Coarsening by Ginzburg-Landau Dynamics

J.-P. Eckmann[1,2] *and J. Rougemont*[1]

[1]Dépt. de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland
[2]Section de Mathématiques, Université de Genève, CH-1211 Genève 4, Switzerland

Abstract. We study slowly moving solutions of the real Ginzburg-Landau equation on the line, by a method due to J. Carr and R.L. Pego. These are functions taking alternately positive or negative values on large intervals. A consequence of our approach is that we can propose a rigorous derivation of a stochastic model of coarsening by successive elimination of the smallest interval, which was described in earlier work by A.J. Bray, B. Derrida and C. Godrèche.

1. Introduction

In a series of papers ([CP1,CP2]), Carr and Pego studied the evolution of multi-kink initial data of the real Ginzburg-Landau equation:

$$\partial_t v \;=\; \partial_x^2 v + v - v^3 \;=\; \partial_x^2 v + V'(v)\;, \tag{1.1}$$

with $v(x,t) \;:\; \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$. These data are for most x very close to the stationary values $v = \pm 1$ with transitions from ± 1 to ∓ 1 at certain points. We call these points 'kinks'. Since an isolated kink moves to the stable stationary solution $\tanh(x - x_0)$ for some $x_0 \in \mathbf{R}$, one expects the dynamics of multi-kink data to be slow. Carr and Pego showed that the dynamics of the position of the kinks can be approximated by a simple potential model, with an exponentially decaying attraction between the kinks. They attract each other weakly, and eventually two opposite kinks collide and 'annihilate'. Carr and Pego proved their results for the equation on an interval subject to Neumann boundary conditions.

In this paper, we extend their method to prove similar results for the equation on the infinite line. First, we show that if the kinks are initially widely separated, then their annihilation goes through a *universal* sequence of shapes. One can then ask (and answer) some questions about the evolution of initial data with an infinity of kinks. Controlling this situation leads to a rigorous derivation of the stochastic model of coarsening studied by Bray *et al.* in [BDG,BD]. There are several interesting technical points in this derivation, for example the problem to show that there can be no 'conspiracy' between kinks which are far apart to change the basic potential evolution between neighboring kinks. Our analysis goes some way towards formulating a fully probabilistic set of 'reasonable' initial conditions. However, we fall short of finding a set of positive measure on the full line of such conditions, but we at least can show that there are local conditions which guarantee the control of the evolution for all times.

Our results hold for equations which are somewhat more general than the Ginzburg-Landau equation, which derives from a potential V: If we introduce the notation

$$V(z) \;=\; \frac{z^2}{2} - \frac{z^4}{4}\;, \tag{1.2}$$

then the r.h.s. of Eq.(1.1) is

$$\mathcal{L}(v) \;=\; \partial_x^2 v + V'(v)\;. \tag{1.3}$$

We can extend our results to all potentials $V \in \mathcal{C}^3$ which satisfy:

$$\begin{gathered} V'(\pm 1) \;=\; V'(0) \;=\; 0\;,\\ V(+x) \;=\; V(-x)\;,\\ V''(\pm 1) \;<0\;, \quad V''(0) \;\geq\; 1\;,\\ V'(x) \;\neq\; 0 \text{ for } x \notin \{\pm 1,\, 0\}\;. \end{gathered}$$

Remark. The condition $V(x) = V(-x)$ can be replaced by the simpler $V(1) = V(-1)$, but then the notation gets somewhat more involved. We have fixed the scale by imposing

$V'(\pm 1) = 0$. Furthermore, to simplify the choice of cutoff functions, we have required $V''(0) \geq 1$, and this could be generalized easily to $V''(0) > 0$.

We start by listing all the bounded stationary—*i.e.,* time-independent—solutions of (1.1). They can be interpreted as trajectories of a free point particle moving in the potential V without friction, with x being the 'time' variable (see Fig. 1). Note that this is an integrable Hamiltonian system. The stationary solutions are:

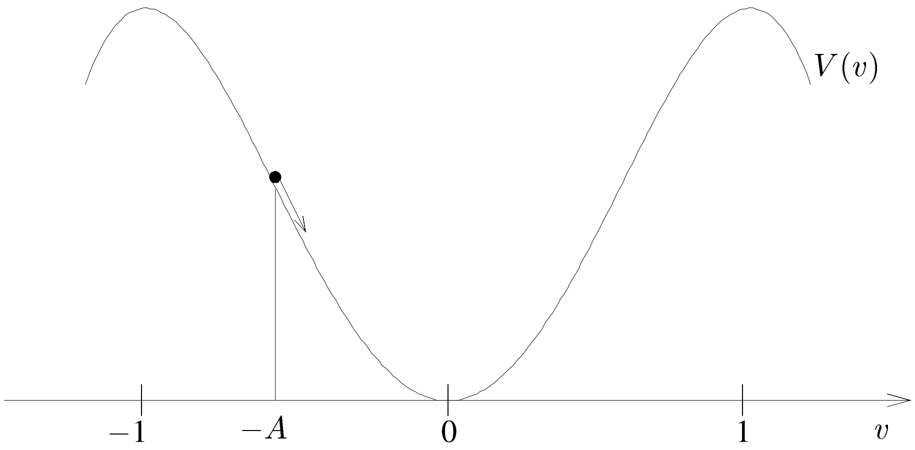

Fig. 1: The mechanical interpretation of the equation $\mathcal{L}(v) = 0$.

— Three constant solutions,

$$u_{\pm}(x) = \pm 1\ , \quad u_0(x) = 0\ .$$

— Two heteroclinic solutions, which we will denote by $\psi(x)$ and $\psi(-x)$. We fix the notation by imposing $x\psi(x) \leq 0$. For V given by Eq.(1.2), such a solution is:

$$\psi(x) = -\tanh\Big(\frac{x}{\sqrt{2}}\Big)\ .$$

— Periodic solutions, which are described in the following proposition (see Fig. 2).

Proposition 1.1. *For every V as described above there is a $P_0 > 0$ such that for every $P \in (P_0, \infty)$, there exists a periodic stationary solution $\varphi_P(x)$ of (1.1) with period $2P$ and amplitude A, and $\varphi_P(x)$ has exactly one maximum and one minimum per period. Furthermore, $\varphi_P \in \mathcal{C}^\infty$ and there is a real analytic bijection between the amplitude $A \in (0,1)$ and $P = P(A)$. We shall assume $\varphi_P(0) = \varphi_P(P) = 0$, $\varphi_P(x) < 0$ for $x \in (0, P)$.*

Remark. In fact, $P_0 = V''(0)\pi$. The result stated in this proposition is certainly not new, but in order to stay self-contained, we give a proof in Section 9.

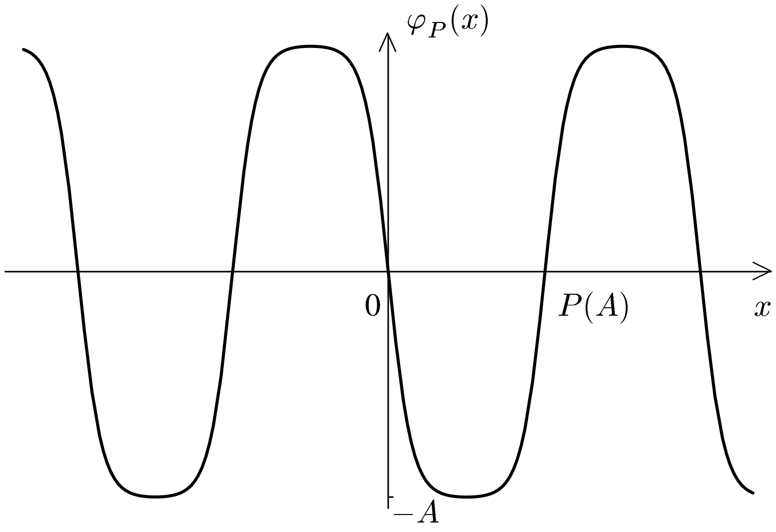

Fig. 2: A periodic solution $\varphi_P(x)$.

Remark. The above list is exhaustive. This can be seen by examining Fig. 1. The initial position $-A$ of the particle must satisfy $A \in [0, 1]$ to keep the orbit bounded. But for any such initial condition, we have exhibited a solution of the equation $\mathcal{L}(v) = 0$. Thus, there are no other bounded stationary solutions of the equation (1.1).

These periodic solutions will play an important role in the sequel. Our aim is to show the existence of 'metastable' states, *i.e.*, states that are unstable, but which creep for a very long time (see the numerical simulations in Fig. 3). It is common knowledge that the solutions $u_\pm$ are stable, that ψ is stable up to an eigenvalue 0 (corresponding to translations) and that all the other stationary solutions are unstable. We want to study the evolution of initial conditions $v(t = 0)$ which are like 'crenelations': We define the set Z of zeros of $v(t = 0)$ as

$$Z = \{z_j \in \mathbf{R}, j \in \mathbf{Z}, z_j < z_{j+1} \text{ and } v(x = z_j, t = 0) = 0 \text{ for all } j\} .$$

We assume $v(x, t = 0)$ is positive for $z_{2j} < x < z_{2j+1}$ and negative for $z_{2j-1} < x < z_{2j}$. We also introduce the interval lengths ℓ_j defined by:

$$\ell_i = z_i - z_{i-1} .$$

Definition. A function which is of the form above will be called *admissible*, and will be denoted u_Z.

If we look only at the zeros of the solution of Eq.(1.1), then we have a reduced system of equations, for the *positions* of the zeros. Thus Z becomes a function of time. One of the difficulties in the infinite domain is to show that there are 'interesting' admissible initial data *which remain admissible for all times* when evolved with the Eq.(1.1).

The evolution of these initial data will look as follows. First, the positive (negative) part of u approaches rapidly $+1$ (-1) and domain walls form in between, which (locally) look like

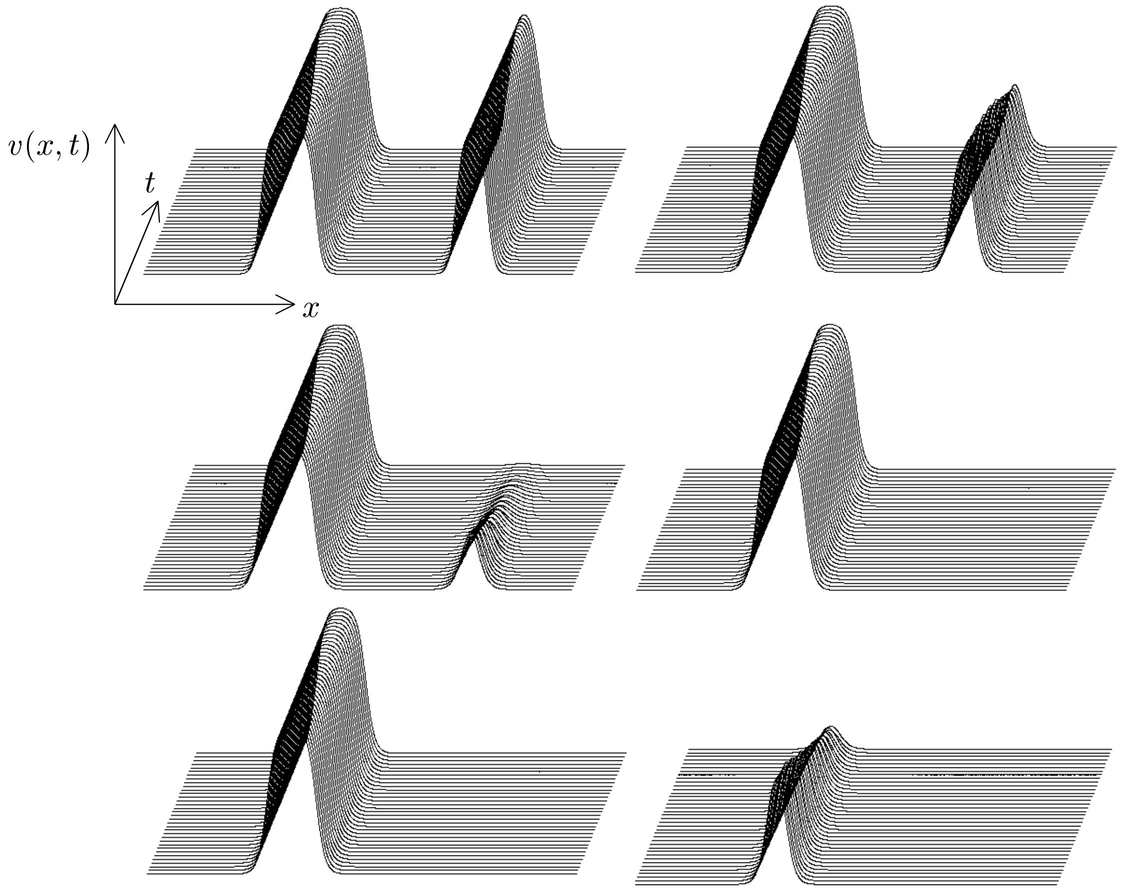

Fig. 3: Numerical simulations. Top left: time 1 to 2.7, top right: time 25 to 29.2, middle left: time 29.2 to 31.3, middle right: time 3668 to 3675, bottom left: time 5043 to 5050 and bottom right: time 5266 to 5269.

$\pm\tanh(x/\sqrt{2})$ (generally, the heteroclinic solutions). Intuitively, ± 1 are stable fixed points, but the domain walls will move. Since there is no reason for $+1$ to be preferred to -1 or vice-versa, the speed of the motion of a domain wall will, to first approximation, only depend on the sizes ℓ_i, ℓ_{i+1} of the two domains adjacent to this wall. Carr and Pego showed, in the finite domain, that the speed of motion of the i^{th} kink is roughly $e^{-\ell_{i+1}} - e^{-\ell_i}$.

We follow the method of Carr and Pego to prove similar results in the infinite domain: to a prescribed set Z, we associate a function $u_Z^{(0)}$ which has Z as the set of its zeros. In each interval (z_i, z_{i+1}), we set $u_Z^{(0)}(x)$ equal to a translate of φ_P with $P = z_{i+1} - z_i$, so that $u_Z^{(0)}$ is a continuous function, alternatively positive and negative between successive zeros. Then we slightly deform this non-differentiable function near each zero to get a smooth function u_Z. (The idea of gluing near the zeros instead of gluing in the middle of the intervals was already present in Carr and Pego and is very fruitful.) This function u_Z is, by construction, equal to a stationary solution of (1.1), except near the set Z. The next step is the study of the stability of these 'almost stationary' functions. We show that the unstable directions are approximately tangent to $\mathcal{M} = \{u_Z \,:\, Z \text{ in some restricted set } \Omega_\Gamma\}$ and the spectrum of the linearized operator corresponding to these unstable modes is contained in a ball of radius $\sup_i e^{-\ell_i}$.

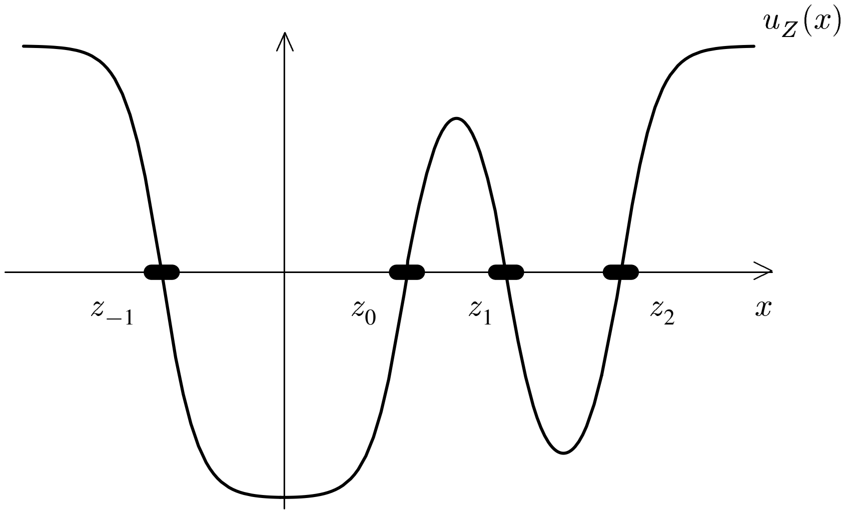

Fig. 4: The function u_Z. The zeros are labeled z_i. Bold lines indicate the regions of gluing.

In the next section, we analyze the behavior of initial conditions close to u_Z, when all kinks are far apart, in particular, we show the existence of an invariant neighborhood of $\mathcal{M}$. In Section 2.4, we provide an explicit formula for the speed of the kinks. In Section 3, we discuss the annihilation of a pair of neighboring kinks. This analysis yields a version of the Bray-Derrida-Godrèche dynamics of intervals which continuously eliminates the smallest interval, replacing it by the union of its neighbors ([BDG]). In Section 4, we construct a set $\mathcal{C}$ of initial conditions which never come to rest, *i.e.,* that 'coarsen' forever in the sense introduced in [BDG]. Most of the proofs are given in Section 5 to Section 9.

Acknowledgments. We are grateful to P. Collet for suggesting this problem to us. We also thank C.-A. Pillet, P. Wittwer, and L. Rey-Bellet for useful comments. This work was supported by the Fonds National Suisse.

2. Dynamics in the dilute state

In this section, we give the basic definitions used throughout the paper and analyze the behavior of a well-separated collection of kinks.

2.1. Definitions

Let $Z = \{z_j\}_{j\in\mathbf{Z}} \in \mathbf{R}^{\mathbf{Z}}$ be a sequence of positive real numbers. Let

$$\begin{aligned} \ell_j &= z_j - z_{j-1} \ , \\ |Z| &= \inf_{j\in\mathbf{Z}} \ell_j \ , \\ c_j &= \tfrac{1}{2}(z_j + z_{j-1}) \ . \end{aligned}$$

Let $\Gamma > P_0$ and suppose that $|Z| > \Gamma$. In particular, this means that $z_{j+1} > z_j$ for all $j \in \mathbf{Z}$. Let Ω_Γ denote the set of such Z:

$$\Omega_\Gamma = \left\{ Z = \{\ldots, z_{-1}, z_0, z_1, \ldots\} \in \mathbf{R}^{\mathbf{Z}} : z_{j+1} - z_j > \Gamma,\, j \in \mathbf{Z} \right\} .$$

We equip Ω_Γ with a probability measure P:

Definition 2.1. *Let $\{\ell_j\}_{j\in\mathbf{Z}}$ be i.i.d. random variables with a probability density $\rho_\Gamma(x)$, $x \in \mathbf{R}^+$, satisfying $\rho_\Gamma(x) > 0$ for $x > \Gamma$ and $\rho_\Gamma(x) = 0$ for $x \le \Gamma$. The probability measure P on Ω_Γ is then induced by choosing, for $Z \in \Omega_\Gamma$,*

$$z_0 \;=\; 0\,, \quad z_j - z_{j-1} \;=\; \ell_j\,.$$

For $Z \in \Omega_\Gamma$, we construct the function $u_Z(x)$ as described in the introduction: Let

$$u^{(j)}(x) \;=\; \begin{cases} \bigl(1 - \Delta(x - z_j)\bigr)\varphi_{\ell_j}(x - z_{j-1}) + \Delta(x - z_j)\varphi_{\ell_{j+1}}(x - z_{j+1})\,, \\ \hfill \text{if } j \text{ is even}\,, \\ \bigl(1 - \Delta(x - z_j)\bigr)\varphi_{\ell_j}(x - z_j) + \Delta(x - z_j)\varphi_{\ell_{j+1}}(x - z_j)\,, \\ \hfill \text{if } j \text{ is odd}\,, \end{cases}$$

where $\Delta(x)$ is a $\mathcal{C}^\infty$ monotone cutoff function satisfying:

$$\Delta(x) \;=\; \begin{cases} 0 & \text{if } x \le -1\,, \\ 1 & \text{if } x \ge 1\,. \end{cases}$$

Then, $u_Z(x)$ is given by the formula

$$u_Z(x) = \sum_{j=-\infty}^{\infty} u^{(j)}(x)\mathbf{1}_j(x)\,, \tag{2.1}$$

where $\mathbf{1}_j$ is the indicator function of the interval $I_j \equiv [c_j, c_{j+1}]$. Note that $u_Z \in \mathcal{C}^\infty$.

Expanding (1.3) around u_Z by setting $v = w + u_Z$ gives:

$$\mathcal{L}(w + u_Z) \;=\; \mathcal{L}(u_Z) - L_Z w + w^2 r(w, u_Z)\,, \tag{2.2}$$

where the linear operator reads:

$$L_Z w \;=\; -\partial_x^2 w - V''\bigl(u_Z\bigr)w\,, \tag{2.3}$$

and the non-linear remainder is given by:

$$r(f, g) \;=\; \int_0^1 ds\,(1 - s)\,V'''(sf + g)\,.$$

2.2. Properties of the linear operator

In this section, we present some properties of the linear operator L_Z defined in Eq.(2.3) acting on $L^2(\mathbf{R}, d\mu)$ where μ is the measure defined in the following:

Definition 2.2. *Let Λ be a compact interval in $\mathbf{R}$ and $\varepsilon > 0$. Let μ be an absolutely continuous measure on $\mathbf{R}$ satisfying*

$$\begin{aligned} \mu(\Lambda) &= 1 - \varepsilon , \\ \mu(\mathbf{R}\backslash\Lambda) &= \varepsilon , \\ \exists C > 0 : \left| \frac{d\mu(x)}{dx} - C \right| &\leq \varepsilon , \quad \text{for } x \in \Lambda . \end{aligned}$$

The corresponding $L^2(\mathbf{R}, d\mu)$-norm is denoted $\| \cdot \|_\Lambda$ and the scalar product $\langle \cdot, \cdot \rangle_\Lambda$.

We first describe the spectrum of L_Z:

Theorem 2.3. *There exist constants $c_1 < \infty$, $M > 0$, and a set $\Omega^* \subset \Omega_{|Z|}$ such that for sufficiently large $|Z|$,*
— $P(\Omega^*) = 1$,
— For all $Z \in \Omega^*$, the $L^2(\mathbf{R}, d\mu)$-spectrum of $L_Z \equiv -\partial_x^2 - V''(u_Z)$ in $(-\alpha, \alpha)$, with $\alpha = \mathcal{O}(e^{-c_1|Z|})$, is pure point with exponentially decaying eigenfunctions e_j, $j \in \mathbf{N}$. The remainder of the spectrum is contained in $[M, \infty)$.

The proof of this result essentially follows the lines of [FSW] or [S], and is sketched in Section 6. There is a corollary to this theorem:

Corollary 2.4. *Let $\varepsilon > 0$ and $\Lambda \subset \mathbf{R}$ be a compact interval. Then there exists an $N_\varepsilon < \infty$ such that for all $w \in L^2(\mathbf{R}, d\mu) \cap L^\infty$,*

$$\sum_{j > N_\varepsilon} |\langle e_j(x), w \rangle_\Lambda| \leq \varepsilon . \tag{2.4}$$

Proof. The l.h.s. of Eq.(2.4) is the projection of w onto a space of functions which have exponentially small tails in Λ. Furthermore, $w \in L^\infty$, hence $\|\mathbf{1}_{\mathbf{R}\backslash\Lambda} w\|_\Lambda \leq \varepsilon$, where $\mathbf{1}_{\mathbf{R}\backslash\Lambda}$ is the indicator function of the complement of Λ.

□

We now define vectors in $L^2(\mathbf{R}, d\mu)$ which 'generate' the translation of the j^{th} kink:

$$\tau_{z_j} = (-1)^j \Theta_j(x) \partial_x u_Z(x) , \tag{2.5}$$

where Θ_j is a (smooth) characteristic function of the interval I_j: $\Theta_j \in \mathcal{C}^\infty$,

$$\Theta_j(x) = \begin{cases} 0 , & \text{if } \operatorname{dist}(x, I_j) > 1 , \\ 1 , & \text{if } x \in I_j , \end{cases}$$

in such a way that all its derivatives are uniformly bounded in j.

Lemma 2.5. *For $|Z|$ sufficiently large and for $\varepsilon > 0$, there is a D_τ, $0 < D_\tau < \infty$, such that for all $k > D_\tau$, one has*

$$\|\tau_{z_k}\|_\Lambda \le \varepsilon .$$

Proof. The claim follows immediately from Definition 2.2 and from the fact that τ_{z_j} has compact support and is uniformly bounded.

□

We denote by $P_{N_\varepsilon} : L^2(\mathbf{R}, d\mu) \to \bigoplus_{j \le N_\varepsilon} \mathcal{H}_{\lambda_j}$ the spectral projector associated with the eigenvalues $\lambda_1, \ldots, \lambda_{N_\varepsilon}$ of L_Z (and $\mathcal{H}_{\lambda_j} \subset L^2(\mathbf{R}, d\mu)$ the corresponding spectral subspaces). Then, if $w \in L^2(\mathbf{R}, d\mu)$ satisfies $\langle w, \tau_{z_j} \rangle_\Lambda = 0$ for all $|j| \le D_\tau$, then its projection onto $\bigoplus_{j \le N_\varepsilon} \mathcal{H}_{\lambda_j}$ is small:

Proposition 2.6. *Let $w \in L^2(\mathbf{R}, d\mu)$. There exist constants $c_1 > 0$, $c_2 > 0$, and $D_\tau > 0$ such that for sufficiently large $|Z|$, if w satisfies $\langle w, \tau_{z_j} \rangle_\Lambda = 0$ for all $|j| \le D_\tau$, then $\|P_{N_\varepsilon} w\|_\Lambda \le c_2 e^{-c_1 |Z|} \|w\|_\Lambda$.*

The proof of this statement can be found in Section 6.

This proposition basically says that a function w which is orthogonal to vectors τ_{z_j} located in Λ is also almost orthogonal to relevant unstable modes. We can infer (see Section 6) the following corollary:

Corollary 2.7. *Let $w \in L^2(\mathbf{R}, d\mu)$. Then, for sufficiently large $|Z|$, there exist constants $M_1 > 0$, $M_2 > 0$, and $M_3 > 0$ such that if w satisfies $\langle w, \tau_{z_j} \rangle_\Lambda = 0$ for $|j| \le D_\tau$, then one has:*

$$\|L_Z w\|_\Lambda^2 \ge M_2 \langle w, L_Z w \rangle_\Lambda \ge M_1 M_2 \|w\|_\Lambda^2 , \tag{2.6}$$

and denoting by χ_Λ a smooth characteristic function of the interval Λ (i.e., $\chi_\Lambda(x) = 1$ if $x \in \Lambda$, χ_Λ has compact support and $|\partial_x \chi_\Lambda(x)| < \frac{1}{2}$ for all x), one has:

$$\|\chi_\Lambda w\|_\infty^2 \le M_3 \langle w, L_Z w \rangle_\Lambda . \tag{2.7}$$

Motivated by the above statement, we can introduce the following norm for perturbations $w \in L^2(\mathbf{R}, d\mu)$ which are orthogonal to $\mathrm{span}\{\tau_{z_j}\}$, $j \in \mathbf{Z}$:

$$\|w\|_Z^2 \equiv \langle w, L_Z w \rangle_\Lambda , \tag{2.8}$$

where $Z \in \Omega_\Gamma$. We will use this last corollary as follows: In the next section, we construct a decomposition of a solution v_t of Eq.(1.1) as $v_t = u_{Z_t} + w_t$ with $Z_t \in \Omega_\Gamma$ and w_t satisfying the hypothesis of Corollary 2.7. Hence w_t essentially decays with rate M_2 and we only have to work out the evolution of Z_t.

2.3. Geometric structure

We next introduce the space of initial conditions for the dynamics given by Eq.(1.1): Let

$$\mathcal{T}_{\Gamma,\sigma} = \left\{ v \in L^\infty(\mathbf{R}) : \|v\|_\infty \le 1, \inf_{Z\in\Omega_\Gamma} \|\chi_\Lambda(v-u_Z)\|_\infty < \sigma, \inf_{Z\in\Omega_\Gamma} \|v-u_Z\|_\Lambda < \infty \right\} , \tag{2.9}$$

with χ_Λ as in Corollary 2.7.

Remark. It is well known [G] that if $\|v_0\|_\infty \le 1$ then the corresponding solution v_t of Eq.(1.1) satisfies $\|v_t\|_\infty \le 1$ for all times $t > 0$. Thus, if $v_t \in \mathcal{T}_{\Gamma,\sigma}$, we have, for some $Z \in \Omega_\Gamma$,

$$\|v_t - u_Z\|_\Lambda \le \|\chi_\Lambda(v_t-u_Z)\|_\infty + \varepsilon\|(1-\chi_\Lambda)(v_t-u_Z)\|_\infty \le \sigma + 2\varepsilon . \tag{2.10}$$

Important terminology. The 'tube' $\mathcal{T}_{\Gamma,\sigma}$ depends on two parameters: Γ and σ. They measure its 'length' and its 'width'. Throughout this paper, we shall use the condition 'for sufficiently small $\mathcal{T}_{\Gamma,\sigma}$' to mean 'for sufficiently large $\Gamma < \infty$ and sufficiently small $\sigma > 0$'.

Proposition 2.8. *For sufficiently small $\mathcal{T}_{\Gamma,\sigma}$ and $v \in \mathcal{T}_{\Gamma,\sigma}$, there exists a differentiable function $Z : \mathcal{T}_{\Gamma,\sigma} \to \Omega_\Gamma$ such that $\langle v - u_{Z(v)}, \tau_{z_j(v)}\rangle_\Lambda = 0$, for $|j| \le D_\tau + 1$. Furthermore, for all $\varepsilon > 0$, there is a $Z^* \in \Omega^*$ such that $\|(L_{Z(v)} - L_{Z^*})w\|_\Lambda < \varepsilon\|w\|_\Lambda$.*

The proof of this proposition is an application of the Implicit Function Theorem and is detailed in Section 7.

Proposition 2.9. *There exists a constant $B > 0$ such that for sufficiently small $\mathcal{T}_{\Gamma,\sigma}$, the following holds: If $v_0 \in \mathcal{T}_{\Gamma,\sigma}$, then as long as $v_t \in \mathcal{T}_{\Gamma,\sigma}$, one has:*

$$(\partial_t + \tfrac{1}{2}M_2)\Big(\|v - u_{Z(v)}\|^2_{Z(v)} - Bg_1^2(Z(v))\Big) \le 0 , \tag{2.11}$$

where M_2 is like in Corollary 2.7, $g_1^2(Z) = \sum_{|j|\le D_\tau} |\langle \mathcal{L}(u_Z), \tau_{z_j}\rangle_\Lambda|^2$, and $\|\cdot\|^2_Z$ is given by Eq.(2.8). Furthermore, $g_1(Z) \to 0$ as $|Z| \to \infty$.

The proof of this result can be found in Section 7.

This result can be converted into a contraction statement as follows: By Proposition 2.9 and Corollary 2.7, we have:

$$\|w\|^2_{Z(v)} \le Bg_1^2(Z(v)) + \Big(\|w\|^2_{Z(v_0)} - Ag_1^2(Z(v_0))\Big)e^{-M_2 t/2} , \tag{2.12}$$

using Gronwall's Lemma. Choosing a number s in the set $\{s \in \mathbf{R}^+ : s^2 + B\sup_{Z\in\Omega_\Gamma} g_1^2(Z) < M_2\sigma^2\}$ which is not empty for $|Z|$ sufficiently large, we can define the following two sets:

$$\begin{aligned} \mathcal{A} &= \big\{ v = w + u_Z \in \mathcal{T}_{\Gamma,\sigma} : \|w\|_Z < s \big\} , \\ \mathcal{Z} &= \big\{ v \in \mathcal{A} : \|w\|^2_Z < Bg_1^2(Z) \big\} . \end{aligned} \tag{2.13}$$

By Eq.(2.12), we see that $\mathcal{A}$ is exponentially attracted towards $\mathcal{Z}$, and Corollary 2.7 implies that $\mathcal{A} \in \mathcal{T}_{\Gamma,\sigma}$. Denoting $v_t \equiv v(\cdot,t)$ a solution of Eq.(1.1), we see that for $v_0 \in \mathcal{Z}$, as long as $|Z(v_t)| > \Gamma$, then $v_t \in \mathcal{Z}$. Hence, the only way to leave $\mathcal{Z}$ is to reach the boundary $|Z| = \Gamma$.

2.4. Speed of the walls

We want to write equations for the time evolution of the function $Z(t) \equiv Z(v_t)$ where v_t is the solution of Eq.(1.1) with an initial condition $v_0 \in \mathcal{Z}$ and for $t < \sup\{t : |Z(v_t)| > \Gamma\}$.

Note first that

$$\partial_t u_{Z(v)} = \sum_{j \in \mathbf{Z}} \left(\partial_{z_j} u_Z\right)\Big|_{Z=Z(v)} \dot{z}_j ,$$

where

$$\dot{z}_j = \partial_t z_j(v(t)) .$$

We shall use the more compact notation

$$\partial_t u_{Z(v)} = \mathrm{D}_Z u_Z \cdot \partial_t Z(v) .$$

Differentiating the identities

$$\langle v - u_{Z(v)},\, (\partial_{z_j} u_Z)\Big|_{Z=Z(v)} \rangle_\Lambda = 0 , \quad \text{for } |j| \le D_\tau + 1 ,$$

with respect to t, we get, for $|j| \le D_\tau + 1$ and $w = v - u_{Z(v)}$:

$$\begin{aligned} \langle \mathcal{L}(v),\, \tau_{z_j(v)} \rangle_\Lambda &= \langle \mathrm{D}_Z u_Z \cdot \partial_t Z(v),\, \tau_{z_j(v)} \rangle_\Lambda - \langle v - u_{Z(v)},\, \mathrm{D}_Z \tau_{z_j(v)} \cdot \partial_t Z(v) \rangle_\Lambda , \\ \partial_t w &= \mathcal{L}(v) - \mathrm{D}_Z u_Z \cdot \partial_t Z(v) . \end{aligned} \tag{2.14}$$

We define the matrix

$$\begin{aligned} \tilde{\mathcal{S}} &= \left(\tilde{\mathcal{S}}_{ij}\right)_{-D_\tau - 1 \le i,j \le D_\tau + 1} \\ &= \left(\langle \partial_{z_j} u_{Z(v)},\, \tau_{z_i(v)} \rangle_\Lambda \;-\; \langle v - u_{Z(v)},\, \partial_{z_j(v)} \tau_{z_i(v)} \rangle_\Lambda\right)_{-D_\tau - 1 \le i,j \le D_\tau + 1} . \end{aligned}$$

We write Eq.(2.14) in the following matrix notation:

$$\tilde{\mathcal{S}} \cdot \dot{Z} = \langle \mathcal{L}(v), \tau_Z \rangle_\Lambda + \dot{Z} \delta_{|j|, D_\tau + 1} \mathcal{O}(\varepsilon) ,$$

where $\delta_{i,j}$ is the Kronecker delta function. We introduce the notation

$$\mathcal{S} = \tilde{\mathcal{S}} - \delta_{|j|, D_\tau + 1} \mathcal{O}(\varepsilon) . \tag{2.15}$$

It will be proved in Section 5 that the matrix $\mathcal{S}$ is invertible. Hence the equations (2.14) become

$$\begin{aligned} \partial_t Z(v) &= \mathcal{S}^{-1} \cdot \langle \mathcal{L}(v),\, \tau_{Z(v)} \rangle_\Lambda , \\ \partial_t w &= \mathcal{L}(w + u_{Z(v)}) - \mathrm{D}_Z u_Z \cdot \mathcal{S}^{-1} \cdot \langle \mathcal{L}(v),\, \tau_{Z(v)} \rangle_\Lambda . \end{aligned} \tag{2.16}$$

Theorem 2.10. *There exist $c_1 > 0$ and $E > 0$ such that for sufficiently small $\mathcal{T}_{\Gamma,\sigma}$, $v_t \in \mathcal{Z}$ and $Z = Z(v_t)$, one has:*

$$\partial_t z_j = E\left(e^{-c_1 \ell_{j+1}} - e^{-c_1 \ell_j}\right) + \mathcal{O}\left(e^{-c_1 \inf_{j \in \mathbf{Z}} \ell_j} \sup_{j \in \mathbf{Z}} \left(e^{-c_1 \ell_{j+1}} - e^{-c_1 \ell_j}\right)\right) + \mathcal{O}(\varepsilon) , \quad z_j \in \Lambda . \tag{2.17}$$

The proof of this theorem is provided in Section 7.

3. Collapse of a domain

The discussion so far followed Carr and Pego quite closely. Now, we are going to use the freedom of working with an infinite line to get a more precise description of the collapsing mechanism. This is possible because *any* distribution of kinks which is sufficiently 'dilute' and does not get stuck inside $\mathcal{T}_{\Gamma,\sigma}$ for all times, leaves $\mathcal{T}_{\Gamma,\sigma}$ through the 'needle hole' at the Γ-end of the tube. This leads to an almost universal shape of the solution in the interval I_j which has length Γ, under the hypothesis that it was sufficiently large at start (larger than $\Gamma_0 \gg \Gamma$). This is illustrated by numerical integration in Fig. 5. Once the universal shape is reached, the kinks will collapse in a time $T_p < \infty$, and the function will have constant sign in the interval I_j.

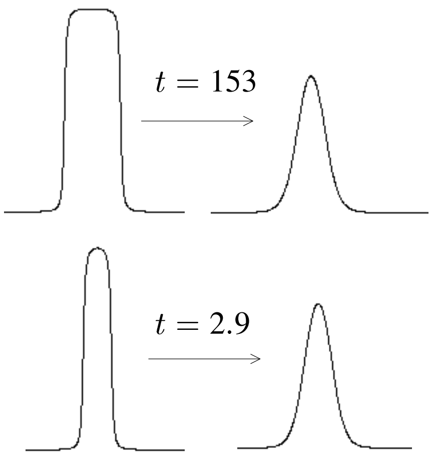

Fig. 5: Two different initial conditions leading to the same shape just before the collapse.

We next give a precise description of this final stage. Suppose that $v_t \in \mathcal{Z}$ for all $t < T$ and that $Z = Z(v_T)$ satisfies $|Z| = \Gamma$. Let $w_t = v_t - u_{Z(v_t)}$. Then, by Proposition 2.6 and Corollary 2.4,

$$\|w_T\|_\Lambda \le \|P_{N_\varepsilon} w_T\|_\Lambda + \|(1 - P_{N_\varepsilon}) w_T\|_\Lambda \le c_2 \big(e^{-c_1\Gamma} + \varepsilon + e^{-MT}\|w_0\|_\Lambda\big) .$$

By the definition of $\mathcal{Z}$ and by Proposition 2.9, $\|w_0\|_\Lambda \le Bg_1(Z) \le c_2 e^{-c_1\Gamma_0}$. Hence $\|w_T\|_\Lambda \le \Lambda(\varepsilon + e^{-c_1\Gamma})$ (see Corollary 5.3 below). In the following theorem, we study the behavior of $v_0 = u_Z$ where Z satisfies: There is a $j \in \mathbf{Z}$ such that $\ell_j = \Gamma$, $\ell_{j\pm1} > \Gamma_0$. In Section 8 we will prove the following theorem:

Theorem 3.1. *For sufficiently small $\mathcal{T}_{\Gamma,\sigma}$ and $\mathcal{T}_{\Gamma_0,\sigma}$, with $\Gamma_0 > \Gamma$, the following holds: Let $v_0 \in \mathcal{T}_{\Gamma_0,\sigma}$. Suppose that for some $T > 0$ and some $i \in \mathbf{Z}$ one has $z_{i+1}(v_T) - z_i(v_T) = \Gamma$, and all other $z_{j+1}(v_T) - z_j(v_T) > \Gamma_0$. Then there is a finite T_p such that $\lim_{t\uparrow T_p} z_{j+1}(v_{t+T}) - z_j(v_{t+T}) = 0$.*

Remark. For large Γ_0 the collapsing time T_p is in fact essentially independent of v_0, and the local shape of the two collapsing kinks is universal (independently of i).

4. Existence of the coarsening dynamics

In this section, we want to describe a probabilistic point of view on the dynamics of the kinks. Since by the above discussion, the Ginzburg-Landau dynamics of many-kink states is essentially specified by the location of these kinks, we will treat a model which implements the dynamics of the (discrete) set of interval lengths.

In the last section, we found an 'effective' equation (Eq.(2.17)) for the coordinates $\{z_j\}_{j\in\mathbf{Z}}$ of the zeros of a solution of the Ginzburg-Landau equation (Eq.(1.1)). Getting rid of the constants and neglecting higher order terms, this equation is:

$$\dot{z}_j \;=\; e^{-(z_{j+1}-z_j)} - e^{-(z_j - z_{j-1})}\;, \quad \text{for } j \in \mathbf{Z}\;.$$

Passing to the variables $\ell_j = z_j - z_{j-1}$ (the interval lengths), we obtain:

$$\dot{\ell}_j \;=\; e^{-\ell_{j+1}} + e^{-\ell_{j-1}} - 2e^{-\ell_j}\;.$$

Introducing $\beta_j = e^{-\ell_j}$ yields:

$$\dot{\beta}_j \;=\; \beta_j\bigl(2\beta_j - \beta_{j+1} - \beta_{j-1}\bigr)\;. \tag{4.1}$$

Furthermore, we define a 'boundary condition': If there exists an index $j \in \mathbf{Z}$ and a time $t > 0$ such that $\beta_j(t-0) = e^{-\Gamma}$ (*i.e.*, $\ell_j(t-0) = \Gamma$) then $\beta_i(t),\ i \in \mathbf{Z}$ is defined by

$$\beta_i(t) \;=\; \begin{cases} \beta_i(t-0) & \text{if } i < j-1\;, \\ \beta_{j-1}(t-0)e^{-\Gamma}\beta_{j+1}(t-0) & \text{if } i = j-1\;, \\ \beta_{i+2}(t-0) & \text{if } i > j-1\;. \end{cases} \tag{4.2}$$

(This corresponds to the merging of the two intervals ℓ_{j-1} and ℓ_{j+1} when ℓ_j vanishes.) The equations (4.1) together with (4.2) define a dynamics on the space $\mathcal{E} = [0, e^{-\Gamma}]^{\mathbf{Z}}$ which we baptize 'coarsening dynamics' in reference to the Bray-Derrida-Godrèche model.

Definition 4.1. *A collapse for $\beta(t)$ satisfying the coarsening dynamics is a time τ such that $\beta(t)$ is discontinuous at $t = \tau$ (i.e., there exists an integer j such that $\beta_j(\tau - 0) = e^{-\Gamma}$).*

We will exhibit a set $\mathcal{C}$ of initial conditions in $\mathcal{E}$ such that the corresponding coarsening dynamics will collapse infinitely often. In terms of the variables z_j, this set can be viewed as a subset of $\mathbf{R}$. Its restriction to any compact subset of $\mathbf{R}$ has positive measure with respect to the probability measure P introduced above.

Remark. In their model, Bray, Derrida and Godrèche describe what should be the asymptotic distribution of interval lengths. Our distribution seems to favor larger intervals than what they expect. However, there are two special features of their model which we do not require here: they allow for only a countable set of interval lengths ($\ell \in \mathbf{N}$) and they study configurations of finite volume (they take finitely many intervals and then study a scale invariant limit, which is maybe equivalent to taking the limit of infinitely many intervals). However, qualitatively, our results are similar to theirs. It would be nice to be able to show that there is a set of initial configurations for Eq.(1.1) of positive measure, such that the evolution does not tend to a stationary state. However, one should keep in mind that the periodic solutions (Proposition 1.1) have stable manifolds. Although these manifolds should be of 'measure zero', we cannot explicitly construct them, and because of the way we estimate the evolution, it is not even obvious to construct a set of initial conditions which is guaranteed not to intersect these manifolds. Controlling the evolution for larger and larger times makes the measure of this set shrink to zero. One should also keep in mind that P is not a measure on the space of initial conditions for Eq.(1.1) but on the space of the positions of the zeros of such an initial condition. There are really many functions whose set of zeros is an element of $\mathcal{C}$, and the analysis of the preceding sections shows that (almost) all such functions have the same long-time behavior. Any initial configuration in $\mathcal{Z}$ for the dynamics of Eq.(1.1) defines an element of $\mathcal{C}$. The evolution in $\mathcal{Z}$ can then be reconstructed from the corresponding evolution in $\mathcal{C}$ under Eq.(4.1), up to terms of order $e^{-\Gamma}$ and terms of order ε, via the function u_Z given by Eq.(2.1).

Let us define the set $\mathcal{C}$:

$$\mathcal{C} = \left\{\beta \in \mathcal{E} \,:\, \exists \{j_n\}_{n\in\mathbf{N}} \subset \mathbf{Z} \text{ s.t. } \forall n \in \mathbf{N},\ \beta_{j_n} \in (\tfrac{e^{-\Gamma}}{2n}, \tfrac{e^{-\Gamma}}{n}),\ \beta_{j_n\pm 1} \in (0, \tfrac{e^{-\Gamma}}{6n})\right\},$$

and state the result:

Theorem 4.2. *Let $t \to \beta(t) \in \mathcal{E}$ be the coarsening dynamics associated with an initial condition $\beta(0) \in \mathcal{C}$. Then there exists an infinite sequence of numbers $0 < \tau_1 < \tau_2 < \ldots < \tau_n < \ldots$, such that $\tau_m \to \infty$ when $m \to \infty$, and, for all $n \in \mathbf{N}$, τ_n is a collapse for $\beta(t)$.*

Proof. Suppose $\beta_0(0) \in (\frac{e^{-\Gamma}}{2n}, \frac{e^{-\Gamma}}{n})$ and $\beta_{\pm 1}(0) < e^{-\Gamma}/(6n)$. Then, by (4.1)

$$\begin{aligned}\dot\beta_0(0) &\geq \tfrac{2e^{-\Gamma}}{3n}\beta_0(0)\,,\\ \dot\beta_{\pm1}(0) &\leq \beta_{\pm1}\big(2\beta_{\pm1} - \beta_0\big) \,<\, 0\,.\end{aligned}$$

In addition, we note that $\dot\beta_0(t) \leq 2\beta_0(t)$. Hence $\beta_{\pm1}(t) < e^{-\Gamma}/(6n)$, which implies

$$\tfrac{e^{-\Gamma}}{2n} e^{2e^{-\Gamma}t/(3n)} \,<\, \beta_0(t) \,<\, \tfrac{e^{-\Gamma}}{n} e^{2t}\,,$$

for all times $t < \sup\{t : \beta_0(t) < e^{-\Gamma}\}$, from which follows that there is a time τ_n in the interval $(\frac{1}{2}\log n, \frac{3}{2} n e^{\Gamma} \log(2n))$ such that $\beta_0(\tau_n) = e^{\Gamma}$. Hence there exists a subsequence τ_{n_j} satisfying the claim. Note that the fact that collapses may occur elsewhere in the meantime is irrelevant, since (apart from shifting the indices) it cannot modify β_0 and it can only make $\beta_{\pm1}$ even smaller.

□

Taking an interval $\Lambda \subset \mathbf{R}$, we define $\mathcal{C}^\Lambda$ as the set $\mathcal{C}$ (viewed as a subset of $\mathbf{R}$ through the correspondence $\{\beta_j\}_{j\in\mathbf{Z}} \leftrightarrow \{z_j\}_{j\in\mathbf{Z}}$) restricted to Λ.

Proposition 4.3. *Let Λ be a compact interval in $\mathbf{R}$ and $|\Lambda|$ be its length. Then there is a $\delta = \delta(|\Lambda|) > 0$ such that*

$$P\big(\mathcal{C}^{\Lambda}\big) \;\geq\; \delta\;,$$

where $P(\cdot)$ is the probability measure defined in Definition 2.1.

Proof. Let $\beta \in \mathcal{C}$ and $\{j_n\}_{n\in\mathbf{N}}$ be the indices such that $\beta_{j_n} \in (\frac{e^{-\Gamma}}{2n}, \frac{e^{-\Gamma}}{n})$ and $\beta_{j_n\pm 1} < \frac{e^{-\Gamma}}{n}$. Let $M^* = \sup\big\{M : \sum_{n=1}^{M} 3\Gamma + \log 2n + 2\log 6n < |\Lambda|\big\}$. The interval Λ cannot contain more than the intervals $\ell_{j_1}, \ldots, \ell_{j_{M^*}}$. Hence

$$P\big(\mathcal{C}^{\Lambda}\big) \;\geq\; \prod_{n=1}^{M^*} \int_{\Gamma+\log n}^{\Gamma+\log 2n} dx\, \rho_{\Gamma}(x) \bigg(\int_{\Gamma+\log 6n}^{\infty} dy\, \rho_{\Gamma}(y) \bigg)^{2}\;.$$

By hypothesis, $\rho(x) > 0$, $\forall x \in (\Gamma, \infty)$, and since Λ is compact, M^* is finite, hence the claim is proved.

$\square$

Let us define the set $\mathcal{C}^*$ which is $\mathcal{C}$ written in the variables z_j:

$$\mathcal{C}^* \;=\; \big\{ Z \in \Omega_{\Gamma} \;:\; \{e^{z_j - z_{j+1}}\}_{j\in\mathbf{Z}} \in \mathcal{C} \big\}\;.$$

We also define $\mathcal{T}^*_{\Gamma,\sigma}$ by replacing Ω_Γ by $\mathcal{C}^*$ in the definition of $\mathcal{T}_{\Gamma,\sigma}$ of Eq.(2.9). The subset $\mathcal{Z}^*$ is then defined by Eq.(2.13), replacing $\mathcal{T}_{\Gamma,\sigma}$ by $\mathcal{T}^*_{\Gamma,\sigma}$. We also denote by $z_j(v)$ the j^{th} zero of the function v, with $z_j(v) < z_{j+1}(v)$.

Theorem 4.4. *For sufficiently small $\mathcal{T}^*_{\Gamma,\sigma}$, for all $v_0 \in \mathcal{T}^*_{\Gamma,\sigma}$, if v_t denotes the solution of Eq.(1.1) associated with the initial condition v_0, there exist a sequence of times $\{t_n\}_{n\in\mathbf{N}}$ and a sequence of indices $\{j_n\}_{n\in\mathbf{N}}$ such that $\lim_{t\to t_n} \big|z_{j_n}(v_t) - z_{j_n-1}(v_t)\big| = 0$, and $\lim_{n\to\infty} t_n = \infty$.*

Proof. Denote by $\{j_n\}_{n\in\mathbf{N}}$ the indices such that $e^{-(z_{j_n} - z_{j_n-1})} \equiv e^{-\ell_{j_n}} \in (\frac{e^{-\Gamma}}{2n}, \frac{e^{-\Gamma}}{n})$ and $e^{-(z_{j_n\pm1} - z_{j_n\pm1-1})} \equiv e^{-\ell_{j_n\pm1}} < \frac{e^{-\Gamma}}{n}$. Choose a set $\{\Lambda_j\}_{j\in\mathbf{Z}}$ of disjoint compact intervals of $\mathbf{R}$, such that for all n, there exists a k with $[z_{j_n-1}, z_{j_n}] \subset \Lambda_k$, *i.e.,* each interval of length ℓ_{j_n} is contained in a single interval Λ_k. Associate with each interval Λ_j a weight μ_j as in Definition 2.2. Then, by Theorem 2.10, the dynamics of the zeros z_{j_n} is given by Eq.(2.17) and their collapse is described by Theorem 3.1. By Theorem 4.2, for sufficiently small $\mathcal{T}^*_{\Gamma,\sigma}$, there exists a sequence of collapsing times. This proves the assertion.

$\square$

5. Miscellaneous bounds

We first give estimates on the behavior of the function u_Z given in Eq.(2.1). In particular, we show that near the set Z, this function is so close to the heteroclinic solution ψ, that it is almost stationary in an L^∞ sense.

Lemma 5.1. *There exist positive K and c_1, such that for sufficiently large $|Z|$, the following holds:*
1) $|\psi((-1)^{j+1}(x-z_j)) - u_Z(x)| \le Ke^{-c_1\min(\ell_j,\ell_{j+1})}$, for $|x-z_j| \le 1$ and for $j \in \mathbf{Z}$.
2) $\|\mathcal{L}(u_Z)\|_\infty \le Ke^{-c_1|Z|}$,

Proof. We first compare φ_P with ψ for fixed P. Let $g(x) = \psi(x) - \varphi_P(x)$, $\alpha = V(\varphi_P(P/2)) = V(-A(P))$, *cf.* Fig. 2, and suppose $x \in [-P/2, P/2]$. If f is a stationary solution of Eq.(1.1) then $f'' + V'(f) = 0$, and thus $f''f' + V'(f)f' = 0$, *i.e.*, $\frac{1}{2}(f')^2 + V(f)$ is constant, and taking x^* with $f'(x^*) = 0$, we get

$$\tfrac{1}{2}(f')^2(x) + V(f(x)) \;=\; V(f(x^*))\;.$$

Therefore the derivative of g satisfies (note that for $x \in [-P/2, P/2]$, $\varphi_P(x)$ is monotone):

$$\begin{aligned}
|g'(x)| &= \sqrt{2}\Big|\sqrt{V(\psi(\infty)) - V(\psi(x))} - \sqrt{\alpha - V(\varphi_P(x))}\,\Big| \\
&= \sqrt{2}\Big|\sqrt{V(\psi(\infty)) - V(\psi(x))} - \sqrt{-(V(\psi(\infty)) - \alpha) + V(\psi(\infty)) - V(\varphi_P(x))}\,\Big| \\
&\le \sqrt{2}\Big|\sqrt{V(\psi(\infty)) - V(\psi(x))} - \sqrt{V(\psi(\infty)) - V(\varphi_P(x))}\Big| + \Big|\sqrt{V(\psi(\infty)) - \alpha}\,\Big| \\
&\le C\big(|\psi(x) - \varphi_P(x)| + e^{-c_1P}\big) \;=\; C\big(|g(x)| + e^{-c_1P}\big)\;.
\end{aligned}$$

In the third line, we have used the inequality $-\sqrt{-a+b} \le \sqrt{a} - \sqrt{b}$, and in the last line, the first term comes from the differentiability of the function V while the second term is a consequence of Eq.(9.3) below. We apply Gronwall's lemma (and $g(0) = 0$) and get

$$|g(x)| \;\le\; K_1e^{-c_1P}\;,$$

and

$$|g'(x)| \;\le\; K_2e^{-c_1P}\;.$$

Now, recalling the definition (2.1) of u_Z, we have $u_Z(x) = \big(1 - \Delta(x-z_j)\big)\varphi_{\ell_j}(x - z_{j-1}) + \Delta(x-z_j)\varphi_{\ell_{j+1}}(x - z_{j+1})$ for $|x - z_j| < \inf(\ell_j, \ell_{j+1})/2$. Hence

$$\min(\varphi_{\ell_j}(x - z_{j-1}), \varphi_{\ell_{j+1}}(x - z_{j+1})) \;\le\; u_Z(x) \;\le\; \max(\varphi_{\ell_j}(x - z_{j-1}), \varphi_{\ell_{j+1}}(x - z_{j+1}))\;.$$

Consequently

$$
\begin{aligned}
&|\psi((-1)^{j+1}(x-z_j)) - u_Z(x)| \\
&= \max\Big(\psi((-1)^{j+1}(x-z_j)) - u_Z(x), \\
&\qquad\qquad u_Z(x) - \psi((-1)^{j+1}(x-z_j))\Big) \\
&\le \max\Big(\psi((-1)^{j+1}(x-z_j)) - \min\big(\varphi_{\ell_j}(x-z_{j-1}), \varphi_{\ell_{j+1}}(x-z_{j+1})\big), \\
&\qquad\qquad \max\big(\varphi_{\ell_j}(x-z_{j-1}), \varphi_{\ell_{j+1}}(x-z_{j+1})\big) - \psi((-1)^{j+1}(x-z_j))\big)\Big) \\
&\le \max\big(K_1 e^{-c_1\ell_j}, K_1 e^{-c_1\ell_{j+1}}\big) ,
\end{aligned}
$$

which proves claim 1).

We write, for $x \in I_j$,

$$
\mathcal{L}(u_Z) \equiv \Delta''(\varphi_{\ell_j} - \varphi_{\ell_{j+1}}) + 2\Delta'(\varphi'_{\ell_j} - \varphi'_{\ell_{j+1}}) - G .
$$

Using the fact that φ_P is a solution of $\mathcal{L}(u) = 0$, we have

$$
G = (1-\Delta)V'(\varphi_{\ell_j}) + \Delta V'(\varphi_{\ell_{j+1}}) - V'\big((1-\Delta)\varphi_{\ell_j} + \Delta\varphi_{\ell_{j+1}}\big) .
$$

We expand G near 0 and look at the coefficient of $V'''(0)/2$, which is the first non-vanishing term:

$$
\begin{aligned}
&(1-\Delta)\varphi_{\ell_j}^2 + \Delta\varphi_{\ell_{j+1}}^2 - \big((1-\Delta)\varphi_{\ell_j} + \Delta\varphi_{\ell_{j+1}}\big)^2 \\
&= (1-\Delta)\varphi_{\ell_j}^2 + \Delta\varphi_{\ell_{j+1}}^2 - (1-\Delta)^2\varphi_{\ell_j}^2 - \Delta^2\varphi_{\ell_{j+1}}^2 - 2(1-\Delta)\Delta\varphi_{\ell_j}\varphi_{\ell_{j+1}} \\
&= (1-\Delta)\Delta\big(\varphi_{\ell_j}^2 + \varphi_{\ell_{j+1}}^2 - 2\varphi_{\ell_j}\varphi_{\ell_{j+1}}\big) \\
&\le (\varphi_{\ell_{j+1}} - \varphi_{\ell_j})^2 .
\end{aligned}
$$

Consequently, $|G| \le \varkappa_3|\varphi_{\ell_{j+1}} - \varphi_{\ell_j}|^2$ and thus, using 1),

$$
|\mathcal{L}(u)| \le \varkappa_1|g(x)| + \varkappa_2|g'(x)| + \varkappa_3|g(x)|^2 \le Ke^{-c_1\min(\ell_j,\ell_{j+1})} ,
$$

which completes the proof of claim 2).

□

We next give estimates related to the vectors τ_{z_j} introduced in Eq.(2.5).

Lemma 5.2. *Let Z in Ω_Γ, $c_j = \frac{1}{2}(z_j + z_{j-1})$, and τ_{z_j} as defined in Eq.(2.5). Then, there exist $K > 0$, $c_1 > 0$, and $c_2 > 0$ such that for sufficiently large Γ, one has:*

$$
\begin{aligned}
\langle\mathcal{L}(u_Z), \tau_{z_j}\rangle_\Lambda &= c_2\big(V(u_Z(c_j)) - V(u_Z(c_{j+1}))\big) + \mathcal{O}(\varepsilon) , \quad \text{for } z_j \in \Lambda , \\
\|L_Z\tau_{z_j}\|_\Lambda &\le Ke^{-c_1\min(\ell_j,\ell_{j+1})} , \text{for } |j| \le D_\tau .
\end{aligned}
$$

Proof. We compute, using Definition 2.2,

$$\begin{aligned}
&\int_{\mathbf{R}} d\mu\, \mathcal{L}\big(u_Z(x)\big)\tau_{z_j}(x) \;=\; C\int_{c_j+1}^{c_{j+1}-1} dx\,\big(\partial_x^2 u_Z + V'(u_Z)\big)\partial_x u_Z + \mathcal{O}(\varepsilon)\\
&= C\Big(\tfrac{1}{2}(\partial_x u_Z)^2 + V(u_Z)\Big)(c_j+1) - C\Big(\tfrac{1}{2}(\partial_x u_Z)^2 + V(u_Z)\Big)(c_{j+1}-1) + \mathcal{O}(\varepsilon)\\
&= C\Big(V(u_Z(c_{j+1}) - V(u_Z(c_j))\Big) + \mathcal{O}(\varepsilon)\;.
\end{aligned}$$

This completes the proof of the first claim. For the second one, we use that τ_{z_j} has compact support and is equal to a stationary solution of Eq.(1.1) in the interval $I_j \cap \{x : |x - z_j| > 1\}$:

$$\begin{aligned}
\|L_Z\tau_{z_j}\|_\Lambda^2 \;&=\; \int_{c_j-1}^{c_{j+1}+1} d\mu\,\left|\partial_x^2\tau_{z_j} + V'(u_Z)\tau_{z_j}\right|^2\\
&\le\; C\int_{z_j-1}^{z_j+1} dx\,\left|\partial_x^2\tau_{z_j} + V'(u_Z)\tau_{z_j}\right|^2\\
&\le\; \Big(C\sup_{|x-z_j|<1}|\mathcal{L}(u_Z)|\Big)^2\;,
\end{aligned}$$

and we use Lemma 5.1 to conclude.

□

Corollary 5.3. *Let* $g_1^2(Z) = \sum_{|j|\le D_\tau} |\langle\mathcal{L}(u_Z),\,\tau_{z_j}\rangle_\Lambda|^2$ *and* $g_2(Z) = \sup\{\frac{\|L_Z\tau\|_\Lambda}{\|\tau\|_\Lambda} : \tau \in \operatorname{span}\{\tau_{z_j} : |j| \le D_\tau\}\backslash\{0\}\}$*. Then, under the hypothesis of Lemma 5.2, there exists* $K_1 > 0$, $K_2 > 0$ *such that:*

$$|g_1(Z)| \;\le\; K_1 e^{-c_1|Z|}\;, \qquad |g_2(Z)| \;\le\; K_2 e^{-c_1|Z|}\;.$$

Proof. The first claim follows from Eq.(9.3) and the second from the following calculation: Let $\tau \equiv \sum_{|j|\le D_\tau} t_j\tau_{z_j}$. Then

$$\begin{aligned}
\|L_Z\tau\|_\Lambda \;&\le\; \sum_{|j|\le D_\tau}|t_j|\|L_Z\tau_{z_j}\|_\Lambda \;\le\; \sum_{|j|\le D_\tau}|t_j|\sup_{|j|\le D_\tau}\|L_Z\tau_{z_j}\|_\Lambda\;,\\
\|\tau\|_\Lambda \;&\ge\; \inf_{|j|\le D_\tau}\|\tau_{z_j}\|_\Lambda\sum_{|j|\le D_\tau}|t_j|\;.
\end{aligned}$$

We finish the proof by noting that $\tau_{z_j}(x)$ is strictly positive in $[c_j + 1, c_{j+1} - 1]$ hence its norm is uniformly bounded from below for $|j| \le D_\tau$. Then we apply Lemma 5.2.

□

Next we prove that certain matrices used in Section 7 have a bounded inverse.

Lemma 5.4. *For sufficiently small $\mathcal{T}_{\Gamma,\sigma}$, and all $v \in \mathcal{T}_{\Gamma,\sigma}$ and for $N < \infty$, the matrices*

$$\begin{aligned}\tilde{\mathcal{S}} &= \left(\tilde{\mathcal{S}}_{ij}\right)_{i,j=1,\ldots,N} = \left(\langle \partial_{z_j} u_Z,\, \tau_{z_i}\rangle_\Lambda - \langle v - u_Z,\, \partial_{z_j}\tau_{z_i}\rangle_\Lambda\right)_{i,j=1,\ldots,N}\,, \\ \mathcal{S}_1 &= \left((\mathcal{S}_1)_{ij}\right)_{i,j=1,\ldots,N} = \left(\langle \partial_{z_j} u_Z,\, \tau_{z_i}\rangle_\Lambda\right)_{i,j=1,\ldots,N}\,,\end{aligned}$$

have uniformly bounded inverse.

Proof. We start by the following remark: since, by our assumption on V, we have $\pi \le P_0 < \Gamma$, the tangent vectors τ_{z_j} and $\tau_{z_{j+2}}$ have disjoint support. Therefore, the matrix $\tilde{\mathcal{S}}$ is tridiagonal and we only have to control the overlap between τ_{z_j} and $\tau_{z_{j\pm1}}$. To prove that $\tilde{\mathcal{S}}$ is invertible, we show that it is diagonally dominant, *i.e.*,

$$|\tilde{\mathcal{S}}_{ii}| > \sum_{j\neq i} |\tilde{\mathcal{S}}_{ij}|\,.$$

1) The diagonal terms are $\tilde{\mathcal{S}}_{ii} = \langle \partial_{z_i} u_Z,\, \tau_{z_i}\rangle_\Lambda - \langle w,\, \partial_{z_i}\tau_{z_i}\rangle_\Lambda$. The first term is uniformly bounded below, by Proposition 2.3 of [CP1]. In fact, it is a consequence of

$$|\langle \partial_{z_i}\varphi_{\ell_i}, \Delta_i \partial_x \varphi_{\ell_i}\rangle_\Lambda| \approx |\langle \partial_{z_i}\psi(\cdot - z_i), \partial_x\psi(\cdot - z_i)\rangle_\Lambda| = \|\partial_x\psi\|_\Lambda^2 > K\,.$$

   The second term in $\tilde{\mathcal{S}}_{ii}$ is $\mathcal{O}(\sigma)$, thus, for sufficiently small $\mathcal{T}_{\Gamma,\sigma}$, the whole expression is bounded below.
2) We next control the off-diagonal terms $\tilde{\mathcal{S}}_{ij} = \langle \partial_{z_j} u_Z,\, \tau_{z_i}\rangle_\Lambda - \langle w,\, \partial_{z_j}\tau_{z_i}\rangle_\Lambda$. The first term is bounded by a constant which goes to zero as Γ goes to infinity, see again Proposition 2.3 of [CP1] (recall that τ_{z_j} has compact support, and the overlap between τ_{z_j} and $\partial_{z_{j\pm1}}\varphi_{\ell_j}$ is very small). The second term is treated as before.
3) The proof for $\mathcal{S}_1$ is a special case of 1) and 2).

□

Remark. Obviously, for ε sufficiently small, the matrix $\mathcal{S}$ defined in Eq.(2.15) is also invertible.

6. Proofs of the properties of the linear operator

The first proof we provide in this section concerns the spectrum of $L_Z = -\partial_x^2 - V''(u_Z)$.

Sketch of the proof of Theorem 2.3. First of all, we show that the operator L_Z with $Z = \{-z, z\}$ (and the convention that $\varphi_\ell(x - z) = \psi(x - z)$ if $\ell = \infty$) has two eigenvalues satisfying the bounds of Theorem 2.3. The function u_Z (with $\mathrm{card}(Z) = 2$) is positive at $|x| \gg 1$ and negative at $x = 0$. We can view L_Z as a Schrödinger operator with a potential $U(x) = -V''(u_Z(x))$, which is a symmetric double well (see Fig. 6), $U_{\min} < U(x) < U_{\max}$. Its spectrum is made up of isolated eigenvalues in $(U_{\min}, U_{\max})$ and absolutely continuous spectrum in $[U_{\max}, \infty)$. When $|Z| \to \infty$, then the lowest eigenvalue is degenerate, and it is (by translation) given as the lowest eigenvalue of $L^* = -\partial_x^2 - V''(\psi)$. In this limit, $\psi'(x)$ is an eigenfunction of L^* with eigenvalue 0 (it corresponds to the invariance under translation). This is the ground state, since ψ' is a positive function. This double eigenvalue splits into $\lambda_- < \lambda_+$ when $|Z| < \infty$. The proof uses the fact that e_- (e_+), the corresponding eigenfunctions, are the even (odd) extensions of the ground state of the same operator with Neumann (Dirichlet) boundary condition at $x = 0$, and the splitting is a consequence of the Dirichlet-Neumann bracketing. Furthermore, the splitting will be exponentially small as $|Z| \to \infty$ (note that $|\varphi_P(x) - \psi(x)| \le e^{-c_1 P}$, for $|x| < P/2$, see Lemma 5.1): one has $\lambda_+ - \lambda_- \le \Lambda e^{-c_1|Z|}$ (see [RS4], p.34, example 6).

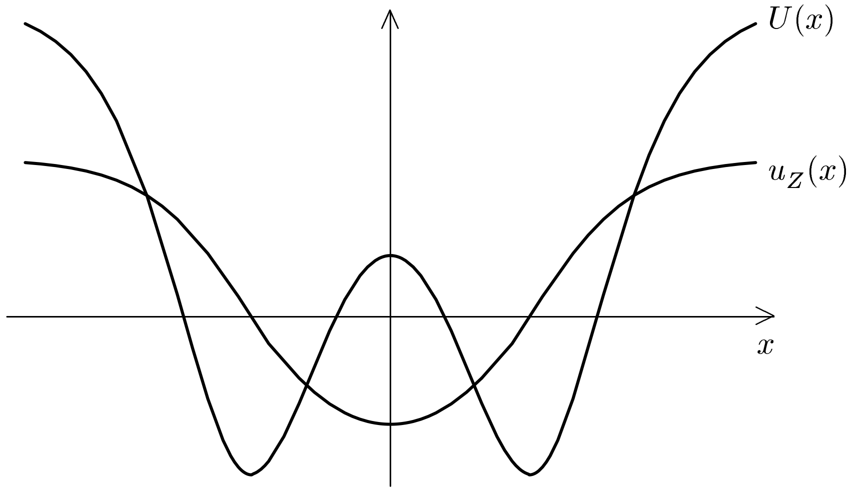

Fig. 6: The potential $U(x)$ together with the function $u_Z(x)$. Note that the minima of U coincide with the zeros of u_Z.

By similar reasoning, for any $N < \infty$, the spectrum of L_Z with Dirichlet boundary conditions on c_{-N} and c_N has $2N + 1$ eigenvalues satisfying the bounds of Theorem 2.3.

Next, we study the spectrum of L_Z on the line, with infinitely many kinks, which can be viewed as a Schrödinger operator with infinitely many potential wells, distributed with the probability P. We write $W(x) \equiv -V''\bigl(u_Z(x)\bigr)$ for this 'disordered' potential. We define the set of intervals

$$\mathfrak{I}_0 = \{I_{0,j} = [z_j - \Gamma/2, z_j + \Gamma/2],\ j \in \mathbf{Z}\}\ .$$

These are intervals of length $d_0 = \Gamma$ centered at $c_{0,j} = z_j$, *i.e.,* at the bottoms of the potential wells. Then, $S_0(Z, \lambda)$ is the set of the intervals in which the potential is smaller than the energy λ plus $\frac{1}{2}$:

$$S_0(Z, \lambda) = \left\{I_{0,j} \in \mathfrak{I}_0 : \forall x \in I_{0,j}\,, W(x) \le \lambda + \tfrac{1}{2}\right\}\ .$$

We then inductively construct a hierarchy of sets $\mathfrak{I}_n$ containing intervals $I_{n,j}$ centered at $c_{n,j}$ and defined as follows:

— If $c_{n,j+1} - c_{n,j} > 3d_n$ then $c_{n+1,j} = c_{n,j}$ and $I_{n+1,j}$ has length $d_{n+1} = 2d_n$.

— If $c_{n,j+1} - c_{n,j} \leq 3d_n$ then $c_{n+1,j} = \frac{1}{2}(c_{n,j} + c_{n,j+1})$ and $I_{n+1,j}$ has length $d_{n+1} + d_n = 3d_n$.

Hence $I_{n+1,j}$ contains at most two intervals $I_{n,j}$ and its endpoints do not belong to an interval $I_{m,k}$ with $m < n$. After renumbering the intervals, we obtain a set $\mathcal{I}_{n+1} = \{I_{n+1,j},\, j \in \mathbf{Z}\}$ of intervals with centers $c_{n+1,j}$ satisfying $c_{n+1,j+1} > c_{n+1,j}$ for all integers j. We call 'singular' the intervals $I_{n,j}$ satisfying:

> **(C1)** The operator $L_Z(I_{n,j})$, *i.e.*, L_Z with Dirichlet boundary conditions on $\partial I_{n,j}$, has an eigenvalue $\lambda_{n,j}$ such that $|\lambda - \lambda_{n,j}| \leq 2^{-n\beta}$,

where β will be fixed later. The 'singular sets' are then defined as :

$$S_n(Z,\, \lambda) \;=\; \bigl\{ I_{n,j} \in \mathcal{I}_n \;:\; I_{n,j} \text{ satisfies } (\mathbf{C1}) \bigr\} \;.$$

Lemma 6.1. *Let $I_{n,j} \in \mathcal{I}_n$. There is a $c_1 > 0$ and a $Z \in \Omega_\Gamma$ such that if $\Lambda^{(n)} = [\Lambda_1^{(n)}, \Lambda_2^{(n)}] \subset (-\alpha, \alpha)$, with $\alpha = \mathcal{O}(e^{-c_1|Z|})$ when $|Z| \to \infty$, we have the following estimate:*

$$P\Bigl(\operatorname{spec}(L_Z(I_{n,j})) \,\cap\, \Lambda^{(n)} \neq \emptyset\Bigr) \;\leq\; (\Lambda_2^{(n)} - \Lambda_1^{(n)})\, 2^{2n}\, \|\rho_\Gamma\|_\infty \;. \tag{6.1}$$

Proof of the lemma. We denote by

$$N_L(x) \;\equiv\; \text{card}\bigl\{ E \leq x \;:\; E \text{ is an eigenvalue of } L \equiv L_Z(I_{n,j}) \bigr\}$$

the integrated density of states and, if A is a random variable over the space of potentials W, then $\mathcal{E}(A)$ is its expectation value. The l.h.s. of (6.1) is bounded by $\mathcal{E}\bigl(N_L(\Lambda_2^{(n)}) - N_L(\Lambda_1^{(n)})\bigr)$. Obviously $N_L(\Lambda_2^{(n)}) \geq N_L(\Lambda_1^{(n)}) \geq 0$, and, if $N_+ \equiv N_L(\Lambda_2^{(n)}) - \mathcal{E}\bigl(N_L(\Lambda_1^{(n)})\bigr)$ and $N_- \equiv N_L(\Lambda_1^{(n)}) - \mathcal{E}\bigl(N_L(\Lambda_1^{(n)})\bigr)$, then

$$\begin{aligned}
\mathcal{E}\bigl(N_L(\Lambda_2^{(n)}) - N_L(\Lambda_1^{(n)})\bigr) \;&=\; \mathcal{E}\bigl(N_+ - N_-\bigr) \;\leq\; \mathcal{E}\bigl(N_+\bigr) \\
&=\; \mathcal{E}\bigl(N_+\bigr) - \mathcal{E}\bigl(N_-\bigr) \;=\; \mathcal{E}\bigl(N_L(\Lambda_2^{(n)})\bigr) - \mathcal{E}\bigl(N_L(\Lambda_1^{(n)})\bigr) \;.
\end{aligned}$$

Thus the l.h.s. of (6.1) is also bounded by $\int_{\Lambda_1^{(n)}}^{\Lambda_2^{(n)}} d\lambda \frac{d\mathcal{E}(N_L)}{d\lambda}$. Note that $I_{n,j}$ contains at most 2^n wells and, since $N_L(\lambda) = N_{L-\lambda}(0)$, we can replace derivatives with respect to the second argument by (minus) the derivatives w.r.t. the first one. Note also that $L_Z(I_{n,j})$ depends only on the interval lengths ℓ_j, $j = 1, \ldots, 2^n$ (up to a translation of the indices), which gives:

$$\begin{aligned}
&\mathcal{E}\bigl(N_L(\Lambda_2^{(n)})\bigr) - \mathcal{E}\bigl(N_L(\Lambda_1^{(n)})\bigr) \\
&=\; \sum_{j=1,\ldots,2^n} \int_{\Lambda_1^{(n)}}^{\Lambda_2^{(n)}} d\lambda \int_\Gamma^\infty \rho_\Gamma(\ell_j) \left(-\frac{\partial N_L}{\partial \ell_j} \right)\, d\ell_j\, \prod_{k\neq j} \rho_\Gamma(\ell_k)\, d\ell_k \\
&\leq\; 2^n (\Lambda_2^{(n)} - \Lambda_1^{(n)}) \|\rho_\Gamma\|_\infty \Bigl(N_L(\ell_j = \Gamma) - N_L(\ell_j = \infty) \Bigr) \\
&=\; 2^{2n} (\Lambda_2^{(n)} - \Lambda_1^{(n)}) \|\rho_\Gamma\|_\infty \;,
\end{aligned}$$

where we have used that there is a number $\alpha > 0$ such that $L_Z(I_{n,j})$ has only 2^n eigenvalues $|\lambda_j| < \alpha$ with $\alpha = \mathcal{O}(e^{-c_1|Z|})$.

$\square$

Taking $\Lambda^{(n)} = [\lambda - 2^{-n\beta}, \lambda + 2^{-n\beta}]$ and $\beta > 4$, the sum over all n of the r.h.s. of (6.1) is a finite number. We can apply the Borel-Cantelli Lemma and conclude that the probability

$$P\Big(\text{card}\big\{n \,:\, \text{spec}(L_Z(I_{n,j})) \cap \Lambda^{(n)} \neq \emptyset\big\} \;=\; \infty\Big)$$

is zero, or, in other words, that there exists almost surely a number $N = N(Z, \lambda) < \infty$ such that all the $I_{N,j}$ violate the condition $\mathbf{(C1)}$.

The remainder of the proof is very similar to the one in [FSW], namely, one shows that the Green's function $G(\lambda, x, y)$ is exponentially decaying at large distances (greater than d_N) in any $N-$regular interval A (it means an interval which does not contain any of the $I_{N,j}$ belonging to $S_N(Z,\, \lambda)$). It is proved by recursion over n as follows:

— If $A \cap S_0 = \emptyset$ it is obvious from the definition of S_0 that $G(\lambda, x, y) \leq e^{-c|x-y|}$ if $|x-y| > d_0$.

— If the Green's function decays exponentially at distances larger than d_n in $n-$regular intervals and $A \cap S_{n+1} = \emptyset$, then one can use the resolvent identity (subscripts I indicate Dirichlet boundary conditions at the endpoints of I):

$$G_A(\lambda, x, y) \;=\; G_{I_{n,j}}(\lambda, x, y) + \sum_{z \in \partial I} \dot{G}_{I_{n,j}}(\lambda, x, z) G_A(\lambda, z, y)\;,$$

where $\dot{G}$ is the derivative with respect to z. It allows us to express G_A in terms of $G_{I_{n,j}}$ to which the recursion hypothesis applies, since the $I_{n,j}$ are of length d_n and we want to prove exponential decay of $G(x, y)$ for x, y distant by more than d_{n+1}.

— There is a similar bound for $\dot{G}$, because one can write G as a function of G_0, the Green's function of the 'free' operator $-\partial_x^2 + W_{\max}$, using a resolvent identity:

$$G \;=\; G_0 + (W_{\max} - W) G\, G_0\;,$$

and G_0 as well as its derivatives behave like $e^{-\sqrt{2-\alpha}|x-y|}$, for all $\lambda < M < 2$.

Given the exponential decay of the Green's function, the behavior of the eigenfunctions follows from the formula:

$$e_j(x) \;=\; \sum_{z \in \partial A} \dot{G}_A(\lambda, x, z) e_j(z)\;.$$

$\square$

Remark. We proved that the Green's function decays exponentially with a certain rate $\varkappa$. Hence this rate is the same for all eigenfunctions e_j, $j \in \mathbf{N}$.

Proof of Proposition 2.6. We use the following notations: $P(\cdot)$ is the spectral measure associated with L_Z, $Z \in \Omega^*$, and $M > 0$ is as in Theorem 2.3. We show that the restriction of P_{N_ε} to the subspace span$\{\tau_{z_j}\,,\, j \in \mathbf{Z}\}\backslash\{0\}$ has empty kernel. Let $\tau \in$ span$\{\tau_{z_j}\,,\, j \in \mathbf{Z}\}\backslash\{0\}$.

$$\begin{aligned} M^2\|(1-P_{N_\varepsilon})\tau\|_\Lambda^2 \;&=\; M^2\int_M^\infty \langle P(d\lambda)\tau,\,\tau\rangle_\Lambda + M^2\sum_{j>N_\varepsilon}\langle e_j,\tau\rangle_\Lambda^2 \\ &\le \int_{\mathbf{R}} \lambda^2\langle P(d\lambda)\tau,\,\tau\rangle_\Lambda + \mathcal{O}(\varepsilon) = \|L_Z\tau\|_\Lambda^2\,, \end{aligned}$$

hence $\|(1-P_{N_\varepsilon})\tau\|_\Lambda^2 \le B^2(Z)\|\tau\|_\Lambda^2$ where $B(Z) \equiv g_2(Z)/M$. By Corollary 5.3, $B(Z) < 1$ hence we have $\inf\|P_{N_\varepsilon}\tau\|_\Lambda \ge \big(1 - B(Z)\big)\|\tau\|_\Lambda \ge \beta\|\tau\|_\Lambda$ with $\beta > 0$. This proves that the map

$$P_{N_\varepsilon} : \text{span}\{\tau_{z_j},\, |j| \le D_\tau\} \to \bigoplus_{k\le N_\varepsilon} \mathcal{H}_{\lambda_k}$$

has trivial kernel. In addition, it is a map between finite-dimensional spaces, and if $D_\tau > N_\varepsilon$, then it is surjective.

Now, define $w = v - u_{Z(v)}$, $\tau \in$ span$\{\tau_{z_j}\,,\, |j| \le D_\tau\}\backslash\{0\}$ such that $P_{N_\varepsilon}w = P_{N_\varepsilon}\tau$. Recall that, by hypothesis, $\langle w, \tau\rangle_\Lambda = 0$, thus:

$$\begin{aligned} \|P_{N_\varepsilon}w\|_\Lambda^2 \;&=\; |\langle w,\, P_{N_\varepsilon}w\rangle_\Lambda| \\ &= |\langle w,\, P_{N_\varepsilon}\tau\rangle_\Lambda| \\ &= |\langle w,\, (P_{N_\varepsilon}-1)\tau\rangle_\Lambda| \\ &\le \|w\|_\Lambda\|(P_{N_\varepsilon}-1)\tau\|_\Lambda \\ &\le \frac{g_2(Z)}{M}\|w\|_\Lambda\|\tau\|_\Lambda \;\le\; \frac{g_2(Z)/M}{1-g_2(Z)/M}\|w\|_\Lambda^2\,, \end{aligned}$$

using $\|w\|_\Lambda \ge \|P_{N_\varepsilon}w\|_\Lambda = \|P_{N_\varepsilon}\tau\|_\Lambda \ge \|\tau\|_\Lambda + \|(1-P_{N_\varepsilon})\tau\|_\Lambda \ge \|\tau\|_\Lambda(1-g_2(Z)/M)$. The proof is complete, since we can use the bound of Corollary 5.3.

□

Proof of Corollary 2.7. We let $Z = Z(v)$ and use the following notations: Z^* is as in Proposition 2.8, $P(\cdot)$ is the spectral measure associated with L_{Z^*} and $\alpha > 0$, $M > 0$ are as in Theorem 2.3.

We start by proving the second inequality of Eq.(2.6). By the Spectral Theorem, we have

$$\begin{aligned}
\langle w,\, L_Z w\rangle_\Lambda \;&=\; \langle w,\, L_{Z^*} w\rangle_\Lambda + \mathcal{O}(\varepsilon)\|w\|_\Lambda^2 \\
&=\; \int_{\mathbf{R}} \lambda \langle P(d\lambda) w,\, w\rangle_\Lambda + \mathcal{O}(\varepsilon)\|w\|_\Lambda^2 \\
&\geq\; M\int_{\mathbf{R}} \langle P(d\lambda) w,\, w\rangle_\Lambda + \int_{-\infty}^{M} \big(\lambda - M\big)\langle P(d\lambda) w,\, w\rangle_\Lambda + \mathcal{O}(\varepsilon)\|w\|_\Lambda^2 \\
&\geq\; M\|w\|_\Lambda^2 + \sum_{j\leq N_\varepsilon} (\lambda - M)\langle w, e_j\rangle_\Lambda^2 + \mathcal{O}(\varepsilon)\|w\|_\Lambda^2 \\
&\geq\; M\|w\|_\Lambda^2 + (-\alpha - M)\|P_{N_\varepsilon} w\|_\Lambda^2 + \mathcal{O}(\varepsilon)\|w\|_\Lambda^2 \,.
\end{aligned}$$

By Proposition 2.6, for $|Z|$ large, $\|P_{N_\varepsilon} w\|_\Lambda \leq c_2(e^{-c_1|Z|})\|w\|_\Lambda$. Thus

$$\langle w,\, L_Z w\rangle_\Lambda \;\geq\; \|w\|_\Lambda^2 \Big(M(1 - \mathcal{O}(e^{-c_1|Z|})) + \mathcal{O}(\varepsilon)\Big) \;\equiv\; M_1 \|w\|_\Lambda^2 \,. \tag{6.2}$$

If $\Gamma = |Z|$ is sufficiently large and ε sufficiently small, M_1 is positive.

To prove the first inequality of Eq.(2.6), we do similar calculations:

$$\begin{aligned}
\langle L_Z w,\, L_Z w\rangle_\Lambda \;&=\; \int_{\mathbf{R}} \lambda^2 \langle P(d\lambda) w,\, w\rangle_\Lambda + \mathcal{O}(\varepsilon)\|w\|_\Lambda^2 \\
&\geq\; M\int_{\mathbf{R}} \lambda\langle P(d\lambda) w,\, w\rangle_\Lambda + \int_{-\infty}^{M} \big(\lambda - M\big)\lambda\langle P(d\lambda) w,\, w\rangle_\Lambda \\
&\quad + \mathcal{O}(\varepsilon)\|w\|_\Lambda^2 \\
&\geq\; M\langle w,\, L_Z w\rangle_\Lambda + (-\alpha - M)M\|P_{N_\varepsilon} w\|_\Lambda^2 + \mathcal{O}(\varepsilon)\|w\|_\Lambda^2 \\
&\geq\; \Big(M(1 - \mathcal{O}(e^{-c_1|Z|}) + \mathcal{O}(\varepsilon))\Big)\langle w,\, L_Z w\rangle_\Lambda \,,
\end{aligned}$$

where we have used Eq.(6.2). If $|Z| = \Gamma$ is sufficiently large and ε sufficiently small, the assertion follows.

We next prove Eq.(2.7). Since $\partial_x(w^2) = 2w\partial_x w \leq (\partial_x w)^2 + w^2$, supposing that $\Lambda = [-K, K]$, we have:

$$\begin{aligned}
\|\chi_\Lambda w\|_\infty^2 \;&=\; \sup_{y\in\mathbf{R}} \int_{-\infty}^{y} dx\, \partial_x(\chi_\Lambda^2 w^2) \\
&=\; \sup_{y\in\mathbf{R}} \int_{-\infty}^{y} dx\, \big(2\chi_\Lambda w^2 \partial_x \chi_\Lambda + \chi_\Lambda \partial_x(w^2)\big) \\
&\leq\; \sup_{y\in\mathbf{R}} \int_{-\infty}^{y} dx\, \chi_\Lambda\big((2\partial_x\chi_\Lambda + 1)w^2 + (\partial_x w^2)\big) \\
&\leq\; C_1 \sup_{y\in\mathbf{R}} \int_{-\infty}^{y} d\mu\, \big(2w^2 + (\partial_x w^2)\big) \\
&\leq\; C_1 \int_{\mathbf{R}} d\mu\, \big(2w^2 + (\partial_x w)^2\big) \,.
\end{aligned}$$

It only remains to prove that for some $C_2 > 0$, $I \equiv \int_{\mathbf{R}} d\mu\, w L_Z w - C_2 \int_{\mathbf{R}} d\mu\, \big((\partial_x w)^2 + w^2\big) > 0$. We have:

$$\begin{aligned}
I \;&=\; \big(1 - C_2\big) \int_{\mathbf{R}} d\mu\, w L_Z w + C_2 \int_{\mathbf{R}} d\mu\, \big(w L_Z w - (\partial_x w)^2 - 2w^2\big) \\
&=\; \big(1 - C_2\big) \int_{\mathbf{R}} d\mu\, w L_Z w - C_2 \int_{\mathbf{R}} d\mu\, \big(2 + V''(u_Z)\big) w^2 \\
&\geq\; (1 - C_2) M_1 \|w\|_\Lambda^2 - C_2 \sup_{|x|\leq 1} |2 + V''(x)| \|w\|_\Lambda^2 \\
&\geq\; \big(M_1 - (M^* + M_1) C_2\big) \|w\|_\Lambda^2 ,
\end{aligned}$$

where we have defined $M^* \equiv \sup_{|x|\leq 1} |2 + V''(x)|$. The proof is complete if one chooses $C_2 < M_1/(K + M_1)$.

□

7. Proofs of the geometric structure

We first prove the existence of an orthogonal coordinate system adapted to the problem.

Proof of Proposition 2.8. We fix z_j, $|j| > D_\tau + 1$ such that $\|v - u_Z\|_\Lambda < \infty$ and we apply the Implicit Function Theorem to the function

$$\Big(\mathcal{F}(v, z_{-D_\tau - 1}, \ldots, z_{D_\tau + 1})\Big)_j \;=\; \langle v - u_Z,\, \tau_{z_j} \rangle \ , \ \text{for } |j| \leq D_\tau + 1 \ .$$

We can check that the hypotheses are satisfied:

1) $\mathcal{F}(u_Z, Z) = 0$,
2) $(\mathrm{D}\mathcal{F}(u_Z, Z)) = -\mathcal{S}$ where $\mathcal{S}$ is as in Lemma 5.4 (with $N = D_\tau + 1$), hence it has bounded inverse.

To prove the second part of the claim, we note that Ω^* has measure one, thus is it dense in Ω_Γ and, for all $\varepsilon' > 0$, for each $Z \in \Omega_\Gamma$, the set $\{\tilde{Z} \in \Omega_\Gamma \,:\, |\tilde{Z} - Z| < \varepsilon'\}$ is an open subset of Ω_Γ and thus contains an $Z^* \in \Omega^*$. By continuity, there is a C such that $\|(L_Z - L_{Z^*}) w\|_\Lambda / \|w\|_\Lambda < C\varepsilon' = \varepsilon$.

□

Before proceeding to the proofs of Proposition 2.9 we put Eq.(2.16) in a more compact form. We have, using Eq.(2.2):

$$\begin{aligned}
\partial_t Z(v) \;&=\; \mathcal{S}^{-1} \langle \mathcal{L}(v),\, \tau_Z \rangle_\Lambda \\
&=\; \mathcal{S}^{-1} \langle \mathcal{L}(w + u_Z),\, \tau_Z \rangle_\Lambda \\
&=\; \mathcal{S}^{-1} \langle \mathcal{L}(u_Z) - L_Z w + w^2 r(w, u_Z),\, \tau_Z \rangle_\Lambda \\
&=\; \mathcal{S}_1^{-1} \langle \mathcal{L}(u_Z), \tau_Z \rangle_\Lambda + (\mathcal{S}^{-1} - \mathcal{S}_1^{-1}) \langle \mathcal{L}(u_Z), \tau_Z \rangle_\Lambda \\
&\quad\; + \mathcal{S}^{-1} \langle -L_Z w + w^2 r(w, u_Z),\, \tau_Z \rangle_\Lambda \\
&\equiv\; \mathcal{P}^{(1)} + \mathcal{P}^{(2)} + \mathcal{P}^{(3)} \ .
\end{aligned} \tag{7.1}$$

The equation for w takes the form

$$\begin{aligned}
\partial_t w &= \mathcal{L}(u_Z) - L_Z w + w^2 r(w, u_Z) - \mathrm{D}_Z u_Z \cdot \partial_t Z(w + u_Z) \\
&= \mathcal{L}(u_Z) - \mathrm{D}_Z u_Z \cdot \big(\mathcal{S}_1^{-1} \langle \mathrm{D}_Z u_Z, \mathcal{L}(u_Z)\rangle_\Lambda\big) \\
&\quad - L_Z w + w^2 r(w, u_Z) \\
&\quad + \mathrm{D}_Z u_Z \cdot \big(-\partial_t Z(w + u_Z) + \mathcal{S}_1^{-1} \langle \mathrm{D}_Z u_Z, \mathcal{L}(u_Z)\rangle_\Lambda\big) \\
&\equiv \mathcal{Q} - L_Z w + w^2 r(w, u_Z) + \mathrm{D}_Z u_Z \cdot \big(-\mathcal{P}^{(2)} - \mathcal{P}^{(3)}\big) \ .
\end{aligned} \tag{7.2}$$

Lemma 7.1. *There exist constants $c_1 > 0$, $c_2 > 0$, c_3, and $c_4 > 0$ such that for sufficiently small $\mathcal{T}_{\Gamma,\sigma}$, the following holds:*

$$|\mathcal{S}_1 - E \cdot 1| \le c_2(\|w\|_\Lambda + e^{-c_1|Z|}) \ , \qquad |\mathcal{S}^{-1} - \mathcal{S}_1^{-1}| \le c_3 \|w\|_\Lambda \ ,$$

where $E = c_4 \int_{-1}^{1} dy\, \sqrt{2V(y)} = c_4 \int_{-\infty}^{\infty} dx\, \big(\psi'(x)\big)^2$.

Proof. The first claim is proved with the following argument: The off-diagonal elements of $\mathcal{S}_1$ are of order $\|w\|_\Lambda$, and the diagonal ones satisfy the bound:

$$\begin{aligned}
|(\mathcal{S}_1)_{ii}| &\le c_4 \int_{c_i^+ + 1}^{c_{i+1}^+ - 1} dx\, |\partial_{z_i} u_Z \partial_x u_Z| \\
&\le c_2 e^{-c_1|Z|} + c_4 \int_{c_i^+ + 1}^{c_{i+1}^+ - 1} dx\, |\partial_x u_Z|^2 \\
&= c_2 e^{-c_1|Z|} + c_4 \int_{c_i^+ + 1}^{c_{i+1}^+ - 1} dx\, \sqrt{2V(u_Z)} \\
&\le c_2 e^{-c_1|Z|} + E \ ,
\end{aligned}$$

using Lemma 7.8 of [CP1] to compare $\partial_{z_j} u_Z$ with $\partial_x u_Z$.

The second claim comes from the following estimation (defining $\mathcal{S}_2 = \mathcal{S}_1 - \mathcal{S}$):

$$\begin{aligned}
\mathcal{S}^{-1} - \mathcal{S}_1^{-1} &= \mathcal{S}^{-1}\big(1 - (\mathcal{S}_1 - \mathcal{S}_2)\mathcal{S}_1^{-1}\big) \\
&= \mathcal{S}^{-1}\mathcal{S}_2\mathcal{S}_1^{-1} \le C\|w\|_\Lambda \ ,
\end{aligned}$$

because of Lemma 5.4 and using that $\big(\mathcal{S}_2\big)_{i,j} = \langle w, \partial_{z_j} \tau_{z_i}\rangle_\Lambda \le C\|w\|_\Lambda$.

$\square$

Lemma 7.1 is used to prove estimates on expressions appearing in Eq.(7.1) and Eq.(7.2):

Lemma 7.2. *There exists constants $c_1 > 0$, $c_2 > 0$, and $c_3 > 0$ such that for sufficiently small $\mathcal{T}_{\Gamma,\sigma}$, the following holds:*

$$\begin{aligned}\sup_{|j|\le D_\tau} |\mathcal{P}_j^{(1)}| &\le c_1 g_1(Z)\,,\\ \sup_{|j|\le D_\tau} |\mathcal{P}_j^{(2)}+\mathcal{P}_j^{(3)}| &\le c_2\|w\|_\Lambda\big(g_1(Z)+g_2(Z)+\|\chi_\Lambda w\|_\infty+\varepsilon\big)\,.\end{aligned}$$

Proof. The bound on $\mathcal{P}^{(1)}$ is obvious from its definition, from the definition of $g_1(Z)$ and from Lemma 7.1.

The bound on $\mathcal{P}^{(2)}+\mathcal{P}^{(3)}$ follows from (see Eq.(2.10)):

$$|\langle L_Z w, \tau_{z_j}\rangle_\Lambda| = |\langle w, L_Z\tau_{z_j}\rangle_\Lambda| \le C\|w\|_\Lambda g_2(Z)\,,$$

$$|\langle w^2 r(w,u_Z), \tau_{z_j}\rangle_\Lambda| \le \|r\|_\infty\|\tau_{z_j}\|_\infty\|w\|_\Lambda^2 \le C(\|\chi_\Lambda w\|_\infty+\varepsilon)\|w\|_\Lambda\,,$$

and from Lemma 7.1.

□

We are now prepared to give the proof of Proposition 2.9:

Proof of Proposition 2.9. We start by expanding the first term of the l.h.s. of Eq.(2.11). Denoting $Z = Z(v)$ and $w = v - u_{Z(v)}$ and using Eq.(2.16) and Eq.(2.2) as well as Corollary 2.7, we get

$$\begin{aligned}\tfrac12\partial_t\|w\|_Z^2 &= \langle\dot w, L_Z w\rangle_\Lambda - \int_{\mathbf{R}} d\mu\, w^2 V''(u_Z)\left(\mathrm{D}_Z u_Z\cdot\dot Z\right)\\ &= \langle \mathcal{Q}, L_Z w\rangle_\Lambda - \|L_Z w\|_\Lambda + \langle w^2 r(w,u_Z), L_Z w\rangle_\Lambda\\ &\quad+\langle L_Z w, D_Z u_Z\cdot(-\mathcal{P}^{(2)}-\mathcal{P}^{(3)})\rangle_\Lambda\\ &\quad-\int_{\mathbf{R}} d\mu\, w^2V''(u_Z)\left(\mathrm{D}_Z u_Z\cdot\big(\mathcal{P}^{(1)}+\mathcal{P}^{(2)}+\mathcal{P}^{(3)}\big)\right)\\ &\le \tfrac14\|L_Z w\|_\Lambda^2+\|\mathcal{Q}\|_\Lambda^2-\|L_Z w\|_\Lambda\\ &\quad+C\big(\|\chi_\Lambda w\|_\infty\|w\|_\Lambda\|L_Z w\|_\Lambda+\|(1-\chi_\Lambda)w\|_\Lambda\|w\|_\Lambda\|L_Z w\|_\Lambda\big)\\ &\quad+C\big(\|L_Z w\|_\Lambda+\|w\|_\infty\|w\|_\Lambda\big)\|w\|_\Lambda\big(g_1(Z)+g_2(Z)+\|\chi_\Lambda w\|_\infty+\varepsilon\big)\\ &\quad+C\|w\|_\Lambda^2 g_1(Z)\\ &\le \|L_Z w\|^2\big(-\tfrac34+C(\|\chi_\Lambda w\|_\infty+\varepsilon)+C(1+\|w\|_\infty)\\ &\quad\times\big(g_1(Z)+g_2(Z)+\|\chi_\Lambda w\|_\infty+\varepsilon\big)\big)+\|\mathcal{Q}\|_\infty^2\,,\end{aligned}$$

where we have used Corollary 2.7 and Lemma 7.2. Taking $\mathcal{T}_{\Gamma,\sigma}$ and ε sufficiently small, we get

$$\tfrac12\partial_t\|w\|_Z^2 \le -\tfrac12\|L_Z w\|_\Lambda^2+\|\mathcal{Q}\|_\infty^2\,. \tag{7.3}$$

We next expand and estimate the time derivative of $g_1(Z)$

$$
\begin{aligned}
\tfrac{1}{2}\partial_t g_1^2(Z) \;=\; & \sum_{-D_\tau \le j,k \le D_\tau} \langle \mathcal{L}(u_Z), \tau_{z_j}\rangle_\Lambda \big(\langle L_Z \partial_{z_k} u_Z, \tau_{z_j}\rangle_\Lambda \\
& + \langle \mathcal{L}(u_Z), \partial_{z_k}\tau_{z_j}\rangle_\Lambda \big) \big(\mathcal{P}_k^{(1)} + \mathcal{P}_k^{(2)} + \mathcal{P}_k^{(3)} \big) \,.
\end{aligned}
$$

We have:

$$
\begin{aligned}
|\langle L_Z \partial_{z_k} u_Z, \tau_{z_j}\rangle_\Lambda| \;&\le\; |\langle \partial_{z_k} u_Z, L_Z \tau_{z_j}\rangle_\Lambda| \;\le\; C g_2(Z) \,, \\
|\langle \mathcal{L}(u_Z), \tau_{z_j}\rangle_\Lambda| \;&\le\; \|\mathcal{L}(u_Z)\|_\Lambda \|\partial_{z_k}\tau_{z_j}\|_\Lambda \;\le\; C g_1(Z) \,.
\end{aligned}
$$

Hence, using Lemma 7.2, we get

$$
\tfrac{1}{2}\partial_t g_1^2(Z) \;\le\; C g_1(Z)\big(g_1(Z)+g_2(Z)\big)\big(g_1(Z) + \|L_Z w\|_\Lambda (g_1(Z)+g_2(Z)+\|\chi_\Lambda w\|_\infty + \varepsilon)\big) \,. \tag{7.4}
$$

Summing Eq.(7.3) and Eq.(7.4) and using $\|\mathcal{Q}\|_\infty \le C g_1(Z)$, we have:

$$
\begin{aligned}
\tfrac{1}{2}\partial_t\big(\|w\|_Z^2 - B g_1^2(Z)\big) \;\le\; & -\tfrac{1}{2}\|L_Z w\|_\Lambda^2 + C g_1^2(Z) + C B g_1(Z)\big(g_1(Z)+g_2(Z)\big) \\
& \times \big(g_1(Z) + \|L_Z w\|_\Lambda (g_1(Z)+g_2(Z)+\|\chi_\Lambda w\|_\infty + \varepsilon)\big) \\
\le\; & -\tfrac{1}{4}\big(\|L_Z w\|_\Lambda^2 - M_2 B g_1^2(Z)\big) \\
& - \frac{M_2}{4} B g_1^2(Z) + C g_1^2(Z)\big(1 + B(g_1(Z)+g_2(Z))\big) \\
& + C B^2 g_1^2(Z)\big(g_1(Z)+g_2(Z)\big)^2\big(g_1(Z)+g_2(Z)+\|\chi_\Lambda w\|_\infty + \varepsilon\big)^2 \\
\le\; & -\tfrac{1}{4}\big(\|L_Z w\|_\Lambda^2 - M_2 B g_1^2(Z)\big) \\
& + C g_1^2(Z)\mathcal{F}(B) \,,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{F}(B) \;&=\; \big(\mathcal{A}_1 B^2 + \mathcal{A}_2 B + 1\big) \,, \\
\mathcal{A}_1 \;&=\; \big(g_1(Z)+g_2(Z)\big)^2\big(g_1(Z)+g_2(Z)+\|\chi_\Lambda w\|_\infty + \varepsilon\big)^2 \,, \\
\mathcal{A}_2 \;&=\; \big(g_1(Z)+g_2(Z)\big) - \frac{M_2}{4C} \,.
\end{aligned}
$$

We take $\mathcal{T}_{\Gamma,\sigma}$ and ε so small that $g_1(Z)+g_2+\|\chi_\Lambda w\|_\infty+\varepsilon < \frac{1}{4}$ and $g_1(Z)+g_2(Z) < M_2/8C$. Thus, $\mathcal{F}(B) \le 1 - M_2 B/(8C) + (M_2/(8C))^2(1/4)^2 B^2$, hence it is negative for $1-\sqrt{3}/2 < M_2 B/(64C) < 1+\sqrt{3}/2$.

□

We finally give proof of Theorem 2.10. Note that in Theorem 2.10, we assume that $v = w + u_{Z(v)} \in \mathcal{Z}$ hence $\|\chi_\Lambda w\|_\infty \le C\|w\|_Z \le C g_1(Z)$ which is smaller than σ.

Proof of Theorem 2.10. We start by recalling Eq.(7.1), with the shorthand $Z = Z(v)$ and with the convention that repeated indices are summed over:

$$
\begin{aligned}
\partial_t Z(v) \;=\; & \big(\mathcal{S}_1^{-1}\big)_{j,i}\langle \mathcal{L}(u_Z),\, \tau_{z_i}\rangle + \big(\mathcal{S}^{-1} - \mathcal{S}_1^{-1}\big)_{j,i}\langle \mathcal{L}(u_Z),\, \tau_{z_i}\rangle \\
& + \big(\mathcal{S}^{-1}\big)_{j,i}\langle -LZw, \tau_{z_i}\rangle + \big(\mathcal{S}^{-1}\big)_{j,i}\langle w^2 r(w,u_Z), \tau_{z_i}\rangle \,.
\end{aligned} \tag{7.5}
$$

Using Lemma 7.1, we find

$$|(\mathcal{S}_1)_{ii}| \leq E + Ce^{-c_1|Z|} .$$

It follows that the first term of Eq.(7.5) is equal to $E\langle \mathcal{L}(u_Z), \tau_{z_j}\rangle + \mathcal{O}(e^{-c_1|Z|})g_1(Z)$. The second term is also estimated using Lemma 7.1

$$\mathcal{S}^{-1} - \mathcal{S}_1^{-1} \leq C\|w\|_2 \leq Cg_1(Z) ,$$

thus, the second term in Eq.(7.5) is equal to $\mathcal{O}(e^{-c_1|Z|})\mathcal{O}(g_1(Z))$. The third term is estimated as in the proof of Lemma 7.2 replacing σ by $g_1(Z)$ everywhere. This completes the proof of Theorem 2.10.

□

8. Proof of Theorem 3.1

First, we study the simpler case of the collapse of a function with two kinks separated by a distance Γ and then, we compare with the evolution of the many-kink solution.

Lemma 8.1. *For sufficiently large $\Gamma > P_0$ and $\Gamma_0 > \Gamma$, there are constants $\varkappa_0 > 0$ and $\varepsilon_0 > 0$ such that the following is true. If $v_0 = u^*_\Gamma$ where u^*_Γ is given by Eq.(2.1) with $z_0 = -\infty$, $z_1 = 0$, $z_2 = \Gamma$, and $z_3 = \infty$ with the convention that $\varphi_P = \psi$ if $P = \infty$. Let v_t be the corresponding solution of Eq.(1.1). Then there is a $T_p < \infty$ such that $v_{T_p}(x) > \varepsilon_0 > 0$ for all $x \in \mathbf{R}$.*

Proof. We use the parabolic maximum principle together with the existence of moving front solutions for the Eq.(1.1). A front is a function $f_s(x - st)$, where s is a fixed number and $f_s(x)$ solves:

$$\partial_x^2 f_s + s\partial_x f_s + V'(f_s) = 0 . \tag{8.1}$$

In the mechanical interpretation shown in Fig. 1, $f_s(x)$ is an oscillating trajectory subject to a constant friction s.

Let $\Gamma_\mathrm{R} > \Gamma$, let $\varphi_{\Gamma_\mathrm{R}} = \varphi_P$ with $P = \Gamma_\mathrm{R}$, and let f_s be the solution of Eq.(8.1) with initial values $f_s(\Gamma_\mathrm{R}) = \varphi_{\Gamma_\mathrm{R}}(\Gamma_\mathrm{R}) = 0$ and $f'_s(\Gamma_\mathrm{R}) = \varphi'_{\Gamma_\mathrm{R}}(0) < 0$ satisfies:

1) $f_s(x) < \varphi_{\Gamma_\mathrm{R}}(x)$ for $0 < x < \Gamma_\mathrm{R}$ and for all $s > 0$.
2) For sufficiently small $s > 0$, there exist $\Gamma_\mathrm{L} \neq \Gamma_\mathrm{C}$ with $\Gamma_\mathrm{L} < -\Gamma_\mathrm{R}$ and $\Gamma_\mathrm{C} < 0$ such that $f_s(\Gamma_\mathrm{L}) = f_s(\Gamma_\mathrm{C}) = 0$.

The first claim is a consequence of the following argument: Integrating Eq.(8.1) from the initial values at $x = \Gamma_\mathrm{R}$, we see that in a small neighborhood of Γ_R, we have $f_s(x) < \varphi_{\Gamma_\mathrm{R}}(x)$. Let A be given by the equation $\Gamma_\mathrm{R} = P(A)$, *cf.* Proposition 1.1, and let x_0 such that $\varphi_{\Gamma_\mathrm{R}}(x_0) = -A$. Then, for $x_0 < x < \Gamma_\mathrm{R}$,

$$\begin{aligned} \varphi_{\Gamma_\mathrm{R}}(x)' &= \sqrt{2}\sqrt{V(A) - V(\varphi_{\Gamma_\mathrm{R}}(x))} , \\ f_s(x)' &> \sqrt{2}\sqrt{V(A) - V(f_s(x))} . \end{aligned} \tag{8.2}$$

If we suppose that there exists an x^*, $x_0 < x^* < \Gamma_{\rm R}$ such that $f_s(x^*) = \varphi_{\Gamma_{\rm R}}(x^*)$, since $f_s(x) < \varphi_{\Gamma_{\rm R}}(x)$ near $x = \Gamma_{\rm R}$, we see that $f_s'(x^*) < \varphi_{\Gamma_{\rm R}}'(x^*)$ which is a contradiction with Eq.(8.2). In the interval $(0, x_0)$ the same argument applies with opposite signs for the square roots in Eq.(8.2). Hence f_s does not intersect $\varphi_{\Gamma_{\rm R}}$ *i.e.*, f_s lies below $\varphi_{\Gamma_{\rm R}}$ in the interval $(0, \Gamma_{\rm R})$.

The second claim follows from the observation that $f_s(x) \to \varphi_{\Gamma_{\rm R}}(x)$ when $s \to 0$ hence, by continuity, there exist such zeros of f_s for small s.

Furthermore, we have that $u_{\Gamma^*}(x) \geq \psi_{\rm R}(x) \equiv \psi(-(x - \Gamma))$ and $u_{\Gamma^*}(x) \geq \psi_{\rm C}(x) \equiv \psi(x - \Gamma_{\rm C})$. Hence, by the maximum principle,

$$h(x, T + t) \;>\; \max\bigl(f_s(x - st), \psi_{\rm L}(x), \psi_{\rm C}(x)\bigr) \;,$$

for all $t > 0$, and for f_s satisfying 1) and 2). In particular, for $T_p = (\Gamma_{\rm R} - \Gamma_{\rm C})/s$, the function $h(x, T + T_p)$ is strictly positive (see Fig. 7, Fig. 8, and [CE], p.149, Example 4).

□

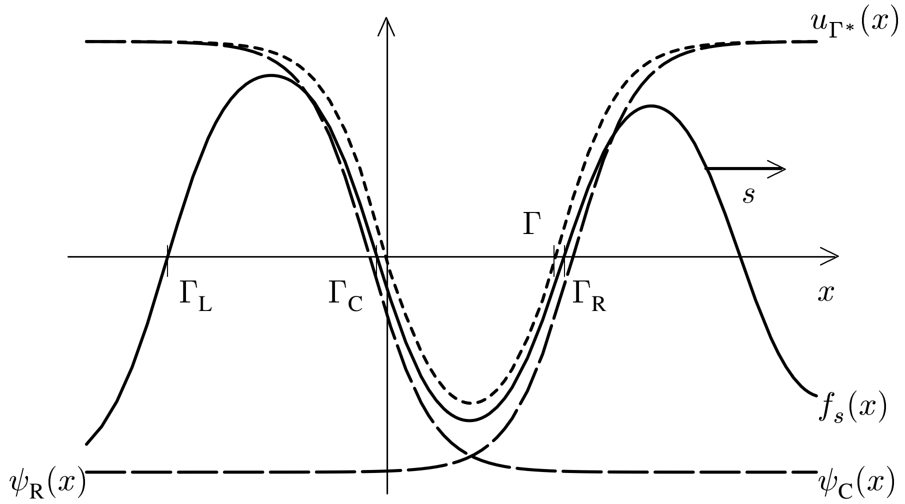

Fig. 7: The moving front f_s, the stationary solutions $\psi_{\rm R}(x)$, $\psi_{\rm C}(x)$, and u_{Γ^*}.

Proof of Theorem 3.1. We first remark that v_0 restricted to the interval $I_1 \equiv [z_j - \Gamma_0/3, z_{j+1} + \Gamma_0/3]$ is close to the two-kinks function $u_{\tilde{Z}}$, with $\tilde{Z} = \{z_j, z_j + \Gamma\}$. The evolution of $\tilde{u}_{\tilde{Z}}$ has been described in Lemma 8.1, and is known to lead to a collapse. We now show that the evolutions of $\tilde{u}_{\tilde{Z}}$ and u_Z remain close to each other for a time longer than the time T_p needed for collapse.

To perform the comparison, we consider the functions $f_0 = v_T = w + u_Z$ and $\tilde{f}_0 = \tilde{w} + u_{\tilde{Z}}$. By Duhamel's Principle:

$$\begin{aligned} f_t(x) - \tilde{f}_t(x) \;&\equiv\; f(x,t) - \tilde{f}(x,t) \\ &=\; \bigl(e^{t\partial_x^2}(f_0 - \tilde{f}_0)\bigr)(x) + \int_0^t ds\,\Bigl(e^{(t-s)\partial_x^2}\bigl(V'(f_s) - V'(\tilde{f}_s)\bigr)\Bigr)(x)\;. \end{aligned} \tag{8.3}$$

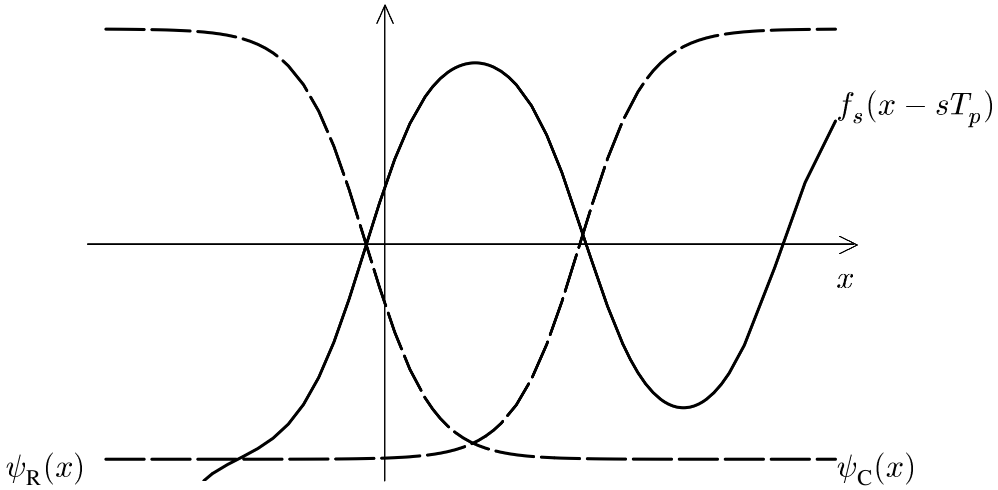

Fig. 8: The position of the front f_s at time $T_p = (\Gamma_{\mathrm{R}} - \Gamma_{\mathrm{C}})/s$, relative to $\psi_{\mathrm{R,C}}(x)$.

Let $I_2 \equiv [z_j, z_j + \Gamma] \subset I_1$. Let μ be an absolutely continuous measure on $\mathbf{R}$ such that $\mu(I_2) = 1 - \varepsilon_0$ and $\mu(\mathbf{R} \backslash I_2) = \varepsilon_0$ with $\varepsilon_0 > 0$ and let $\| \cdot \|_r$ be the $L^r(\mathbf{R}, d\mu)-$norm. If Δ_t denotes the l.h.s of Eq.(8.3), we have that $\Delta_0(x) = 0$ if $x \in I_1$ and $|V'(f_s) - V'(\tilde{f}_s)| \leq \varkappa |\Delta_s|$ for some $\varkappa > 0$ because V' is in $\mathcal{C}^1$. Using that $e^{t\partial_x^2}$ is an L^p-contractive semi-group (see [RS2], p.255), we have that

$$\begin{aligned}
\|\Delta_t\|_r &\leq \|e^{t\partial_x^2}\Delta_0\|_r + \varkappa \int_0^t ds\, \|e^{(t-s)\partial_x^2}\Delta_s\|_r \\
&\leq \|\Delta_0\|_r + \varkappa \int_0^t ds\, \|\Delta_s\|_r \\
&\leq \delta(\Gamma_0) + \varkappa \int_0^t ds\, \|\Delta_s\|_r \\
&\leq \delta(\Gamma_0) e^{\varkappa t} ,
\end{aligned}$$

for all $p \geq 1, t < T_p$, and with $\delta(\Gamma_0) \to 0$ when $\Gamma_0 \to \infty$. The last line follows from Gronwall's Lemma. Thus, for Γ_0 sufficiently large, $\|\delta_{T_p}\|_r < \varepsilon/2$ for all $r \geq 1$. Let $\mathbf{1}_{I_2}$ be the indicator function of the interval I_2, then by Hölder's Inequality:

$$\begin{aligned}
\left(\int_{I_2} d\mu\, |\Delta_{T_p}|^r\right)^{1/r} &= \left(\int_{\mathbf{R}} d\mu\, \mathbf{1}_{I_2} |\Delta_{T_p}|^r\right)^{1/r} \\
&\leq \left(\int_{\mathbf{R}} d\mu\, \mathbf{1}_{I_2}\right)^{1/q} \left(\int_{\mathbf{R}} d\mu\, |\Delta_{T_p}|^s\right)^{1/s} \\
&< (1 - \varepsilon_0)^{1/q} \varepsilon/2 ,
\end{aligned}$$

if $1 \leq r < q$. Thus, since $\|f\|_r \to \|f\|_\infty$ when $r \to \infty$ (see [R] p.71), we get that $\sup_{x \in I_2} |\Delta_{T_p}| < \varepsilon/2$, and, since $|f^*_{T_p}(x)| > \varepsilon$ by Theorem 3.1, we find $|f_{T_p}(x)| > \varepsilon/2$ for $x \in I_2$.

□

9. Proof of Proposition 1.1

Instead of the convention settled in Proposition 1.1, we shall choose the more symmetric definition: $\varphi_P(x)$ has a minimum at $x = 0$, *i.e.*, $\varphi_P(0) = -A$. We seek particular solutions of the equation $\mathcal{L}(u) = 0$. In the mechanical interpretation of a free particle moving in the potential V without friction, $u(x)$ is the position of the particle at time x (see Fig. 1). Intuitively, it is clear that if the particle starts at rest from a position $u(0)$, with $-1 < u(0) = -z < 0$, its trajectory will oscillate around 0 with a certain period $2P$. Looking for a relation between P and A, we show that if u solves the initial value problem

$$\begin{aligned} u'' &= -V'(u)\ , \\ u(0) &= -A\ ,\ u'(0) = 0\ , \end{aligned} \tag{9.1}$$

then there exists a (minimal) $P(A)$ such that $u(P(A)/2) = 0$. We can transform the equation (9.1) into: $\frac{1}{2}((u')^2)' = -(V(u))'$ (supposing $u' \not\equiv 0$), which, after integration, becomes $(u')^2 = -2(V(u) - V(-A))$, where the integration constant was set to $V(-A)$ in order to match the condition $u'(0) = 0$. When $-A \le u < 0$, $V(-A) \ge V(u)$, hence we can take the square root:

$$u' = \sqrt{2}\sqrt{V(-A) - V(u)}\ .$$

The r.h.s. is invertible if $-A < u \le 0$, yielding an equation for the inverse function $x(u)$:

$$x'(u) = \frac{1}{\sqrt{2}\sqrt{V(-A) - V(u)}}\ .$$

There will be a solution satisfying the boundary condition $u(0) = -A$ and the periodicity condition $u(P(A)/2) = 0$ if and only if the integral

$$\frac{1}{\sqrt{2}} \int_{-A}^{0} \frac{ds}{\sqrt{V(-A) - V(s)}} = x(0) - x(-A) = P(A)/2 - 0 \tag{9.2}$$

exists. This is an elliptic integral of the first kind, which is an analytic bijection from $(0, 1)$ onto (P_0, ∞) (see Fig. 9 and, *e.g.*, [A], p.322-324).

We have described the solution u of Eq.(9.1) on the interval $[0, P(A)/2]$, and we can indeed check that it extends to a periodic function. The equation $\frac{1}{2}(u')^2 + V(u) = V(-A) = V(A)$ together with $u'' = -A'(u)$ leads to the existence of a number $P^*(A)$ where

$$u\big(P^*(A)/2\big) = A\ ,\ u'\big(P^*(A)/2\big) = 0\ ,\ u''\big(P^*(A)/2\big) < 0\ ,$$

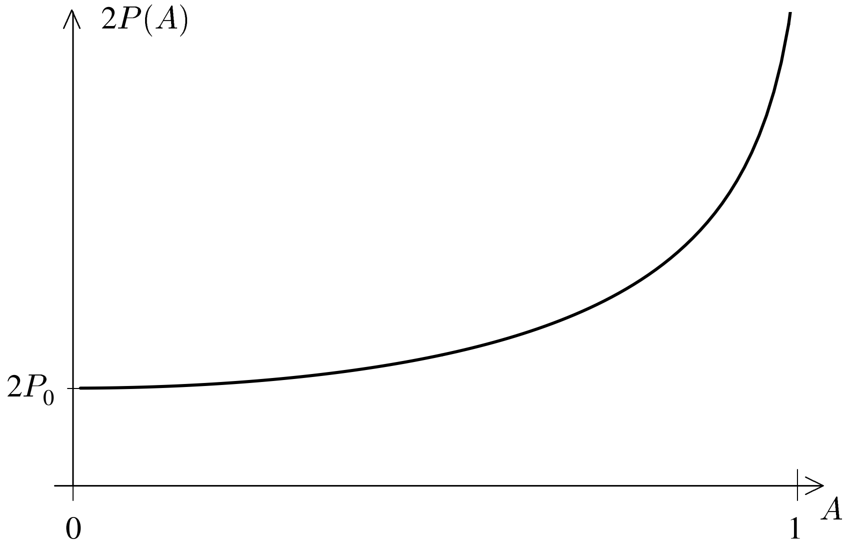

Fig. 9: The period $2P$ as a function of the amplitude A.

i.e., a maximum of u of height A. This number $P^*(A)$ is determined by (recall that V is even):

$$x(z) - x(0) \;=\; P^*(A)/2 - P(A)/2 \;=\; \frac{1}{\sqrt{2}} \int_0^z \frac{ds}{\sqrt{V(A) - V(s)}}$$
$$= \frac{1}{\sqrt{2}} \int_z^0 \frac{ds}{\sqrt{V(A) - V(s)}} \;=\; P(A)/2\,.$$

The initial value problem (9.1) is invariant under $x \to -x$, hence the solution is even. Altogether, we found the behavior of u on the interval $[-P(A), P(A)]$ and since $u\bigl(-P(A)\bigr) = u\bigl(P(A)\bigr) = A$ and $u'\bigl(-P(A)\bigr) = u'\bigl(P(A)\bigr) = 0$, periodic copies will match together.

$\square$

Remark. An additional property of the integral (9.2) that is useful for our analysis is the following: For sufficiently large P, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$1 - A(P) \;\le\; c_2 e^{-c_1 P}\,. \tag{9.3}$$

References

[A] Arfken, G.: *Mathematical Methods for Physicists, 3d edition*, Orlando, Academic Press (1985).

[BDG] Bray, A.J., Derrida, B. and C. Godrèche: Non-trivial Algebraic Decay in a Soluble Model of Coarsening. Europhys. Lett. **27** (3), 175-180 (1994).

[BD] Bray, A.J. and B. Derrida: Exact Exponent λ of the Autocorrelation Function for a Soluble Model of Coarsening. Phys. Rev. E **51** (3), 1633-1636 (1995).

[CE] Collet, P. and J.-P. Eckmann: *Instabilities and Fronts in Extended Systems*, Princeton, Princeton University Press (1990).

[CP1] Carr, J. and R.L. Pego: Metastable Patterns in Solutions of $u_t = \varepsilon^2 u_{xx} - f(u)$. Comm. Pure. Appl. Math. **XLII**, 523-576 (1989).

[CP2] Carr, J. and R.L. Pego: Invariant Manifolds for Metastable Patterns in $u_t = \varepsilon^2 u_{xx} - f(u)$. Proc. Roy. Soc. Edinburgh **116A**, 133-160 (1990).

[FSW] Fröhlich, J., Spencer, T. and P. Wittwer: Localization for a Class of One Dimensional Quasi-Periodic Schrödinger Operators. Comm. Math. Phys. **132**, 5-25 (1990).

[G] Gallay, Th.: *Existence et stabilité des fronts dans l'équation de Ginzburg-Landau à une dimension*, PhD Thesis, University of Geneva (1994).

[R] Rudin,W.: *Real and Complex Analysis*, Singapore, McGraw-Hill (1987).

[RS2] Reed, M. and B. Simon: *Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-Adjointness*, San Diego, Academic Press (1975).

[RS4] Reed, M. and B. Simon: *Methods of Modern Mathematical Physics, IV: Analysis of Operators*, San Diego, Academic Press (1978).

[S] Spencer, T.: The Schrödinger Equation with a Random Potential. In *Critical Phenomena, Random Fields, Gauge Theories. Les Houches 1984*. Amsterdam, North Holland (1986).