily equivalent to the multiplication operator $\hat{M}$, i.e. $\hat{M} = FDF^{-1}$, the set $\hat{\mathcal{D}}_0 = F\mathcal{D}_0$ is a maximal 0-stability domain of $\hat{M}$. Let us describe this stability domain $\hat{\mathcal{D}}_0$. First of all let us compute $\hat{T} = FTF^{-1} : L^2(\mathbf{R}^1_+) \longrightarrow L^2(\mathbf{R}^1_-)$. One has

$$4.62(FM(\Omega)F^{-1}\hat{f})(t) = \int_{-\infty}^{+\infty} ds\, \hat{\Omega}(t-s)\hat{f}(s), \qquad \hat{f} \in L^2(\mathbf{R}^1), \tag{3.69}$$

where

$$4.63\hat{\Omega}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\, e^{-ix\xi}\Omega(x), \qquad \xi \in \mathbf{R}^1. \tag{3.70}$$

Since $|\Omega(x)| = |\Theta(x)|$ a.e. in $\mathbf{R}^1$, the function $\hat{\Omega}(x)$ is well-defined by (4.12). Moreover, $\Omega \in C^\infty(\mathbf{R}^1) \cap L^2(\mathbf{R}^1)$. Hence, the integral operator (4.62) is well-defined, too. Using (4.62) we find

$$4.64(Fe^{-D_+}M_+(\Omega)^*F^{-1}\hat{f}_+)(t) = \chi_{\mathbf{R}^1_+}(t)e^{-t}\int_0^{+\infty} ds\, \overline{\hat{\Omega}(s-t)}\hat{f}_+(s), \qquad \hat{f}_+ \in L^2(\mathbf{R}^1_+). \tag{3.71}$$

Since $\hat{J} = FJF^{-1} : L^2(\mathbf{R}^1) \longrightarrow L^2(\mathbf{R}^1)$ acts as

$$4.65(\hat{J}\hat{f})(t) = \hat{f}(-t), \qquad \hat{f}_+ \in L^2(\mathbf{R}^1), \quad t \in \mathbf{R}^1, \tag{3.72}$$

one gets for operator $\hat{T}$ (cf. (4.49)) that

$$4.66(\hat{T}\hat{f}_+)(t) = \chi_{\mathbf{R}^1_-}(t)\frac{1}{1-t}e^t \int_0^{+\infty} ds\, \overline{\hat{\Omega}(s+t)}\, \sqrt{s}\, \hat{f}_+(s) \quad \hat{f}_+ \in \operatorname{dom}(\hat{M}) \cap L^2(\mathbf{R}^1_+), \tag{3.73}$$

So the elements $\hat{g}$ of $\hat{\mathcal{D}}_0$ can be represented as

$$4.67\hat{g}(t) = \tag{3.74}$$
$$\chi_{\mathbf{R}^1_-}(t)\frac{1}{1-t}e^t \int_0^{+\infty} ds\, \overline{\hat{\Omega}(s+t)}\, \sqrt{s}\, \hat{f}_+(s) + \chi_{\mathbf{R}^1_+}(t)\hat{f}_+(t), \quad \hat{f}_+ \in \operatorname{dom}(\hat{M}) \cap L^2(\mathbf{R}^1_+).$$

The relation (4.67) gives an explicit description of domains $\hat{\mathcal{D}}_0$ and $\mathcal{D}_0$.

# 4 Functions of Momentum Operator

Let $\phi(\cdot) : \mathbf{R}^1 \longrightarrow \mathbf{R}^1_+$ be a measurable function which obeys (1.8) and (1.9). Let $\mathcal{H}_\pm = H^2_\pm(\mathbf{R}^1)$ as above. We set

$$5.1\phi_+ = \phi|\mathbf{R}^1_+ \qquad \text{and} \qquad \phi_- = -\phi|\mathbf{R}^1_-. \tag{4.1}$$

Note that $\phi_\pm(t) \geq 0$ for a.e. in $\mathbf{R}^1_\pm$. Then we define

$$5.2H_+ = \phi_+(D_+) \qquad \text{and} \qquad H_- = \phi_-(-D_-) + I_{\mathcal{H}_-} \tag{4.2}$$

and set

$$5.3H = -H_- \oplus H_+. \tag{4.3}$$

Obviously, one has

$$5.4H = \phi(D) - P_{\mathcal{H}_-}, \tag{4.4}$$

where $P_{\mathcal{H}_-}$ is orthogonal projection on $\mathcal{H}_-$. Since $\alpha > 1$ (see (1.9)), the consideration can be simplified due the fact that

$$5.9W^{\alpha/2,2}(\mathbf{R}^1) \subseteq C(\mathbf{R}^1) \cap L^2(\mathbf{R}^1). \tag{4.5}$$

Let $J$ be given by (4.11) and let $\{r_n\}_{n=1}^\infty$ be an enumeration of the rational numbers of $\mathbf{R}^1$. We set

$$5.10q(x) \stackrel{\text{def}}{=} \sum_{n=1}^\infty \frac{c_n}{\sqrt{|x-r_n|}}, \qquad x \in \mathbf{R}^1, \tag{4.6}$$

where $c_n > 0$, $n = 1, 2, \ldots$, and

$$5.11 C \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} c_n < +\infty. \tag{4.7}$$

Since $q(x) \geq 0$ for a.e. in $\mathbf{R}^1$ and

$$5.12 \int_{a-1}^{a+1} dx \, \frac{1}{\sqrt{|x - r_n|}} \leq 4, \qquad a \in \mathbf{R}^1, \tag{4.8}$$

we obtain

$$5.13 \int_{a-1}^{a+1} dx \, q(x) \leq 4 \sum_{n=1}^{\infty} c_n = 4C, \qquad a \in \mathbf{R}^1, \tag{4.9}$$

which shows that $q \in L^1_{loc}(\mathbf{R}^1)$. Since always

$$5.14 \int_{a-1}^{a+1} dx \, q(x)^2 = +\infty, \qquad a \in \mathbf{R}^1, \tag{4.10}$$

one has $q \notin L^2_{loc}(\mathbf{R}^1)$. We define

$$5.15 \Theta_*(x) \stackrel{\text{def}}{=} \frac{1}{1 + q(x)} e^{-\sqrt{|x|}}, \qquad x \in \mathbf{R}^1. \tag{4.11}$$

The function $\Theta_*(x)$ obeys

$$5.16 0 \leq \Theta_*(x) \leq 1 \quad \text{a.e. in} \quad \mathbf{R}^1. \tag{4.12}$$

Moreover, one has

$$5.17 \int_{-\infty}^{+\infty} dx \, \frac{|\log(\Theta_*(x)|}{1 + x^2} = \tag{4.13}$$
$$\int_{-\infty}^{+\infty} dx \, \frac{\log(1 + q(x)) + \sqrt{|x|}}{1 + x^2} \leq \int_{-\infty}^{+\infty} dx \, \frac{q(x)}{1 + x^2} + \int_{-\infty}^{+\infty} dx \, \frac{\sqrt{|x|}}{1 + x^2}.$$

Since

$$5.18 \int_{-\infty}^{+\infty} dx \, \frac{q(x)}{1 + x^2} = \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} dx \, \frac{1}{\sqrt{|x - r_n|}} \frac{1}{1 + x^2} \leq (4 + \pi) C \tag{4.14}$$

one gets

$$5.19 \int_{-\infty}^{+\infty} dx \, \frac{|\log(\Theta_*(x)|}{1 + x^2} < +\infty. \tag{4.15}$$

Replacing $\Theta$ by $\Theta_*$ in (4.20) we define the outer function $\Omega_*(z)$, $z \in \mathbf{C}_+$. In accordance with (4.28) we introduce contraction

$$5.20 \Pi_* \stackrel{\text{def}}{=} J M_+(\Omega_*)^*. \tag{4.16}$$

To verify $\mathrm{ima}(\Pi_*^*) \cap \mathrm{dom}(\sqrt{H_+}) = \{0\}$ we assume that

$$5.21 h \stackrel{\text{def}}{=} \Pi_*^* f_- \in \mathrm{dom}(\sqrt{H_+}) \subseteq W^{\alpha/2,2}(\mathbf{R}^1). \tag{4.17}$$

By (5.9) one has $h \in C(\mathbf{R}^1) \cap L^2(\mathbf{R}^1)$ which yields

$$5.22\Theta_*(x)|f_+(x)| = |h(x)| \in C(\mathbf{R}^1), \quad f_+ \stackrel{\mathrm{def}}{=} Jf_- \in H^2_+(\mathbf{R}^1), \quad x \in \mathbf{R}^1. \tag{4.18}$$

Hence

$$5.23e^{-\sqrt{|x|}}|f_+(x)| = (1+q(x))|h(x)| \in L^2(\mathbf{R}^1), \qquad x \in \mathbf{R}^1. \tag{4.19}$$

which implies

$$5.24\frac{1}{\sqrt{|x-r_n|}}|h(x)| \in L^2(\mathbf{R}^1), \qquad n = 1, 2, \ldots. \tag{4.20}$$

However, this is possible only if $h(r_n) = 0$ for $n = 1, 2, \ldots$. Since $h \in C(\mathbf{R}^1)$ one gets $h \equiv 0$ which proves that $\mathrm{ima}(\Pi^*_*) \cap \mathrm{dom}(\sqrt{H_+}) = \{0\}$. Therefore, collection of elements

$$5.25g = H_-^{-1}JM_+(\Omega_*)^*\sqrt{H_+}f_+ \oplus f_+, \qquad f_+ \in \mathrm{dom}(H_+) \tag{4.21}$$

gives a maximal 0-stability domain $\mathcal{D}_{0,*}$ for $H$. From (5.25) one deduces the corresponding pointwise representation

$$5.26g(x) = \int_{-\infty}^{+\infty} dy\, E_*(x+y)\overline{\Omega(y)}(\sqrt{\phi_+(D_+)}f_+)(y) + f_+(x), \qquad f_+ \in \mathrm{dom}(\phi_+(D_+)), \tag{4.22}$$

where

$$5.27E_*(\xi) = \frac{1}{2\pi}\int_0^{+\infty} dt\, e^{-i\xi t}\frac{1}{1+\phi_-(-t)}. \tag{4.23}$$

By reasonings of Section 3 we find that $\mathcal{D}_{0,*}$ is a maximal 0-stability domain not only for $H = \phi(D) - P_{\mathcal{H}_-}$ but also for $\phi(D)$.

Since $\phi(D)$ via the Fourier transform is unitarily equivalent to multiplication operator $\hat{M}(\phi)$,

$$\mathrm{dom}(\hat{M}(\phi)) = \{\hat{f} \in L^2(\mathbf{R}^1) : \phi(t)\hat{f}(t) \in L^2(\mathbf{R}^1)\}, 5.28 \tag{4.24}$$

$$(\hat{M}(\phi)\hat{f})(t) = \phi(t)\hat{f}(t), \qquad \hat{f} \in \mathrm{dom}(\hat{M}(\phi)), 5.29 \tag{4.25}$$

i.e. $\hat{M}(\phi) = F\phi(D)F^{-1}$, the set $\hat{\mathcal{D}}_{0,*} = F\mathcal{D}_{0,*}$ is a maximal 0-stability domain for $\hat{M}(\phi)$. The coresponding vectors have the form

$$\hat{g}(t) = 5.30\chi_{\mathbf{R}^1_-}(t)\frac{1}{1-\phi(t)}\int_0^{+\infty} ds\, \overline{\hat{\Omega}_*(s+t)}\sqrt{\phi(s)}\hat{f}_+(s) + \chi_{\mathbf{R}^1_+}(t)\hat{f}_+(t), \tag{4.26}$$

$$\hat{f}_+ \in \mathrm{dom}(\hat{M}(\phi)) \cap L^2(\mathbf{R}^1_+),$$

where

$$5.31\hat{\Omega}_*(\xi) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\, e^{-i\xi x}\Omega_*(x). \tag{4.27}$$

Since $|\Omega_*(x)| = \Theta_*(x)$ a.e. in $\mathbf{R}^1$ the Fourier transform $\hat{\Omega}_*$ is well-defined.

## 5 Concluding Remarks

Let $\mathcal{D}_0$ be a maximal 0-stability domain given by (4.54-1) and let $D_0 = D|\mathcal{D}_0$. The operator $D_0$ is symmetric and non-negative. Let us show that $D_0$ is closed. Since $D \supseteq D_0$, the operator $D_0$ is closable. Let $\overline{D_0}$ be its closure. Obviously, we have $\mathrm{dom}(\overline{D_0}) \supseteq \mathcal{D}_0$ and

$$6.1(\overline{D_0}f, f) \geq 0, \qquad f \in \mathrm{dom}(\overline{D_0}). \tag{5.1}$$

Since $D \supseteq \overline{D_0}$, the domain $\mathrm{dom}(\overline{D_0})$ is a 0-stability domain of $D$. However, $\mathcal{D}_0$ is maximal. Hence $\mathrm{dom}(\overline{D_0}) = \mathcal{D}_0$, i.e., $D_0$ is closed.

Further we would like to note that the deficiency indices of $D_0$ are equal and, moreover, equal to infinity. Otherwise any self-adjoint extension $\tilde{D}$ of $D_0$ would be semibounded from below. Since $D$ is a self-adjoint extension of $D_0$ which is not semibounded from below, deficiency indices of $\mathcal{D}_0$ have to be infinite.

Next few remarks about spectral properties of self-adjoint extensions of $D_0$. Denote the point, singular continuous and absolutely continuous spectra by $\sigma_p(\cdot)$, $\sigma_{sc}(\cdot)$ and $\sigma_{ac}(\cdot)$, respectively.

**Theorem 5.1** *VI.1 Let $\tilde{D}$ be a self-adjoint extension of $D_0$. Then up to unitary equivalence the operator $D_+$ is always contained in $\tilde{D}$, in particular, one has*

$$6.2\sigma_{ac}(\tilde{D}) \supseteq \overline{\mathbf{R}^1_+}. \tag{5.2}$$

*Proof:* Introduce identification operator $K_\Delta : \mathcal{H}_+ \longrightarrow \mathcal{H}$,

$$6.3K_\Delta f_+ \stackrel{\mathrm{def}}{=} KE_{D_+}(\Delta)f_+, \qquad f_+ \in \mathcal{H}_+, \tag{5.3}$$

where $K : \mathcal{H}_+ \longrightarrow \mathcal{H}$ is given by (4.54-1) and $E_{D_+}(\Delta)$ is spectral measure of $D_+$ for a bounded interval $\Delta \subseteq \mathbf{R}^1_+$. Since always $E_{D_+}(\Delta)f_+ \in \mathrm{dom}(\sqrt{H_+})$, $f_+ \in \mathcal{H}_+$, the definition (6.3) makes sense. Using (4.54-1) one gets

$$6.4(K_\Delta f_+)(x) = (T_\Delta f_+)(x) + E_{D_+}(\Delta)f_+, \qquad f_+ \in \mathcal{H}_+, \tag{5.4}$$

where

$$6.5(T_\Delta f_+)(x) = \int_{-\infty}^{+\infty} dy\, E(y+x)\overline{\Omega(y)}(\sqrt{D_+}E_{D_+}(\Delta)f_+)(y), \qquad f_+ \in \mathcal{H}_+. \tag{5.5}$$

Let us introduce kernels

$$6.6T_1(x,y) = E(y+x)\overline{\Omega(y)}e^{\frac{1}{2}\sqrt{|y|}}, \qquad x, y \in \mathbf{R}^1, \tag{5.6}$$

and

$$6.7T_{2,\Delta}(y,t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sqrt{|y|}} e^{iyt} \sqrt{t}\, \chi_\Delta(t), \qquad y \in \mathbf{R}^1, \quad t \in \mathbf{R}^1_+. \tag{5.7}$$

Denoting by $T_1 : L^2(\mathbf{R}^1) \longrightarrow L^2(\mathbf{R}^1)$ and $T_{2,\Delta} : L^2(\mathbf{R}^1_+) \longrightarrow L^2(\mathbf{R}^1)$ the corresponding integral operators we find representation

$$6.8T_\Delta = T_1 T_{2,\Delta} F \tag{5.8}$$

for (6.5). Since $T_1$ and $T_{2,\Delta}$ are Hilbert-Schmidt operators, one gets that $T_\Delta$ is a nuclear operator for each bounded interval $\Delta \subseteq \mathbf{R}^1_+$. Since

$$6.9\mathrm{ima}(K_\Delta) \subseteq \mathcal{D}_0, \tag{5.9}$$

one gets

$$6.10\tilde{D}K_\Delta f_+ - K_\Delta D_+ f_+ = DK_\Delta f_+ - K_\Delta D_+ f_+, \qquad f_+ \in \mathrm{dom}(D_+). \tag{5.10}$$

By (6.4) we obtain

$$\begin{aligned} &6.11\tilde{D}K_\Delta f_+ - K_\Delta D_+ f_+ = \\ &\quad D_- T_\Delta f_+ \oplus D_+ E_{D_+}(\Delta) f_+ - (T_\Delta D_+ f_+ \oplus E_{D_+}(\Delta) D_+ f_+) = (D_- T_\Delta f_+ - T_\Delta D_+ f_+) \oplus 0. \end{aligned} \tag{5.11}$$

Since $T_\Delta D_+ = T_\Delta E_{D_+}(\Delta) D_+$ the operator $T_\Delta D_+$ is a nuclear one. To check that operator $D_- T_\Delta$ has the same property we consider operator $D_- T_1$. One has

$$6.12(D_- T_1 f)(x) = -i\frac{d}{dx} \int_{-\infty}^{+\infty} dy\, E(y+x) \overline{\Omega(y)} e^{\frac{1}{2}\sqrt{|y|}} f(y), \qquad f \in L^2(\mathbf{R}^1). \tag{5.12}$$

By (4.53) we find

$$6.13 - i\frac{d}{dx} E(y+x) = -\frac{1}{2\pi} \int_0^{+\infty} dt\, e^{-i(y+x)t} e^{-t} \frac{t}{1+t} = \frac{1}{2\pi} \frac{1}{1+i(y+x)} + E(y+x). \tag{5.13}$$

Hence

$$6.14D_- T_1 = T_0 + T_1, \tag{5.14}$$

where $T_0$ is integral operator with kernel

$$6.15T_0(x,y) = \frac{1}{2\pi} \frac{1}{1+i(y+x)} \overline{\Omega(y)} e^{\frac{1}{2}\sqrt{|y|}}, \qquad x, y \in \mathbf{R}^1. \tag{5.15}$$

Therefore, $T_0$ is a Hilbert-Schmidt operator. So $D_- T_\Delta$ is a nuclear operator. Hence the operator

$$6.16\dot{C}_\Delta f_+ \stackrel{\mathrm{def}}{=} \tilde{D}K_\Delta f_+ - K_\Delta D_+ f_+, \qquad f_+ \in \mathrm{dom}(\dot{C}_\Delta) = \mathrm{dom}(D_+), \tag{5.16}$$

is closable and its closure $C_\Delta$ is a nuclear operator. By the trace class existence theorem, cf. [4], the wave operators $W_\pm(\tilde{D}, D_+; K_\Delta)$,

$$6.17W_\pm(\tilde{D}, D_+; K_\Delta) = s - \lim_{t \to \pm\infty} e^{it\tilde{D}} K_\Delta e^{-itD_+}, \tag{5.17}$$

exist for each bounded interval $\Delta \in \mathbf{R}^1_+$. Furthermore, we have

$$6.18\|W_\pm(\tilde{D}, D_+; K_\Delta f_+\|^2 = \lim_{t\to\pm\infty} \|K_\Delta e^{-itD_+} f_+\|^2 = \lim_{t\to\pm\infty}\{\|T_\Delta e^{-itD_+} f_+\|^2 + 2\Re\mathrm{e}(T_\Delta e^{-itD_+} f_+, E_{D_+}(\Delta) e^{-itD_+} f_+) + \|E_{D_+}(\Delta) f_+\|^2\}, \qquad f_+ \in \mathcal{H}_+. \tag{5.18}$$

Since $T_\Delta$ is compact, by Theorem VI.45 of [4] we immediately find that

$$6.19\|W_\pm(\tilde{D}, D_+; K_\Delta) f_+\| = \|E_{D_+}(\Delta) f_+\|, \qquad f_+ \in \mathcal{H}_+, \tag{5.19}$$

for each bounded interval $\Delta \subseteq \mathbf{R}^1_+$. This property yields that there are isometries $W_\pm : \mathcal{H}_+ \longrightarrow \mathcal{H}$ such that

$$6.20 W_\pm | E_{D_+}(\Delta) f_+ = W_\pm(\tilde{D}, D_+; K_\Delta) | E_{D_+}(\Delta) f_+, \qquad f_+ \in \mathcal{H}_+. \tag{5.20}$$

Obviously, one has

$$6.21 e^{it\tilde{D}} W_\pm = W_\pm e^{itD_+}, \qquad t \in \mathbf{R}^1, \tag{5.21}$$

which shows that

$$6.22\tilde{D} \supseteq W_+ D_+ W_+^*. \tag{5.22}$$

Hence $D_+$ is unitarily equivalent to a part of $\tilde{D}$. In particular, this proves (6.2). □

Theorem VI.1 shows that the relation $\sigma_{ac}(\tilde{D}) \supseteq \overline{\mathbf{R}^1_+}$ is fixed and cannot be changed by variation of extensions of $D_0$. However, the spectrum below the threshold zero can be chosen more or less freely.

Choosing the Friedrichs extension $D_F$ of $D_0$ one gets

$$6.22 - 1\sigma(D_F) \cap \mathbf{R}^1_- = \emptyset, \tag{5.23}$$

i.e. we have no negative spectrum. This strikes imagination as far as it concerns the momentum operator. Let $\mathcal{E}$ be a countable subset of $\mathbf{R}^1_-$ and let $n(\cdot) : \mathcal{E} \longrightarrow \mathbf{N}$, $\mathbf{N} = \{1, 2, \ldots\}$. By [5] there is a self-adjoint extension $\tilde{D}$ of $D_0$ such that

$$6.23\sigma_p(\tilde{D}) \cap \mathbf{R}^1_- = \mathcal{E} \quad \text{and} \quad \sigma_{sc}(\tilde{D}) \cap \mathbf{R}^1_- = \sigma_{ac}(\tilde{D}) \cap \mathbf{R}^1_- = \emptyset \tag{5.24}$$

and $n(\lambda)$ coincides with the multiplicity of the eigenvalue $\lambda \in \mathcal{E}$. Further, by [6] and [2] for any closed Borel subset $S \subseteq \overline{\mathbf{R}^1_-}$, which can be the support of some absolutely continuous measure, there is a self-adjoint extension $\tilde{D}$ of $D_0$ such that

$$6.24\sigma_{ac}(\tilde{D}) \cap \overline{\mathbf{R}^1_-} = S \quad \text{and} \quad \sigma_p(\tilde{D}) \cap \mathbf{R}^1_- = \sigma_{sc}(\tilde{D}) \cap \mathbf{R}^1_- = \emptyset. \tag{5.25}$$

Moreover, there is extension such that $\tilde{D}E_{\tilde{D}}(S)$ is unitarily equivalent to $DE_D(S)$. Finally, by [7] for any Borel subset $S \subseteq \overline{\mathbf{R}^1_-}$, which is the closure of an open subset $S_0 \subseteq \mathbf{R}^1_-$, there is a self-adjoint extension $\tilde{D}$ of $D_0$ such that

$$6.25\sigma_{sc}(\tilde{D}) \cap \overline{\mathbf{R}^1_-} = S \quad \text{and} \quad \sigma_p(\tilde{D}) \cap \mathbf{R}^1_- = \sigma_{ac}(\tilde{D}) \cap \mathbf{R}^1_- = \emptyset. \tag{5.26}$$

At last, by [2] there is always an extension which mix all these three types of spectra.

Warning! It is false to conjecture that we have a total control of spectral properties of self-adjoint extensions $\tilde{D}$ of $D_0$. We have control only on the negative spectral part of extensions $\tilde{D}$. Concerning the non-negative part we just know that $\sigma_{ac}(\tilde{D}) \cap \overline{\mathbf{R}^1_+}$ is unchanged. For example, extensions $\tilde{D}$ can change the spectrum $\sigma_p(\tilde{D}) \cap \overline{\mathbf{R}^1_+}$ and $\sigma_{sc}(\tilde{D}) \cap \overline{\mathbf{R}^1_+}$ as well as the multiplicity of $\sigma_{ac}(\tilde{D}) \cap \overline{\mathbf{R}^1_+}$.

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