SEMICLASSICS OF THE QUANTUM CURRENT

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Abstract

We give the semi-classical asymptotics of the quantum current in 2 cases: First for $T > 0$ (T is the absolute temperature). Here we get a complete asymptotics. Then for $T = 0$. Here it vanishes to the accessible orders.

Contents

*Partially supported by the European Union, grant FMRX-960001.

1 Introduction

Since the classical work of Weyl estimating the counting function for the number of eigenvalues has been one of the central themes in semiclassical analysis. In the present paper we study Schrödinger operators of the form $H(h) = H(h, \vec{A}, V) = (-ih\nabla - \vec{A})^2 + V$, where V is a potential and $\vec{A}$ is a vector potential generating a magnetic field $\vec{B} = \nabla \times \vec{A}$. In this case, the Weyl estimate corresponds to the fact that the *counting function* $\operatorname{tr}\left[1_{(-\infty,\lambda]}(H(h))\right]$ is given to leading order, in the limit $h \to 0$, by the classical phase space integral $\iint 1_{(-\infty,\lambda]}\Big((hp - \vec{A}(x))^2 + V(x)\Big)\, dp\, dx$.

More detailed information is given by the *density* or *local counting function* ρ, that is given, as a distribution, by

$$\int \rho\psi\, dx = \operatorname{tr}[\psi 1_{(-\infty,\lambda]}(H(h))],$$

for all $\psi \in C_0^\infty$. Clearly the counting function is the total integral $\int \rho\, dx$.

The physical interpretation of ρ is that it is the density of a non-interacting Fermi gas in the fields V and $\vec{B}$, and with chemical potential λ. An equally important physical quantity is the *quantum current* $\vec{j}$. The density is the response in the total Fermi energy to a variation in the potential, i.e.,

$$\int \rho\psi\, dx = \frac{\partial}{\partial\varepsilon}\operatorname{tr}[(H(h, \vec{A}, V + \varepsilon\psi) - \lambda)1_{(-\infty,\lambda]}(H(h, \vec{A}, V + \varepsilon\psi))]\Big|_{\varepsilon=0}.$$

Likewise, the current is the response in the total Fermi energy to a variation in the vector potential, i.e.,

$$\int \vec{j} \cdot \vec{a}\, dx = \frac{\partial}{\partial\varepsilon}\operatorname{tr}\left[(H(h, \vec{A} + \varepsilon\vec{a}, V) - \lambda)1_{(-\infty,\lambda]}\Big(H(h, \vec{A} + \varepsilon\vec{a}, V)\Big)\right]\Bigg|_{\varepsilon=0}.$$

The quantum current has largely been ignored in semiclassical analysis. The reason is quite simply that classically there is no current. The reason being that there is no classical diamagnetism, i.e., the classical energy

$$\iint (hp - \vec{A}(x))^2 + V(x) dp\, dx$$

is independent of $\vec{A}$, which is easily seen by a change of variables.

The aim of this paper is to prove that the semiclassical limit of the quantum current indeed vanishes. Moreover, we explore to which order in the semiclassical parameter h the current vanishes.
Formally we have (and it can be justified under the assumptions that we will impose below):

$$\begin{aligned}\int \vec{a}\cdot\vec{j}\,dx &= \frac{\partial}{\partial\varepsilon}\mathrm{tr}\left[(H(h,\vec{A}+\varepsilon\vec{a},V)-\lambda)1_{(-\infty,\lambda]}\left(H(h,\vec{A}+\varepsilon\vec{a},V)\right)\right]\Big|_{\varepsilon=0}\\ &= 2\mathrm{tr}\left[(\vec{a}\cdot(-ih\nabla-\vec{A})-\frac{ih}{2}\mathrm{div}(\vec{a}))1_{(-\infty,\lambda]}\left(H(h,\vec{A},V)\right)\right].\end{aligned}$$

Notice that since the operator multiplying the characteristic function is symmetric, we get,

$$\begin{aligned}\int \vec{a}\cdot\vec{j}\,dx &= \Re\{\mathrm{tr}\left[\vec{a}\cdot(-ih\nabla-\vec{A})1_{(-\infty,\lambda]}\left(H(h,\vec{A},V)\right)\right]\}\\ &= \Re\{\mathrm{tr}\left[B(h)1_{(-\infty,\lambda]}\left(H(h,\vec{A},V)\right)\right]\},\end{aligned}$$

where $B(h)=\vec{a}\cdot(-ih\nabla-\vec{A})$, and where $\Re(z)$ denotes the real part of z. This is the quantity we will study semi-classically.
We will denote by a_0 the Weyl-symbol of $H(h)$. It is easy to see that

$$a_0(x,p)=(p-\vec{A}(x))^2+V(x). \tag{1.1}$$

To assure that a_0 satisfies:

$$|\partial^\alpha a_0|\leq c_\alpha|a_0|,$$

for all $\alpha\in\mathbf{N}^n$, we will impose the following assumption on $\vec{A}$ and V.

Assumption 1.1.

- $V\in C^\infty(\mathbf{R}^n)$ and $\min_{x\in\mathbf{R}^n}V(x)>1$.
- $\forall\alpha\in\mathbf{N}^n$, $\exists c_\alpha>0:\ \forall x\in\mathbf{R}^n$
  $|\partial_x^\alpha V(x)|\leq c_\alpha V(x)$.
- $\exists C,M>0$ such that: $\forall x,y\in\mathbf{R}^n$
  $|V(x)|\leq CV(y)(1+|x-y|)^M$.

- $\vec{A} \in C^\infty(\mathbf{R}^n, \mathbf{R}^n)$.
- $\forall \alpha \in \mathbf{N}^n$ with $|\alpha| \geq 1\, \exists c_\alpha \in \mathbf{R} : \|\partial^\alpha A\| \leq c_\alpha V^{1/2}$.

These will be standing assumptions throughout the paper. Notice that this includes the case of constant magnetic field. That $V \geq 1$ means in reality only that it is bounded below, for we can add a constant to the potential without changing the current - we only have to change λ correspondingly.

Under the above assumptions we know from [Rob87, Thm III-4] that $\exists h_0 > 0$ so that $H(h)$ is essentially self-adjoint on $\mathcal{S}(\mathbf{R}^n)$ (the Schwartz space of rapidly decreasing smooth functions) and uniformly bounded below for $h \in (0, h_0]$. These rather strong asumptions on V, $\vec{A}$ (compare [AHS78]) are made to be in a position to apply the functional calculus of Helffer and Robert [HR83] (see also [Rob87]).
For notational convenience we will introduce the operator:

$$P_A = -ih\nabla - \vec{A}.$$

and write

$$B(h) = \vec{a} \cdot P_A.$$

Then $B(h)$ has Weyl symbol

$$\sigma[B(h)](x,p) = \vec{a} \cdot (p - \vec{A}) - \frac{h}{2i}\mathrm{div}(\vec{a}), \tag{1.2}$$

which we will write as $b_0 + hb_1$.

Let us take λ_0 such that $a_0^{-1}((-\infty, \lambda_0])$ is compact. This λ_0 will be fixed throughout the text. Our plan is to calculate the asymptotics of

$$\mathrm{tr}[B(h)f(H(h))] \quad \text{for } h \searrow 0, \tag{1.3}$$

first for $f \in C_0^\infty(-\infty, \lambda_0)$ and later for $f = 1_{(-\infty,\lambda]}$ with $\lambda < \lambda_0$.
The organization of the paper is the following: In sections 2 and 3 we prove that the operator $B(h)$ can be included in the functional calculus by Helffer and Robert. Thus we get for $f \in C_0^\infty(-\infty, \lambda_0)$ that $B(h)f(H(h))$ is h-admissible, and we get a complete expansion of $\mathrm{tr}[B(h)f(H(h))]$ in powers of h. These two sections follow very closely the presentation in Robert's book [Rob87].

As a reasonably easy application of the results in section 3, we calculate the semi-classical asymptotics for the quantum current $\vec{j}$, for a Fermi gas at $T > 0$ (T absolute temperature) in section 4. Here $\vec{j}$ is given by

$$\int_{\mathbf{R}^n} \vec{a} \cdot \vec{j}\, dx = \Re\{\mathrm{tr}[B(h)f(H(h))]\},$$

where $V(x)$ is supposed to go to infinity for $|x| \to \infty$, and f is the function:

$$f(t) = \frac{e^{-\beta(t-\alpha)}}{1 + e^{-\beta(t-\alpha)}},$$

with β playing the role of an inverse temperature.

Theorem 1.2. *Let $H(h)$ satisfy Assumption 1.1 and let furthermore $V(x)$ satisfy:*

$$\exists\, c, s > 0 \text{ so that } V(x) \geq c|x|^s.$$

Let $c(h) = \sum h^j c_j$ be the symbol (calculated formally) of the operator $B(h)f(H(h))$. Then the c_j are in $L^1(\mathbf{R}^n_x \times \mathbf{R}^n_\xi)$ and we have:

$$(2\pi h)^n \mathrm{tr}[B(h)f(H(h))] = \sum_{j=0}^{N} h^j \iint c_j(x, \xi)\, dx d\xi + O(h^{N+1}) \quad \text{for } h \searrow 0.$$

Remark 1.3. The first term in the expansion is zero, and the next is purely imaginary, so we have to go to order h^{2-n} to get a contribution to the current.

In section 5 we go on to calculate the semiclassical asymptotics for the quantum current at zero temperature. Here $\vec{j}$ is defined by the equation:

$$\int_{\mathbf{R}^n} \vec{a} \cdot \vec{j}\, dx = \Re\{\mathrm{tr}[B(h)1_{(-\infty,\lambda]}(H(h))]\}. \tag{1.4}$$

Theorem 1.4. *Let $H(h)$ satisfy Assumption 1.1. Let $\lambda < \lambda_0$ and assume λ not to be a critical value for a_0* [1]*. Then*

$$(2\pi h)^n \mathrm{tr}[B(h)1_{]-\infty,\lambda]}(H(h))] = \iint_{\{a_0(x,p) \leq \lambda\}} \vec{a} \cdot (p - \vec{A})\, dx\, dp + O(h). \tag{1.5}$$

[1] This is independent of $\vec{A}$ because the critical values of a_0 are the critical values of V. Notice also that the assumption on λ_0, $a_0^{-1}((-\infty, \lambda_0])$ is compact, is independent of $\vec{A}$.

We thus have $\vec{j} = O(h^{1-n})$ because the integral on the right hand side in eq. (1.5) vanishes. By the change of variables $q = p - \vec{A}(x)$, we get indeed

$$\begin{aligned}\iint_{\{(p-\vec{A})^2+V(x)\le\lambda\}} \vec{a}\cdot(p-\vec{A})\,dxdp &= \iint_{\{q^2-V(x)\le\lambda\}} \vec{a}\cdot q\,dxdq \\ &= \int_{x\in\mathbf{R}^n} \vec{a}(x)\cdot\Big(\int_{\{q^2\le\lambda-V(x)\}} q\,dq\Big)\,dx \\ &= 0.\end{aligned}$$

The problem in proving Theorem 1.4 is that the function

$$\tau \mapsto \mathrm{tr}[B(h)1_{]-\infty,\tau]}(H(h))],$$

is not monotone and therefore the standard Tauberian argument does not work. Our argument uses the relative boundedness of $B(h)$ with respect to $H(h)$ to reduce to the counting function $\mathrm{tr}[1_{]-\infty,\tau]}(H(h))]$ where standard arguments can be applied.
In section 6 we will improve Theorem 1.4 in the case where the set of classical closed orbits on the energy surface

$$\Sigma_\lambda = \{(x,p)|a_0(x,p) = \lambda\}$$

is of surface measure zero.

Theorem 1.5. *We maintain the assumptions from Theorem 1.4. Suppose:*

$$S_\lambda(\{(x,p)\in\Sigma_\lambda|\exists t\neq 0\, s.t.\, \Phi^t_{a_0}(x,p) = (x,p)\}) = 0, \tag{1.6}$$

where S_λ is the surface measure on Σ_λ and Φ_{a_0} is the Hamiltonian flow associated with a_0. Then

$$(2\pi h)^n \mathrm{tr}[B(h)1_{(-\infty,\lambda]}(H(h))] = h\gamma_1 + o(h) \quad \textit{for } h\searrow 0,$$

where γ_1 is:

$$\gamma_1(\lambda) = \iint_{\{a_0(x,p)\le\lambda\}} b_1\ dxdp + \frac{1}{2i}\int_{\Sigma_\lambda} \{a_0,b_0\}_P \frac{dS_\lambda}{|\nabla a_0|}.$$

Here $\{a_0,b_0\}_P = \partial_p a_0(x,p)\partial_x b_0(x,p) - \partial_x a_0(x,p)\partial_p b_0(x,p)$, is the Poisson bracket.

As γ_1 is purely imaginary the conclusion of this paper is that the quantum current at zero temperature vanishes to all accessible orders. We believe that the next term in the expansion of the current (the term of order $O(h^{1-n})$) may be oscillating. A simple calculation shows that for the counting function the term of order $O(h^{-1})$ is indeed oscillating for the harmonic oscillator in dimension 2 (i.e. $\vec{a} = 0, V = x^2 + 1$). Unfortunately in the case of the harmonic oscillator with a constant magnetic field in dimension 2, i.e. $\vec{A} = (-x_2, 0)$, $V(x) = x^2 + 1$, the contributions to the current within an energy level cancel to such an extent that the total current is $O(1)$ - not $O(h^{-1})$. One thus has to search for strong current in more complicated settings.

Notations

Since the first sections are very close to the book [Rob87], we will use the notations therein. Thus $Op_h^w a$ denotes the h-pseudodifferential operator with Weyl symbol a. We will also use the symbol classes and theorems on continuity and composition from the same book. Finally we write

$$\int_{\mathbf{R}^n} f(p) d_h p = \frac{1}{(2\pi h)^n} \int_{\mathbf{R}^n} f(p) dp.$$

2 Functional Calculus

Let $f \in C_0^\infty(-\infty, \lambda_0)$ real valued. We want to prove that $B(h)f(H(h))$ is h-admissible. That will be the result of Prop. 2.5 below.

Lemma 2.1. *We have the following inclusion of domains (as closures of quadratic forms starting from $\mathcal{S}(\mathbf{R}^n)$):*

- $\mathcal{D}(V) \supseteq \mathcal{D}(H(h))$,
- $\mathcal{D}(P_A^2) \supseteq \mathcal{D}(H(h))$.

Lemma 2.2.

$$\|B(h)f(H(h))\|_{\mathcal{L}(L^2)} = O(1) \text{ for } h \searrow 0,$$

where $\mathcal{L}(L^2)$ is the space of bounded operators on L^2.

Proof. We know [Rob87, Thm.III-4, Prop.III-13] that $f(H(h))$ is self-adjoint, bounded and of finite rank.
Let $v \in L^2$, $\|v\| = 1$. Then we write

$$v = \sum_{j=1}^{N(h)} \alpha_j(h)\phi_j(h) + w(h),$$

where $\langle \phi_j, \phi_k \rangle = \delta_{j,k}$, $H(h)\phi_j(h) = \lambda_j(h)\phi_j(h)$ and $f(H(h))w(h) = 0$.
Then

$$\begin{aligned}
\sum_{l=1}^{n} \|(P_A)_l f(H(h))v\|^2 &\leq \langle \sum_{j=1}^{N} \alpha_j f(\lambda_j)\phi_j, (P_A)^2 \sum_{j=1}^{N} \alpha_j f(\lambda_j)\phi_j \rangle \\
&= \sum_{j,k} \bar{\alpha_j}\alpha_k f(\lambda_k)\overline{f(\lambda_j)}\langle \phi_j, (\lambda_k - V(x))\phi_k \rangle \\
&= \sum |\alpha_j|^2 f(\lambda_j)^2 \lambda_j - \langle f(H(h))v, V(x) f(H(h))v \rangle \\
&\leq \sum |\alpha_j|^2 f(\lambda_j)^2 \lambda_j - \inf V \|f(H(h))v\|^2 \\
&\leq \|f\|_\infty^2 \lambda_0,
\end{aligned}$$

where we have used Lemma 2.1. □

Let $f \in C_0^\infty(-\infty, \lambda_0)$. Then $f(H(h))$ is h-admissible by the functional calculus (see [Rob87]). Therefore for all $N \in \mathbf{N}$ sufficiently big there exist $a_{f,1}, a_{f,2}, ..., a_{f,N}$ such that:

$$f(H(h)) = \sum_{j=0}^{N} h^j Op_h^w a_{f,j} + h^{N+1} D_{f,N+1}(h),$$

where $D_{f,N+1}$ is uniformly bounded in $\mathcal{L}(L^2)$. Moreover we have:

Lemma 2.3. *$B(h)D_{f,N+1}$ is uniformly bounded in $\mathcal{L}(L^2)$ and satisfies*

$$\|B(h)D_{f,N+1}\|_{\mathcal{L}(L^2)} \leq \|\vec{a}\|_\infty p(f),$$

where p is a seminorm in $\mathcal{S}$. Here p does not depend on h, $\vec{a}$ and f.

Proof. We recall that $1 \leq \inf\{\bigcup_{h\in(0,h_0]} \mathrm{Spec}(H(h))\}$. By the spectral theorem we may thus assume that $f \in C_0^\infty(0,\lambda_0)$. We define the continuous function $\theta(s)$ for $s \in \mathbf{C}$ as:

$$\theta(s) = \begin{cases} \frac{1}{|\Im(s)|} & \text{if } |\Im(s)| \geq 2/\pi \\ \pi/2 & \text{otherwise} \end{cases},$$

where $\Im(s)$ denotes the imaginary part of s. Let us define, for $s \in \mathbf{C}$, the path $Z_{\theta(s)}$ in the complex plane as the union of the following three paths:

- $\gamma_\pm = \{(t+1/2)e^{\pm i\theta(s)} | t \geq 0\}$
- $\gamma_0 = \{e^{iv}/2 | -\theta(s) \leq v \leq \theta(s)\}$.

When we integrate over $Z_{\theta(s)}$ we will look upon it as a continuous path starting with γ_-.
We will denote the Mellin transform of f by $\mathcal{M}[f]$:

$$\mathcal{M}[f](s) = \int_0^{+\infty} t^{s-1} f(t)\, dt,$$

defined on all of $\mathbf{C}$. We remember the following properties of the Mellin transform:

- $\mathcal{M}[f](s)$ is holomorhic on all of $\mathbf{C}$.
- $\forall\, \rho \in \mathbf{R}$ we have: $\mathcal{M}[f](s)$ decreases rapidly on the line $\{s \in \mathbf{C} | \Re(s) = \rho\}$.
- $\forall\, \rho \in \mathbf{R}$ we have: $f(t) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \mathcal{M}[f](s) t^{-s} ds$.

Now we are ready to start the proof. We have by construction ([Rob87, eq.(46),p.143 and Thm.III-10)])

$$D_{f,N+1} = \frac{1}{(2\pi)^2}\int_{\rho-i\infty}^{\rho+i\infty} \mathcal{M}[f](s) \int_{Z_{\theta(s)}} z^{-s}(H(h)-z)^{-1}\Delta_{z,N+1}(h)\, dz\, ds, \tag{2.1}$$

where $\Delta_{z,N+1}(h)$, again by construction, satisfies

$$||\Delta_{z,N+1}(h)||_{\mathcal{L}(L^2)} = O\left(\left(\frac{|z|}{d(z)}\right)^{q(N)}\right)$$

([Rob87, Thm. III-9]), where $d(z)$ =dist$(z,[1,\infty[)$, $\rho \in \mathbf{R}$ and $q(N)$ is a sufficiently big integer independent of z and h. Because of Lemma 2.4 we can apply the differential operator $B(h)$ under the two integral signs, and we obtain the desired estimate, because for ρ sufficiently big the double integral becomes absolutely convergent:

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} |\mathcal{M}[f](\rho+is)| \\
&\qquad \times \int_{Z_{\theta(\rho+is)}} |z^{-(\rho+is)}| \| B(h)(H(h)-z)^{-1}\Delta_{z,N+1}(h)\|_{\mathcal{L}(L^2)}\, dz\, ds \\
&\leq \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} |\mathcal{M}[f](\rho+is)| \Big\{\ c\|\vec{a}\|_\infty e^{\theta|s|} \\
&\qquad + c\sin(\theta(\rho+is))^{-q(N)-1/2} e^{\theta|s|}\int_0^\infty (r+1/2)^{-\rho}\, dr\Big\}\ \ ds \\
&\leq c\|\vec{a}\|_\infty \int_{-\infty}^{\infty} |\mathcal{M}[f](\rho+is)| [1+(\sin(\theta(s))^{-q(N)-1/2}] ds \\
&\leq c\|\vec{a}\|_\infty \int_{-\infty}^{\infty} |\mathcal{M}[f](\rho+is)(1+s^2)^M| \frac{[\theta(s)+(\sin(\theta(s))^{-q(N)-1/2}]}{(1+s^2)^M} ds.
\end{aligned}$$

As $\frac{1}{\sin(\theta(\rho+is))} = O(s)$ for $s \to \infty$ and $\mathcal{M}[f](\rho+is)$ decreases rapidly on the line of integration, we get that the integral converges (to assure convergence of the r integral we can take $\rho = 2$). If we investigate the f dependence, which comes from the term

$$\sup_s |\mathcal{M}[f](\rho+is)(1+s^2)^M|,$$

we get the seminorm of f that we wanted. $\square$

Lemma 2.4. *For $z \in \gamma_0$ we have:*

$$\|B(h)(H(h)-z)^{-1}\|^2_{\mathcal{L}(L^2)} \leq 4n\|\vec{a}\|^2_\infty.$$

For $z \in \gamma_\pm$ we have:

$$\|B(h)(H(h)-z)^{-1}\|^2_{\mathcal{L}(L^2)} \leq \frac{n\|\vec{a}\|^2_\infty}{\sin\theta}.$$

Proof. Because $H(h)$ is essentially self-adjoint on $\mathcal{S}$, $\{\phi \in L^2 | (H(h)-z)^{-1}\phi \in \mathcal{S}\}$ is dense in L^2 for $z \notin \mathbf{R}$. Let ϕ be in the above set, $z \in Z_{\theta(s)}$, and let $\psi = (H(h) - z)^{-1}\phi$. Then we have:

$$\begin{aligned} \|B(h)(H(h)-z)^{-1}\phi\|^2 &= n\|\vec{a}\|_\infty^2 \langle \psi, P_A^2 \psi \rangle \\ &\leq n\|\vec{a}\|_\infty^2 \langle \psi, H(h)\psi \rangle \\ &= n\|\vec{a}\|_\infty^2 \langle \phi, [H(h)-\bar{z}]^{-1} H(h) [H(h)-z]^{-1}\phi \rangle. \end{aligned}$$

So now we just need to prove that the function $g(x,z) = \frac{x}{(x-\bar{z})(x-z)} = \frac{x}{|x-z|^2}$ is bounded by 4 for $(x,z) \in [1,\infty) \times \gamma_0$ and by $\frac{1}{\sin\theta}$ for $(x,z) \in [1,\infty) \times \gamma_\pm$. The inequality for $z \in \gamma_0$ is obvious, and, for $z \in \gamma_\pm$, we get by simple geometric considerations that:

$$\frac{x}{|x-z|^2} \leq \frac{x}{x^2 \sin\theta} \leq \frac{1}{\sin\theta}.$$

□

We conclude the following:

Proposition 2.5. *The operator $B(h)f(H(h))$ is h-admissible, and its symbol can be calculated as follows:*

$$\sigma[B(h)f(H(h))] \equiv \sigma[B(h)H_{f,N}] \pmod{h^{N+1}},$$

where $H_{f,N} = \sum_{j=0}^N h^j Op_h^w(a_{f,j})$ is the sum of the first $N+1$ terms in the expansion of $f(H(h))$ as an h-admissible operator.

Proof. This is an easy consequence of the Lemma 2.3. Let us indeed write $f(H(h)) = H_{f,N} + h^{N+1}D_{N+1}(h)$. Then $B(h)H_{f,N}$ is h-admissible and strongly h-admissible because the symbol of $H_{f,N}$ has compact support. Thus, we only have to prove that $B(h)D_{N+1}(h)$ is bounded in $\mathcal{L}(L^2)$, and that is the statement of Lemma 2.3. □

3 Traces with smooth functions

Our task in this section will be to find the asymptotics of

$$\operatorname{tr}(B(h)f(H(h))) \quad \text{for } h \searrow 0, \tag{3.1}$$

for $f \in C_0^\infty(-\infty, \lambda_0)$. This is given by Theorem 3.4 below.

Lemma 3.1.

$$\|B(h)f(H(h))\|_{tr} = O(h^{-n}).$$

Proof. Let $g \in C_0^\infty(-\infty, \lambda_0)$, $g \equiv 1$ on $\operatorname{supp} f$. We write:

$$\begin{aligned} g(H(h)) &= H_{g,N} + h^{N+1} D_{g,N+1}(h), \\ \text{where } H_{g,N} &= \sum_{j=0}^{N} h^j Op_h^w(a_{g,j}). \end{aligned}$$

We can then calculate:

$$\begin{aligned} B(h)f(H(h)) &= B(h)f(H(h))g(H(h)) \\ &= B(h)f(H(h))H_{g,N} + B(h)f(H(h))h^{N+1}D_{g,N+1}(h). \end{aligned}$$

Thus

$$B(h)f(H(h))[I - h^{N+1}D_{g,N+1}(h)] = B(h)f(H(h))H_{g,N}(h). \tag{3.2}$$

Here [...] is invertible for h small enough. So we get:

$$\begin{aligned} \|B(h)f(H(h))\|_{tr} \leq\ & \|B(h)f(H(h))\|_{\mathcal{L}(L^2)} \\ & \|I - h^{N+1}D_{g,N+1}(h)\|_{\mathcal{L}(L^2)}^{-1} \|H_{g,N}\|_{tr}. \end{aligned}$$

The first factor on the right is estimated by Lemma 2.2 and gives a constant. The second factor converges to 1, and the third factor is estimated by [Rob87, Thm. II-49] and gives an $O(h^{-n})$-term. ◻

Lemma 3.2. *We take the expansion of $f(H(h))$ as an h-admissible operator, as well, and we get:*

$$\|B(h)f(H(h)) - B(h)H_{f,N}H_{g,N}\|_{tr} = O(h^{N+1-n}).$$

Proof. Since $f(H(h))g(H(h)) = f(H(h))$, we get

$$\begin{aligned} \|\dots\|_{tr} \leq h^{N+1}\Big[&\|B(h)f(H(h))\|_{tr}\|D_{g,N+1}\|_{\mathcal{L}(L^2)} \\ &+ \|B(h)D_{f,N+1}(h)\|_{\mathcal{L}(L^2)}\|H_{g,N}\|_{tr}\Big]. \end{aligned}$$

Here the trace norms on the right are known to be $O(h^{-n})$. The $\mathcal{L}(L^2)$-norms are known to be bounded in h, see Lemma 2.3. ◻

Lemma 3.3.

$$\|B(h)H_{f,N}H_{g,N} - B(h)H_{f,N}\|_{tr} = O(h^{N+1-n}). \tag{3.3}$$

Proof. We have:

$$f(H(h))g(H(h)) = f(H(h)) = H_{f,N} + h^{N+1}D_{f,N+1}.$$

But on the other hand we have as well:

$$\begin{aligned} f(H(h))g(H(h)) &= [H_{f,N} + h^{N+1}D_{f,N+1}][H_{g,N} + h^{N+1}D_{f,N+1}] \\ &= H_{f,N}H_{g,N} + h^{N+1}[\text{bounded terms}], \end{aligned}$$

so $H_{f,N} = H_{f,N}H_{g,N}$ modulo $O(h^{N+1})$.
NOTE: Here we have used that the symbols of $H_{f,N}$ and $H_{g,N}$ have compact supports and thus give rise to uniformly (in h) bounded operators on L^2 . We can therefore write:

$$B(h)[H_{f,N}H_{g,N} - H_{f,N}] = h^{N+1}B(h)Op_h^w(\delta(h)).$$

Where $\delta(h)$ is an h-admissible function in the class denoted by $\Sigma_1^{(1+x^2+p^2)^{1/2}}$ in [Rob87] . This comes from using the composition theorem for strongly admissible operators on $H_{f,N}H_{g,N}$ and then noticing that the first N terms must be killed by the terms in $H_{f,N}$. The composition $B(h)Op_h^w(\delta(h))$ is thus a composition of strongly admissible operators with symbols in $\Sigma_1^{(1+x^2+p^2)^{1/2}}$. We find:

$$\|B(h)Op_h^w(\delta(h))\|_{tr} = O(h^{-n}), \tag{3.4}$$

by [Rob87, Thm.II-49] and the composition theorem. □

We now have:

Theorem 3.4. *Let the h-admissible operator $B(h)f(H(h))$ have the (formal) symbol*
$\sum_{j=0}^{\infty} h^j c_j$*. Then we have the following asymptotics:*

$$(2\pi h)^n \mathrm{tr}[B(h)f(H(h))] = \sum_{j=0}^{\infty} h^j \iint_{\mathbf{R}^{2n}} c_j(x,p)\, dx\, dp.$$

Proof. The proof is easy given the observations, because we have:

$$\|B(h)f(H(h)) - B(h)H_{f,N}(h)\|_{tr} = O(h^{N+1-n}), \tag{3.5}$$

and the trace of $B(h)H_{f,N}(h)$ is easy to calculate because both operators in the composition are strongly h-admissible. $\square$

Remark 3.5. We can easily calculate the first terms of the symbol $\sum h^j c_j$:

$$\begin{aligned} \sigma[B(h)] &= b_0 + hb_1, \\ \sigma[f(H(h))] &= f(a_0) - h^2\frac{1}{24}f''(a_0)\sum_{j,k}\left(\frac{\partial^2 a_0}{\partial p_j \partial p_k}\frac{\partial^2 a_0}{\partial x_j \partial x_k} - \frac{\partial^2 a_0}{\partial x_j \partial p_k}\frac{\partial^2 a_0}{\partial p_j \partial x_k}\right) \\ &+ O(h^3). \end{aligned}$$

Thus

$$\begin{aligned} c_0 &= b_0 f(a_0), && (3.6) \\ c_1 &= b_1 f(a_0) + \frac{1}{2i}\{b_0, f(a_0)\}_P. && (3.7) \end{aligned}$$

4 Current for positive temperature

We define

$$f(x) = \frac{e^{-\beta(x-\alpha)}}{1+e^{-\beta(x-\alpha)}},$$

where β is to be understood as an inverse temperature, and α is a constant[2].

To assure that the expression $tr[B(h)f(H(h))]$ makes sense, we make the following assumption:

$$\exists\, c, s > 0 \text{ so that } V(x) \geq c|x|^s. \tag{4.1}$$

Under this assumption the spectrum of $H(h)$ is purely discrete. We write $\lambda_k(h)$ (resp. $\phi_k(h)$) for the sequence of eigenvalues (resp. eigenvectors).

Observation 4.1. Because we only need f on $\mathrm{Spec}(H(h))$ for the abstract definition of $f(H(h))$, we may assume that $\mathrm{supp}\, f \subseteq (0,\infty)$, $f \in \mathcal{S}$ and $\exists u \in \mathcal{S}$ so that $f = u^2$.

[2] α is included because we have translated our operator to have $\inf V(x) > 1$.

Then we want to prove Theorem 1.2.

Lemma 4.2. *There exists $N \in \mathbf{N}$ only depending on s in (4.1) so that*

$$\|B(h)f(H(h))\|_{tr} = c\sup_{\lambda}\{(|u(\lambda)| + |u'(\lambda)|)(1+\lambda^2)^N\}h^{-n}.$$

Proof. We write

$$B(h)f(H(h)) = [B(h)u(H(h))]u(H(h)),$$

and we will prove that the two operators are Hilbert-Schmidt.We have,

$$\begin{aligned}
\|B(h)u(H(h))\|_{HS}^2 &= \sum_{k=1}^{\infty}\|B(h)\phi_k(h)\|^2 f(\lambda_k(h)) \\
&\leq \sum_{k=1}^{\infty} c(\sum_{l}\|P_{A,l}\phi_k(h)\|)^2 f(\lambda_k(h)) \\
&\leq \sum_{k=1}^{\infty} c(1 + \langle P_A^2\phi_k(h), \phi_k(h)\rangle) f(\lambda_k(h)) \\
&\leq \sum_{k=1}^{\infty} c(1+\lambda_k(h)) f(\lambda_k(h)).
\end{aligned}$$

So we only have to prove, that if $\tilde{u}$ is Schwartz, then $\sum_{k=0}^{\infty}\tilde{u}(\lambda_k(h))$ converges and is $O(h^{-n})$. But this follows easily from the CLR-estimate([Sim79]): $\exists c > 0$ so that:

$$N_h(\lambda) \leq ch^{-n}\iint_{\{a_0(x,p)\leq\lambda\}} dxdp,$$

where $N_h(\lambda)$ is the number of eigenvalues of $H(h)$ smaller than λ. From this (remember that $V(x) \geq c|x|^s$) we get

$$N_h(\lambda) \leq ch^{-n}\lambda^{n\frac{2+s}{2s}}.$$

This implies that $N_h(\cdot)$ defines a tempered distribution. We have

$$N_h'(\lambda) = \sum_k \delta(\lambda - \lambda_k(h)),$$

which implies:

$$\begin{aligned}\sum_{k=0}^{\infty} \tilde{u}(\lambda_k(h)) &= \int N_h'(\lambda)\tilde{u}(\lambda)\, d\lambda \\ &= -\int \tilde{u}'(\lambda) N_h(\lambda)\, d\lambda.\end{aligned}$$

This finishes the proof. ◻

Let us take a partition of unity and multiply it by f to get a family of functions $\{f_j^2\}_{j=0}^{\infty}$ so that $f_j^2 \equiv f$ on $[j+1/4, j+3/4]$, $\operatorname{supp} f_j \subseteq [j-1/4, j+5/4]$. Then it is clear from the above lemma that $\sum B(h) f_j^2(H(h))$ converges to $B(h)f(H(h))$ in trace norm. The idea is now to try to take the limit $h \searrow 0$ inside the sum. To be able to do this we need the following lemma:

Lemma 4.3. *$B(h)f_j^2(H(h))$ is h-admissible, so we write*

$$B(h) f_j^2(H(h)) = \sum_{k=0}^{N} h^k Op_h^w(c_{j,k}) + h^{N+1} R_{j,N+1}(h),$$

where $\{R_{j,N+1}(h) | h \leq h_0\}$ is bounded in $\mathcal{L}(L^2)$, then

$$\|R_{j,N+1}(h)\|_{tr} \leq c h^{-n} p(f_j^2) P(j)$$

where $P(j)$ is a polynomial in j and where $p(f)$ is a seminorm of f in the Schwartz space.

Proof. We take $g \in C_0^{\infty}$, $g \equiv 1$ on $[-1/4, 5/4]$. Let $g_j(t) = g(t-j)$, then $g_j f_j^2 = f_j^2$. We write

$$\begin{aligned} h^{N+1}\|R_{j,N+1}(h)\|_{tr} \leq\; & \|B(h)f_j^2(H(h)) - B(h)H_{f_j,N}H_{g_j,N}\|_{tr} \\ & + \|B(h)H_{f_j,N}H_{g_j,N} - B(h)H_{f_j,N}\|_{tr} \\ & + \|B(h)H_{f_j,N} - C_{j,N}(h)\|_{tr}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} f_j^2(H(h)) &= H_{f_j,N} + h^{N+1} D_{f_j,N+1} \\ g_j(H(h)) &= H_{g_j,N} + h^{N+1} D_{g_j,N+1} \\ C_{j,N} &= \sum_{k=0}^{N} h^k Op_h^w(c_{j,k}). \end{aligned}$$

We try to estimate each of the three terms on the right hand side in (4.2)
The first:

$$\begin{aligned} & \|B(h)f_j^2(H(h)) - B(h)H_{f_j,N}H_{g_j,N}\|_{tr} \\ \leq\ & h^{N+1}\|B(h)f_j^2(H(h)D_{g_j,N+1}(h)\|_{tr} \\ & +h^{N+1}\|B(h)D_{f_j,N+1}(h)H_{g_j,N}(h)\|_{tr} \\ \leq\ & h^{N+1}\|B(h)f_j^2(H(h))\|_{\mathcal{L}}\|D_{g_j,N+1}(h)\|_{\mathcal{L}}\|1_{\mathrm{supp}\, f_j}(H(h))\|_{tr} \\ & +h^{N+1}\|B(h)D_{f_j,N+1}(h)\|_{\mathcal{L}}\|H_{g_j,N}(h)\|_{tr}. \end{aligned}$$

We now remember the following inequalities:

$$\begin{aligned} \|B(h)f_j^2(H(h))\|_{\mathcal{L}} &\leq c\|f_j^2\|_\infty(j+1) && \text{from the proof of Lemma 2.2} \\ \|D_{g_j,N+1}(h)\|_{\mathcal{L}} &\leq c.p(g_j) && \text{from [Rob87, Eq.(51) p.144] .} \\ \|1_{\mathrm{supp}\, f_j}(H(h))\|_{tr} &\leq N_h(j+1) && \\ &\leq ch^{-n}(j+1)^{n\frac{2+s}{2s}} && \\ \|B(h)D_{f_j,N+1}(h)\|_{\mathcal{L}} &\leq c.p(f_j^2) && \text{from Lemma 2.3} \\ \|H_{g_j,N}(h)\|_{tr} &\leq c.p(g_j) && \text{from [Rob87, II.49]} \end{aligned}$$

Now we only have left to notice that $p(g_j) \leq (1+j^2)^M p(g)$.
The second:
Here we just use the composition theorem for operators with symbols in the class $\Sigma_1^{(1+x^2+p^2)^{1/2}}$ and notice that

$$H_{f_j,N}H_{g_j,N} - H_{f_j,N} = h^{N+1}Op_h^w(\delta(h)).$$

Since

$$\|B(h)Op_h^w(\delta(h))\|_{tr} \leq c.p(f_j^2)$$

by the composition theorem and [Rob87, II-49] we get the desired estimate for this term. The third:

$$\|B(h)H_{f_j,N} - C_{j,N}(h)\|_{tr} = h^{N+1}\|Op_h^w(\delta(h)\|_{tr},$$

where $B(h)H_{f_j,N} = \sum h^k Op_h^w(c_{j,k}) + h^{N+1}Op_h^w(\delta(h))$. By the composition theorem for strongly admissible operators in $\Sigma_1^{(1+x^2+p^2)^{1/2}}$ and [Rob87, II-49] we get:

$$\|Op_h^w(\delta(h))\|_{tr} \leq c.p(f_j^2).$$

$\square$

Now we can prove the theorem of this section (Theorem 1.2):

Proof.

$$\lim_{h\searrow 0} h^{-(N+1)}[(2\pi h)^n \mathrm{tr}[B(h)f(H(h))] - \sum_{k=0}^{N} h^k \iint c_k(x,\xi)\, dx d\xi] =$$
$$\lim_{h\searrow 0} \sum_{j=0}^{\infty} (2\pi h)^n h^{-(N+1)} \mathrm{tr}[B(h)f_j(H(h)) - C_{j,N}]$$

and by the lemma

$$\sum_{j=0}^{\infty} (2\pi h)^n h^{-(N+1)} \|B(h)f_j(H(h)) - C_{j,N}\|_{tr}$$

converges, so we get:

$$\begin{aligned}
\lim_{h\searrow 0} h^{-(N+1)}[(2\pi h)^n \mathrm{tr}[B(h)f(H(h))] - \sum_{k=0}^{N} h^k \iint c_k(x,\xi)\, dx d\xi] \\
&= \sum_{j=0}^{\infty} \lim_{h\searrow 0} (2\pi h)^n h^{-(N+1)} \mathrm{tr}[B(h)f_j(H(h)) - C_{j,N}] \\
&= \sum_{j=0}^{\infty} \iint c_{j,N+1}(x,\xi)\ dx d\xi \\
&= \iint c_{N+1}(x,\xi)\ dx d\xi.
\end{aligned}$$

In the first and the last line we used that $\sum c_{j,N+1}$ converges to c_{N+1} in L^1. This is because we have (by the functional calculus and the composition theorem) that:

$$\begin{aligned}
c_{j,N} &= \sum_k Q_k f_j^{(k)}(a_0), \\
c_N &= \sum_k Q_k f^{(k)}(a_0),
\end{aligned}$$

where Q_k is a polynomial in b_0, b_1, a_0 and their derivatives (b_j are the principal and sub-principal symbols of $B(h)$). Now

$$(p_j - A_j) = \frac{1}{2}\partial_{p_j} a_0,$$

and we have the estimate:

$$|\partial^\alpha a_0| \leq c_\alpha |a_0|,$$

so we can dominate Q_k by a polynomial in a_0. Then the result follows by dominated convergence. $\square$

5 Current for $T = 0$

We now fix a $\lambda < \lambda_0$ We suppose that λ is not a critical value for the principal symbol a_0 of $H(h)$. We will now prove Theorem 1.4.

Let $\phi_j(h)$ and $\lambda_j(h)$ be the eigenvectors and eigenvalues of $H(h)$ respectively with $\lambda_j \leq \lambda_0$. There will be a finite number of these for each h.

Lemma 5.1.

$$\|B(h)1_{(-\infty,\lambda]}(H(h))\| \leq \lambda_0.$$

Proof. The proof is exactly as for Lemma 2.2. $\square$

Lemma 5.2.

$$\|B(h)1_{(-\infty,\lambda]}(H(h))\|_{tr} = O(h^{-n}). \tag{5.1}$$

Proof.

$$\begin{aligned}
\|B(h)1_{(-\infty,\lambda]}(H(h))\|_{tr} &\leq \lambda_0 \|1_{(-\infty,\lambda]}(H(h))\|_{tr} \\
&= \lambda_0 N_h(\lambda),
\end{aligned}$$

and the lemma follows from the CLR estimate. $\square$

We will now take a partition of unity such that:

- $f_1^2 + f_2^2 = 1$ on $[0, \lambda\ ]$, $f_1, f_2 \in C_0^\infty(\mathbf{R})$.
- supp $f_2 \subset [\lambda - \epsilon/2, \lambda + \epsilon/2]$
  where $\lambda + \epsilon < \lambda_0$ and $a_0^{-1}(\lambda - \epsilon, \lambda + \epsilon)$ is noncritical for a_0.
- supp $f_1 \subset (-\infty, \lambda - \epsilon/4]$.

We can then split our problem in the analysis of two terms, by:

$$\begin{aligned}\mathrm{tr}[B(h)1_{(-\infty,\lambda]}(H(h))] &= \mathrm{tr}[B(h)f_1^2(H(h))] \\ &+ \sum_{\lambda_j(h)\le\lambda} \langle\phi_j(h), B(h)\phi_j(h)\rangle f_2^2(\lambda_j(h)). \quad (5.2)\end{aligned}$$

The first term on the right is nicely taken care of by Theorem 3.4. It thus remains to handle the second term which I will denote $M(\lambda, h)$. It is obvious that $M(\cdot, h)$ extends to a function on $\mathbf{R}$ constant "below" and "above" the support of f_2. We will now smoothen out the function $M(\lambda, h)$ by convolution with W_h (see below) and then use a Tauberian theorem to compare $M(\lambda, h)$ with $[M(\cdot, h) * W_h](\lambda)$ (see Theorem 5.5).

Definition 5.3. Let $\theta \in C_0^\infty(\mathbf{R})$, supp $\theta \subset (-T', T')$, where T' is sufficiently small[3], $\theta \geq 0$ and $\theta(0) = 1$. We assume: θ even and $\hat{\theta}(0) > 0$. We now define $W_h(\tau) = \frac{1}{2\pi h}\hat{\theta}(\frac{-\tau}{h})$.

We calculate:

$$\begin{aligned}\frac{d}{d\tau}[M(\cdot, h) * W_h](\tau) &= \sum \langle\phi_j(h), B(h)\phi_j(h)\rangle f_2^2(\lambda_j(h))[\delta_{\lambda_j(h)} * W_h](\tau) \\ &= \frac{1}{2\pi h}\mathrm{tr}\Big[B(h)f_2^2(H(h)) \int e^{ih^{-1}t\tau}\theta(t)U_h(t)\, dt\Big], \end{aligned} \quad (5.3)$$

where:

$$U_h(t) = exp[-ih^{-1}tH(h)]. \quad (5.4)$$

We get:

[3]The hypothesis on T' are the following:

- $T' < T$ where T is the time interval in which the Hamilton- Jacobi equation (5.9) has a unique solution.
- Further, if $(x, p) \in a_0^{-1}(suppf_2)$, $|t| < T'$ and $\Phi_{a_0}^t(x, p) = (x, p)$ (where Φ_{a_0} is the Hamiltonian flow associated to a_0) then $t = 0$.

It is standard to show that such a T' exists.

Proposition 5.4. *The leading term in the asymptotics of*

$$\mathrm{tr}[B(h)f_2^2(H(h))\int e^{ih^{-1}t\tau}\theta(t)U_h(t)\ dt] = J(\tau,h) + O(h^{2-n}), \tag{5.5}$$

for $\tau \in [\lambda - \epsilon/4, \lambda + \epsilon/4]$, *is:*

$$J(\tau,h) = (2\pi h)^{1-n} f_2^2(\tau) \int_{\{a_0=\tau\}} \vec{a}\cdot(\vec{q}-\vec{A})\,\frac{dS_\tau}{|\nabla a_0|} + O(h^{2-n}). \tag{5.6}$$

Proof. The proof is standard and will only be sketched (See for example [Rob87])
First we localize the analysis. Let $\chi \in C_0^\infty(\lambda-\epsilon, \lambda+\epsilon)$ with $\chi \equiv 1$ on $\mathrm{supp} f_2$. Then we have:

$$f_2^2(H(h))U_h(t) = f_2^2(H(h))\tilde{U}_h(t), \tag{5.7}$$

where

$$\tilde{U}_h(t) = exp[-ih^{-1}tH(h)\chi(H(h))]. \tag{5.8}$$

Let $\sum h^j q_j$ be the Weyl-symbol of $H(h)\chi(H(h))$. Then there is a compact subset K $(K \subseteq a_0^{-1}(\mathrm{supp}\,\chi))$ of $\mathbf{R}^{2n}$ so that $\mathrm{supp}\,q_j \subseteq K\ \forall j \in \mathbf{N}$. This is why we prefer to work with $\tilde{U}_h(t)$.
Then we use an approximation of $\tilde{U}_h(t)$ to order h^M as a Fourier Integral Operator (this construction is for example carried through in [Rob87, Chapt.IV,sect.6]). Let S be the solution of the Hamilton-Jacobi equation:

$$\begin{aligned} \partial_t S(t,x,q) + q_0(x, \partial_x S(t,x,q)) &= 0 \\ S(0,x,q) &= x\cdot q. \end{aligned} \tag{5.9}$$

$S \in C^\infty((-T,T)\times\mathbf{R}^n\times\mathbf{R}^n)$, where T is chosen small enough for the system to have a unique solution $\forall (x,q) \in \mathbf{R}^n \times \mathbf{R}^n$. The desired approximation of $\tilde{U}_h(t)$ is

$$\tilde{U}_h^{(M)}(t) = J_h(\sum_{j=0}^{M} h^j u_j, S), \tag{5.10}$$

where

$$[J_h(u,S)\psi](x) = \iint e^{ih^{-1}[S(t,x,q)-yq]}u(t,x,q)\psi(y)\,dy\,d_hq. \tag{5.11}$$

We will not pass through the details of this construction, but note the following:

- $\sup_{|t|<T} \|\tilde{U}_h^{(M)}(t) - \tilde{U}_h(t)\|_{\mathcal{L}(L^2)} = O(h^M)$.
- $\exists\, K_2 \subset \mathbf{R}^{2n}$ compact such that $(x,q) \notin K_2$
$\Rightarrow u_j(t,x,q) = \begin{cases} 1 & \text{if } j=0 \\ 0 & \text{if } j>0 \end{cases}$.
- $u_j(0,x,q) = \begin{cases} 1 & \text{if } j=0 \\ 0 & \text{if } j>0 \end{cases}$

Composition of $B(h) f_2^2(H(h))$ with $\tilde{U}_h^{(M)}(t)$ gives us to leading order $J_h(d_0, S)$, where

$$d_0(x,q) = c_0(x, \partial_x S(t,x,q)) u_0(t,x,q).$$

Here $c_0(x,q) = \vec{a}(q - \vec{A}(x)) f_2^2(a_0(x,q))$. Now we evaluate the trace by integrating the kernel over the diagonal. This gives us

$$J(\tau,h) = \iiint e^{ih^{-1}[t\tau + S(t,x,q) - xq]} \theta(t) c_0(x, \partial_x S(t,x,q)) u_0(t,x,q)\ dx\, d_h q\, dt, \tag{5.12}$$

which, by the stationary phase theorem, is

$$J(\tau,h) = (2\pi h)^{1-n} \theta(0) \int_{\{q_0=\tau\}} c_0(x,q) u_0(0,x,q) \frac{dS_\tau}{|\nabla q_0|} + O(h^{2-n}). \tag{5.13}$$

Now, using that

$$\begin{aligned} \theta(0) &= 1, \\ u_0(0,x,q) &= 1, \\ c_0(x,q) &= \vec{a} \cdot (\vec{q} - \vec{A}) f_2^2(a_0(x,q)), \\ q_0 &= a_0 \chi(a_0), \end{aligned} \tag{5.14}$$

we get

$$J(\tau,h) = (2\pi h)^{1-n} f_2^2(\tau) \int_{\{a_0=\tau\}} \vec{a} \cdot (\vec{q} - \vec{A})\, \frac{dS_\tau}{|\nabla a_0|} + O(h^{2-n}). \tag{5.15}$$

$\square$

We thus have

$$\frac{d}{d\tau}[M(\cdot,h) * W_h](\tau) = O(h^{-n}), \tag{5.16}$$

for $\tau \in [\lambda - \epsilon/2, \lambda + \epsilon/2]$ (we remind the reader of the equation (5.3)). On the other hand it follows from the properties,
$M(\tau,h) = O(h^{-n})$, $\hat{\theta} \in \mathcal{S}(\mathbf{R})$, $\operatorname{supp} M(\cdot,h) \subset [\lambda - \epsilon/2, \lambda + \epsilon/2]$
and the definition of W_h, that $[M(\cdot,h) * W_h](\tau)$ as well as its derivatives are $O(h^\infty)$ for $|\tau - \lambda| > \epsilon$, e.g. if $|\tau - \lambda| \geq \epsilon$, we get:

$$\begin{aligned} |[M(\cdot,h) * W_h](\tau)| &\leq c\|M(\cdot,h)\|_{L^1} \sup_{|\nu|>\epsilon/2} \{h^{-1}|\hat{\theta}(\frac{\nu}{h})|\} \\ &\leq ch^{-(n+1)}h^{2N} \cdot \frac{1}{(h^2+\epsilon^2/4)^N}. \end{aligned}$$

Therefore we can integrate and obtain:

$$\begin{aligned} [M(\cdot,h) * W_h](\tau) &= \int_{\lambda-\epsilon}^{\tau} \frac{d}{d\tau'}[M(\cdot,h) * W_h](\tau')\, d\tau' + O(h^\infty) \\ &= (2\pi h)^{-n} \int_{\lambda-\epsilon}^{\tau} \int_{\{a_0=\tau'\}} f_2^2(\tau')\vec{a}\cdot(\vec{q}-\vec{A})\frac{dS_{\tau'}}{|\nabla a_0|} d\tau' + O(h^{1-n}) \\ &= (2\pi h)^{-n} \iint_{\{a_0(x,q)\leq\tau\}} \vec{a}\cdot(\vec{q}-\vec{A}) f_2^2(a_0(x,q))\; dxdq + O(h^{1-n}). \end{aligned} \tag{5.17}$$

Now we know a lot about the asymptotics of $[M(\cdot,h) * W_h](\lambda)$. The next result relates this to the asymptotics of $M(\lambda,h)$.

Theorem 5.5.

$$|M(\lambda,h) - [M(\cdot,h) * W_h](\lambda)| = O(h^{1-n}).$$

Proof. Remember that we have chosen $\lambda < \lambda_0$. We have:

$$
\begin{aligned}
& |M(\lambda,h) - [M(\cdot,h) * W_h](\lambda)| \\
\leq\ & \frac{1}{2\pi h}\int_{\{|\mu|<\lambda_0-\lambda\}} |M(\lambda,h) - M(\lambda-\mu,h))|\hat{\theta}(\frac{-\mu}{h})\, d\mu \\
& +\frac{1}{2\pi h}\int_{\{|\mu|\geq\lambda_0-\lambda\}} |M(\lambda,h) - M(\lambda-\mu,h))|\hat{\theta}(\frac{-\mu}{h})\, d\mu \\
\leq\ & \frac{1}{2\pi}\int_{\{|\mu|<\frac{\lambda_0-\lambda}{h}\}} \sum_{\lambda<\lambda_j(h)\leq\lambda+h\mu} |\langle\phi_j(h), B(h)\phi_j(h)\rangle| f_2^2(\lambda_j(h))|\hat{\theta}(\mu)|\, d\mu \\
& +ch^{-1-n}\int_{\{|\mu|\geq\lambda_0-\lambda\}} \hat{\theta}(\frac{-\mu}{h})\, d\mu \\
\leq\ & \|B(h)1_{(-\infty,\lambda_0]}(H(h))\|_{\mathcal{L}(L^2)} \int_{\mathbf{R}} |\sigma_h(\lambda+h\mu) - \sigma_h(\lambda)||\hat{\theta}(\mu)|\, d\mu \\
& +O(h^\infty),
\end{aligned}
$$

where $\sigma_h(\tau) = \sum_{\lambda_j(h)\leq\tau} f_2^2(\lambda_j(h))$.
Now $\sigma_h(\tau)$ satisfies the hypothesis of [Rob87, Thm.V-13] (in fact it is the family of functions he wants to study there), so we conclude from his Lemma V-14:

$$\exists\Gamma > 0 : |\sigma_h(\lambda+h\mu) - \sigma_h(\lambda)| \leq \Gamma(1+|\mu|)h^{1-n}.$$

Inserting this inequality in the equation above and using the fact that $\hat{\theta} \in \mathcal{S}(\mathbf{R})$, we get the Theorem 5.5. □

Now we get Theorem 1.4 from Theorem 5.5 and equation (5.17) combined with Theorem 3.4.

6 The case of no periodic orbits

It is known from various sources that the classical hamiltonian flow has an influence on the spectrum of the quantized operator. On compact manifolds results in this direction have been proved by Duistermaat and Guillemin [DG75] and Colin de Verdiere [dV79].
We use the following assumption where we write $\Sigma_\tau = \{(x,p) \in \mathbf{R}^{2n} : a_0(x,p) = \tau\}$ for $\tau \in \mathbf{R}$.

Assumption 6.1. Let $\lambda < \lambda_0$ be a non-critical value for a_0. Suppose:

$$S_\lambda(\{(x,p) \in \Sigma_\lambda | \exists t \neq 0 \, s.t. \, \Phi^t_{a_0}(x,p) = (x,p)\}) = 0, \tag{6.1}$$

where S_λ is the surface measure on Σ_λ and Φ_{a_0} is the Hamiltonian flow associated with a_0.

Petkov and Robert [PR85] used this assumption to obtain good error estimates in the semi-classical limit on the number of eigenvalues in an interval. This assumption is also used in some of V. Ivrii's work [Ivred].
Under the above additional assumption we want to prove:

$$(2\pi h)^n \mathrm{tr}[B(h) 1_{(-\infty,\lambda]}(H(h))] = h\gamma_1 + o(h), \tag{6.2}$$

where γ_1 will be calculated explicitly as:

$$\gamma_1(\lambda) = \iint_{\{a_0(x,p) \leq \lambda\}} b_1 \; dxdp - \frac{1}{2i} \int_{\Sigma_\lambda} \{a_0, b_0\}_P \frac{dS_\lambda}{|\nabla a_0|}.$$

Here $\{a_0, b_0\}_P = \partial_p a_0(x,p) \partial_x b_0(x,p) - \partial_x a_0(x,p) \partial_p b_0(x,p)$, is the Poisson bracket.This is the statement of Theorem 1.5.

We take the partition of unity f_1^2 and f_2^2 as in section 5, and write:

$$\mathrm{tr}[B(h) 1_{(-\infty,\lambda]}(H(h))] = \mathrm{tr}[B(h) f_1^2(H(h))] + M(\lambda, h), \tag{6.3}$$

where

$$M(\lambda, h) = \sum_{\lambda_j(h) \leq \lambda} \langle \phi_j(h), B(h) \phi_j(h) \rangle f_2^2(\lambda_j(h)). \tag{6.4}$$

The term $\mathrm{tr}[B(h) f_1^2(H(h))$ has a complete asymptotics given by Theorem 3.4, so we are left with the analysis of $M(\lambda, h)$. This will be the subject of the next two subsections. In the first we prove that $M(\lambda, h)$ and it's smoothed out version $[M(\cdot, h) * W_h](\lambda)$ (where W_h was defined in Definiton 5.3) are equal up to an error of size $o(h^{1-n})$. Since $[M(\cdot, h) * W_h](\tau)$ is a continuous function in τ it is uniquely determined as a function once it is known as a distribution. This will be used in the last subsection to calculate γ_1.

6.1 The technical part

In this subsection we shall prove the following improvement of Theorem 5.5:

Theorem 6.2. *Under the assumptions 1.1 and 6.1 we have*

$$|M(\lambda,h) - [M(\cdot,h) * W_h](\lambda)| = o(h^{1-n}), \tag{6.5}$$

where W_h was defined in Definition 5.3.

We define for $L > 1$:

$$K_L = \{(x,p) \in a_0^{-1}([\lambda-\epsilon_0, \lambda+\epsilon_0]) : \exists t \neq 0, |t| \leq LT'; \Phi^t_{a_0}(x,p) = (x,p)\}. \tag{6.6}$$

Here T' comes from the definition of W_h. We will now make a partition of unity in a neighborhood of the energy surface Σ_λ such that one part contains the closed orbits (i.e. K_L) but is of very small measure.

Observation 6.3. We may assume ϵ_0 so small that K_L is compact.

We take Ω_L to be an open neighborhood of K_L. One should think of Ω_L as being very "thin", i.e. of very small measure. Then we take functions $\omega_1, \omega_2 \in C_0^\infty(\mathbf{R}^n)$ with

- $\omega_1^2 + \omega_2^2 = 1$ on $a_0^{-1}([\lambda-\epsilon/2, \lambda+\epsilon/2])$,
- $\text{supp}(\omega_1^2+\omega_2^2) \subset a_0^{-1}(\lambda-\epsilon, \lambda+\epsilon)$,
- $\omega_1 \equiv 1$ on K_L and $\text{supp}(\omega_1) \subset \Omega_L$.

Now, for k=1,2, we set

$$\begin{aligned} M_k(\tau,h) &= \sum_{\lambda_j(h)\leq\tau} \langle \phi_j(h), Op_h^w(\omega_k)B(h)Op_h^w(\omega_k)f_2^2(H(h))\phi_j(h)\rangle \\ &= \sum_{\lambda_j(h)\leq\tau} \langle Op_h^w(\omega_k)\phi_j(h), B(h)Op_h^w(\omega_k)\phi_j(h)\rangle f_2^2(\lambda_j(h)). \end{aligned}$$

Lemma 6.4.

$$|M(\lambda,h) - M_1(\lambda,h) - M_2(\lambda,h)| = O(h^{2-n}).$$

Proof. Let $v \in C_0^\infty(\mathbf{R})$ $v \equiv 1$ on the a_0-image of the supports of the ω_j. Then $vf_2^2 = f_2^2$ and we get:

$$\begin{aligned}
&M(\lambda,h) - M_1(\lambda,h) - M_2(\lambda,h) \\
&= \sum_{\lambda_j(h)\le\tau} \langle \phi_j(h), B(h)[v(H(h)) - Op_h^w(\omega_1)^2 - Op_h^w(\omega_2)^2]\phi_j(h)\rangle f_2^2(\lambda_j(h)) \\
&- \sum_{\lambda_j(h)\le\tau} \langle \phi_j(h), ([Op_h^w(\omega_1), B(h)]Op_h^w(\omega_1) \\
&\qquad + [Op_h^w(\omega_2), B(h)]Op_h^w(\omega_2))\phi_j(h)\rangle f_2^2(\lambda_j(h)),
\end{aligned} \tag{6.7}$$

where $[,]$ is the commutator bracket. The sums have $O(h^{-n})$ terms, so it is enough to estimate inside the sums.

The first term in (6.7):

By the composition rules for (strongly) h-admissible operators we get that

$$Op_h^w(\omega_k)^2 = Op_h^w(\omega_k^2) + h^2 R_k(h),$$

where $\{R_k(h) | h \in]0, h_0]\}$ is a uniformly bounded family of operators on $L^2(\mathbf{R}^n)$. The composition $B(h)R_k(h)$ is bounded by composition rules for strongly admissible operators so we do not have to worry about the remainder term. We then look at $v(H(h))$. By the functional calculus we have

$$v(H(h)) = Op_h^w(v(a_0)) + O(h^2),$$

because $H(h)$ has no sub-principal symbol. Here, as well, the composition of $B(h)$ with the remainder term can be neglected, so the first term in (6.7) is:

$$\sum_{\lambda_j(h)\le\tau} \langle \phi_j(h), B(h)[Op_h^w(v(a_0) - \omega_1^2 - \omega_2^2)] f_2^2(H(h))\phi_j(h)\rangle + O(h^{2-n}).$$

Now $B(h)[Op_h^w(v(a_0) - \omega_1^2 - \omega_2^2)]$ is a composition of strongly h-admissible operators and as such a strongly h-admissible operator with symbol given by [Rob87, Thm.II-32]. We can compose the resulting operator with $f_2^2(H(h))$ (because both operators are h-admissible and their symbols have compact supports) and the result is an h-admissible operator with symbol $\equiv 0$ because the supports of the symbols are disjoint. Therefore the first term in (6.7) is $O(h^{2-n})$.

The second term in (6.7):

¿From the symbolic calculus for strongly h-admissible operators, we get that this term is:

$$\sum_{\lambda_j(h)\leq\tau} \langle\phi_j(h), Op_h^w(g) f_2^2(H(h))\phi_j(h)\rangle + O(h^{2-n}), \tag{6.8}$$

with

$$\begin{aligned} g &= \frac{1}{i}\{\omega_1, b_0\}_P\omega_1 + \frac{1}{i}\{\omega_2, b_0\}_P\omega_2 \\ &= \frac{1}{2i}\{\omega_1^2+\omega_2^2, b_0\}_P, \end{aligned}$$

where $\{,\}_P$ is the Poisson bracket from classical mechanics.
So g has support disjoint from the support of the symbol of $f_2^2(H(h))$ and thus the first term in (6.8) is $O(h^\infty)$. $\square$

To know the first terms of $M(\lambda,h)$ it thus suffices to know the first terms of the $M_k(\lambda,h)$. This is what we will calculate next. We use the function W_h, as above, but with a parameter:

Definition 6.5. Let $\theta\in C_0^\infty(\mathbf{R})$, supp $\theta\subset(-T',T')$ be the function used in Definition 5.3. For $l\geq 1$ we define $W_{h,l}(\tau)=\frac{l}{2\pi h}\hat{\theta}(\frac{-l\tau}{h})$.

We will now analyze $[M_k(\cdot,h)*W_{h,l}](\tau)$. This analysis and a Tauberian result will lead us to the proof of Theorem 6.2.
As in section 5 we calculate:

$$\begin{aligned} &\frac{d}{d\tau}[M_k(\cdot,h)*W_{h,l}](\tau) \\ =& \sum_j f_2^2(\lambda_j(h))\langle\phi_j(h), Op_h^w(\omega_k)B(h)Op_h^w(\omega_k)\phi_j(h)\rangle[\delta_{\lambda_j(h)}*W_{h,l}](\tau) \\ =& \sum_j f_2^2(\lambda_j(h))\langle\phi_j(h), Op_h^w(\omega_k)B(h)Op_h^w(\omega_k)\phi_j(h)\rangle W_{h,l}(\tau-\lambda_j(h)) \\ =& \int_{\mathbf{R}}\sum_j f_2^2(\lambda_j(h))\langle\phi_j(h), Op_h^w(\omega_k)B(h)Op_h^w(\omega_k)\phi_j(h)\rangle \\ &\times\frac{l}{2\pi h}e^{-\frac{il}{h}(\lambda_j(h)-\tau)t}\theta(t)\,dt \\ =& \int_{\mathbf{R}}\mathrm{tr}[Op_h^w(\omega_k)B(h)Op_h^w(\omega_k)f_2^2(H(h))e^{-i\frac{lt}{h}H(h)}]\frac{l}{2\pi h}e^{\frac{ilt\tau}{h}}\theta(t)\,dt. \end{aligned} \tag{6.9}$$

As usual, we take a $\chi \in C_0^\infty(\mathbf{R}),\ \chi \equiv 1$ on supp f_2 and we may then use

$$\tilde{U}_h(t) = \exp(-i\frac{t}{h}H(h)\chi(H(h))),$$

instead of $e^{-i\frac{t}{h}H(h)}$.
We have that

$$C_k(h) \equiv Op_h^w(\omega_k)B(h)Op_h^w(\omega_k),$$

is a strongly h-admissible operator with symbol $\sum h^j c_j^{(k)}$ where each term has support in the compact set: supp ω_k.

We end up having to calculate:

$$\frac{1}{2\pi h}\int \mathrm{tr}[C_k(h) f_2^2(H(h))\tilde{U}_h(t)]e^{-ih^{-1}t\tau}\theta(\frac{t}{l})\,dt. \tag{6.10}$$

Proposition 6.6. *We have the following asymptotics:*

$$\begin{aligned}(2\pi h)^n\frac{d}{d\tau}[M_1(\cdot,h)*W_{h,1}](\tau) &= f_2^2(\tau)\int_{\{a_0(x,p)=\tau\}} c_0^{(1)}(x,p)\frac{dS_\tau(x,p)}{|\nabla a_0(x,p)|}\\ &\quad +h\gamma_1^{(1)}(\tau)+O(h^2),\end{aligned} \tag{6.11}$$

Where the O is uniform in τ for $\tau \in \mathrm{supp}\, f_2^2$. And $\gamma_1^{(1)} \in C_0^\infty(\mathbf{R})$ with support contained in the support of f_2.

Proof. The proof is as for Prop. 5.4. □

Proposition 6.7. *We have, $\forall\, l$: $1 \le l \le L$:*

$$\frac{d}{d\tau}[M_2(\cdot,h)*(W_{h,1}-W_{h,l})](\tau) = O(h^{2-n}) \tag{6.12}$$

Where the O is uniform in τ for $\tau \in \mathrm{supp}\, f_2^2$.

Proof. The proof is the same as the proof of [PR85, Prop.4.3]. □

We can write the statement of Proposition 6.7 like this:

$$(2\pi h)^n\frac{d}{d\tau}[M_2(\cdot,h)*W_{h,l}](\tau) = \gamma_0^{(2)}(\tau)+h\gamma_1^{(2)}(\tau)+O(h^{2-n}), \tag{6.13}$$

where the γ's are continuous and independent of l: $1 \le l \le L$, and where the O is uniform for $\tau \in \mathrm{supp}\, f_2$. We now need a finer version of the Tauberian theorem:

Theorem 6.8. *There exists a constant $c \in \mathbf{R}$ so that:*

$$|M_2(\tau,h) - [M_2(\cdot,\tau) * W_{h,l}](\tau)| \leq \frac{c}{l}\gamma_0^{(2)}(\tau)h^{1-n} + c_2(l,\tau)h^{2-n},$$

where c_2 is continuous.

The proof is based on the following lemma:

Lemma 6.9. *$\exists k_0, k_1 \in \mathbf{R}$ such that for all $\phi_j(h)$ eigenvectors with $\lambda_j(h) \in [\lambda-\epsilon, \lambda-\epsilon]$ we have:*

$$|\langle \phi_j(h), Op_h^w(\omega_2)B(h)Op_h^w(\omega_2)\phi_j(h)\rangle| \leq k_0\|Op_h^w(\omega_2)\phi_j(h)\|^2 + k_1 h.$$

Proof. Let $\chi \in C_0^\infty(\mathbf{R})$, $\chi \equiv 1$ on $[\lambda-\epsilon, \lambda+\epsilon]$, and $\operatorname{supp}\chi \subset [\lambda-2\epsilon, \lambda+2\epsilon]$. Let $g = \chi(a_0)$. Then

$$\chi(H(h)) = Op_h^w(g) + h^2 R(h),$$

where $\{\|R(h)\| \; | h \in (0,h_0]\}$ is a bounded set. Here we used that $H(h)$ has no sub-principal symbol. Therefore we get:

$$\phi_j(h) = \chi(H(h))\phi_j(h) = Op_h^w(g)\phi_j(h) + h^2 R(h)\phi_j(h).$$

So

$$\begin{aligned}
& \langle \phi_j(h), Op_h^w(\omega_2)B(h)Op_h^w(\omega_2)\phi_j(h)\rangle \\
= \; & \langle \phi_j(h), Op_h^w(\omega_2)B(h)Op_h^w(\omega_2)Op_h^w(g)\phi_j(h)\rangle + O(h^2) \\
= \; & \langle \phi_j(h), Op_h^w(\omega_2)(B(h)Op_h^w(g))Op_h^w(\omega_2)\phi_j(h)\rangle \\
& + \langle \phi_j(h), Op_h^w(\omega_2)B(h)[Op_h^w(\omega_2); Op_h^w(g)]\phi_j(h)\rangle + O(h^2) \\
= \; & \langle Op_h^w(\omega_2)\phi_j(h), (B(h)Op_h^w(g))Op_h^w(\omega_2)\phi_j(h)\rangle + O(h),
\end{aligned}$$

where we have used the composition rules for strongly h-admissible operators to conclude that the commutator $[Op_h^w(\omega_2); Op_h^w(g)]$ is a bounded operator of norm $O(h)$ and that $B(h)Op_h^w(g)$, $B(h)Op_h^w(\omega_j)$ are bounded operators, with operator norms uniformly bounded in h. □

Now we can prove Theorem 6.8:

Proof. If we define:

$$\begin{aligned}
\sigma_h(\tau) &= \sum_{\lambda_j(h)\le\tau} \langle \phi_j(h), Op_h^w(\omega_2)^2\phi_j(h)\rangle f_2^2(\lambda_j(h)) \\
&= \sum_{\lambda_j(h)\le\tau} \|Op_h^w(\omega_2)\phi_j(h)\|^2 f_2^2(\lambda_j(h),
\end{aligned}$$

then $\sigma_h(\tau)$ satisfies the hypothesis of [PR85, Prop.3.2]:

1. $\sigma_h(\tau)$ is monotone, increasing in τ.
2. $\sigma_h(\tau) \equiv 0$ for $\tau \le \lambda - \epsilon$.
   $\sigma_h(\tau)$ is constant in τ for $\tau \ge \lambda + \epsilon$.
3. $\sigma_h(\tau) \le Kh^{-n}$ for all h sufficiently small and all τ.
4. $h^n \frac{d}{d\tau}[\sigma_h * W_{h,l}](\tau) = \gamma_0(\sigma,\tau) + \gamma_2(\sigma,\tau)h + O_{\tau,l}(h^2)$,
   where the O is (locally) uniform in τ and where
   $\tau \mapsto \gamma_0(\sigma,\tau)$ and $\tau \mapsto \gamma_1(\sigma,\tau)$ are of class C^1 on $\mathbf{R}$.

Only the last statement is not obvious. The proof of this statement follows the same lines as the proof of Proposition 6.7 and will thus be omitted. Therefore we can conclude from [PR85, Prop.3.2], that there exists $\tilde\gamma \ge 0$, and $\tilde C(L,\tau)$ a locally bounded, positive function of such that:

$$|\sigma_h(\tau + h\mu) - \sigma_h(\tau)| \le \tilde\gamma(1/L + |\mu|)h^{1-n}\gamma_0(\sigma,\tau) + \tilde C(L,\tau)(1+|\mu|^2)h^{2-n}. \tag{6.14}$$

Thus we get:

$$\begin{aligned}
& |M_2(\tau,h) - [M_2(\cdot,h) * W_{h,l}](\tau)| \\
\le\ & \frac{l}{2\pi h}\int |M_2(\tau,h) - M_2(\tau-\mu,h)||\hat\theta(\frac{-l\mu}{h})|\,d\mu \\
=\ & \frac{l}{2\pi}\int |M_2(\tau,h) - M_2(\tau+h\mu,h)||\hat\theta(l\mu)|\,d\mu \\
=\ & \frac{l}{2\pi}\int |\sum_{\tau\le\lambda_j(h)\le\tau+h\mu} \langle\phi_j(h), Op_h^w(\omega_2)B(h)Op_h^w(\omega_2)\phi_j(h)\rangle f_2^2(\lambda_j(h))||\hat\theta(l\mu)|\,d\mu \\
\le\ & \frac{k_0 l}{2\pi}\int |\sigma_h(\tau+h\mu) - \sigma_h(\tau)||\hat\theta(l\mu)|\,d\mu + O(h^{2-n}) \\
\le\ & \frac{k_0 l}{2\pi}\int [\tilde\gamma(1/L+|\mu|)h^{1-n}\gamma_0(\sigma,\tau) + \tilde C(L,\tau)(1+|\mu|^2)h^{2-n}|\hat\theta(l\mu)|]\,d\mu \\
& +O(h^{2-n}).
\end{aligned}$$

So modulo an error of order h^{2-n} we get:

$$\begin{aligned}
& |M_2(\tau,h) - [M_2(\cdot,h) * W_{h,l}](\tau)| \\
\leq\ & c.l \int_{|\mu|\leq 1/l} (1/l + |\mu|)\gamma_0(\sigma,\tau)|\hat{\theta}(l\mu)|\, d\mu.h^{1-n} \\
+\ & c.l \int_{|\mu|\geq 1/l} (1/l + |\mu|)\gamma_0(\sigma,\tau)|\hat{\theta}(l\mu)|\, d\mu.h^{1-n} \\
\equiv\ & Term1 + Term2.
\end{aligned}$$

It is easy to handle $Term1$:

$$Term1 \leq \frac{c}{l}\gamma_0(\sigma,\tau)\sup(|\hat{\theta}|)h^{1-n}.$$

$Term2$ is calculated as (by the change of variables:$\mu' = l\mu$):

$$\begin{aligned}
Term2 &= c\gamma_0(\sigma,\tau)\int_{|\mu|\geq 1} \frac{1+|\mu|}{l}|\hat{\theta}(\mu)|\, d\mu h^{1-n} \\
&= c\frac{\gamma_0(\sigma,\tau)}{l}h^{1-n}\int_{|\mu|\geq 1}(1+|\mu|)|\hat{\theta}(\mu)|\, d\mu.
\end{aligned}$$

□

We can now prove Theorem 6.2:

Proof. We have:

$$\begin{aligned}
|M(\lambda,h) - [M(\cdot,h) * W_h](\lambda)| \ \leq\ & |M(\lambda,h) - M_1(\lambda,h) - M_2(\lambda,h)| \\
+\ & |[M(\cdot,h) - M_1(\cdot,h) - M_2(\cdot,h)] * W_h(\lambda)| \\
+\ & |M_1(\lambda,h) - [M_1(\cdot,h) * W_h](\lambda)| \\
+\ & |M_2(\lambda,h) - [M_2(\cdot,h) * W_h](\lambda)|.
\end{aligned}$$

Here the first two terms on the right are taken care of by Lemma 6.4. By def. of ω_1, Prop. 6.6, Prop. 6.7 and Thm 6.8:

$$\begin{aligned}
& |M_1(\lambda,h) - [M_1(\cdot,h) * W_h](\lambda)| + |M_2(\lambda,h) - [M_2(\cdot,h) * W_h](\lambda)| \\
=\ & |M_1(\lambda,h) - [M_1(\cdot,h) * W_{h,1}](\lambda)| \\
& + |M_2(\lambda,h) - [M_2(\cdot,h) * W_{h,L}](\lambda)| + O(h^{2-n}) \\
\leq\ & \gamma\gamma_0(1,\lambda)h^{1-n} + C_1(1,1,\lambda)h^{2-n} \\
& + \frac{\gamma}{L}\gamma_0(2,\lambda)h^{1-n} + C_1(1,2,\lambda)h^{2-n} + O(h^{2-n}).
\end{aligned} \tag{6.15}$$

Let $\epsilon > 0$. Let us choose L so big that

$$\frac{\gamma}{L}\int_{\Sigma_\lambda}\frac{dS_\lambda}{|\nabla a_0|} \leq \epsilon/2.$$

Now we construct an open set $U_\lambda \subset \mathbf{R}^{2n}$ satisfying:

- $K_L \cap \Sigma_\lambda \subset U_\lambda$.
- $\int_{\Sigma_\lambda \cap U_\lambda} \frac{dS_\lambda}{|\nabla a_0|} \leq \epsilon/2$.
- $\exists \alpha > 0$ such that $K_L \cap \Sigma_\tau \subset U_\lambda$ for all τ satisfying $|\tau - \lambda| \leq \alpha$.

(This construction is made in [PR85].) We can now define Ω_L by:

$$\Omega_L = U_\lambda \cup a_0^{-1}(\lambda - \epsilon_0, \lambda - \alpha) \cup a_0^{-1}(\lambda + \alpha, \lambda + \epsilon_0). \tag{6.16}$$

We thus see that the h^{1-n} term in (6.15) can be made as small as we want (Ω_L was defined just below Observation 6.3). $\square$

6.2 The calculation of γ_1

Let us write:

$$(2\pi h)^n[M(\cdot, h) * W_h](\tau) = C_0(\tau) + hC_1(\tau) + O(h^2),$$

where the O is locally uniform i τ. We will now prove:

Theorem 6.10.

$$\begin{aligned} C_1(\tau) \;\; &= \iint_{\{a_0 \leq \tau\}} f_2^2(a_0) b_1 \, dx dp + \frac{1}{2i} f_2^2(\tau) \int_{\Sigma_\tau} \{a_0, b_0\}_P \frac{dS_\tau}{|\nabla a_0|} \\ &\quad - \frac{1}{2i} \int_{\{a_0 \leq \tau\}} (f_2^2)'(a_0) \{a_0 . b_0\}_P \, dx dp, \end{aligned} \tag{6.17}$$

for all $\tau \in (\lambda - \epsilon/2, \lambda + \epsilon/2)$.

Since $C_0(\tau)$ and $C_1(\tau)$ are continuous, they are determined uniquely by the distributions they define. That will make it easy for us to find C_1. Let $\phi \in C_0^\infty(\lambda - \epsilon/2, \lambda + \epsilon/2)$.

Lemma 6.11.

$$\int \phi(\tau)[M(\cdot,h)*W_h](\tau)\,d\tau = \mathrm{tr}[B(h)f_2^2(H(h))\Phi(H(h))] + O(h^{2-n}), \tag{6.18}$$

where $\Phi(\mu) = \int_\mu^\infty \phi(t)\,dt$.

Proof.

$$\begin{aligned}\int \phi(\tau)[M(\cdot,h)*W_h](\tau)\,d\tau &= \iint \phi(\tau+\mu)W_h(\tau)M(\mu,h)\,d\mu d\tau\\ &= \int M(\mu,h)[\int \phi(\tau+\mu)W_h(\tau)\,d\tau]\,d\mu.\end{aligned}$$

Now, by taking the Taylor series for ϕ we get:

$$\begin{aligned}\int \phi(\tau+\mu)W_h(\tau)\,d\tau &= \frac{1}{2\pi h}\int \phi(\tau+\mu)\hat{\theta}(\frac{-\tau}{h})\,d\tau\\ &= \phi(\mu)+O(h^2),\end{aligned}$$

because $\hat{\theta}$ is an even function (θ is assumed even), and $\theta(0)=1$.
So

$$\int \phi(\tau)[M(\cdot,h)*W_h](\tau)\,d\tau = \int \phi(\mu)M(\mu,h)\,d\mu + O(h^{2-n}).$$

But now:

$$\begin{aligned}\int \phi(\mu)M(\mu,h)\,d\mu &= \int \phi(\mu)\sum_{\lambda_j(h)\le\mu}\beta_j(h)f_2^2(\lambda_j(h))\,d\mu\\ &= \sum_j \beta_j(h)f_2^2(\lambda_j(h))\int_{\lambda_j(h)}^\infty \phi(\mu)\,d\mu\\ &= \mathrm{tr}[B(h)(f_2^2\Phi)(H(h))],\end{aligned}$$

where $\beta_j(h) = \langle\phi_j(h), B(h)\phi_j(h)\rangle$. $\square$

We will now prove Theorem 6.10.

Proof. We can calculate $\mathrm{tr}[B(h)f_2^2(H(h))\Phi(H(h))]$ using Theorem 3.4:

$$\begin{aligned}(2\pi h)^n\mathrm{tr}[B(h)f_2^2(H(h))\Phi(H(h))] &= \iint b_0(f_2^2\Phi)(a_0)\,dxdp\\ +h\iint[b_1(f_2^2\Phi)(a_0) &+ \frac{1}{2i}\{b_0,(f_2^2\Phi)(a_0)\}_P]\,dxdp + O(h^2)\end{aligned}$$

The term of order one is the sum of two terms:

$$\iint [b_1(f_2^2\Phi)(a_0)]\, dxdp = \int_{\mathbf{R}} (f_2^2\Phi)(\mu) \int_{\{a_0=\mu\}} b_1 \frac{dS_\mu}{|\nabla a_0|}\, d\mu, \tag{6.19}$$

and

$$\begin{aligned} & -\frac{1}{2i} \iint \partial_x b_0 \partial_p (f_2^2\Phi)(a_0) - \partial_p b_0 \partial_x (f_2^2\Phi)(a_0)\, dxdp \\ = \quad & \frac{1}{2i} \int_{\mathbf{R}} (f_2^2\Phi)(\mu) \frac{d}{d\mu} \int_{\{a_0=\mu\}} \{a_0, b_0\}_P \frac{dS_\mu}{|\nabla a_0|}\, d\mu. \end{aligned} \tag{6.20}$$

We have used [Rob87, Lemma V-9]. Both have the form: $\int g(\mu)\Phi(\mu)\, d\mu$, where $g \in C_0^\infty(\mathbf{R})$. We get:

$$\int g(\mu)\Phi(\mu)\, d\mu = \int_{\mathbf{R}} \int_\mu^\infty g(\mu)\phi(\tau) d\tau d\mu = \int_{\mathbf{R}} \phi(\tau) \int_{-\infty}^\tau g(\mu) d\mu d\tau$$

So we get:

$$C_1(\tau) = \int_{-\infty}^\tau [g_1(\mu) + g_2(\mu)]\, d\mu,$$

where g_j are the functions defined by (6.19) and (6.20):

$$\begin{aligned} C_1(\tau) &= \int_{-\infty}^\tau f_2^2(\mu) [\int_{\{a_0=\mu\}} b_1 \frac{dS_\mu}{|\nabla a_0|} + \frac{1}{2i} \frac{d}{d\mu} \{ \int_{\{a_0=\mu\}} \{a_0, b_0\}_P \frac{dS_\mu}{|\nabla a_0|} \}]\, d\mu \\ &= \iint_{\{a_0 \le \tau\}} f_2^2(a_0) b_1\, dxdp + \frac{1}{2i} f_2^2(\tau) \int_{\Sigma_\tau} \{a_0, b_0\}_P \frac{dS_\tau}{|\nabla a_0|} \\ & \quad -\frac{1}{2i} \int_{\{a_0 \le \tau\}} (f_2^2)'(a_0) \{a_0, b_0\}_P\, dxdp. \end{aligned} \tag{6.21}$$

□

Theorem 6.10, Lemma 6.11, Theorem 6.2 together with equation (6.3) now prove Theorem 1.5.

ACKNOWLEDGEMENTS

I wish to thank CIMAT for help and hospitality during the academic year 1995-1996. I also thank B. Helffer for letting me study some unpublished notes by him on this problem. Finally I wish to thank J.P. Solovej for help and advice.

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