On the point and continuous spectra for coupled quantum waveguides and resonators

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June 20, 1997

Abstract

A system of quantum resonators connected through small apertures with a quantum waveguide and a system of waveguides coupled through small windows are considered in the case of Dirichlet boundary condition. Solvable model based on the operator extension theory in Pontryagin space is suggested for the description of trapped modes imbedded in the continuous spectrum for the systems. The model allows one not only to prove the existence of such modes but also to suggest an effective and simple algorithm for its determination. A system of quantum waveguides coupled laterally through small windows is considered. The existence of eigenvalue imbedded in the continuous spectrum is shown. The case of periodic set of windows is studied in the framework of the model. The dispersion equation is obtained in an explicit form. The existence of bands imbedded in the continuous spectrum is proved, and an algorithm for its determination is described.

1 Introduction

The problem of eigenvalues imbedded in the continuous spectrum is studied now actively both from mathematical and physical points of view. Mathematicians are trying to reveal general features of the phenomenon by studying of concrete problems (see, for example, [1]). The problem is important for physicists because it is closely related with the description of transport properties of many physical systems in acoustics, quantum mechanics, fluid mechanics [2], [3], [4]. We shall describe systems of quantum resonators connected with a waveguide and waveguides coupled through small windows. Note that the problem of eigenvalues which are less then lower bound of the continuous spectrum is essentially more simple and has been studied better (see, for example, [5]).

Analogous (from mathematical point of view) systems were considered in fluid mechanics [6], [7], [3], [8], [4]. The conventional proofs of existence of eigenvalues imbedded in the continuous spectrum are based on using of variational inequalities. This limitation allows one to look for the effect in the case of the Neumann boundary condition only. As a result, important case of the Dirichlet boundary condition has not been considered. But it is the Dirichlet condition that takes place for quantum waveguide. There is no results about trapped modes imbedded in continuous spectrum for quantum waveguides.

We use another approach. To describe complicated problem for system of resonators coupled with a waveguide through small openings we construct a model in which these small apertures are replaced by point-like ones. The model is based on the theory of self-adjoint extensions of symmetric operators. It allows one to obtain the dispersion equation in an explicit form, and, hence, to find the eigenvalue in question. It has been proved earlier that the model solution is the main term of the asymptotics in the width of the aperture of the solution of corresponding realistic problem. To construct the model for the Dirichlet boundary condition we have to work in the space with indefinite metric (see [14]). The model is analogous to well-known zero-range potential approach [13], [12].

In the first section we describe briefly the model. In the second part of the Letter we deal with system of resonators connected with a waveguide. It is related with the problem of nanoelectronics, particularly with the investigation of ballistic transport in narrow channels [9]. The extension theory model for more simple case of Neumann boundary condition for resonators connected with a waveguide was constructed in [15], [16].

The third section is devoted to waveguides coupled through small windows (see, for example, [19]). The existence of eigenvalues which are less then the continuous spectrum for this problem was shown in [20]. In our model this fact is rather simple. We study the situation when eigenvalues are imbedded in the continuous spectrum. The model allows us to investigate more complicated case of periodic system of windows (see [10] for corresponding example in nanoelectronics) and to show that there exist a band imbedded in the continuous spectrum.

It may be mentioned that the problem of eigenvalues imbedded in the band spectrum for two-dimensional periodic array of quantum dots in constant magnetic field which is orthogonal to the system plane is considered in [11].

2 Description of the model

Describe briefly the model for the case of using an indefinite metric space. Let $\Delta^{1,2}$ be the Laplace operators with the Dirichlet boundary conditions in the domains $\Omega^{1,2}$ with smoooth boundaries having common point r_0. Conventional "restriction- extension" procedure for constructing the model is the following. One restricts the initial self-adjoint operator on the set of functions vanishing near the point r_0. The obtained symmetric operator has self-adjoint extensions which gives us the model in question [14]. But in the case of the Dirichlet boundary condition the restriction of the operator on the set of smooth functions vanishing near the point r_0 is essentially self-adjoint operator, i.e. the procedure fails. To construct the model we have to use extended space with indefinite metric [14]. Consider how to take into account the dipole moment. It is necessary to add the function $h_{-1}^{1,2} = \frac{\partial}{\partial n_y} G^{1,2}(r, r', k_0)|_{r'=r_0}$ to the initial space L_2. Here $G^{1,2}$ is Green function for the Dirichlet problem in $\Omega^{1,2}$. Let $A^{1,2}$ be the following set of functions:

$$A^{1,2} = \left\{ f(r) : f \in L_2(\Omega^{1,2}), \int_{\Omega^{1,2}} f(r)|r - r_0|^{-2} dr - converges \right\}.$$

Note that these functions have roots of corresponding multiplicity at r_0. Let us introduce the following chain of functions:

$$h_{-1}^{1,2}, h_1^{1,2} = (-\Delta^{1,2} - \lambda_0)^{-1} h_{-1}^{1,2}.$$

Here k_0 is some imaginary number ($k_0 = \lambda_0$ is a regular point of the operator $-\Delta^{1,2}$). Let us consider the set $\tilde{A}^{1,2}$:

$$\tilde{A}^{1,2} = \left\{ \tilde{f}^{1,2} : \tilde{f}^{1,2} = f^{1,2} + C_1^{1,2} h_1^{1,2} + C_{-1}^{1,2} h_{-1}^{1,2} \right\}.$$

Here $f^{1,2} \in A^{1,2}$. We define a scalar product in $\tilde{A}^{1,2}$ by the following manner:

$$\begin{gathered}(\tilde{f}, \tilde{g})_{\tilde{A}^{1,2}} = (f, g)_{L_2} + \int_{\Omega^{1,2}} f(r) (\overline{C_1^g h_1^{1,2} + C_{-1}^g h_{-1}^{1,2}}) dr + \\ \int_{\Omega^{1,2}} \overline{g(r)} (C_1^f h_1^{1,2} + C_{-1}^f h_{-1}^{1,2}) dr + C_1^f \overline{C_1^g} (h_1^{1,2}, h_1^{1,2})_{L_2} + \\ (C_1^f \overline{C_{-1}^g} + C_{-1}^f \overline{C_1^g})(h_{-1}^{1,2}, h_1^{1,2})_{L_2}.\end{gathered}$$

The set $\tilde{A}^{1,2}$ is imbedded into Pontryagin space Π_1 by conventional way [17].

Let $\tilde{\Delta}^1$ be the Laplace operator with the domain

$$D(\tilde{\Delta}^1) = \left\{ \tilde{f} : \tilde{f} \in \tilde{A}^1, f \in W_2^{2,loc}(\Omega^1), f = f_1 + C h_1^1 \right\}.$$

Here $f_1, (-\tilde{\Delta}^1 - \lambda_0) f_1 \in A^1$. On the set A^1 the operator $\tilde{\Delta}^1$ acts as the Laplace operator and on the chain h_p^1 the operator $(-\tilde{\Delta}^1 - \lambda_0)$ is a shift operator: $(-\tilde{\Delta}^1 - \lambda_0) h_1^1 = h_{-1}^1$. The operator $-\tilde{\Delta}^1$ is self-adjoint one.

Restrict this operator onto the following set:

$$D(\Delta_0^1) = \left\{ f : f \in D(\tilde{\Delta}^1), ((-\tilde{\Delta}^1 - \lambda_0) f, h_{-1}^1) = 0 \right\}.$$

The obtained operator Δ_0^1 is a symmetric one and has deficiency indices $(1, 1)$. The construction for Ω^2 is analogous. Then the operator $\Delta_0 = \Delta_0^1 \oplus \Delta_0^2$ is symmetric and has deficiency indices $(2, 2)$. It has self-adjoint extensions. Note that the domain of the adjoint operator is the following:

$$D(\tilde{\Delta}_0^*) = \left\{ \tilde{f} : \tilde{f} \in \tilde{A}^1, f = f_1 + C h_1^1 \right\}.$$

Here $f_1, (-\tilde{\Delta}^1 - \lambda_0) f_1 \in A^1$. To describe the domain of the extensions it is necessary to find a linear set of elements from $D(\Delta_0^*)$ which satisfy the condition of the annihilation of the following boundary form:

$$J(f, g) = ((-\tilde{\Delta}_0^* - \lambda_0) \tilde{f}, \tilde{g}) - (\tilde{f}, (-\tilde{\Delta}_0^* - \lambda_0) \tilde{g}).$$

It easy to show that $D(\Delta_0)$ consists of all elements from $D(\tilde{\Delta})$ which satisfy the condition $C_1^1 = C_1^2 = 0$. Using this fact and known asymptotics of the Green function near the point of the boundary, one obtains after brief calculation:

$$J(f, g) = C_1^{f,1} \overline{C_{-1}^{g,1}} - C_{-1}^{f,1} \overline{C_1^{g,1}} + C_1^{f,2} \overline{C_{-1}^{g,2}} - C_{-1}^{f,2} \overline{C_1^{g,2}}.$$

Using conventional procedure of construction of Lagrange planes from simplectic geometry, we come to the following theorem.

Theorem 1. *The set of self-adjoint extensions of the operator $-\Delta_0$ consists of the sets of operators $\{-\Delta_A\}, \{-\Delta_B\}$, the domains of which consist of all elements from $D(-\Delta_0^*$ satisfying the conditions*

$$\begin{pmatrix} C_1^1 \\ C_1^2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} C_{-1}^1 \\ C_{-1}^2 \end{pmatrix}$$

or

$$\begin{pmatrix} C_{-1}^1 \\ -C_1^2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} C_1^1 \\ C_{-1}^2 \end{pmatrix},$$

where A, B are Hermitian matrices.

3 Waveguide with coupled resonators

Let there are two identical resonators $\Omega_{0\pm}$ connected with the waveguide $\Omega_w, \Omega_w = \{(x, y) : |y| < d\}$, the boundary of which is represented in Cartesian coordinates (x, y) by two parallel lines $\Gamma_\pm$. The points of connections are $r_{0\pm}$. Let the system is symmetric about the centreline $y = 0$ of the waveguide. We consider the Laplace operator with Dirichlet boundary condition in the space $L_2(\Omega_w \oplus \Omega_{0+} \oplus \Omega_{0-})$. We construct model operator $-\Delta$ using "restriction - extension" procedure described above, i.e. using the extension of the initial space. The extended space $\mathcal{H}$ can be represented in a form of orthogonal sum $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_a$, where $\mathcal{H}_s(\mathcal{H}_a)$ is the corresponding subspace of symmetric (antisymmetric) functions in respect to variable y. These subspaces are invariant in respect to the operator $-\Delta$. Hence,

$$-\Delta = -\Delta|_{\mathcal{H}_a} \oplus (-\Delta|_{\mathcal{H}_s}).$$

Note that functions from the domain of the operator $-\Delta|_{\mathcal{H}_a}$ satisfy Dirichlet condition on the centreline $y = 0$. Consequently, the continuous spectrum are the following

$$\sigma_c(-\Delta|_{\mathcal{H}_a}) = [\pi^2 d^{-2}, \infty), \sigma_c(-\Delta|_{\mathcal{H}_s}) = [\pi^2(2d)^{-2}, \infty), \sigma_c(-\Delta) = [\pi^2(2d)^{-2}, \infty).$$

In accordance with theorem 1, there is a family of model operators, but we shall deal with one fixed extension which corresponds to the following matrix $B : b_{11} = b_{22} = 0, b_{12} = b_{21} = -1$. It was shown [18] that this choice of the extension gives us correct correlation between the model and "realistic" problems.

The following representation for the Green function for Dirichlet Laplacian in the waveguide is well-known:

$$G_w(x, y, x^*, y^*, k) = \sum_{n=0}^{\infty} \frac{\sin(\pi n(y + d)(2d)^{-1}) \sin(\pi n(y^* + d)(2d)^{-1})}{2p_n} e^{-p_n|x-x^*|},$$

$$p_n(\lambda) = \begin{cases} \sqrt{(\pi n/(2d))^2 - k^2} & for \quad (\pi n/(2d))^2 > k^2, \\ i\sqrt{k^2 - (\pi n/(2d))^2} & for \quad (\pi n/(2d))^2 < k^2. \end{cases}$$

Using this expression and analogous one for the Green function for the resonator $\Omega_\pm$ and taking into account "the boundary condition at the points $r_{0\pm}$" for chosen extension one comes to the following dispersion equation for the model operator $-\Delta|_{\mathcal{H}_a}$:

$$\sum_{n,m=1}^{\infty} \frac{1}{(\lambda_{nm} - \lambda)(\lambda_{nm} - \lambda_0)} = \sum_{n=1}^{\infty} (\frac{1}{2p_n(\lambda)} - \frac{1}{p_n(\lambda_0)}), \tag{1}$$

where λ_{nm} are eigenvalues for the resonator Ω_{0+}.

Theorem 2. *If*

$$\pi^2/(4d^2) < min_{m,n}\lambda_{nm} < \pi^2/(d^2). \tag{2}$$

then for sufficiently great $|\lambda_0|$ the model operator $-\Delta$ has an eigenvalue imbedded in the continuous spectrum.

Really, condition (2) leads to the fact that equation (1) has real root less then the lower bound of the continuous spectrum $\pi^2/(d^2)$. Hence, we have an eigenvalue for the operator $-\Delta|_{\mathcal{H}_a}$ If λ_0 tends to infinity (i.e. for sufficiently small diameter of connection aperture (see [18])) then the root tends to $\pi^2/(d^2)$. Hence, for sufficiently great λ_0 it becames greater then $\pi^2/(2d)^2$. Consequently, we obtain an eigenvalue imbedded in the continuous spectrum for the operator $-\Delta$.

Note that in the framework of conventional approach authors deal with Neumann Laplacian and can prove only that there exists an eigenvalue of the operator $-\Delta|_{\mathcal{H}_a}$ less then $\pi^2/(d^2)$. Namely, they show that there exists test function ψ such that

$$\frac{\int_{\Omega_w \cup \Omega_{0+} \cup \Omega_{0-}} |\nabla\psi|^2 dS}{\int_{\Omega_w \cup \Omega_{0+} \cup \Omega_{0-}} |\psi|^2 dS} < \pi^2/(4d^2).$$

Hence, in accordance with variational principle there exists an eigenvalue in question. In such a way, they get a proof of existence of an eigenvalue imbedded in the continuous spectrum for Neumann Laplacian which has the continuous spectrum is $[0, \infty)$ (but they have no effective algorithm for its determination). Moreover, the conventional approach allows one to consider the Neumann boundary condition only. The reason is the following. If one proves that there exist an eigenvalue of the operator $-\Delta|_{\mathcal{H}_a}$ less then the lower bound of the continuous spectrum of this operator he does not conclude that there is an eigenvalue imbedded in the continuous spectrum because the continuous spectrum of the full operator in the case of Dirichlet condition is not the half axis $[0, \infty)$, but only $[\pi^2/(d^2), \infty)$. Our approach allows one not only to prove the existence of such eigenvalues, but also to determine it approximately using simple procedure. One can see that our approach allows one to consider both Dirichlet and Neumann cases (for Neumann case the model is more simple (see [16]) because there is not necessity to extend the initial space L_2). One can consider the corresponding periodic system of coupled resonators but we shall deal with another situation.

4 Waveguides coupled through windows

Let us consider two identical waveguides $\Omega_{w\pm}$ coupled through point-like window at the point r_0. Using the model described above for Dirichlet Laplacian for the system we come to the following dispersion equation:

$$\sum_{n=1}^{\infty}\left(\frac{1}{p_n(\lambda)} - \frac{1}{p_n(\lambda_0)}\right) = 0, \tag{3}$$

If $\lambda < \lambda_1$ (λ_1 is the minimal eigenvalue for the cross-section) then there exist an eigenvalue λ_* less then λ_1. It tends to the lower bound of the continuous spectrum when $\lambda_0 \to \infty$. Note that corresponding result for "realistic" problem (when there is a window of finite width) is in [20]. One can use the comparison between the model and "realistic" results to ensure an appropriate correlation between model and "realistic" solutions by choosing the model parameter λ_0. Note that the condition $\lambda_0 \to \infty$ corresponds to the fact that the width of the window in the corresponding "realistic" problem tends to zero. It is shown in [20] that there are several eigenvalues but if the width of the window tends to zero, there is only one eigenvalue. In our model we have one eigenvalue which is less then the lower bound of the continuous spectrum. But we can modify the model by using so-called "zero-range potentials with internal structure" (see, for example, [21]) to obtain several eigenvalues.

Consider now three waveguides $\Omega_{1,2,3}$ coupled through point-like windows (fig. 1a). We construct our model for the Dirichlet Laplacian and represent the model operator in the form

$$-\Delta = -\Delta|_{\mathcal{H}_a} \oplus (-\Delta|_{\mathcal{H}_s})$$

as above. Note that to construct the model for the operator $-\Delta|_{\mathcal{H}_a}$ is to construct the model for the Dirichlet Laplacian for the system shown on figure 1b. Hence, one gets for the operator $-\Delta|_{\mathcal{H}_a}$ the following dispersion equation:

$$\sum_{n=1}^{\infty}\left(\frac{1}{p_n^1(\lambda)} - \frac{1}{p_n^1(\lambda_0)}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{\tilde{p}_n^2(\lambda)} - \frac{1}{\tilde{p}_n^2(\lambda_0)}\right) = 0, \tag{4}$$

where

$$p_n^1(\lambda) = \begin{cases} \sqrt{(\pi n/(2d_1))^2 - k^2} & for \quad (\pi n/(2d_1))^2 > k^2, \\ i\sqrt{k^2 - (\pi n/(2d_1))^2} & for \quad (\pi n/(2d_1))^2 < k^2, \end{cases}$$

$$\tilde{p}_n^2(\lambda) = \begin{cases} \sqrt{(\pi n/(d_2))^2 - k^2} & for \quad (\pi n/(d_2))^2 > k^2, \\ i\sqrt{k^2 - (\pi n/(d_2))^2} & for \quad (\pi n/(d_2))^2 < k^2. \end{cases}$$

Let $d_1 < d_2 \le 2d_1$. If $\lambda < \pi^2/(2d_1)^2$ then there is a root of equation (4), i.e. an eigenvalue of the operator $-\Delta|_{\mathcal{H}_a}$. For sufficiently large $|\lambda_0|$ it is an eigenvalue imbedded in the continuous spectrum $[(\pi n/(2d_2))^2, \infty)$ for the full operator $-\Delta$.

Let us consider a situation when there is a periodic system (with the period a) of point-like windows at the points $r_{j\pm}, j = 0, \pm 1, \pm 2, ...,$ (fig. 1c) (note, that the model for Neumann Laplacian for periodic system of resonators connected with a half plane is studied in [22]). Let for simplicity $d_2 = 2d_1$. In this situation we have for the Dirichlet case the following representation for an eigenfunction of the model operator $-\Delta|_{\mathcal{H}_a}$:

$$\psi(x,k) = \begin{cases} \sum_{j=-\infty}^{\infty} \alpha_j^1 \frac{\partial G_1}{\partial n}(r, r_{j+}, k), & r \in \Omega_1, \\ \sum_{j=-\infty}^{\infty} \alpha_j^w \frac{\partial G_w}{\partial n}, & r \in \Omega_w, \end{cases}$$

The periodicity of the system leads to Bloch's condition for the function ψ:

$$\psi(x + a, k) = \exp{(ipa)}\psi(x, k),$$

where p is quasimomentum. Hence, one obtains the additional condition for the coefficients $\alpha_j^w(\alpha_j^1)$:

$$\alpha_j^w = \exp{(ipaj)}\alpha_0^w. \tag{5}$$

Using the choice of the extension operator described above, condition $d_2 = 2d_1$ and (5) one gets the dispersion equation in the form:

$$(G_w(r, r_0, \lambda) - G_w(r, r_0, \lambda_0))\Big|_{r=r_0} + \sum_{j\neq 0, j=-\infty}^{\infty} \exp{(ipaj)} G_w(r_0, r_j, k) = 0.$$

Using the expression for G_w, one gets

$$\sum_{n=0}^{\infty} (p_n^1(\lambda_0) - p_n^1(\lambda)(2p_n^1(\lambda)p_n^1(\lambda_0))^{-1} + \sum_{n=0}^{\infty} 2exp(-p_n a) \frac{\cos pa - \exp{(-p_n a)}}{1 - 2\exp{(-p_n a)}\cos pa + \exp{(-2p_n a)}}, \tag{6}$$

One can see that if $\lambda < \pi^2/(2d_1)^2$ then there is a band for the operator $-\Delta|_{\mathcal{H}_a}$, i.e. the band imbedded in the continuous spectrum of the operator $-\Delta$. One can determine its parameters by solving equation (6). The analogous consideration is for the Neumann boundary condition.

5 Conclusion

A problem of eigenvalues and bands imbedded in the continuous spectrum is considered. We deals with a system of quantum resonators connected through small orifices with a quantum waveguide and waveguides coupled through small windows. The approach is based on the theory of self-adjoint extensions of symmetric operators. Note that conventional approaches (see Introduction) allows one to use variational methods only. As a result, the case of the Neumann boundary condition only has been considered in previous works and the existence only of the eigenvalues imbedded in the continuous spectrum has been proved. The advantage of our approach is that it allows one to obtain asymptotics of the corresponding solutions (and, consequently, the asymptotics for the eigenvalues) instead of variational inequalities for the eigenvalues. Hence, we have an effective instrument to compute eigenvalues and bands imbedded in the continuous spectrum. The suggested approach may be used for the investigation of other spectral problems.

Acknowledgements

The author is grateful to Prof. A.G.Kostyuchenko for stimulating questions, to Prof. L.Parnovski, Prof. P.Exner and Prof. S.Vugalter for the discussion. The work was partly supported by RFBR grants No 96-01-00074 and No 95-01-00439, Soros Foundation, and EC-Russia Collaboration Contract ESPRIT P9282 ACTCS.

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