# COMPLETE ANALYTICITY OF THE 2D POTTS MODEL ABOVE THE CRITICAL TEMPERATURE

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*Dedicated to the memory of*
**Roland**
**L'vovich**
**Dobrushin**

Abstract. We investigate the complete analyticity (CA) of the two-dimensional $q$-state Potts model for large values of $q$. We are able to prove it for every temperature $T > T_{cr}(q)$, provided we restrict ourselves to nice subsets, their niceness depending on the temperature $T$. Contrary to this restricted complete analyticity (RCA), the full CA is known to fail for some values of the temperature above $T_{cr}(q)$.

Our proof is based on Pirogov-Sinai theory and cluster expansions for the Fortuin-Kasteleyn representation, which are available for the Potts model at all temperatures, provided $q$ is large enough.

## 1. Introduction.

In this paper we are dealing with the two-dimensional $q$-state Potts model, which is the statistical mechanics model on $\mathbb{Z}^2$ with formal Hamiltonian

$$H(\sigma) = -\sum_{\{x,y\}} \delta_{\sigma(x),\sigma(y)}, \tag{1.1}$$

where $\sigma(x) = 1, \dots, q$ is the spin variable at the site $x \in \mathbb{Z}^2$, $\delta_{\sigma(x),\sigma(y)}$ is 1 for $\sigma(x) = \sigma(y)$ and is 0 otherwise, the summation is taken over nearest neighbors, and $q > 1$ is an integer. The case $q = 2$ is the well known Ising model.

It is known that the Potts model undergoes a first-order phase transition at a certain transition temperature $T_{cr} = T_{cr}(q)$, provided $q$ is large enough. Namely, the model has $q$ different Gibbs states for temperatures $T < T_{cr}$, $q + 1$ states at $T = T_{cr}$ and one state for $T > T_{cr}$ (see, for example, [KS]). We are going to study

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the Potts model in this last high temperature regime, and we want to investigate the problem of whether the unique Gibbs state the model has in that regime is completely analytic.

The notion of "Complete Analyticity" (CA) of an interaction $U$ was introduced in [DS2] and [DS3] for lattice spin systems. It can be defined in many different equivalent ways. According to one of them one has to consider an arbitrary finite subset $\Lambda$ of $\mathbb{Z}^d$, and to compare the conditional Gibbs measures in $\Lambda$ defined by $U$ and two boundary conditions which only differ at a single site $y$. Namely, one asks for the distance in total variation between the restrictions of these Gibbs measures to an arbitrary subset $\Lambda' \subset \Lambda$ to decay exponentially with the Euclidean distance from $y$ to $\Lambda'$. In the papers [DS2] and [DS3] complete analyticity was shown to be equivalent to several properties of the conditional Gibbs measures corresponding to finite subsets of the lattice and arbitrary boundary conditions. All these properties are in the form of some estimates which are uniform, both in all the finite subsets of the lattice and in all the corresponding boundary conditions. They include the analytic dependence of the logarithm of the (conditional) partition function on the interaction parameters $U = \{U_A, A \subset \mathbb{Z}^d, |A| < \infty\}$, the representation of the logarithm of the partition function as a sum of a volume term and a boundary term, the exponential decay of the truncated correlation functions, etc. Later, Stroock and Zegarlinski showed in [SZ] that complete analyticity is also equivalent to some statements about the various corresponding Glauber-type dynamics (i.e., reversible spin flip dynamics) and their corresponding Dirichlet forms – including logarithmic Sobolev inequalities, and exponential convergence to equilibrium. Again all the statements were uniform over all boundary conditions and all finite subsets of the lattice.

It is natural to ask in the case of concrete models, like the Ising model, for which values of the parameters one has all these nice properties. It was realized that the notion of complete analyticity as originally defined, uniform over all finite subsets of the lattice, is actually too strong to hold in certain cases in which one still expects the system to have a very decent behavior. In particular, it is violated for the Ising model at low temperature and small nonzero field (for uncountably many curves in dimension $D = 2$ and for an open region of the $(T, h)$-plane for $D \geq 3$, see [EFS], pp. 1010–15). Another explicit two-dimensional counterexample, due to one of us, was described in [MO2]. In this example the Hamiltonian considered is just slightly more complicated than the one of the Ising model, but the analysis is much simpler. The idea behind these examples is that if one considers arbitrary subsets of $\mathbb{Z}^2$, then pathologies should not be unexpected, simply because the subsets may have boundaries which are comparable in size to the sets themselves. From the point of view of the physics involved in such problems, one is ready to compromise over such weird shapes and be satisfied with a condition of complete analyticity restricted to "reasonable" subsets of the lattice, including sufficiently thick rectangles, say. A project of this type was carried out by Martinelli and Olivieri in [MO2], [MO3] and related results appeared also in [LY]. In these papers results similar to those of Stroock and Zegarlinski were proven, in the form of equivalences between statements of complete analyticity, properties of reversible spin-flip dynamics and logarithmic Sobolev inequalities, uniformly only over certain subsets of the lattice, including all (sufficiently large) cubes. This weaker property was called "strong mixing for cubes" in [MO2], [MO3], [MOS] and "restricted complete analyticity" (RCA) in [SS].

As was remarked by Roland Dobrushin, the notion of restricted complete analyticity for a given lattice model is equivalent to (full) complete analyticity of another model, obtained from the initial one by partitioning the lattice into cubic blocks of a certain size and considering the spin configurations of the initial system in the blocks to be the spin values of the new system. Therefore one can formulate the notion of restricted complete analyticity in as many equivalent ways as it is possible for the usual complete analyticity.

The introduction of the notion of restricted complete analyticity turns out to be meaningful once one is able to establish this property beyond the region where the standard (full) complete analyticity is known to hold. The first such result was obtained in [MO2], where it was proven that the $d$-dimensional Ising model is RCA for any nonzero value of the magnetic field $h$, provided the temperature $T$ is low enough: $T \leq T(d,h)$. This includes values where the standard CA property is violated ([EFS]). It was actually this result which prompted the remark quoted in the preceding paragraph. Indeed, if one considers the model obtained by partitioning the Ising model into cubic blocks of size $l$, then there exists a scale $l(h)$ such that for $l \geq l(h)$ the block model satisfies the so-called $\widetilde{B}(0)$ property, introduced in [DS1], which means that the finite volume ground state configurations should not depend on the boundary conditions for any volume. From that property the low-temperature RCA follows as a direct corollary of results in [DS2]. The next result in this direction was obtained in [MOS], where the authors showed, among other things, that RCA holds for the two-dimensional zero-field Ising model down to the critical temperature. Later some of us extended these results in [SS] by proving that RCA for 2D Ising model holds everywhere on the $(T,h)$-plane except for the phase transition segment $((T=0,h=0),(T=T_c,h=0))$ and the critical point $(T=T_c,h=0)$. It should be mentioned that it is not known to which extent CA holds in the above region. As mentioned above, in [EFS] this property was shown to be violated at low temperatures and small fields. It is not known whether or not CA holds at all temperatures above $T_c$.

In the present paper we study the same problem for the 2D $q$-state Potts model. Our main result is that if the number of states $q$ is large enough, then the model is RCA for all temperatures $T$ above the transition temperature $T_c = T_c(q)$. The validity of the RCA property for the Potts model turns out to be of more importance than for the zero-field Ising model: while CA for the Ising model still might hold at all temperatures above the critical one, it definitely fails for the Potts model for some supercritical temperatures, as was shown in [EFK]. For an estimate of the temperatures where CA does hold, see [L1,L2].

More precisely, we are proving the following result. Let $l$ be an integer, and consider a natural partition of the lattice $\mathbb{Z}^2$ by $2l \times 2l$ squares. Consider the collection of all finite subsets $\Lambda \subset \mathbb{Z}^2$, which can be represented as a finite union of these squares. Such subsets $\Lambda$ will be called *$l$-regular*. Then the following result holds:

**Theorem 1: Restricted Complete Analyticity for the 2D Potts model.**

*Consider the two-dimensional $q$-state Potts model with $q$ large enough, at any* *temperature $T > T_{cr}(q)$. Then each of the 12 equivalent properties of Complete Analyticity, formulated in [DS3], is valid for every $l$-regular box $\Lambda$, provided $l \geq l(T)$, where $l(T)$ is some finite function.*

The analysis of the result in [EFK] leads us to expect that the function $l(T)$ has to diverge as $T \downarrow T_{cr}(q)$ for such a theorem to hold.

We point out that our results hold for all $q$ for which Theorem A below applies. While in its original form this Theorem does not provide the expected range for $q$, presumably Theorem A (or some version of it) will eventually be proven for all $q$ for which there is a first-order phase transition in the temperature.

The strategy of the proof of the main result is the following: we first note that from the results of [MOS] it follows that for the two-dimensional systems the RCA property holds provided the system satisfies the weak mixing condition. The latter is defined by saying that for an arbitrary finite subset $\Lambda$ of $\mathbb{Z}^2$, if we compare the Gibbs measures with any two boundary conditions, then the distance in total variation between the restrictions of the corresponding Gibbs measures to an arbitrary set $\Lambda' \subset \Lambda$ decays exponentially with the Euclidean distance between $\Lambda'$ and $\partial\Lambda$. (We want to stress that in higher dimensions it may happen that weak mixing holds while RCA is violated – one example being the Czech models, studied in [Shl] – and that is the reason why our results are necessarily restricted to the two-dimensional case.) To show the weak mixing we use the Edwards-Sokal (ES) coupling between the $q$-state Potts model and the corresponding Fortuin-Kasteleyn random cluster model (or FK model). The latter is described in detail below; for the box $\Lambda$ it is a probability distribution on the set of all partitions of $\Lambda$ into connected components, which are called clusters. The conditional distribution of the Potts model in $\Lambda$, given the configuration of FK clusters, which is defined by the ES coupling, is remarkably simple: for every cluster independently one has to choose one of the values $1, \ldots, q$ with equal probability $\frac{1}{q}$, and to put all spins in that cluster to be equal to the value chosen. The exception comes from the clusters attached to the boundary, where the (common) value for the spins in the cluster is defined by the boundary conditions. So, roughly speaking, the conditional distribution in $\Lambda' \subset \Lambda$ under the condition that there are no clusters connecting $\Lambda'$ with the boundary $\partial\Lambda$ does not depend on boundary conditions. As a result, we have weak mixing, provided we can show that the probability that there exists a cluster, connecting $\Lambda'$ with $\partial\Lambda$, decays exponentially with the Euclidean distance between $\Lambda'$ and $\partial\Lambda$.

The analogues of the last statement were obtained for different models by different methods. One was proven by Martirosyan [Mar2] for the low temperature Ising model with magnetic field (in arbitrary dimension). This result was strengthened (and the proof simplified) by one of us in [Sch]. Another result of this kind was obtained by one of us for the so called Czech models in [Shl]. Here we prove it by using the cluster expansion and Pirogov-Sinai theory for the large-$q$ FK model, obtained in [LMMRS]. The specific feature of the large-$q$ FK model is that it admits a cluster expansion which converges for all temperatures (and not only for low or high temperatures, like the Ising model). That enables us to obtain the weak mixing for all temperatures above the critical one.

Once we know the weak mixing property, we can claim, by invoking the result of

[MOS], that the $q$-state Potts model has the following properties above the critical temperature $T_{cr}(q)$:
(i) – Restricted complete analyticity, in the sense that the sets $\Lambda$ in the definition that we reviewed above are restricted to be $l$-regular, $l = l(T)$.
(ii) – Exponential convergence to equilibrium of the associated Glauber dynamics uniformly over $l$-regular subsets, uniformly over boundary conditions and over initial conditions.
(iii) – Positive lower bound for the spectral gap of the generator of the associated Glauber dynamics, uniform over $l$-regular subsets with arbitrary boundary conditions.
(iv) – Finite upper bound for the logarithmic Sobolev constant of the generator of the associated Glauber dynamics, uniform over $l$-regular subsets with arbitrary boundary conditions.
(v) – A constructive condition for uniqueness of the Gibbs measure in infinite volume which was introduced by Dobrushin and one of us in [DS1] is satisfied.

In fact, these nice properties were stated in [MOS] to hold only for square subsets, but they are valid for all $l$-regular subsets of the lattice and also all rectangles. We refer the reader to [MOS] for the precise statements and the various necessary definitions.

Another consequence of RCA is ([MO1]) that a sufficiently often iterated decimation transformation (how often depends on how close one is to the transition temperature) acting on the infinite volume Gibbs measure results in a Gibbs measure, even though applying the decimation transformation only a few times can result in a non-Gibbsian measure [EFK]. Similarly, the restriction of the RCA Gibbs measure to the spins on a line will be a Gibbs measure, compare [L1,L2].

It should be noted, that our restriction to the two-dimensional case comes only from the fact that in the final step of the proof we apply the results of [MOS], which are essentially two-dimensional. However, all the results of the present paper which do not rely on [MOS] hold in any dimension.

In the next section we introduce the necessary notation and review some known results. The proof of our main statement is given in Section 3.

## 2. Notation and terminology.

*The lattice* : The cardinality of a set $\Lambda \subset \mathbb{Z}^2$ will be denoted by $|\Lambda|$. The expression $\Lambda \subset\subset \mathbb{Z}^2$ will mean that $\Lambda$ is a finite subset of $\mathbb{Z}^2$. For each $x \in \mathbb{Z}^2$, we define the usual norm $\|x\| = \max\{|x_1|, |x_2|\}$. The distance between two sets $A, B \in \mathbb{Z}^2$ will be denoted by

$$\operatorname{dist}(A, B) = \inf\{||x - y|| : x \in A, y \in B\}.$$

The (interior) boundary of a set $\Lambda \subset \mathbb{Z}^2$ will be denoted by

$$\partial\Lambda = \{x \in \Lambda : \|x - y\| = 1 \text{ for some } y \notin \Lambda\}.$$

For lattice squares centered at the origin, we will use the notation

$$\Lambda(l) = \mathbb{Z}^2 \cap [-l, l]^2.$$

We will consider also layers

$$L(l) = \Lambda(l) \setminus \Lambda(l - 1). \tag{2.1}$$

The graph of bonds, i.e., (unordered) pairs of nearest neighbors is defined as

$$\mathbb{B} = \{\{x,y\} : x, y \in \mathbb{Z}^2 \text{ and } \|x-y\| = 1\}.$$

Given a set $\Lambda \subset\subset \mathbb{Z}^2$ we define also

$$\mathbb{B}_\Lambda = \{\{x,y\} : x \text{ or } y \in \Lambda \text{ and } \|x-y\| = 1\},$$

$$\partial\mathbb{B}_\Lambda = \{\{x,y\} : x \in \Lambda, y \notin \Lambda \text{ and } \|x-y\| = 1\}.$$

A chain is a sequence of distinct sites $x_1, \ldots, x_n$, with the property that for $i = 1, \ldots, n-1$, $||x_i - x_{i+1}|| = 1$. The sites $x_1$ and $x_n$ are called the end-points of the chain $x_1, \ldots, x_n$, and $n$ is its length. A chain is said to connect two sets if it has one end-point in each set.

*The configurations, observables and measures* : At each site in $\mathbb{Z}^2$ there is a spin which can take values $1, \ldots, q$, where $q$ is an integer. The spin configurations will therefore be elements of the set $\{1, \ldots, q\}^{\mathbb{Z}^2} = \Omega$. Given $\sigma \in \Omega$, we write $\sigma(x)$ for the spin at the site $x \in \mathbb{Z}^2$. For $A \subset \mathbb{Z}^2$ we denote by $\sigma_A$ the restriction of $\sigma$ to $A$. Likewise, this restriction $\sigma_A$ can be viewed as a subset of the set of all configurations:

$$\sigma_A = \{\sigma \in \Omega : \sigma|_A = \sigma_A\}. \tag{2.2}$$

The single spin space, $\{1, \ldots, q\}$ is endowed with the discrete topology and $\Omega$ is endowed with the corresponding product topology. The following definition will be important when we introduce finite systems with boundary conditions later on; given $\Lambda \subset\subset \mathbb{Z}^2$ and a configuration $\eta \in \Omega$, we introduce

$$\Omega_{\Lambda,\eta} = \{\sigma \in \Omega : \sigma(x) = \eta(x) \text{ for all } x \notin \Lambda\}.$$

Real-valued functions with domain in $\Omega$ are called observables. Local observables are those which depend only on the values of finitely many spins; more precisely, $f : \Omega \to \mathbb{R}$ is a local observable if there exists a set $S \subset\subset \mathbb{Z}^2$ such that $f(\sigma) = f(\eta)$ whenever $\sigma(x) = \eta(x)$ for all $x \in S$. The smallest $S$ with this property is called the support of $f$, denoted $\mathrm{supp}(f)$. The topology introduced above on $\Omega$, has the nice feature that it makes the set of local observables dense in the set of all continuous observables.

We will also use bond variables. For every bond $\{x,y\} \in \mathbb{B}$ we introduce the bond variable $n_{xy}$, taking values 0 and 1. The bond configurations $n$ will be the elements of the set $\mathcal{B} = \{0,1\}^{\mathbb{B}}$. We call the bond $\{x,y\}$ open with respect to the configuration $n$, if $n_{xy} = 1$, and closed otherwise. Two bonds open with respect to $n$ will be called connected by $n$, if there is a chain of endpoints of bonds open with respect to $n$, joining the endpoints of these bonds. A maximal connected component of bonds will be called a (open) cluster of $n$. A single site, not connected to any other site, forms a cluster by definition.

We introduce now the sets of bond configurations which are compatible with (site) boundary conditions. Namely, for every $\Lambda \subset\subset \mathbb{Z}^2$ and every configuration $\eta \in \Omega$, we introduce

$$\begin{aligned} \mathcal{B}_{\Lambda,\eta} = \{n \in \mathcal{B} : n_{xy} = 0 \text{ for all } \{x,y\} \notin \mathbb{B}_\Lambda, \\ \eta(u) = \eta(v) \text{ for all } u, v \notin \Lambda, \text{ connected by } n\}. \end{aligned} \tag{2.3}$$

We denote by $\mathcal{B}_\Lambda$ the larger family of all bond configurations, which are indifferent to the boundary conditions:

$$\mathcal{B}_\Lambda = \{n \in \mathcal{B} : n_{xy} = 0 \text{ for all } \{x,y\} \notin \mathbb{B}_\Lambda\}.$$

We endow $\Omega$ also with the Borel $\sigma$-algebra corresponding to the topology introduced above. In this fashion, each probability measure $\mu$ on this space can be identified by the corresponding expected values $\int f d\mu$ of all the local observables $f$.

*The Gibbs measures* : We will consider always the formal Hamiltonian (1.1). In order to give precise definitions, we define, for each set $\Lambda \subset\subset \mathbb{Z}^2$, each boundary condition $\eta \in \Omega$ and each $\sigma \in \Omega_{\Lambda,\eta}$

$$H_{\Lambda,\eta}(\sigma) = -\sum_{\{x,y\}\in\mathbb{B}_\Lambda} \delta_{\sigma(x),\sigma(y)}. \tag{2.4}$$

Given $\Lambda \subset\subset \mathbb{Z}^2$, $\eta \in \Omega$, and $E \subset \Omega$, we write

$$Z^P_{\Lambda,\eta,T}(E) = \sum_{\sigma\in\Omega_{\Lambda,\eta}\cap E} \exp(-\beta H_{\Lambda,\eta}(\sigma)), \tag{2.5}$$

where $\beta = 1/T$ is the inverse temperature, and the superscript $P$ stands for "Potts". We abbreviate $Z^P_{\Lambda,\eta,T} = Z^P_{\Lambda,\eta,T}(\Omega)$.

The Gibbs (probability) measure in $\Lambda$ with boundary condition $\eta$ at temperature $T$ is now defined on $\Omega$ as

$$\mu_{\Lambda,\eta,T}(\sigma) = \begin{cases} \dfrac{\exp(-\beta H_{\Lambda,\eta}(\sigma))}{Z^P_{\Lambda,\eta,T}}, & \text{if } \sigma \in \Omega_{\Lambda,\eta}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

*The Fortuin-Kasteleyn model* : For every bond configuration $n \in \mathcal{B}_{\Lambda,\eta}$ we define the total number of open bonds by

$$|n| = \sum_{\{x,y\}\in\mathbb{B}_\Lambda} n_{xy}.$$

For every bond configuration $n \in \mathcal{B}_{\Lambda,\eta}$ we define the number $\mathcal{C}_\Lambda(n)$ of inner clusters in $\Lambda$ to be the number of open clusters of $n$, which contain no points outside $\Lambda$.

The FK (probability) measure in $\Lambda$ with boundary condition $\eta$ at temperature $T$ is defined on $\mathcal{B}$ by

$$\mu_{\Lambda,\eta,T}(n) = \begin{cases} \dfrac{(e^\beta - 1)^{|n|} q^{\mathcal{C}_\Lambda(n)}}{Z^{FK}_{\Lambda,\eta,T}}, & \text{if } n \in \mathcal{B}_{\Lambda,\eta}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.7}$$

where the FK partition function

$$Z^{FK}_{\Lambda,\eta,T} = \sum_{n \in \mathcal{B}_{\Lambda,\eta}} (e^{\beta} - 1)^{|n|} q^{\mathcal{C}_{\Lambda}(n)}. \tag{2.8}$$

*The Edwards-Sokal coupling* : Finally we remind the reader of the construction of the ES coupling between the Potts model and the Fortuin-Kasteleyn random cluster model.

Consider the box $\Lambda$, and for any bond $\{x, y\} \in \mathbb{B}_{\Lambda}$ let us introduce a new variable $n_{xy}$, taking values 0 and 1. Using now the identity $\exp\{\beta\delta_{u,v}\} = 1 + (e^{\beta} - 1)\delta_{u,v}$, we can rewrite the expression for the Gibbs factor $\exp(-\beta H_{\Lambda,\eta}(\sigma))$ in the formula for the Gibbs distribution (2.6). Namely, for every $\sigma \in \Omega_{\Lambda,\eta}$ we have

$$\begin{aligned} \exp(-\beta H_{\Lambda,\eta}(\sigma)) &= \exp\{\beta \sum_{\{x,y\} \in \mathbb{B}_{\Lambda}} \delta_{\sigma(x),\sigma(y)}\} \\ &= \sum_{n_{xy} \in \mathcal{B}_{\Lambda}} \prod_{\{x,y\} \in \mathbb{B}_{\Lambda}} [\delta_{n_{xy},0} + (e^{\beta} - 1)\delta_{n_{xy},1}\delta_{\sigma_x,\sigma_y}] \\ &= \sum_{n_{xy} \in \mathcal{B}_{\Lambda,\eta}} \prod_{\{x,y\} \in \mathbb{B}_{\Lambda}} [\delta_{n_{xy},0} + (e^{\beta} - 1)\delta_{n_{xy},1}\delta_{\sigma_x,\sigma_y}]. \end{aligned} \tag{2.9}$$

The second equality is straightforward, and the third one holds because the bond configurations from $\mathcal{B}_{\Lambda} \setminus \mathcal{B}_{\Lambda,\eta}$ are not contributing to the sum. So we can now introduce the Edwards-Sokal probability distribution on the pairs $(\sigma, n) \in \Omega_{\Lambda,\eta} \times \mathcal{B}_{\Lambda,\eta}$ by

$$\begin{aligned} \mu_{\Lambda,\eta,T}(\sigma, n) &= \frac{\prod_{\{x,y\} \in \mathbb{B}_{\Lambda}} [\delta_{n_{xy},0} + (e^{\beta} - 1)\delta_{n_{xy},1}\delta_{\sigma_x,\sigma_y}]}{Z^{P}_{\Lambda,\eta,T}} \\ &\equiv \frac{\prod_{\{x,y\} \in \mathbb{B}_{\Lambda}} [\delta_{n_{xy},0} + (e^{\beta} - 1)\delta_{n_{xy},1}\delta_{\sigma_x,\sigma_y}]}{Z^{ES}_{\Lambda,\eta,T}}, \end{aligned} \tag{2.10}$$

where the Edwards-Sokal partition function $Z^{ES}_{\Lambda,\eta,T}$ is defined by:

$$Z^{ES}_{\Lambda,\eta,T} = \sum_{n_{xy} \in \mathcal{B}_{\Lambda}, \sigma \in \Omega_{\Lambda,\eta}} \prod_{\{x,y\} \in \mathbb{B}_{\Lambda}} [\delta_{n_{xy},0} + (e^{\beta} - 1)\delta_{n_{xy},1}\delta_{\sigma_x,\sigma_y}] \equiv Z^{P}_{\Lambda,\eta,T}. \tag{2.11}$$

The statement that the formula (2.10) indeed introduces a probability measure as well as the equality of the two partition functions follow immediately from the identity (2.9). A straightforward check shows that the marginal distribution of the $n$-variables under the measure (2.10) is nothing else than the FK measure (2.10), and therefore the three partition functions are equal:

$$Z^{P}_{\Lambda,\eta,T} = Z^{ES}_{\Lambda,\eta,T} = Z^{FK}_{\Lambda,\eta,T}. \tag{2.12}$$

## 3. The proof of Theorem 1.

**3.1. The general strategy.**

As we already said in the introduction, the RCA property will be established once we check the weak mixing property for the Potts model. This property is the following estimate on the variation distance:

$$Var(\mu_{\Lambda,\eta^1,T}|_A, \mu_{\Lambda,\eta^2,T}|_A) \leq C \sum_{x\in A, y\notin V} \exp\{-c \operatorname{dist}(x,y)\}, \tag{3.1}$$

which should hold for every $A, V$, such that $A \subset V \subset\subset \mathbb{Z}^2$, and where $\mu_{\Lambda,\eta,T}|_A$ is a restriction of the measure, while $C, c$ are positive constants, which do not depend on $A, V, \eta^1, \eta^2$.

Actually, the proof of the implication {weak mixing} $\Rightarrow$RCA, given in [MOS], does not use the full strength of the weak-mixing property (3.1). It is enough to know (3.1) only for some special $\Lambda$-s and $V$-s. Namely, let some $k$ be fixed, and consider the subsets $\Lambda$, which can be obtained by taking unions, intersections and complements of at most $k$ lattice rectangles of sizes not greater than $l$, for some real $l$. Let $0 < p < 1$ be some real number, and let $A = A(\Lambda, l, p) = \{x \in \Lambda : \operatorname{dist}_2(x, \Lambda^c) \geq l^p\}$. In that case (3.1) boils down to

$$Var(\mu_{\Lambda,\eta^1,T}|_A, \mu_{\Lambda,\eta^2,T}|_A) \leq \exp\{-cl^p\}, \tag{3.2}$$

(with smaller $c$). To use [MOS] one needs to know (3.2) for all $l$ sufficiently big and $k \leq 2$. In order to check (3.2) it is enough to prove for all such $\Lambda, A$ the estimate

$$\left| \frac{\mu_{\Lambda,\eta^1,T}(\sigma_A)}{\mu_{\Lambda,\eta^2,T}(\sigma_A)} - 1 \right| \leq \exp\{-cl^p\} \tag{3.3}$$

for every configuration $\sigma_A$, uniformly in $\eta^1, \eta^2$, which clearly implies (3.2). We will give the proof only for the case of the square box $\Lambda(l)$; for the reader who will read it, the generalization will be obvious.

Without loss of generality we can suppose that $\eta^1, \eta^2 \in \sigma_A$, (see (2.2)) as is evident from the definitions (2.4), (2.6). In this case the ratio in (3.3) can be rewritten as

$$\frac{Z^P_{\Lambda\setminus A,\eta^1,T} Z^P_{\Lambda,\eta^2,T}}{Z^P_{\Lambda,\eta^1,T} Z^P_{\Lambda\setminus A,\eta^2,T}}. \tag{3.4}$$

Let us explain why one should expect the last ratio to be close to one in the regime $T > T_{cr}$. The Potts model has one Gibbs state in that regime, which is called the chaotic state. Therefore the typical configuration of the system in the box $\Lambda$ under the boundary condition $\eta$ is the following: near the boundary it is dictated by the boundary condition $\eta$, whereas somewhere close to the boundary $\partial\Lambda$ there is a long contour $\Gamma$, separating the boundary layer from the rest of the box, where the system behaves chaotically. So the partition function can be written as a sum over such contours,

$$Z^P_{\Lambda,\eta,T} = \sum_\Gamma Z^P_{\Lambda,\eta,T}(\Gamma).$$

Within the precision we need, we can rewrite it as

$$Z^P_{\Lambda,\eta,T} \approx \sum_{\Gamma}{}^{(')} Z^P_{\Lambda,\eta,T}(\Gamma), \tag{3.5}$$

where the summation is restricted to those $\Gamma$ which are close to the boundary $\partial\Lambda$. The partition function $Z^P_{\Lambda\setminus A,\eta,T}$ can be written in the same way. However, the boundary of the box $\Lambda \setminus A$ is not connected, so the analogue of (3.5) is the following:

$$Z^P_{\Lambda\setminus A,\eta,T} \approx \sum_{\Gamma,\bar\Gamma}{}^{(')} Z^P_{\Lambda\setminus A,\eta,T}(\Gamma,\bar\Gamma), \tag{3.6}$$

where again the summation is restricted to $\Gamma$ lying close to the boundary $\partial\Lambda$ and to $\bar\Gamma$ lying close to the boundary $\partial A$. We have, therefore, approximate equalities:

$$Z^P_{\Lambda,\eta^i,T} \approx \sum_{\Gamma_i}{}^{(')} Z^P_{\Lambda,\eta^i,T}(\Gamma_i), \tag{3.7}$$

$$Z^P_{\Lambda\setminus A,\eta^i,T} \approx \sum_{\Gamma_i,\bar\Gamma}{}^{(')} Z^P_{\Lambda\setminus A,\eta^i,T}(\Gamma_i,\bar\Gamma). \tag{3.8}$$

So it is enough to estimate the ratio

$$\frac{Z^P_{\Lambda\setminus A,\eta^1,T}(\Gamma_1,\bar\Gamma) Z^P_{\Lambda,\eta^2,T}(\Gamma_2)}{Z^P_{\Lambda,\eta^1,T}(\Gamma_1) Z^P_{\Lambda\setminus A,\eta^2,T}(\Gamma_2,\bar\Gamma)}$$

for every triple $(\Gamma_1,\Gamma_2,\bar\Gamma)$ of contours, which are close to the corresponding parts of the boundaries of our subsets. To see the desired cancellation we observe that the logarithm of the partition function $Z^P_{\Lambda\setminus A,\eta,T}(\Gamma,\bar\Gamma)$ can be represented as a volume term plus a boundary term, and if the two boundaries $\partial\Lambda$, $\partial A$ are well separated, this boundary term is nearly a sum of two terms corresponding to the contours $\Gamma,\bar\Gamma$ (again with the same precision). This is the strategy we are going to follow.

There are different options to study these partition functions. One way is to use the variant of the Pirogov-Sinai theory for the Potts model, developed in [Mar1]. Technically however it is easier to pass first to the FK representation, introduced above, and then use the Pirogov-Sinai theory for it, developed in [LMMRS]. To implement this program we rewrite (3.4) with the help of the identity (2.12) as

$$\frac{Z^P_{\Lambda\setminus A,\eta^1,T} Z^P_{\Lambda,\eta^2,T}}{Z^P_{\Lambda,\eta^1,T} Z^P_{\Lambda\setminus A,\eta^2,T}} = \frac{Z^{FK}_{\Lambda\setminus A,\eta^1,T} Z^{FK}_{\Lambda,\eta^2,T}}{Z^{FK}_{\Lambda,\eta^1,T} Z^{FK}_{\Lambda\setminus A,\eta^2,T}}. \tag{3.9}$$

We want to express the above partition functions in terms of more familiar FK partition functions with free and wired boundary conditions. We then want to use the corresponding contour models to treat the latter. In order to proceed we need some more notation, notions and results which we borrow from [LMMRS].

**3.2. Pirogov-Sinai theory of the FK model and cluster expansions.**

We call a *plaquette* $p$ any four-tuple of bonds in $\mathbb{B}$, which form an elementary cell. We call two bonds adjacent, if they share a vertex, and we call them coadjacent if they belong to the same plaquette. These definitions lead to natural notions of connectedness and coconnectedness of a subset of $\mathbb{B}$. Let $X \subset \mathbb{B}$ be a subgraph with no isolated sites. We denote by $v(X) \subset \mathbb{Z}^2$ the set of its vertices, and by $|X|$ the number of its bonds. The subset $v_I(X) \subset v(X)$ of *inner* vertices consists of all vertices which belong to four bonds of $X$. The bond $b \in X$ belongs to the boundary $\partial X \subset X$ iff $b \in p$, where $p$ is a plaquette such that $p \not\subset X$. The bond $b \notin X$ belongs to the coboundary $\delta X \subset X^c$ iff $v(b) \cap v(X) \neq \emptyset$. We denote by $C(X)$ the number of connected components of the graph $X$.

Let now $V \subset \mathbb{B}$ be a finite subgraph without isolated vertices. We introduce the partition functions with free and wired boundary conditions by

$$Z^f(V) = \sum_{X \subset V, X \cap \delta V^c = \emptyset} (e^\beta - 1)^{|X|} q^{C(X) + |v_I(V) \setminus v(X)|},$$

$$Z^w(V) = \sum_{X \subset V, \partial V \subset X} (e^\beta - 1)^{|X|} q^{C(X) - C(V) + |v_I(V) \setminus v(X)|}.$$

The following limits exist and are equal:

$$\lim_{V \to \mathbb{B}} (1/|V|) \ln Z^f(V) = \lim_{V \to \mathbb{B}} (1/|V|) \ln Z^w(V) = f(\beta).$$

A coconnected subset $\gamma \subset \mathbb{B}$ is called a *contour*, if it is a coboundary of some $X \subset \mathbb{B}$. If $\gamma$ is finite, then either $X$ or $X^c$ is finite. The unique infinite component of $\mathbb{B} \setminus \gamma$ is called the exterior of $\gamma$ and is denoted by $\mathrm{Ext}(\gamma)$. We also introduce $V(\gamma) = \mathbb{B} \setminus \mathrm{Ext}(\gamma)$, and $\mathrm{Int}(\gamma) = V(\gamma) \setminus \gamma$. For $b \in \delta X$ we introduce $d(b)$ as the number of endpoints of $b$, which belong to $X$, and we define the length of the contour $\gamma$ by

$$||\gamma|| = \sum_{b \in \gamma} d(b).$$

If $X$ is finite, then $\gamma$ is called a *contour of the free class*, and if $X^c$ is finite, then $\gamma$ is called a *contour of the wired class*. Note, that some of the contours belong to both classes. For each of the classes one introduces in the standard way the notions of compatible contours and external contours.

For a family $\theta = \{\gamma_1, \dots, \gamma_n\}$ of mutually compatible external contours in $V$ we introduce $V(\theta) = \cup_i V(\gamma_i)$, $\mathrm{Int}(\theta) = V(\theta) \setminus \theta$, $\mathrm{Ext}(\theta) = \mathbb{B} \setminus V(\theta)$, $\mathrm{Ext}_V(\theta) = V \setminus V(\theta)$.

With these definitions we obtain the following relations between the partition functions:

$$Z^f(V) = \sum_{\theta_f \subset V} q^{|v_I(V \setminus \mathrm{Int}(\theta_f))|} Z^w(\mathrm{Int}(\theta_f)), \tag{3.10}$$

where the sum is over the families $\theta_f$ of mutually compatible external $f$-contours in $V$, and

$$Z^w(V) = \sum_{\theta_w \subset V} (e^\beta - 1)^{|\mathrm{Ext}_V(\theta_w)|} Z^f(V(\theta_w)), \tag{3.11}$$

where the sum runs over the families $\theta_w$ of mutually compatible external $w$-contours in $V$, which do not intersect with the boundary $\partial V$.

*A contour model* is specified by assigning weights $\varphi(\gamma)$ to contours. The corresponding partition function is defined by

$$\mathcal{Z}(V|\varphi) = \sum_{\partial \subset V} \prod_{\gamma \in \partial} \varphi(\gamma), \tag{3.12}$$

where the sum is over admissible families $\partial$ of contours in $V$. We are going to consider contour models both for $f$- and $w$-contours; in the first case admissibility means that contours $\gamma_f$ are compatible and are in $V$, while in the second case it means that contours $\gamma_w$ are compatible, are in $V$ and moreover $\gamma_w \cap \partial V = \emptyset$.

For every family $\partial$ of admissible contours we introduce the subset $\theta(\partial) \subset \partial$ as the collection of all external contours in $\partial$. Evidently,

$$\mathcal{Z}(V|\varphi) = \sum_{\theta \subset V} \prod_{\gamma \in \theta} \varphi(\gamma) \mathcal{Z}(\mathrm{Int}(\gamma)|\varphi), \tag{3.13}$$

where the summation is over all families $\theta$ of external contours.

We are going to consider the probability distribution $\nu_{V,\varphi}$ on the ensemble of the admissible contours in $V$, corresponding to the contour functional $\varphi$. Namely, we define the probability to observe the family $\partial$ by

$$\nu_{V,\varphi}(\partial) = \frac{\prod_{\gamma \in \partial} \varphi(\gamma)}{\mathcal{Z}(V|\varphi)}. \tag{3.14}$$

By applying the Peierls transformation one gets immediately from this definition, that the probability of a given contour $\gamma$ to appear in $V$ satisfies the Peierls estimate:

$$\nu_{V,\varphi}\{\partial : \gamma \in \partial\} \le \varphi(\gamma). \tag{3.15}$$

*The contour model with a parameter* $a \ge 0$ is defined by the following partition function:

$$\mathcal{Z}(V|\varphi, a) = \sum_{\theta \subset V} \prod_{\gamma \in \theta} e^{a|V(\gamma)|} \varphi(\gamma) \mathcal{Z}(\mathrm{Int}(\gamma)|\varphi), \tag{3.16}$$

where the sum runs over all families $\theta$ of external contours.

We introduce also the probability distribution $\nu_{V,\varphi,a}(\partial)$ for the contours of the contour model with parameter by modifying the definition (3.14) in an obvious way. The important difference is that once $a > 0$, then the estimate (3.15) is no longer valid in general.

A contour model with parameter is in fact associated to an "unstable phase" or "wrong" boundary condition. The presence of a parameter $a > 0$ favors the formation of a "large" contour representing a flip into a "stable phase", taking place very close to the boundary (Lemma 1 below).

The advantage of contour models lies in the fact that they can be treated by means of the cluster expansion technique. However, that is possible only for those contour models, whose contour functional satisfies the estimate

$$|\varphi(\gamma)| \le e^{-\tau||\gamma||},$$

with $\tau$ reasonably big. In that case the functional is called a $\tau$-functional, following [PS1], [PS2], [Sin]. This ensures the existence of the free energy per bond:

$$f(\varphi) = \lim_{V \to \mathbb{B}} (1/|V|) \ln \mathcal{Z}(V|\varphi).$$

Actually, it implies much more. Namely, one has the following formula for the partition function:

$$\ln \mathcal{Z}(V|\varphi) = \sum_{B \subset V} \phi(B),$$

where the sum runs over all connected subsets of $V$, and $\phi$ is a $\varphi$-dependent function, which satisfies the bound

$$\phi(B) \le e^{-\frac{\tau}{2} d(B)},$$

where $d(B)$ is the number of bonds in the smallest connected set which contains all boundary bonds of $B$.

In particular, one has the following formula for the logarithm of the partition function:

$$\ln \mathcal{Z}(V|\varphi) = |V| f(\varphi) + \sum_{b \in \partial V} g_\varphi(b, V), \tag{3.17}$$

where the function $g_\varphi(b, V)$ is defined for every pair consisting of a bond $b$ and a box $V$, such that $b \in \partial V$, and has the following regularity properties:

$$|g_\varphi(b, V)| \le C e^{-\frac{\tau}{2}}, \tag{3.18}$$

$$|g_\varphi(b, V_1) - g_\varphi(b, V_2)| \le C e^{-\frac{\tau}{2} \operatorname{dist}(b, V_1 \triangle V_2)} \tag{3.19}$$

for $b \in \partial V_1 \cap \partial V_2$, where $V_1 \triangle V_2$ stands for the symmetric difference. (The above statements are standard from the point of view of the theory of cluster expansions and can be found, for example, in [DKS], sect. 3.11.)

In [LMMRS] the contour functionals, which describe the FK model (in a sense which will be explained later) were constructed. We will need the following result, which is part of the main result of [LMMRS]:

**Theorem A.** *Consider the two-dimensional FK model for the $q$-state Potts model, $q$ being large enough, in the regime when $\beta < \beta_{cr}(q)$. Then there exist $\tau$-functionals $\varphi_f, \varphi_w$ and a real parameter $a = a(\beta) > 0$ such that*

$$Z^f(V) = q^{|v_I(V)|} \mathcal{Z}(V|\varphi_f), \tag{3.20}$$

$$Z^w(V) = (e^\beta - 1)^{|V|} \mathcal{Z}(V|\varphi_w, a). \tag{3.21}$$

*The following relations hold:*

$$a + \ln(e^\beta - 1) + f(\varphi_w) = \frac{1}{2} \ln q + f(\varphi_f) = f(\beta), \tag{3.22}$$

$$\varphi_f(\gamma_f) \mathcal{Z}(\operatorname{Int}(\gamma_f)|\varphi_f) = q^{-|v(\operatorname{Int}(\gamma_f))|} Z^w(\operatorname{Int}(\gamma_f)), \tag{3.23}$$

$$\varphi_w(\gamma_w) \mathcal{Z}(\operatorname{Int}(\gamma_w)|\varphi_w) = e^{-a|V(\gamma_w)|} (e^\beta - 1)^{-|V(\gamma_w)|} Z^f(V(\gamma_w)). \tag{3.24}$$

*The parameter $\tau$ can be chosen arbitrarily large, provided $q$ is sufficiently large.*

The relation between the contour models and the initial FK model comes from comparing the formulas (3.10), (3.11) with (3.12), (3.16), (3.20) and (3.21): the distribution of the external contours of the FK model in the box $V$ with free b.c. coincides with the distribution of the external contours in $V$ defined by the contour model with contour functional $\varphi_f$, while that of the FK model with wired b.c. coincides with the distribution of the contour model with the functional $\varphi_w$ and parameter $a$. Indeed, in both cases the partition function is written as a sum of products of terms, corresponding to compatible external contours. Since the formulas (3.10), (3.11), (3.12), (3.16), (3.20) and (3.21) are valid for all volumes, it implies that the factors corresponding to external contours are actually the same.

**3.3. The boundary clusters.**

We are ready now to rewrite the ratio (3.9) with the help of the partition functions introduced above. We will consider first the case when $\Lambda$ is the square box $\Lambda(l)$. Let $n \in \mathcal{B}_{\Lambda(l),\eta}$, and consider all open clusters $K$ of $n$, which have sites in $\Lambda(l)^c$. Such clusters will be called *boundary clusters*. By $\mathcal{K} = \mathcal{K}(n)$ we denote the collection of all boundary clusters $K$ of $n$. The set of all possible collections of boundary clusters $\mathcal{K}$ of configurations in $\mathcal{B}_{\Lambda(l),\eta}$ will be denoted by $\mathcal{S}_\eta$. Denote by $O = O(\mathcal{K})$ the complement

$$O = \mathbb{B}_{\Lambda(l)} \setminus \cup_{K\in\mathcal{K}} K.$$

It is immediate to see that

$$Z^{FK}_{\Lambda(l),\eta,T} = \sum_{\mathcal{K}\in\mathcal{S}_\eta} Z^f(O(\mathcal{K}))(e^\beta - 1)^{\sum_{K\in\mathcal{K}}|K|}. \tag{3.25}$$

Let us introduce the shorthand notation $\Lambda(l, l^p)$ for the annulus $\Lambda(l) \setminus \Lambda(l - l^p)$. Then for every configuration $n \in \mathcal{B}_{\Lambda(l,l^p),\eta}$ we can introduce the set of its boundary clusters in the same manner as it was done above. This set splits into two families: the family $\mathcal{K}$ of boundary clusters which are attached to the exterior boundary of the annulus $\Lambda(l, l^p)$ and the family $\bar{\mathcal{K}}$ of boundary clusters which are attached to the interior boundary of $\Lambda(l, l^p)$ and are disjoint from the exterior one. The set of all such pairs $(\mathcal{K}, \overline{\mathcal{K}})$ will be denoted by $\widetilde{\mathcal{S}}_\eta$. In the obvious notation one has the following analogue of the formula (3.25):

$$Z^{FK}_{\Lambda(l,l^p),\eta,T} = \sum_{(\mathcal{K},\overline{\mathcal{K}})\in\widetilde{\mathcal{S}}_\eta} Z^f(O(\mathcal{K}\cup\overline{\mathcal{K}}))(e^\beta - 1)^{\sum_{K\in\mathcal{K}\cup\overline{\mathcal{K}}}|K|}. \tag{3.26}$$

Let us introduce the subset $\mathcal{S}'_\eta \subset \mathcal{S}_\eta$ formed by all families $\mathcal{K}$, such that every $K \in \mathcal{K}$ has a *height*

$$\begin{aligned} \mathrm{he}(K, \partial\Lambda(l)) &:= \max\Big\{\mathrm{dist}\big(u, \partial\Lambda(l)\big) \,:\, u \in K\Big\} \\ &\le l^p/3\,. \end{aligned}$$

In the same way we define the subset $\widetilde{\mathcal{S}}'_\eta \subset \widetilde{\mathcal{S}}_\eta$ as the collection of all pairs $(\mathcal{K}, \overline{\mathcal{K}})$ with heights $\mathrm{he}(K, \partial\Lambda(l)) \le l^p/3$ and $\mathrm{he}(\overline{K}, \partial\Lambda(l - l^p)) \le l^p/3$. If we denote by $\overline{\mathcal{S}}'_\eta$

the set of all families $\overline{\mathcal{K}}$ of boundary clusters $\overline{K}$ satisfying the last restriction, then clearly

$$\widetilde{\mathcal{S}}'_\eta = \mathcal{S}'_\eta \times \overline{\mathcal{S}}'_\eta. \tag{3.27}$$

Suppose now for a moment that we are able to show that

$$Z^{FK}_{\Lambda(l),\eta,T} = \Big[\sum_{\mathcal{K}\in\mathcal{S}'_\eta} Z^f(O(\mathcal{K}))(e^\beta-1)^{\sum_{K\in\mathcal{K}}|K|}\Big](1+Ce^{-\widetilde{\tau}l^p}), \tag{3.28}$$

and that

$$Z^{FK}_{\Lambda(l,l^p),\eta,T} = \Big[\sum_{(\mathcal{K},\overline{\mathcal{K}})\in\widetilde{\mathcal{S}}'_\eta} Z^f(O(\mathcal{K}\cup\overline{\mathcal{K}}))(e^\beta-1)^{\sum_{K\in\mathcal{K}\cup\overline{\mathcal{K}}}|K|}\Big](1+C'e^{-\widetilde{\tau}l^p}), \tag{3.29}$$

where the constants $C=C(l,p,\eta), C'=C'(l,p,\eta)$ are uniformly bounded in $l$ and $\eta$, and $\widetilde{\tau}=\widetilde{\tau}(\tau)>0$ is independent of $l$ and $\eta$. We claim that in such a case the relation (3.3) follows from the expansion (3.17) and the relation (3.19). Indeed, let us insert the expansions (3.28) and (3.29) into (3.9), with $\Lambda=\Lambda(l), A=\Lambda(l^p)$. Using (3.27), we have

$$\frac{Z^{FK}_{\Lambda(l,l^p),\eta^1,T}Z^{FK}_{\Lambda(l),\eta^2,T}}{Z^{FK}_{\Lambda(l),\eta^1,T}Z^{FK}_{\Lambda(l,l^p),\eta^2,T}} = \frac{\displaystyle\sum_{\substack{\mathcal{K}_1\in\mathcal{S}'_{\eta_1},\mathcal{K}_2\in\mathcal{S}'_{\eta_2}\\ \overline{\mathcal{K}}\in\overline{\mathcal{S}}_{\eta'_1}(=\overline{\mathcal{S}}_{\eta'_2})}} Z^f(O(\mathcal{K}_1\cup\overline{\mathcal{K}}))(e^\beta-1)^{\sum_{K\in\mathcal{K}_1\cup\overline{\mathcal{K}}}|K|}Z^f(O(\mathcal{K}_2))(e^\beta-1)^{\sum_{K\in\mathcal{K}_2}|K|}}{\displaystyle\sum_{\substack{\mathcal{K}_1\in\mathcal{S}'_{\eta_1},\mathcal{K}_2\in\mathcal{S}'_{\eta_2}\\ \overline{\mathcal{K}}\in\overline{\mathcal{S}}_{\eta'_2}(=\overline{\mathcal{S}}_{\eta'_1})}} Z^f(O(\mathcal{K}_1))(e^\beta-1)^{\sum_{K\in\mathcal{K}_1}|K|}Z^f(O(\mathcal{K}_2\cup\overline{\mathcal{K}}))(e^\beta-1)^{\sum_{K\in\mathcal{K}_2\cup\overline{\mathcal{K}}}|K|}} \times (1+C''e^{-\widetilde{\tau}l^p}).$$

Consider the ratio of the corresponding terms:

$$\frac{Z^f(O(\mathcal{K}_1\cup\overline{\mathcal{K}}))(e^\beta-1)^{\sum_{K\in\mathcal{K}_1\cup\overline{\mathcal{K}}}|K|}Z^f(O(\mathcal{K}_2))(e^\beta-1)^{\sum_{K\in\mathcal{K}_2}|K|}}{Z^f(O(\mathcal{K}_1))(e^\beta-1)^{\sum_{K\in\mathcal{K}_1}|K|}Z^f(O(\mathcal{K}_2\cup\overline{\mathcal{K}}))(e^\beta-1)^{\sum_{K\in\mathcal{K}_2\cup\overline{\mathcal{K}}}|K|}}.$$

Note that the total sets of the boundary clusters $K$, appearing in the numerator or in the denominator, are the same, and each is equal to $\mathcal{K}_1\cup\mathcal{K}_2\cup\overline{\mathcal{K}}$. Hence all the factors $(e^\beta-1)^{\sum_*}$ cancel out. Now, the sets $O(\mathcal{K}_*\cup\overline{\mathcal{K}}), O(\mathcal{K}_*)$ are in general not connected, so the corresponding partition functions split into products, and the factors which appear both in the numerator and in the denominator also cancel. A moment's thought leads to the conclusion that what is left equals the ratio

$$\frac{Z^f(\widetilde{O}(\mathcal{K}_1\cup\overline{\mathcal{K}}))Z^f(\widetilde{O}(\mathcal{K}_2))}{Z^f(\widetilde{O}(\mathcal{K}_1))Z^f(\widetilde{O}(\mathcal{K}_2\cup\overline{\mathcal{K}}))},$$

where $\widetilde{O}(\mathcal{K}_* \cup \overline{\mathcal{K}})$, $\widetilde{O}(\mathcal{K}_*)$ are those connected components of the sets $O(\mathcal{K}_* \cup \overline{\mathcal{K}})$, $O(\mathcal{K}_*)$, which contain the whole "middle level", i.e. the set $\partial\Lambda(l - \frac{1}{2}l^p)$. The application of the expansion (3.17) and the relation (3.19) implies immediately, that the last ratio is equal to $1 + Ce^{-\tilde{\tau}l^p}$ with $C = C(\mathcal{K}_1, \mathcal{K}_2, \overline{\mathcal{K}}, l, p)$ uniformly bounded in $\mathcal{K}_1, \mathcal{K}_2, \overline{\mathcal{K}}, l$, which proves our statement (3.3).

The above argument shows, that the only things that remain to be proven are the relations (3.28), (3.29). We will do this in the next subsection.

The reason why our project is bound to succeed is that above the critical temperature $\beta_{cr}^{-1}(q)$ the FK model (as well as the Potts model) has a unique state — the chaotic one — which is characterized by the appearance of a large amount of small connected clusters. So the boundary conditions, fixed around some box $V$, are unable to influence the behavior of the system in the bulk. More precisely, no matter which boundary conditions we choose, there will be a contour in the vicinity of the boundary $\partial V$, separating the boundary influenced behavior outside it from the chaotic one inside. We start the rigorous proof of this picture by considering the wired b.c. In that case the formula (3.21) tells us that the corresponding distribution of the external contours coincides with the one for the contour model with parameter. In light of that the appearance of the following statement is natural:

**Lemma 1. Estimate on the volume of the unstable phase.**
*Let $\theta_w = \{\gamma_1, \dots, \gamma_n\}$ be a family of mutually compatible external $w$-contours in $V$. Consider the event that the contours $\{\gamma_1, \dots, \gamma_n\}$ are the only external contours in the ensemble defined by the contour $\tau$-functional $\varphi_w$ with parameter $a$. That is, we consider the probability distribution*

$$\nu_{V,\varphi_w,a}(\theta_w) = \nu_{V,\varphi_w,a}(\gamma_1, \dots, \gamma_n) = \frac{\prod_1^n e^{a|V(\gamma_i)|}\varphi_w(\gamma_i)\mathcal{Z}(\mathrm{Int}(\gamma_i)|\varphi_w)}{\mathcal{Z}(V|\varphi_w, a)}. \tag{3.30}$$

*Introduce the random variable $u_V = u_V(\theta_w) = |\mathrm{Ext}_V(\theta_w)|$.*
*Then*

$$\nu_{V,\varphi_w,a}(u_V \geq N) \leq \exp\{-aN + C|\partial V|\}, \tag{3.31}$$

*where $C = C(\tau, \beta)$.*

*Note.* It is worth noting that our statement *does not* hold for an arbitrary contour model with parameter, even for large $\tau$. The reason is that when one discusses general contour models, one asks for the upper bound $|\varphi(\gamma)| \leq e^{-\tau||\gamma||}$ only, and so one does not rule out the possibility that $\varphi(\gamma)$ is actually much smaller and even vanishes for some contours. But in such a case the number of sites in the box $V$ which stay outside all external contours is of the order of $|V|$, and the estimate (3.31) breaks down. However for the situation at hand we have also the lower bound

$$\varphi(\gamma) \geq e^{-\bar{\tau}||\gamma||} \tag{3.32}$$

for some real $\bar{\tau}$, and this is enough to prove the estimate (3.31).

*Proof of Lemma 1.* The idea of the proof of the upper bound is to replace the partition function in the denominator of (3.30) by a lower bound which has the form of one of the factors of the numerator of (3.30). To do this we consider the collection

$\Theta_w(V) = \{\Gamma_1, \dots, \Gamma_k, k = k(V)\}$ of mutually compatible external $w$-contours in $V$ which minimizes the variable $u_V$. It is clear that $u_V(\Theta_w(V)) = C|\partial V|$ for some $C$. Then

$$\nu_{V,\varphi_w,a}(u_V \geq N) = \sum_{\theta_w \subset V: u_V(\theta_w) \geq N} \nu_{V,\varphi_w,a}(\theta_w) =$$
$$\frac{\sum_{\theta_w \subset V: u_V(\theta_w) \geq N} \prod_{\gamma \in \theta_w} e^{a|V(\gamma)|} \varphi_w(\gamma) \mathcal{Z}(\mathrm{Int}(\gamma)|\varphi_w)}{\mathcal{Z}(V|\varphi_w, a)} \leq$$
$$\frac{\sum_{\theta_w \subset V: u_V(\theta_w) \geq N} \prod_{\gamma \in \theta_w} e^{a|V(\gamma)|} \varphi_w(\gamma) \mathcal{Z}(\mathrm{Int}(\gamma)|\varphi_w)}{e^{a(|V| - u_V(\Theta_w(V)))} \prod_{\Gamma \in \Theta_w(V)} \varphi_w(\Gamma) \mathcal{Z}(\mathrm{Int}(\Gamma)|\varphi_w)} \leq$$
$$\leq e^{a(u_V(\Theta_w(V)) - N)} \frac{1}{\prod_{\Gamma \in \Theta_w(V)} \varphi_w(\Gamma)} \frac{\mathcal{Z}(V|\varphi_w)}{\prod_{\Gamma \in \Theta_w(V)} \mathcal{Z}(\mathrm{Int}(\Gamma)|\varphi_w)}.$$

We now claim that each of the last two factors admits an upper bound of the order of $\exp\{C|\partial V|\}$ for some $C$. For the last factor this follows from the expansion (3.17), since the complement $V \setminus \cup_{\Gamma \in \Theta_w(V)} \mathrm{Int}(\Gamma)$ is contained in the neighborhood of $\partial V$ of radius 2. For the first one we use (3.24) and (3.20) to express the contour functional $\varphi_w$ via partition functions $\mathcal{Z}(*|\varphi_w)$, $\mathcal{Z}(*|\varphi_f)$ of contour models (with no parameters). We obtain that

$$\varphi_w(\gamma_w) = e^{-a|V(\gamma_w)|}(e^\beta - 1)^{-|V(\gamma_w)|} q^{|v_I(V(\gamma_w))|} \frac{\mathcal{Z}(V(\gamma_w)|\varphi_f)}{\mathcal{Z}(\mathrm{Int}(\gamma_w)|\varphi_w)}.$$

We then use the expansion (3.17) to write each partition function as an exponent of the volume term and the boundary term and the relation (3.22) to observe that all volume terms cancel out. (The fact that we are dealing not with just an abstract contour model, but with a specific one which admits the lower bound (3.32) on the contour functional is made explicit by our use of the relation (3.24), which implies in particular the strict positivity of the contour functional.) ∎

### 3.4. Fingers of the boundary clusters and their surgeries.

In what follows we are proving the relation (3.29) for the case of the square box $\Lambda(l)$. The relation (3.28) is easier and can be proven by the same argument with simpler notation.

In the following statement we estimate the probability of the event that the boundary cluster goes deep inside the box.

**Lemma 2. Estimate of the probability of a long finger.**
*Let $q$ be such that Theorem A above holds. Fix a real number $0 < p < 1$ and consider the event*

$$\pi(l, p, \eta) = \{n \in \mathcal{B}_{\Lambda(l,l^p),\eta} : K \cap \Lambda(l - l^p/3, l^p + l^p/3) \neq \emptyset \text{ for some } K \in \mathcal{K}(n)\}. \tag{3.33}$$

*Then*

$$\mu_{\Lambda(l,l^p),\eta,T}(\pi(l,p,\eta)) \leq C \exp\{-b\tau l^p\}, \tag{3.34}$$

*where $C = C(p,T) > 0$ and $b > 0$ is an absolute constant (e.g. $1/6$).*

*Proof of Lemma 2.* The idea of the proof is to study "fingers", which are protruding parts of the boundary clusters. The finger can be either attached to the exterior boundary of $\Lambda(l, l^p)$ or it joins the exterior and the interior boundaries of $\Lambda(l, l^p)$. If the finger is "thin" somewhere — which means that its length is of higher order that its thickness — then one can cut across it, obtaining an exterior contour of the length of the order $l^p$, which implies the estimate needed. If the finger is "fat" everywhere, that implies that the number of open bonds inside it is much larger than the perimeter, so one can hope to control the situation by using the estimate (3.31).

To implement this program we start by defining fingers and their parameters.

For a boundary cluster $K$ and fixed numbers $0 < k, h < l^p/6$, we define the set $F_{k,h} \subset K$ —the $(k,h)$-*finger*— and the sets of bonds $B_k, B_h \subset K$ —the *bases* of the finger— by the following properties:

i) $F_{k,h} \cap \Big[L\big(l-(l^p/3)\big) \cup L\big(l^p + l^p/3\big)\Big] \neq \emptyset$ [see (2.1)],

ii) $B_k \subset L(l^p + k) \cap K$, $B_h \subset L(l-h) \cap K$.

iii) $F_{k,h}$ is a connected component of $K \setminus \big(B_k \cup B_h\big)$,

iv) there is no path in $F_{k,h}$, connecting $L\big(l-(l^p/3)\big) \cup L\big(l^p + l^p/3\big)$ to the boundary $\partial\Lambda(l, l^p)$,

v) $B_k \cup B_h$ is the smallest set of bonds, satisfying ii), iii) and iv). (Either $B_k$ or $B_h$ can be empty.)

The proof is based on the following three ingredients:

*1) Surgery of a finger.* This is a map that to each configuration $n$ exhibiting a finger $F_{k,h}$ with bases $B_k$, $B_h$, associates the configuration $n' = n'(n)$ which is obtained from $n$ by declaring all bonds in $B_k$, $B_h$ to be closed. This configuration $n'$ is characterized by an exterior contour $\kappa_{k,h}$ (of the wired class) delimiting the finger $F_{k,h}$. The map is many-to-one, with the multivaluedness coming from the number of ways to choose the $|B_k|+|B_h|$ bonds in a proper place in layers $L(l^p+k), L(l-h)$. For our purposes it is enough to take the rough bound

$$(2l^2)^{|B_k|+|B_h|} \tag{3.35}$$

for the number of preimages. On the other hand, the relation between the probabilities $\mu_{\Lambda(l,l^p),\eta,T}(n)$ and $\mu_{\Lambda(l,l^p),\eta,T}(n')$ is the following:

$$\frac{\mu_{\Lambda(l,l^p),\eta,T}(n)}{\mu_{\Lambda(l,l^p),\eta,T}(n')} = \frac{(e^\beta - 1)^{|B_k|+|B_h|}}{q}. \tag{3.36}$$

The numerator comes from the number of connections severed by the surgery, and the denominator $q$ arises from the one extra cluster, $F_{k,h}$, obtained after the surgery. As a consequence, the probability of an event $E(F_{k,h}, B_k, B_h)$ that the given finger appears, satisfies the inequality

$$\begin{aligned} &\mu_{\Lambda(l,l^p),\eta,T}\Big(E(F_{k,h}, B_k, B_h)\Big) \\ &\quad \leq (2l^2)^{|B_k|+|B_h|}\, \frac{(e^\beta - 1)^{|B_k|+|B_h|}}{q}\, \mu_{\Lambda(l,l^p),\eta,T}\Big(E(F_{k,h}, \kappa_{k,h})\Big), \end{aligned} \tag{3.37}$$

where $E(F_{k,h}, \kappa_{k,h})$ — the event to observe the contour $\kappa_{k,h}$ delimiting the cluster $F_{k,h}$ — is obtained from $E(F_{k,h}, B_k, B_h)$ through surgery.

*2) Thin-finger estimation.* A finger $F_{k,h}$ will be called $(l^s, \gamma)$-*thin*, for some $s > 0$ and $0 \le \gamma < 1$ if

$$|\kappa_{k,h}| \ge \frac{1}{3} l^s \quad \text{and} \quad |B_k| + |B_h| \le 2 l^{\gamma s}.$$

The $\mu_{\Lambda(l,l^p),\eta,T}$-probability of the union of all configurations $n'$, which have the contour $\kappa_{k,h}$ among their external contours, is at most $\exp\{-\tau ||\kappa_{k,h}||\}$. Hence we can use (3.37) plus a Peierls estimate (3.15) to obtain the following bound:

$$\begin{aligned} &\mu_{\Lambda(l,l^p),\eta,T}\{n \in \pi(l,p,\eta) : \text{ some } K \text{ in } \mathcal{K}(n) \text{ contains a } (l^s,\gamma)\text{-thin finger}\} \\ &\quad \le (2l^2)^{2l^{\gamma s}} \frac{(e^\beta - 1)^{2l^{\gamma s}}}{q} \Big[ \sum_{j \ge l^s/3} (2l)^2 4^{2j} \exp\{-\tau j\} \Big] \\ &\quad \le \exp\{-\tau l^s/6\} , \end{aligned} \tag{3.38}$$

provided $l$ is large enough. Here the combinatorial factor $(2l)^2 4^{2j}$ estimates the number of contours of length $j$ that can be drawn inside our box.

*3) Fat-finger estimation.* A finger $F_{k,h}$ will be called $(l^s, \gamma)$-*fat*, for some $s > 0$ and $0 \le \gamma < 1$ if

$$|\kappa_{k,h}| \le l^{\gamma s} \quad \text{and } |F_{k,h}| \ge \frac{1}{6} l^s.$$

If we apply the estimate (3.31) with $V$ to be the interior of $\kappa$, we obtain, via inequality (3.37) and the fact that $|B_k| + |B_h| \le |\kappa_{k,h}|$, the bound

$$\begin{aligned} &\mu_{\Lambda(l,l^p),\eta,T}\{n \in \pi(l,p,\eta) : \text{ some } K \text{ in } \mathcal{K}(n) \text{ contains a } (l^s,\gamma)\text{-fat finger}\} \\ &\quad \le (2l^2)^{2l^{\gamma s}} \frac{(e^\beta - 1)^{l^{\gamma s}}}{q} \nu_{\mathrm{Int}(\kappa_{k,h}),\varphi_w,a}\big(u_{\mathrm{Int}(\kappa_{k,h})} \ge \frac{1}{6} l^s\big) \\ &\quad \le (2l^2)^{2l^{\gamma s}} \frac{(e^\beta - 1)^{l^{\gamma s}}}{q} \exp\{-\frac{a}{6} l^s + C l^{\gamma s}\} \\ &\quad \le \exp\{-a l^s/10\} , \end{aligned} \tag{3.39}$$

for $l$ large.

With these ingredients, the proof of (3.34) proceeds as follows. We fix a positive real number $0 < \alpha < 1$, such that $1 - \alpha$ is sufficiently small to guarantee that

$$\frac{\alpha}{1-\alpha} p > 1 , \tag{3.40}$$

and perform the following finite sequence of steps:

*Step 1.* We consider first the configurations $n \in \pi(l,p,\eta)$ which for some $k, h < l^p/6$ have a finger $F_{k,h}$ with both bases having less than $l^{\alpha p}$ bonds:

$$\max\Big(|B_k|, |B_h|\Big) \;\le\; l^{\alpha p} . \tag{3.41}$$

The length of the contour $\kappa_{k,h}$ is at least $l^p/3$ because it penetrates at least a distance $l^p/3$ inside $\Lambda(l, l^p)$ while the bases are at most at a distance $l^p/6$ from the boundary $\partial\Lambda(l, l^p)$. Hence, such a finger is $(l^p, \alpha)$-thin and the bound (3.38) shows that the configurations considered in this step have a probability of occurrence not exceeding

$$\exp\{-\tau l^p/6\} \ . \tag{3.42}$$

*Step 2*. For the remaining configurations the condition (3.41) is violated. We consider a part of them: those for which for all $k, h \leq l^p/6$ both bases have less than $l^{\alpha(\alpha p+p)}$ bonds. That is, either

$$|B_k| > l^{\alpha p} \text{ for all } 0 \leq k \leq l^p/6 \tag{3.43}$$

*or*

$$|B_h| > l^{\alpha p} \text{ for all } 0 \leq h \leq l^p/6 \tag{3.44}$$

and, moreover,

$$\max\Big(|B_k|, |B_h|\Big) \ \leq \ l^{\alpha(\alpha p+p)} \text{ for all } 0 \leq h, k \leq l^p/6 \ . \tag{3.45}$$

To bound their contribution we consider two cases:

*Case 2.1*. If the length of the contour $\kappa_{k,h}$ is of an order larger than the size of the bases:

$$|\kappa_{k,h}| \geq l^{s_2} \quad \text{with} \quad s_2 = (\alpha p + p)(1+\alpha)/2 \ ,$$

then the finger in question is $(l^{s_2}, \widetilde{\alpha})$-thin, with $\widetilde{\alpha} = \frac{2\alpha}{1+\alpha}$. The thin-finger bound (3.38) tells us that the probability of these configurations is bounded above by

$$\exp\{-\tau l^{s_2}/6\} \ .$$

*Case 2.2*. In the opposite case we have

$$|\kappa_{k,h}| \leq l^{s_2} \ ,$$

which implies that the finger is $(l^{\alpha p+p}, \widetilde{\widetilde{\alpha}})$-fat, with $\widetilde{\widetilde{\alpha}} = \frac{1+\alpha}{2}$. Indeed, given that either (3.43) or (3.44) is satisfied, the finger contains at least $l^{\alpha p} \times l^p/6$ bonds. Applying (3.39) we conclude that we are dealing with configurations whose probability is at most

$$\exp\{-a l^{\alpha p+p}/10\} \ .$$

We proceed by induction and we arrive to

*Step m*. Introduce the quantity

$$r_m = \sum_{i=1}^{m-1} \alpha^i p \ .$$

During the $m$-th step we treat the portion of configurations not treated before — namely, those which have fingers such that for all $0 \leq h, k \leq l^p/6$ the maximum base-width is in the range

$$l^{r_m} \ < \ \max\Big(|B_k|, |B_h|\Big) \ \leq \ l^{\alpha(r_m+p)} \ (\equiv l^{r_{m+1}}) \ .$$

The l.h.s. inequality implies that either $|B_k| > l^{r_m}$ for all $0 \leq k \leq l^p/6$, or $|B_h| > l^{r_m}$ for all $0 \leq h \leq l^p/6$ (or both). We have two cases:

*Case m.1*. If the order of the length of the contour $\kappa_{k,h}$ exceeds that of the size of the bases:

$$|\kappa_{k,h}| \geq l^{s_m} \quad \text{with} \quad s_m = (r_m + p)(1+\alpha)/2 \;,$$

then the finger is $(l^{s_m}, \widetilde{\alpha})$-thin. ¿From (3.38) the probability of the corresponding configurations is bounded by

$$\exp\{-\tau l^{s_m}/6\} \;. \tag{3.46}$$

*Case m.2*. In the opposite case, when

$$|\kappa_{k,h}| \leq l^{s_m} \;,$$

we use the fact that the finger contains at least $l^{r_m} \times l^p/6$ bonds. Hence the finger is $(l^{r_m+p}, \widetilde{\widetilde{\alpha}})$-fat and (3.39) implies that the event formed by these configurations has a probability bounded by

$$\exp\{-al^{r_m+p}/10\} \;. \tag{3.47}$$

This procedure terminates after a finite number of steps, because condition (3.40) ensures that there exists a $m_0$ —independent of $l$— such that, for all $l$ we have $l^{\sum_{i=1}^{m_0} \alpha^i p}$ exceeds $4l$, which is the maximum possible size for $|B_k|$ and $|B_h|$. The sum of the (finitely many) estimates (3.46)–(3.47) proves the bound (3.34). We see that the leading contribution comes at the first step, which was taken care of in (3.42).■

As was mentioned above, the result of Lemma 2 implies the relations (3.28), (3.29), which in turn imply Theorem 1.

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