On the Spectral Theory of Dispersive N-Body Hamiltonians

by

Mondher DAMAK

Université Paris 7, UMR 9994 Laboratoire de Physique

Mathématique et Géométrie, Case 7012

2, Place Jussieu, F-75251 Paris cedex 05

Abstract

In this work we describe a general class of dispersive N-body hamiltonians for which we prove the HWZ theorem and a Mourre estimate outside a closed and countable set of energies called thresholds. As a consequence of the Mourre estimate we prove a strong form of the limiting absorption principle which implies absence of singular continuous spectrum and gives criteria of local smoothness.

1 Introduction

The basic hamiltonian describing N non-relativistic interacting particles is:

$$H = \sum_{i=1}^{N} \frac{-1}{2m_i} \Delta_{x_i} + \sum_{i<j} V_{ij}(x_i - x_j)$$

where $x_i \in \mathbb{R}^3$ and $m_i > 0$ are the position and mass of particle number i. Important results concerning the spectral theory of this type of hamiltonians have been obtained during the last years (see [DG]). One of the oldest (but fundamental) results, the theorem of Hunziker, van Winter and Zislin concerning the essential spectrum of such operator has been established in [H], [W] and [Z], see also [RS] and [CFKS]. This theorem says that the bottom of the essential spectrum of a non-relativistic multi-particle operator is determined by the two-cluster decompositions of the system of particles. In [LSV] Lewis, Siedentop and Vugalter show that the result still holds for multi-particle relativistic hamiltonians too, in this case the kinetic energy $\frac{-1}{2m_i}\Delta_{x_i}$ being replaced by $(-\Delta_{x_i}+m_i^2)^{1/2}$.

A fundamental step in the study of more subtle spectral properties of N-body Schrödinger operators H is the proof that they obey a Mourre estimate at all non-threshold points. The first proof of this result is due to Mourre [M] for $N=3$ and to Perry, Sigal and Simon [PSS] for general N; these proofs were simplified afterwards by Froese-Herbst [F-H].

Our purpose here is to prove HWZ theorem and the Mourre estimate (outside a set of points called thresholds) for a rather general class of N-body dispersive systems, i.e. for hamiltonians of the form

$$H = h(P) + \sum_{i<j} V_{ij}(x_i - x_j)$$

where $P = -i\nabla$ and h is a continuous and divergent at infinity function.

In particular, the relativistic and non-relativistic hamiltonians are included. The class of interactions that we consider is sensibly more general then usual, e.g. they can be non local and quite singular. In the case of relativistic operators the Coulomb potentials are allowed.

The first proof that a dispersive N-body operator satisfies a Mourre estimate is due to Derezinski [D] but his hypotheses contain implicit conditions which are difficult to check on examples. After this Gerard [G] gave a new proof of the estimate for what he called "regular dispersive systems". Both proofs are geometric in nature being natural extensions of the proof of Froese and Herbst. Also the proof of the HWZ theorem in the relativistic case due to Lewis, Siedentop and Vugalter follows the geometric ideas introduced by Zislin in this context (see [CFKS]).

In this paper we use quite different methods, namely the C^*-algebra techniques introduced in [BG1], [BG2] that require neither partitions of unity in configuration space nor equations of Weinberg-van Winter type. Then not only the results that we obtain are more general than the previous one but also the proofs seem to us simpler and less technical.

As a consequence of the Mourre estimate we get absence of singular continuous spectrum, discreteness of eigenvalues outside the thresholds and local (outside the thresholds) H-smoothness of operators decaying like $< x >^{-s}$ for all $s > 1/2$.

The organization of this paper is as follows. In section 2 we introduce the algebraic framework in which our investigations will be done. The main results of section 3 are Theorem 3.1, which is a general criterion for an operator to be affiliated to the C^*-algebra $\mathscr{T}$, and the dispersive HWZ theorem. We also show that the N-body relativistic hamiltonians with Coulomb interactions are covered by our general results. Section 4 is devoted to the proof of the Mourre estimate for a rather general class of N-body dispersive systems. Our conditions (both on the kinetic energy and on the interactions) are of the same nature as those of C. Gerard but are simpler and more general.

2 Preliminaries

2.1. Throughout this paper $\mathscr{L}$ denotes a finite lattice, *i.e.* a finite partially ordered set such that for any two elements $a, b \in \mathscr{L}$ their upper and lower bounds $a \vee b$, $a \wedge b$ exist in $\mathscr{L}$. In the usual setting of the N-body problem $\mathscr{L}$ is the lattice of partitions of the set $\{1, 2, ..., N\}$. The order relation in $\mathscr{L}$ is denoted $a \leq b$ and $a < b$ means strict inequality; a_{min} (resp. a_{max}) are the least (resp. the largest) element of $\mathscr{L}$. We denote by $\mathscr{L}(2)$ the set of maximal elements of $\mathscr{L} \setminus \{a_{max}\}$(in the N-body case this is the set of two-cluster partitions).

2.2. Let X be a finite dimensional real vector space equipped with a volume element dx, i.e. dx is a translation invariant positive Radon measure on X. Equivalently, one may give a non zero element e of the exterior product $\bigwedge^n X$ $(n = dimX)$ and if $(e_1, ..., e_n)$ is a basis in X such that $e_1 \wedge ... \wedge e_n = e$ then one sets $\int f dx = \int_{\mathbb{R}^n} f(x_1 e_1 + ... + x_n e_n) dx_1 ... dx_n$. Then $\mathscr{H}(X)$ will be the Hilbert space $L^2(X; dx)$ and we denote $\mathbb{B}(X)$ (resp. $\mathbb{K}(X)$) the C^*-algebra of bounded (resp. compact) operators in $\mathscr{H}$(X). Note that $\mathbb{B}(X)$ and $\mathbb{K}(X)$ do not depend on the choice of the volume element.

Let X^* be the dual space of X. The Fourier transformation of a function $f : X \to \mathbb{C}$ is the function $\mathscr{F} f \equiv \widehat{f} : X^* \to \mathbb{C}$ defined by $\widehat{f}(k) = \int \exp[-i < x, k >] f(x) dx$. There is a unique volume element dk on X^* such that $\mathscr{F}$ extends to an isometric operator of $L^2(X; dx)$ onto $L^2(X^*; dk)$. We shall always equip X^* with this volume element and we set $\mathscr{H}(X^*) = L^2(X^*; dk)$.

If $f : X \to \mathbb{C}$ is a Borel function we denote $f(Q)$ the operator of multiplication by f in the space $\mathscr{H}(X)$. If $g : X^* \to \mathbb{C}$ is a Borel function then $g(P)$ is the operator in $\mathscr{H}$(X) such that $\mathscr{F} g(P) \mathscr{F}^*$ is the operator of multiplication by g in $\mathscr{H}(X^*)$. Let $\mathbb{T}(X)$ be the C^*-subalgebra of $\mathbb{B}(X)$ consisting of operators of the form $f(P)$ where $f : X^* \to \mathbb{C}$ is continuous and convergent to zero at infinity.

2.3. If Y is a subspace of X provided with a volume element dy, then the Hilbert space $\mathscr{H}(Y)$, the Fourier transform $\mathscr{F}^Y$, the C^*-algebras $\mathbb{B}(Y)$, $\mathbb{K}(Y)$ and $\mathbb{T}(Y)$ are well defined. By convention, if $Y = 0 \equiv \{0\}$ we put $\mathscr{H}(0) = \mathbb{K}(0) = \mathbb{T}(0) = \mathbb{C}$.

Now let Z be a subspace of X supplementary to Y, so $X = Y \bigoplus Z$. Then there is a unique volume element dz on Z such that $dx = dy \otimes dz$. Indeed, if e and u are the elements of $\bigwedge^n \mathrm{X}$ and $\bigwedge^k Y$ $(k = dimY)$ which define the volume elements dx and dy respectively, then there is a unique $v \in \bigwedge^m Z$ $(m = dimZ)$ such that $e = u \wedge v$; dz will be the volume element on Z associated to v. We get then a canonical isomorphism

$$\mathscr{H}(X) \cong \mathscr{H}(Y) \otimes \mathscr{H}(Z) \cong L^2(Z; \mathscr{H}(Y))$$

The decomposition $X = Y \bigoplus Z$ induces also a canonical identification $X^* = Y^* \bigoplus Z^*$, and if we make a Fourier transformation in the variable $z \in Z$ only we get then a canonical isomorphism

$$\mathscr{H}(X) \cong \mathscr{H}(Y) \otimes \mathscr{H}(Z^*) \cong L^2(Z^*; \mathscr{H}(Y))$$

2.4. Let $\{X^a, X_a\}_{a \in \mathscr{L}}$ be a family of couples of vector subspace of X equipped with volume elements dx^a, dx_a such that the subspaces X^a and X_a are supplementary in X and $dx = dx^a \otimes dx_a \quad \forall a \in \mathscr{L}$. We suppose that :

i) if $b \leq a$ then $X^b \subset X^a$ and $X_a \subset X_b$;

ii) $X^a + X^b = X^{a \vee b}$ and $X_a \cap X_b = X_{a \vee b}$;

iii) $X^{a_{max}} = X_{a_{min}} = X$ and $X^{a_{min}} = X_{a_{max}} = \{0\}$.

Then for each a we have a canonical factorization

$$\mathscr{H}(X) \cong \mathscr{H}(X^a) \otimes \mathscr{H}(X_a) \cong L^2(X_a; \mathscr{H}(X^a))$$

If $S \in \mathbb{B}(X^a)$, $T \in \mathbb{B}(X_a)$ we shall denote by $S \otimes_a T$ the operator in $\mathscr{H}(X)$ which corresponds to $S \otimes T$ by the preceding isomorphism. If $\mathscr{C}_1 \subset \mathbb{B}(X^a)$ and $\mathscr{C}_2 \subset \mathbb{B}(X_a)$ are C^*-subalgebras, we denote by $\mathscr{C}_1 \otimes_a \mathscr{C}_2$ the C^*-subalgebra of $\mathbb{B}(X)$ generated by the operators of the form $S \otimes_a T$ with $S \in \mathscr{C}_1$, $T \in \mathscr{C}_2$.

For each $a \in \mathscr{L}$,we introduce now the following C^*-subalgebra of $\mathbb{B}(X)$:

$$\mathscr{T}(a) := \mathbb{K}(X^a) \otimes_a \mathbb{T}(X_a)$$

and for each $a \in \mathscr{L}$ we define:

$$\mathscr{T}_a := \sum_{b \leq a, b \in \mathscr{L}} \mathscr{T}(b), \qquad \mathscr{T} = \mathscr{T}_{a_{max}} = \sum_{a \in \mathscr{L}} \mathscr{T}(a)$$

The sums which appear in the r.h.s have to be interpreted in the vector space sense. Note the following particular cases : $\mathscr{T}(a_{min}) = \mathbb{T}(X)$, $\mathscr{T}(a_{max}) = \mathbb{K}(X)$.

The main property of these spaces are: $\mathscr{T}$ is a C^*-algebra, $\mathscr{T}(a)\mathscr{T}(b) \subset \mathscr{T}(a \vee b)$ for each $a, b \in \mathscr{L}$, the preceding sums are direct in the Banach space sense and each $\mathscr{T}_a$ is a C^*-subalgebra of $\mathscr{T}$.

These assertions can be proved without difficulty by mimicking the proof of Theorem 2.1 from [BGS] (or see chapter 9 in [ABG]). These facts can also be deduced from the more general formalism developed in [BG3] by observing that there are canonical isomorphisms between the algebras $\mathscr{T}(a)$ and the algebras denoted $\mathscr{C}(X_a^\sigma)$ in section 2 of [BG3] (see pages 35-36). This is interesting because it shows that *the C^*-algebras $\mathscr{T}(a)$ depend only on the choice of the subspace X_a of X; if other choices for the volume element on X_a and for the supplementary space X^a are made then we get the same algebras $\mathscr{T}(a)$.*

If $b \leq a$ then $X^b \subset X^a$ and we put $X_b^a = X^a \cap X_b$ so $X^a = X^b \oplus X_b^a$. We equip X_b^a with the volume element dx_b^a associated to dx^a and dx^b by the rule $dx^a = dx^b \otimes dx_b^a$ and we define:

$$\mathscr{T}^a(b) = \mathbb{K}(X^b) \otimes \mathbb{T}(X_b^a), \qquad \mathscr{T}^a = \sum_{b \leq a, b \in \mathscr{L}} \mathscr{T}^a(b).$$

The tensor product here is relative to the factorization $L^2(X^a) \cong L^2(X^b) \otimes L^2(X_b^a)$. We then have the following relation : $\mathscr{T}_a = \mathscr{T}^a \otimes_a \mathbb{T}(X_a)$.

2.5. Let $\mathscr{S}(X)$ be the space of rapidly decreasing test function and $\mathscr{S}^*(X)$ its adjoint space, the space of temperate distributions on X. The operators $f(Q)$ and $g(P)$ have natural extensions to the space $\mathscr{S}^*(X)$ if f and g are of class C^∞ and are polynomially bounded together with all their derivatives. We denote $|\,.\,|$ some euclidean norms on X or X^* and we put $< x >= (1 + |\,x\,|^2)^{\frac{1}{2}}$. This allows us to define the operators $< Q >^s$ and $< P >^s$ for any $s \in \mathbb{R}$; they act in the space $\mathscr{S}^*(X)$. Then we define the Sobolev spaces $\mathscr{H}^s(X) = \{f \in \mathscr{S}^*(X) \,|\, < P >^s f \in \mathscr{H}(X)\}$ with the natural topology. These spaces do not depend on the chosen norms.

We introduce now a class of weighted Sobolev spaces. Let $\theta \in C_0^\infty(X)$ be such that $\theta(x) > 0$ if $2^{-1} < |\,x\,| < 2$ and $\theta(x) = 0$ otherwise. Choose one more function $\eta \in C_0^\infty(X)$ such that $\eta(x) > 0$ if $|\,x\,| < 1$. Then for any $s, t \in \mathbb{R}$ and $1 \leq p \leq \infty$ let $\mathscr{H}_{t,p}^s$ be the space of distributions u which locally belong to $\mathscr{H}^s$ and such that

$$\|\theta(Q)u\|_{\mathscr{H}^s} + \left[\int_1^\infty \left\|r^t\theta(r^{-1}Q)u\right\|_{\mathscr{H}^s}^p \frac{dr}{r}\right]^{1/p} < \infty$$

If $p = \infty$ the second term is interpreted as $\sup_{r\geq 1} \|r^t\theta(r^{-1}Q)u\|_{\mathscr{H}^s}$. The left hand side above is a norm on $\mathscr{H}^s_{t,p}$ which provides this space with a Banach space structure. We set $\mathscr{H}^s_t \equiv \mathscr{H}^s_{t,2}$, these are the usual weighted Sobolev spaces defined by the norms $\|< P >^s< Q >^t u\|$. If $t_1 < t < t_2$, $t = (1-\lambda)t_1 + \lambda t_2$ and $p, p_1, p_2 \in [1,\infty], s \in \mathbb{R}$ then:

$$\mathscr{H}^s_{t,p} = (\mathscr{H}^s_{t_1,p_1}, \mathscr{H}^s_{t_2,p_2})_{\lambda,p}$$

Moreover, if $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then $(\mathscr{H}^s_{t,p})^* = \mathscr{H}^{-s}_{-t,p'}$.

3 A Dispersive HWZ Theorem

We describe now some explicit classes of self -adjoint operators affiliated to $\mathscr{T}$. We shall keep the notations introduced above. Let us fix a continuous function $h : X^* \to \mathbb{R}$ such that $h(k) \longrightarrow +\infty$ as $|\,k\,| \to \infty$. We denote by H_0 the self-adjoint operator $h(P)$ in the Hilbert space $\mathscr{H}(X)$. Since h is bounded from below H_0 is bounded from below too.

Recall the canonical isomorphism $X^* = X^{a*} \oplus X^*_a$. For each $a \in \mathscr{L}$, we define $h^a : X^{a*} \to \mathbb{R}$ by $h^a(k^a) = \inf\{h(k^a + k_a) \,|\, k_a \in X^*_a\}$. Observe that h^a is continuous and divergent at infinity. Then let $\mathscr{H}^h(X^a)$ be the form domain in $\mathscr{H}(X^a)$ of the operator $h^a(P^a)$ (we denote by P^a the momentum operator in $\mathscr{H}(X^a)$) equipped with the graph norm $< f, (1+|\,h^a(P^a)\,|)f >^{1/2}$. After identifying $\mathscr{H}(X^a)$ and $\mathscr{H}(X^a)^*$ we get:

$$\mathscr{H}^h(X^a) \subset \mathscr{H}(X^a) \subset \mathscr{H}^h(X^a)^*$$

We shall consider hamiltonians H of the form $H = H_0 + V$ with perturbations of the form $V = \sum_a V^a \otimes_a 1$ where $\otimes_a$ is the tensor product determined by the factorization $\mathscr{H}(X) \cong \mathscr{H}(X^a) \otimes \mathscr{H}(X_a)$ and V^a is a sesquilinear form in $\mathscr{H}(X^a)$. Since $\mathscr{H}(X^{a_{min}}) = \mathbb{C}$, $V^{a_{min}}$ is a real constant.

The next Lemma allows us to define the total hamiltonian for the interactions we have in mind.

Lemma 3.1 *Let V^a be a symmetric form on $\mathscr{H}(X^a)$ which is $h^a(P^a)$-form bounded with relative bound $\nu \geq 0$. Then $V_a = V^a \otimes_a 1$ is a symmetric form in $\mathscr{H}(X)$, $h(P)$-form bounded with relative bound equal to ν.*

Proof. By hypothesis, V^a is a continuous sesquilinear form on the Hilbert space $\mathscr{H}^h(X^a)$ and there is a constant $c < \infty$ such that $|< u, V^a u >| \leq \nu < u, h^a(P^a)u > +c\|u\|^2$ for all $u \in \mathscr{H}^h(X^a)$. Then $V_a = V^a \otimes_a 1$ is a continuous symmetric form on $\mathscr{H}^h(X^a) \otimes \mathscr{H}(X_a)$. $\mathscr{H}^h(X)$ is the form domain of $h(P)$ and $\mathscr{H}^h(X) \subset \mathscr{H}^h(X^a) \otimes \mathscr{H}(X_a)$. We work in the representation $\mathscr{H}(X) \cong L^2(X_a; \mathscr{H}(X^a))$ and we make a Fourier transformation in the variable $x_a \in X_a$. Then we have $(V_a \widehat{f})(k_a) = V^a \widehat{f}(k_a)$ for $\widehat{f} \in L^2(X_a^*; \mathscr{H}^h(X^a))$ and $k_a \in X_a^*$, while the operator $h(P)$ becomes the operator of multiplication by the operator valued-function $k_a \mapsto h(P^a + k_a)$. Then for each $\varepsilon > 0$ there is a constant $c_\varepsilon < \infty$ such that

$$
\begin{aligned}
\pm < f, V_a f > &= \pm \int_{X_a^*} < \widehat{f}(k_a), V^a \widehat{f}(k_a) > dk_a \\
&\leq (\nu + \varepsilon) \int_{X_a^*} < \widehat{f}(k_a), h^a(P^a) \widehat{f}(k_a) > dk_a + c_\varepsilon \int_{X_a^*} \left\| \widehat{f}(k_a) \right\|^2 dk_a \\
&\leq (\nu + \varepsilon) \int_{X_a^*} < \widehat{f}(k_a), h(P^a + k_a) \widehat{f}(k_a) > dk_a + c_\varepsilon \int_{X_a^*} \left\| \widehat{f}(k_a) \right\|^2 dk_a \\
&\leq (\nu + \varepsilon) < f, h(P) f > + c_\varepsilon \|f\|^2 \quad for \quad all \quad f \in \mathscr{H}^h(X).
\end{aligned}
$$

This proves the lemma. ∎

Let us recall the notion of smallness at infinity which is the weak decay assumption that we need.

Definition 3.1 *Let $s, t \in \mathbb{R}$, let Y be a vector space and let $L : \mathscr{H}^s(Y) \to \mathscr{H}^t(Y)$ a linear continuous operator. We shall say that L is small at infinity if it satisfies one (and therefore all) of the following equivalent conditions:*
a) $L : \mathscr{H}^s(Y) \to \mathscr{H}^{t'}(Y)$ is compact for some $t' < t$;
a′) the preceding assertion holds for all $t' < t$;
b) there is $\theta \in C^\infty(Y)$ with $\theta(y) = 0$ (resp. 1) if $|y| \leq 1$ (resp. $|y| \geq 2$) and there is $t' < t$ such that $\lim_{r \to \infty} \left\| \theta(Qr^{-1}) L \right\|_{\mathbb{B}(\mathscr{H}^s, \mathscr{H}^{t'})} = 0$;
b′) L has the preceding property for all $\theta \in C^\infty(Y)$ with $\theta(y) = 0$ (resp. 1) near zero (resp. near ∞) and all $t' < t$.

For the proof of the equivalence see Definition 3.5 in [BGS]. If $t = -s$ then giving a continuous linear operator $\mathscr{H}^s \to \mathscr{H}^{-s}$ is equivalent with giving a continuous sesquilinear form on $\mathscr{H}^s$. In particular we get the notion of sesquilinear form on $\mathscr{H}^s$ small at infinity.

Now we state and prove a criterion for an operator to be affiliated to $\mathscr{T}$. Observe that V^a may be non-local and that it could be a pseudo-differential operator of the same order as $h(P)$.

Let $\mu = 2s$, $s > 0$. From now on we suppose that: *there are two constants* c_1 *and* c_2 *such that*

$$c_1 < k >^{\mu} \leq h(k) \leq \; c_2 < k >^{\mu} \textit{ for all } k \in X^*$$

Then we also have $c_1' < k^a >^{\mu} \leq h^a(k^a) \leq \; c_2' < k^a >^{\mu}$ for all $k^a \in X^{a*}$. It is clear that $\mathscr{H}^h(X^a) = \mathscr{H}^s(X^a)$.

Theorem 3.1 *Assume that for each* $a \in \mathscr{L}$ *a continuous symmetric and small at infinity sesquilinear form* V^a *with domain* $\mathscr{H}^s(X^a)$ *is given such that* $\sum_a \mu_a < 1$ *where* μ_a *is the relative form bound of* V^a *with respect to* $h^a(P^a)$. *Identify* V^a *with an operator* $\mathscr{H}^s(X^a) \to \mathscr{H}^{-s}(X^a)$ *and assume that there is a* $t > s$ *such that* $V^a \mathscr{H}^t(X^a) \subset \mathscr{H}^{t-2s}(X^a)$. *Then* $H := h(P) + \sum_{a \in \mathscr{L}} V_a$ *is a self-adjoint operator in* $\mathscr{H}(X)$ *with form domain equal to* $\mathscr{H}^s(X)$. *The same assertion holds for each* $H_a = h(P) + \sum_{b \leq a} V_b$. *The operator* H *is affiliated to* $\mathscr{T}$ *and* H_a *is affiliated to* $\mathscr{T}_a$.

Proof. Our proof is similar to that of Proposition 3.6 in [BGS]. By the closed graph theorem, V^a is continuous when considered as operator $V^a : \mathscr{H}^t(X^a) \to \mathscr{H}^{t-2s}(X^a)$. Without loss of generality we assume $s < t \leq 2s$. For any $\varepsilon > 0$ let us define
$V^a_\varepsilon = (1 + \varepsilon h^a(P^a))^{-1} V^a (1 + \varepsilon h^a(P^a))^{-1}$. Since $V^a : \mathscr{H}^{2s}(X^a) \to \mathscr{H}^{-s}(X^a)$
is a compact symmetric operator we have $V^a_\varepsilon \in \mathbb{K}(X^a)$ and V^a_ε is symmetric too. Since $(1+\varepsilon h^a(P^a))^{-1} \to 1$ as $\varepsilon \to 0$ strongly in each $\mathscr{H}^r(X^a)$, we see that $V^a_\varepsilon \to V^a$ strongly in $B(\mathscr{H}^r(X^a), \mathscr{H}^{r-2s}(X^a))$ for all $r \in [2s-t, t]$ and in norm in $B(\mathscr{H}^r(X^a), \mathscr{H}^{r-2s-\delta}(X^a))$ for $r \in [2s-t, t]$ and $\delta > 0$. Let $H_\varepsilon := h(P) + V_\varepsilon$ with $V_\varepsilon = \sum_a (V_a)_\varepsilon$ and $(V_a)_\varepsilon = V^a_\varepsilon \otimes_a 1$. Then $(H_\varepsilon)_a := h(P) + \sum_{b \leq a} (V_b)_\varepsilon$. Proposition 9.3.4 in [ABG] implies that H_ε is affiliated to $\mathscr{T}$. Hence it is enough to show that $\lim\limits_{\varepsilon \to 0} H_\varepsilon = H$ in norm-resolvent sense. For this we use the identity ($\lambda > 0$ large enough)

$$(\lambda + H_\varepsilon)^{-1} - (\lambda + H)^{-1} = (\lambda + H_\varepsilon)^{-1} [V - V_\varepsilon] (\lambda + H)^{-1}.$$

Since $(\lambda + H)^{-1}$ and $(\lambda + H_\varepsilon)^{-1}$ are isomorphisms of $\mathscr{H}^{-s}(X)$ onto $\mathscr{H}^s(X)$ and $V - V_\varepsilon : \mathscr{H}^s(X) \to \mathscr{H}^{-s}(X)$ the identity holds in $B(\mathscr{H}^{-s}(X), \mathscr{H}^s(X))$. We shall show two things: $(\lambda + H)^{-1}\mathscr{H}(X) \subset \mathscr{H}^r(X)$ for some $r > s$ and $\|(\lambda + H_\varepsilon)^{-1}\|_{B(\mathscr{H}^{-s}, \mathscr{H}^s)} \leq const$ independently of ε; here λ is large enough and also independent of ε. Assuming these facts are true, we get

$$\left\|(\lambda+H_\varepsilon)^{-1}-(\lambda+H)^{-1}\right\|_{B(\mathscr{H},\mathscr{H}^s)}\le c\,\|V-V_\varepsilon\|_{B(\mathscr{H}^r,\mathscr{H}^{-s})}\to 0$$

because we may take $\delta=r-s>0$ above. Observe that $V^a_\varepsilon\to V^a$ in norm in $B(\mathscr{H}^r(X^a),\mathscr{H}^{-s}(X^a))$ which implies $(V_a)_\varepsilon\to V_a$ in norm in $B(\mathscr{H}^r(X),\mathscr{H}^{-s}(X))$.

Now we prove the assertion concerning $(\lambda+H_\varepsilon)^{-1}$. Using the equality $(\lambda+H_\varepsilon)^{-1}=(\lambda+h(P))^{-1/2}[1+(\lambda+h(P))^{-1/2}V^\varepsilon(\lambda+h(P))^{-1/2}]^{-1}(\lambda+h(P))^{-1/2}$ it is enough to show that there are λ and $\mu<1$ independent of ε , such that $\left\|(\lambda+h(P))^{-1/2}V_\varepsilon(\lambda+h(P))^{-1/2}\right\|\le\mu<1$. But $(\lambda+h(P))^{-1/2}V_\varepsilon(\lambda+h(P))^{-1/2}=$

$$\sum_a(\lambda+h(P))^{-1/2}(\lambda+\varepsilon h^a(P^a))^{-1}V^a(\lambda+\varepsilon h^a(P^a))^{-1}\otimes_a 1.(\lambda+h(P))^{-1/2}$$

hence

$$\begin{aligned}\left\|(\lambda+h(P))^{-1/2}V_\varepsilon(\lambda+h(P))^{-1/2}\right\| &\le \sum_a\left\|(\lambda+h(P))^{-1/2}V_a(\lambda+h(P))^{-1/2}\right\|\\ &\le \sum_a(\mu_a+\frac{c}{\lambda})\end{aligned}$$

where c/λ can be made as small as we want by choosing λ large enough. Finally, the proof of the assertion concerning $(\lambda+H_\varepsilon)^{-1}$ is the same as in [BGS]. ■

The main result of this section is the following generalization of the HWZ Theorem.

Theorem 3.2 *For each $a\in\mathscr{L}$ define $\tau_a=\inf\sigma(H_a)$ and let $\tau=\min_{a\in\mathscr{L}(2)}\tau_a$. If the hypotheses of Theorem 3.1 hold, then $\sigma_{ess}(H)=[\tau,\infty)$.*

Proof. By Theorem 3.1 H is affiliated to $\mathscr{T}$ and H_a is affiliated to $\mathscr{T}_a$. According to Proposition 8.4.2 in [ABG] (see also Theorem 3.2 in [BGS]) we shall then have $\sigma_{ess}(H)=\cup_{a\in\mathscr{L}(2)}\sigma(H_a)$. So it suffices to show that for each $a\in\mathscr{L}(2)$, $\sigma(H_a)$ is a half line. Since H is bounded from below we may assume, without loss of generality, that H is positive; this implies $H_a\ge 0$. $\mathscr{T}_a$ is the tensor product of $\mathscr{T}^a$ with the abelian C^* -algebra $\mathbb{T}(X_a)$. Working in the spectral representation of $\mathbb{T}(X_a)$ we see that there is a proper continuous family (see page 378 in [ABG]) $\{H^a(k_a),k_a\in X_a^*\}$ of hamiltonians $H^a(k_a)$ affiliated to $\mathscr{T}^a$ such that $\mathscr{F}_a(H_a+1)^{-1}\mathscr{F}_a^*$ (where $\mathscr{F}_a=1\otimes_a\mathscr{F}^{X_a}$ is the operator of Fourier transformation in the variable $x_a\in X_a$ only) is the operator of multiplication by the operator-valued function $k_a\longmapsto(H^a(k_a)+1)^{-1}\equiv F(k_a)$ (see remark 9.2.6 in [ABG]).

This function is norm continuous, tends in norm to zero at infinity, and $0 < F(k_a) \leq 1$. So $m(k_a) \equiv \sup F(k_a) = \|F(k_a)\| \to 0$ as $k_a \to \infty$. But $m(k_a)$ is the upper bound of the spectrum of $F(k_a)$ and depends continuously on k_a (by the norm-continuity of F). Since X_a^* is a connected set we see that the range of m has to be an interval. Finally, one has $\sigma(H_a) = \bigcup_{k_a \in X_a^*} \sigma(H^a(k_a))$ (see section 8.2.4 in [ABG]), which clearly shows that $\sigma(H_a)$ is an interval which has to be a half-line. This finishes the proof. ■

Application: Relativistic Hamiltonians with Coulomb Potentials

We consider the quantum mechanical many-body problem of electrons and fixed nuclei interacting via Coulomb forces with a relativistic form for the kinetic energy. The HWZ theorem for this model has been established in [LSV]; here we show that this situation is covered by our general Theorem 3.2.

Suppose we have N electrons with coordinates $x_1, ..., x_N \in \mathbb{R}^3$ and k nuclei with coordinates $R_1, ..., R_k$ in $\mathbb{R}^3$ (here $R_1, ..., R_k$ are fixed parameters) and with positive charges $Z_1, ..., Z_k$. We consider the following relativistic Hamiltonian:

$$H = \sum_{i=1}^{N} \sqrt{|\, P_i \,|^2 + m^2} + \sum_{1 \leq i < j \leq N} \alpha \,|\, x_i - x_j \,|^{-1} - \sum_{i}^{N} \sum_{j}^{k} \alpha Z_j \,|\, x_i - R_j \,|^{-1}$$

where m is the mass of the electron and $\alpha = e^2$. Now $\mathscr{L}$ is the lattice of partitions of the set $\{1, 2, ..., N\}$ and $X = \{x \in \mathbb{R}^{3N} \mid \sum_{i=1}^{N} x_i = 0\}$. For each cluster decomposition $a \in \mathscr{L}$ we take:
$X^a = \{x \in X \mid \sum_{k \in C} x_k = 0 \text{ for each } C \in a\}$
$X_a = \{x \in X \,|\, x_i = x_j \text{ if } i, j \text{ belong to the same cluster of } a\}$.

As we explained in 2.4 only the choice of the spaces X_a is really important.

Theorem 3.3 *If $\alpha Z_i < \frac{2}{\pi}$ for all $i \in \{1, ..., k\}$ then $\sigma_{ess}(H) = [\tau, \infty)$, where $\tau = \min_{a \in \mathscr{L}(2)} \inf \sigma(H_a)$.*

Proof. We define the function $h : X^* \to \mathbb{R}$ by $h(k) = \sum_{i=1}^{N} \sqrt{|\, k_i \,|^2 + m^2}$ where $X^* = \{k \in \mathbb{R}^{3N} \mid \sum_{i=1}^{N} k_i = 0\}$ is the dual (momentum) space of X. Then h is continuous and divergent at infinity, $\mathscr{H}^h(X^a) = \mathscr{H}^{1/2}(X^a)$ and $\mathscr{H}^h(X^a)^* = \mathscr{H}^{-1/2}(X^a)$, $(s = 1/2)$. Since, in our case H contains only two-body forces (i.e $V^a \neq 0$ only if $a = l$ is a pair) each $V^a : \mathscr{H}^{1/2}(X^a) \to \mathscr{H}^{-1/2}(X^a)$ is small at infinity. Indeed, if $V^a \neq 0$ then $X^a \cong \mathbb{R}^3$ and each V^a is of the form $\gamma / \,|\, x \,|$ $(\gamma \in \mathbb{R})$ which be may written as $V^a(x) = V_1(x) + V_2(x)$

with $V_1 \in L^2(\mathbb{R}^3)$ and $V_2 \in C_\infty(\mathbb{R}^3)$. We have also that $V^a \mathscr{H}^1(\mathbb{R}^3) \subset \mathscr{H}(\mathbb{R}^3)$ $(t = 1 > 1/2)$. The H_0-boundedness of V with relative bound less than one is a straightforward consequence of Theorem 1 in [LY]. So all our hypotheses are satisfied. ■

4 The Mourre Estimate

In this section we prove that for N-body dispersive systems the Mourre estimate holds outside a set of points (called thresholds) which can be described explicitly in terms of eigenvalues of some subhamiltonians. The conjugate operator is the generator of dilations $D = \frac{1}{2}(< Q, P > + < P, Q >)$ on X. We will denote by D^a the generator of dilations on X^a $(D = D^{a_{max}})$. We shall keep the notations introduced above.

4.1. We recall some results concerning the conjugate operator method (see [ABG] for details). Let $\mathscr{H}$ be a Hilbert space and A a self-adjoint, densely defined operator in $\mathscr{H}$. If $S \in B(\mathscr{H})$, we denote $\mathscr{W}_\tau(S) = e^{-iA\tau} S e^{-iA\tau}$, then $\{\mathscr{W}_\tau\}_{\tau \in \mathbb{R}}$ is a group of automorphisms of $B(\mathscr{H})$. The derivative $\frac{d}{d\tau}\mathscr{W}_\tau(S) \mid_{\tau=0} = i[S, A]$ exists as a sesquilinear form on $D(A)$ and it exists in the strong topology of $B(\mathscr{H})$ if and only if the sesquilinear form $[S, A]$ extends to a bounded operator in $\mathscr{H}$ (which will be denoted by the same symbol). In this case we say that S is of class $C^1(A)$ and we write $S \in C^1(A)$. If the derivative exists in the norm topology of $B(\mathscr{H})$, we say that S is of class $C^1_u(A)$. For the development of the theory, for example in order to show that the limiting absorption principle holds, one has to require more regularity than C^1_u (see [ABG] and references there). We say that $S \in B(\mathscr{H})$ is of class $\mathscr{C}^{1,1}(A)$ if $\int_0^1 \|(\mathscr{W}_\tau - 1)^2 S\| \frac{d\tau}{\tau^2} < \infty$. If S is of class $\mathscr{C}^{1,1}(A)$ it will be automatically of class $C^1_u(A)$.

Now let H be a self-adjoint operator in $\mathscr{H}$. We shall say that H is of class $C^1(A), C^1_u(A)$ or $\mathscr{C}^{1,1}(A)$ if $R(z) = (H - z)^{-1}$ has the corresponding property for some $z \in \mathbb{C} \setminus \sigma(H)$.

The proof of the Mourre estimate is simplified by the introduction of two functions $\rho_H \equiv \rho^A_H$ and $\tilde{\rho}_H \equiv \tilde{\rho}^A_H$ defined on $\mathbb{R}$ with values in $(-\infty, +\infty]$. Let H be any self-adjoint operator of class $C^1(A)$. For any $\lambda \in \mathbb{R}$ we denote:

$$\begin{aligned} \rho_H(\lambda) = \sup\{\alpha &\in \mathbb{R} \mid \exists \varphi \in C_0^\infty(\mathbb{R})\text{real}, \varphi(\lambda) \neq 0, \text{such that} \varphi(H)[H, iA]\varphi(H) \\ &\geq \alpha\varphi(H)^2\} \\ \tilde{\rho}_H(\lambda) = \sup\{\alpha &\in \mathbb{R} \mid \exists \varphi \in C_0^\infty(\mathbb{R})\text{real}, \varphi(\lambda) \neq 0, \text{and} K \text{a compact operator}, \\ &\quad \text{such that} \varphi(H)[H, iA]\varphi(H) \geq \alpha\varphi(H)^2 + K\} \end{aligned}$$

For a detailed description of the properties of the functions ρ and $\tilde{\rho}$ see [ABG].

We say that A is conjugate to H at the point $\lambda \in \mathbb{R}$ if $\tilde{\rho}_H(\lambda) > 0$. Let $J \subset \mathbb{R}$ a Borel set; if $\tilde{\rho}_H(\lambda) > 0$ for all $\lambda \in J$ we say that A is locally conjugate to H on J.

4.2. For each $a \in \mathscr{L}$, we define the operator $H^a = h(P^a) + \sum_{b \leq a} V_b^a$ where $V_b^a = V^b \otimes 1$ the tensor product being determined by the factorization $L^2(X^a) \cong L^2(X^b) \otimes L^2(X_b^a)$. We use the notation $\sigma_p(H)$ for the set of eigenvalues of a self-adjoint operator H. If H is a dispersive N-body hamiltonian, one associates to it two sets: the set of thresholds of H, defined as $\tau(H) := \bigcup_{a < a_{max}} \sigma_p(H^a)$, and the critical set of H, defined as $\kappa(H) := \bigcup_{a \in \mathscr{L}} \sigma_p(H^a)$. We define also the set $\mu^D(H) = \mathbb{R} \setminus \kappa(H)$.

The next lemma contains a technical result that will be needed in the proof of the main theorem of this section.

Lemma 4.1 *Let X be a vector space, A a self-adjoint operator in a Hilbert space $\mathscr{H}$ and $\{H_x\}_{x \in X}$ a family of self-adjoint operators of class $C^1(A)$. Assume that for some (hence for all) $z \in \mathbb{C} \setminus \mathbb{R}$ the maps $x \longmapsto (H_x - z)^{-1}$ and $x \longmapsto [(H_x - z)^{-1}, A]$ are norm continuous. Let $\lambda \in \mathbb{R}$, then:*
(a) The function $x \longmapsto \rho_{H_x}(\lambda)$ is lower semicontinuous.
(b) Let U be a compact subset of X and $c_0 > 0$ a constant such that $c_0 < \rho_{H_x}(\lambda)$ for all $x \in U$. For each $c_1 \in (0, c_0)$ there exists $\varphi \in C_0^\infty(\mathbb{R})$ real, with $\varphi(\lambda) \neq 0$, such that for all $x \in U$: $\varphi(H_x)[H_x, iA]\varphi(H_x) \geq c_1 \varphi^2(H_x)$.

Proof. (a) By using, for example, the Helfer-Sjöstrand formula (or see (8.1.1) in [ABG]) one easily sees that the functions $x \longmapsto \varphi(H_x)$ and $x \longmapsto [\psi(H_x), iA]$ are continuous in norm for all $\varphi, \psi \in C_0^\infty(\mathbb{R})$. On the other hand $\varphi(H_x)[H_x, iA]\varphi(H_x) = \varphi(H_x)[\psi(H_x), iA]\varphi(H_x)$ where ψ is any function in $C_0^\infty(\mathbb{R})$ such that $\psi(x) = x$ on supp φ, so the function $x \longmapsto \varphi(H_x)[H_x, iA]\varphi(H_x)$ is also norm-continuous.

Let $x_0 \in X$ and $r \in \mathbb{R}$ be such that $\rho_{H_{x_0}}(\lambda) > r$. We must show that there is a neighborhood of x_0 on which $\rho_{H_x}(\lambda) > r$. There exist $a_2 > a_1 > r$ and $\varphi_0 \in C_0^\infty(\mathbb{R})$ real with $\varphi_0(\mu) = 1$ on a neighborhood of λ and such that $\varphi_0(H_{x_0})[H_{x_0}, iA]\varphi_0(H_{x_0}) \geq a_2 \varphi_0^2(H_{x_0})$.

Let $\varepsilon_0 > 0$. By a continuity argument there is a neighborhood V_0 of x_0 in X such that $\varphi_0(H_x)[H_x, iA]\varphi_0(H_x) \geq a_2 \varphi_0^2(H_x) - \varepsilon_0$ for all $x \in V_0$. Let $\psi_0 \in C_0^\infty(\mathbb{R})$ be real with $\psi_0(\lambda) \neq 0$ and $\varphi_0(\mu) = 1$ on supp ψ_0. Multiplying left and right the preceding inequality by $\psi_0(H_x)$ we get $\psi_0(H_x)[H_x, iA]\psi_0(H_x) \geq (a_2 - \varepsilon_0)\psi_0^2(H_x)$ for all $x \in V_0$. Finally, choose ε_0 such that $a_2 - \varepsilon_0 = a_1$. Then $\psi_0(H_x)[H_x, iA]\psi_0(H_x) \geq a_1 \psi_0^2(H_x)$ hence $\rho_{H_x}(\lambda) \geq a_1 > r$

for all $x \in V_0$.
(b) As in the proof of (a) we show that each $x_0 \in U$ has a neighborhood $U(x_0)$ in U such that there is $\varphi_0 \in C_0^\infty(\mathbb{R})$ real with $\varphi_0(\mu) = 1$ on a neighborhood of λ with the property:

$$x \in U(x_0) \Rightarrow c_1 \varphi_0^2(H_x) \leq \varphi_0(H_x)[H_x, iA]\varphi_0(H_x).$$

Now the proof can be completed as follows. Since U is compact, on may choose a finite number of neighborhoods $U(x_1), ..., U(x_n)$ covering U. Let $\varphi_1, ..., \varphi_n$ be the corresponding functions constructed as above. Then for each $k \in \{1, ..., n\}$ $c_1\varphi_k^2(H_x) \leq \varphi_k(H_x)[H_x, iA]\varphi_k(H_x)$ for all $x \in U(x_k)$. Finally, let $\varphi \in C_0^\infty(\mathbb{R})$ real with $\varphi(\lambda) \neq 0$ and $\varphi_k(\mu) = 1$ on supp φ for all k. Multiplying left and right the preceding inequality by $\varphi(H_x)$ we get $c_1\varphi^2(H_x) \leq \varphi(H_x)[H_x, iA]\varphi(H_x)$ for all $x \in U$. ■

We now state the main result of this section. Note that we only need that H to be of class $C_u^1(A)$. For physically interesting and non-trivial functions h satisfying all the conditions below (e.g. relativistic 3-body kinetic energy) see [G].

Theorem 4.1 *Besides the hypotheses of Theorem 3.1, we assume that h is of class C^1, $|\, kh'(k)\, | \leq c(1+|\, h(k)\, |)$ for all $k \in X^*$, and for each $a \in \mathscr{L}$, $k_a h'(k^a + k_a) > 0$ for $k_a \neq 0$. Moreover, assume that $[D^a, V^a]$ is a compact operator $\mathscr{H}^{2s}(X^a) \to \mathscr{H}^{-2s}(X^a)$. Then D is locally conjugate to H on $\mathbb{R} \setminus \tau(H)$.*

Proof. (a) By a simple calculation one shows that $e^{-iDt}\mathscr{T}(a)e^{iDt} \subset \mathscr{T}(a)$ $\forall a \in \mathscr{L}$ and $t \in \mathbb{R}$. Moreover, one can see easily that $[(H+1)^{-1}, iD] \in \mathscr{T}$, which implies that H is of class $C_u^1(D)$ (see page 420 of [ABG]). Hence from Theorem 5.10 in [BGS] (or see Th 8.4.3 in [ABG]) it follows that $\tilde{\rho}_H(\lambda) = \min_{a \in \mathscr{L}(2)} \rho_{H_a}(\lambda)$ so it is sufficient to show that if $\lambda \notin \tau(H)$ then $\rho_{H_a}(\lambda) > 0$ for all $a \in \mathscr{L}(2)$.

(b) Using the relation $D = D^a \otimes_a 1 + 1 \otimes_a D_a$ (where D_a is the generator of dilations on X_a) we write $[H_a, iD] = [H_a, iD^a \otimes_a 1] + [H_a, i(1 \otimes_a D_a)]$. If we work in the representation $\mathscr{H}(X) \cong L^2(X_a; \mathscr{H}(X^a))$ and we make a Fourier transformation in the variable $x_a \in X_a$, then for each real function $\varphi \in C_0^\infty(\mathbb{R})$ the operator $\varphi(H_a)[H_a, iD]\varphi(H_a)$ becomes the operator of multiplication by the operator-valued function:

$$k_a \longmapsto \varphi(H^a(k_a))\{[H^a(k_a), iD^a] + k_a h'(P^a + k_a)\}\varphi(H^a(k_a)) \tag{1}$$

Since the family $\{H^a(k_a)\}_{k_a \in X_a^*}$ is proper $(a \neq a_{max})$ we have $\varphi(H^a(k_a)) = 0$ if k_a is large enough and if φ has support in a fixed compact. Let as fix $\lambda \notin \tau(H)$ and a compact neighborhood K of λ. Then for each $a \in \mathscr{L}(2)$ there is a compact set U_a in X such that

$\varphi(H^a(k_a)) = 0$ if $k_a \notin U_a$ and supp $\varphi \subset K$. So it is enough to show that there are $c_0 > 0$ and $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(\lambda) \neq 0$ and supp $\varphi \subset K$ such that the r.h.s of (1) is larger than $c_0\varphi^2(H^a(k_a))$ for all $k_a \in U_a$.

(c) Since $\lambda \notin \tau(H)$ the inequality $\rho_{H^a(0)}(\lambda) > 0$ is a straightforward consequence of the definition of $\tau(H)$. Note that $H^a(0) = H^a$. From (a) of the preceding lemma we conclude that the function $k_a \longmapsto \rho_{H^a(k_a)}$ is lower semi continuous. So there exists a compact neighborhood $\mathscr{V}_a$ of 0 in X_a^* and a constant $c_1 > 0$ such that $\rho_{H^a(k_a)}(\lambda) \geq c_1$ for each $k_a \in \mathscr{V}_a$. Now, from (b) of the preceding lemma, if we choose $c_2 \in (0, c_1)$, then there exists $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(\lambda) \neq 0$ and supp $\varphi \subset K$ such that:

$$\forall k_a \in \mathscr{V}_a, \varphi(H^a(k_a))[H^a(k_a), iD^a]\varphi(H^a(k_a)) \geq c_2\varphi^2(H^a(k_a)) \tag{2}$$

(d) By induction over a and by using the equality $\tilde{\rho}_H(\lambda) = \min_{a\in\mathscr{L}(2)} \rho_{H_a}(\lambda)$ we easily get that $\tilde{\rho}_H(\lambda) \geq 0$. From the decomposition $\mathscr{T}_a = \mathscr{T}^a \otimes_a \mathbb{T}(X_a)$ we see that $H^a(k_a)$ has in $\mathscr{H}(X^a)$ the same properties as H in $\mathscr{H}(X)$. It follow that $\tilde{\rho}_{H^a(k_a)}(\lambda) \geq 0$. Now as a consequence of lemma 7.2.12 in [ABG] we get: $\forall\varepsilon > 0$, $\exists\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(\lambda) \neq 0$ such that

$$\forall k_a \in U_a,\ \varphi(H^a(k_a))[H^a(k_a), iD^a]\varphi(H^a(k_a)) \geq -\varepsilon\varphi^2(H^a(k_a)) \tag{3}$$

(e) From our hypotheses it follows that there is $c_3 > 0$ such that $k_a h'(P^a + k_a) \geq c_3 > 0$ for each $k_a \notin \mathscr{V}_a$.

Finally, if we take $\varepsilon_0 \leq \inf(c_2, c_3)$ and $\varphi_0 \in C_0^\infty(\mathbb{R})$ with $\varphi_0(\lambda) \neq 0$ such that (3) holds for ε_0 and φ_0, we get:

$$\varphi_0(H^a(k_a))\{[H^a(k_a), iD^a] + k_a h'(P^a + k_a)\}\varphi_0(H^a(k_a)) \geq \inf(c_2, c_3)\varphi_0^2(H^a(k_a))$$

this completes the proof of the theorem. ∎

4.3. We summarize in the next theorem some consequences concerning the spectral properties of the dispersive N-body hamiltonian H. Here we need that H be of class $\mathscr{C}^{1,1}(D)$.

Theorem 4.2 *Besides the hypotheses of Theorem 3.1, we assume that h is of class C^2, $(|\, kh'(k)\,| + |\, k^2h''(k)\,|) \leq c(1 + |\, h(k)\,|)$ for all $k \in X^*$, and for each $a \in \mathscr{L}$, $k_a h'(k^a + k_a) > 0$ for $k_a \neq 0$. Moreover, assume that V^a can be decomposed into a sum $V^a = V_S^a + V_L^a$, where $V_S^a : \mathscr{H}^s(X^a) \to \mathscr{H}^{-s}(X^a)$ (the short-range component) and $V_L^a : \mathscr{H}^s(X^a) \to \mathscr{H}^{-s}(X^a)$ (the long-range component) are symmetric operators satisfying the following decay conditions at infinity. Choose $\theta \in C_0^\infty(X)$ such that $\theta(x) > 0$ if $0 < 1 < |\, x\,| < 2$*

and $\theta(x) = 0$ otherwise and denote $\|.\|_{\mathscr{X}}$ is the norm in $\mathscr{X} \equiv B(\mathscr{H}^s(X^a), \mathscr{H}^{-s}(X^a))$. Then we assume $[V_L^a, iD^a] \in \mathscr{X}$ and

$$\int_1^\infty \left\{ \|\theta(\frac{Q^a}{r})V_S^a\|_{\mathscr{X}} + \|\frac{1}{r}\theta(\frac{Q^a}{r})[V_L^a, iD^a]\|_{\mathscr{X}} \right\} dr < \infty$$

Then $\tau(H)$ is a closed countable set, the eigenvalues of H outside $\tau(H)$ are of finite multiplicity and can accumulate only at points belonging to $\tau(H)$, and H has no singularly continuous spectrum. The limits $\lim_{\mu \to \pm 0}(H - \lambda - i\mu)^{-1}$ exist in the weak* topology of $B(\mathscr{H}^{-s}_{1/2,1}, \mathscr{H}^s_{-1/2,\infty})$ uniformly in λ on each compact subset of the open real set $\mu^D(H)$. Finally, if $\mathscr{K}$ is a Hilbert space and $T : \mathscr{H}(X) \to \mathscr{K}$ is a linear operator which is continuous when $\mathscr{H}(X)$ is equipped with the topology induced by $\mathscr{H}^s_{-1/2,\infty}$, then T is locally H-smooth on $\mu^D(H)$.

Proof. The first assertion will hold inductively given the conclusion of the second assertion for subsystems and the definition of thresholds. Our hypotheses imply that H is of class $\mathscr{C}^{1,1}(D)$; for this we use Proposition 7.5.7 and Theorem 7.5.8 in [ABG] with $\Lambda =< Q >$. Since H is bounded from below, it has a spectral gap. Then the second and the third assertion are consequences of Theorem 3.1 and Theorem 7.4.2 of [ABG] and also of the first assertion. The last assertion follows from Theorem 7.4.1 and Proposition 7.4.4 of [ABG]. ■

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References

[ABG] W. Amrein, A. Boutet de Monvel and V. Georgescu, *C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Birkhauser, Progress in Math. Ser., nr. 135, 1996.

[BG1] A. Boutet de Monvel and V. Georgescu, *Graded C^*-Algebras in the N-Body Problem*, J. Math. Phys. 32, (1991) 3101-3110.

[BG2] A. Boutet de Monvel and V. Georgescu, *Graded C^*-Algebras and Many-Body Perturbation Theory: II The Mourre estimate*, in "Méthodes semi-classiques", vol. 2, Colloque international, Nantes, juin 1991, Astérisque, vol. 210, Société Mathématique de France, 1992, pp. 75-96.

[BG3] A. Boutet de Monvel and V. Georgescu, *Graded C^*-Algebras Associated to Symplectic Spaces and Spectral Analysis of Many-Channel Hamiltonians,* in *Dynamics of Complex and Irregular Systems*, Bielefeld encounters in Math. Physics VIII, 1991, World Scientific, 1993, pp. 22-66.

[BGS] A. Boutet de Monvel V. Georgescu and A. Soffer, *N-Body Hamiltonians with Hard Core Interactions*, Reviews in Math. Physics 6. (1994), no. 3, 1-82.

[CFKS] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry*, Springer, Berlin, 1987.

[D] J. Derezinski, *The Mourre Estimate For Dispersive N-Body Schrödinger Operators*, Trans. Amer. Math. Soc 317 (1990), no 2, 773-798.

[DG] J. Derezinski and Ch. Gerard, *Scattering Theory of Classical and Quantum N-Particle Systems*, Springer, 1997.

[F-H] R. G. Froese and I. Herbst, *A New Proof of the Mourre Estimate*, Duke Math. J., 49 (1982) pp. 1075-1085.

[G] Ch. Gerard, *The Mourre Estimate For Regular Dispersive Systems*, Ann. Inst. H. Poincare, Physique Theorique. 54 (1991), 59-88.

[H] W. Hunziker, *On the Spectra of Schrödinger Multiparticle Hamiltonians*, Helv. Phys. Acta, 39: 451-462, 1966.

[LSV] R. T. Lewis, H. Siedentop and A. Vugalter, *The Essential Spectrum Of Relativistic Multi-Particle Operators*, preprint, 1996.

[LY] Elliott H. Lieb and Horng-Tzer Yau, *The Stability and Instability of Relativistic Matter*, Commun. Math. Phys, 118 (1988) 177-213.

[M] E. Mourre, *Absence of Singular Continuous Spectrum for Certain Self-Adjoint Operators*, Comm. Math. Phys., 78 (1981) pp. 391-408.

[PSS] P. Perry, I. M. Sigal and B. Simon, *Spectral Analysis of N-Body Schrödinger Operators*, Ann. Math. 114 (1981), 519-567.

[RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Volume 4: Analysis of Operators*, Academic Press, New York, 1978.

[W] C. van Winter, *Theory of Finite Systems of Particles I. The Green Function*, Mat. Fys Dan. Vid. Selsk, 2(8): 1-60, 1994.

[Z] G. M. Zislin, A Study of the Spectrum of the Schrödinger Operator for a System of several particles, Trudy Maskov. Mat. Obsc, 9: 81-120, 1960.