A KAM type theorem for systems with round-off errors

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Abstract. Perturbations due to round-off errors in computer modeling are discontinuous and therefore one cannot use results like KAM theory about smooth perturbations of twist maps. We elaborate a special approximation scheme to construct two smooth periodic on the angle perturbations of the twist map, bounding the discretized map from above and from below. Using the well known Moser's theorem we prove the existence of invariant curves for these smooth approximations. As a result we are able to prove that any trajectory of the discretized twist map is eventually periodic. We discuss also some questions, concerning the application of the intersection property in Moser's theorem and the generalization of our results for the twist map in Lobachevski plane.

1 Introduction

In recent years an increasing interest has been addressed to problems related to numerical simulations of complex dynamics. Classical mathematical results in this field provide the theoretical foundation for the numerical modeling of stable dynamical systems, while the analysis of mathematical problems arising due to round-off errors was started relatively recently. Probably the first theoretical study was published in 1984 in [1]. The results of this paper show that the influence of arbitrary small round-off errors can change the behavior of a complex (chaotic) system drastically, and that the resulting behavior depends very sensitively on the fine structure of the phase space discretization corresponding to these errors. It is enough to mention here the period multiplication phenomenon [1], and the localization under the action of such perturbations [1]-[4], when trajectories that should normally be dense in the phase space remain confined to a small region. Starting from 1984 a large number of publications were devoted to the mathematical analysis of problems related to numerical simulations of complex dynamics (see, for example, [2]–[7] and references therein).

One of the most complex and intriguing problem arising in this field is the analysis of the influence of round-off errors for systems whose trajectories are neither stable nor unstable, or such that they have local components of this type. A typical example here is a rotation map $F: (\varphi, r) \to (\varphi + \alpha, r)$ of the two-dimensional plane $\mathbb{R}^2$ around the origin. The study of the discretized version of this map was started in [3]. If the angle $\alpha \in [0, 1)$ is irrational then any trajectory of the map F is dense on a circle. However the action of round-offs disrupts these invariant circles, and a'priori it is not clear that there are no trajectories going to infinity. Moreover in [3] an example of the rotation-like map with this property was constructed. Starting from that time at least 10 papers were dedicated to the discussion of this problem, however it was not solved.

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Discretized plane rotation

Figure 1: Invariant set (black dots) of the 0.0035-discretized pure plane rotation

Numerical simulations (see Figure 1) show that the invariant set of the discretized rotation looks very much like invariant tori in the Kolmogorov - Arnold - Moser (KAM) theory. Therefore in [4] it was conjectured that in spite of the discontinuities of the perturbations due to the round-off errors some kind of KAM theory can be constructed for this problem, namely to prove that any trajectory of the discretized rotation is eventually periodic. In this paper we follow this idea and the main result is the following theorem.

Definition 1.1 A map F of the annulus

$$\mathcal{A}(r_-,r_+) := \{(\varphi,r) \in \mathbb{R}^2 :\ 0 \le r_- \le r \le r_+ \le \infty\}$$

into $\mathbb{R}^2$ is said to be *twist* if in polar coordinates ($0 \le \varphi < 1,\ 0 \le r < \infty$) it can be written as

$$\begin{cases} \varphi \to \varphi + \Phi(r) \pmod 1 \\ r \to r \end{cases} \tag{1.1}$$

where the function $\Phi(r)$ is C^l-smooth for a sufficiently large l ($l = 6$ is enough) and satisfies the inequality

$$|d\Phi(r)/dr| > 0.$$

Notice, that in principle, the constant r_+ might be equal to ∞, in which case the annulus $\mathcal{A}(0,\infty)$ coincides with the entire plane $\mathbb{R}^2$.

Theorem 1.1 *For any constants $0 < \hat{r}_- < r_- < r_+ < \hat{r}_+ < \infty$ there exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$ if $r_- < r < r_+$ then $\hat{r}_- < (D_\varepsilon \circ F)^n(\varphi,r) < \hat{r}_+$ for any $n \in \mathbb{Z}^+$ and any angles $\varphi \in [0,1)$. Moreover, if we additionally assume that $|d\Phi(r)/dr| > \mathrm{Const}/r$ for any $r > 0$, then for each $\varepsilon > 0$ any trajectory of the ε-discretized twist map (1.1) is eventually periodic.*

Perturbations due to round-off errors in computer modeling are discontinuous and therefore one cannot use results like KAM theory about smooth perturbations of twist maps. We elaborate a special approximation scheme to construct two smooth periodic on φ perturbations of the twist map, bounding the discretized map from above and from below. Using the well known Moser's theorem [9] we prove the existence of invariant curves for these smooth approximations. The existence of the invariant curves yields by some special shadowing argument the boundedness of the trajectories of the discretized twist map.

2 Twist maps with C^∞-smooth perturbations

Consider a family of C^∞-perturbed twist maps of the annulus $\mathcal{A}(r_-,r_+)$

$$\begin{cases} \varphi \to \varphi + \Phi(r) + \varepsilon h(\varphi,r,\varepsilon) \pmod 1 \\ r \to r + \varepsilon g(\varphi,r,\varepsilon) \end{cases} \tag{2.1}$$

depending on the small parameter $\varepsilon \ge 0$.

In this section we intensively use the following well known result due to Moser about the existence of invariant curves for the system (2.1).

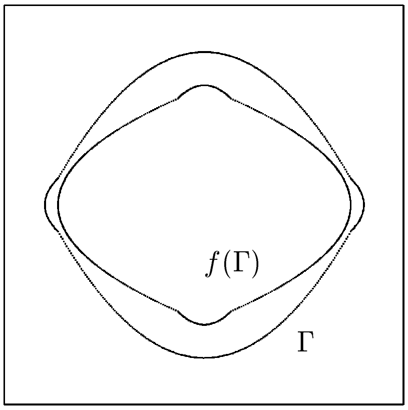

Figure 2: Counterexample for the intersection property.

Theorem 2.1 *[9] Let us assume that*

- $|d\Phi(r)/dr| > 0$ *for any* $r_- \le r \le r_+$, $r_+ - r_- > 1$.
- *The functions* $\Phi, h, g \in C^\infty$ *with respect to all variables, and* h *and* g *are* 1*-periodic with respect to the angle variable* φ.
- *For a sufficiently small* ε_0 *for any* $|\varepsilon| \le \varepsilon_0$ *any closed curve around the origin intersects with its image under the action of the map (2.1) (intersection property).*

Then for any $0 \le \varepsilon \le \varepsilon_0$ *in the annulus* A *there exists an (2.1)-invariant closed curve.*

Once again the assumption about the C^∞-smoothness can be weaken to the C^l-smoothness for a sufficiently large l ($l = 6$ is enough).

The last of the above assumptions is called the *intersection property.* Actually in the proof of this theorem a weaker condition is verified. Namely, it is enough to check up this condition only for closed curves $\{(\varphi, r) : r = r(\varphi)\}$ satisfying the inequality $|dr/d\varphi| < \varepsilon^{3/8}$ (i.e., curves C^1-close to circles). This follows from the estimate of the smoothness of the invariant curve and the fact that the intersection property is used in the proof only to check up that after the change of variables any circle intersects with its image (see [9] p.2,15). Therefore we can use this weaker condition as a definition of the intersection property.

Moreover it is straightforward to show that if any circle centered at the origin intersects with its image than any closed curve intersecting two concentric circles with the distance 2ε between them also intersects with its image under the action of the above map.

It seems that using this argument one can weaken the intersection property even more to the property that any circle centered at the origin intersects with its image. Moreover one might think that the intersection property for the circles yields this property for an arbitrary curve close to a circle. However, the following lemma shows that this is not true.

Lemma 2.1 *There exists a* C^∞ *map* f *of the annulus* $\mathcal{A}$*, for which the intersection property holds for any circle centered at the origin, but there is a closed curve arbitrary close to a circle which does not intersect with its image.*

Proof. Let the map f be a superposition of the rotation by $\pi/2$ around the origin and the linear map S, contracting with the coefficient $1 - a$ on the vertical coordinate ($0 < a \ll 1$). Notice that we first apply the rotation. Clearly under the action of this map any circle centered at the origin intersects with its image at points with angles 0 and π. Consider now a curve Γ, coinciding with a circle everywhere except small neighborhoods of points with angles 0 and π. In these neighborhoods we consider very small outward perturbations of the circle (see Figure 2). The image of the curve Γ is the same as the image of the circle, except the neighborhoods of the points with the angles $\pi/2$ and $3\pi/2$. Clearly if our perturbation is less than a, then the image of the curve Γ does not intersect with the curve itself. ■

Notice that in the above example the intersections of the circles centered at the origin with their images occur at points with angles 0 and π, and they are not transversal. If the intersection is transversal, then there exists a neighborhood of the circle such that for any closed curve in this neighborhood the intersection property holds. However this neighborhood is rather thin and it is simple to change the construction in the proof of Lemma 2.1 to make the intersections to be transversal by adding some expansion along the horizontal coordinate with the coefficient $(1+b)$, provided $a > b > 0$. It is also straightforward to construct the twist map with the same property by the superposition of the map f with the rotation by the angle γr, providing the constant γ is small enough.

It is worth noting that the twist property is very essential for the existence of the invariant curve and one cannot extend above results for the case of the pure rotation, i.e., when $\Phi(r) = \text{Const}$.

Lemma 2.2 *Let $\Phi(r) \equiv \alpha = \text{Const}$. Then for any sufficiently small $\varepsilon > 0$ there exist a pair of C^∞-smooth functions h, g satisfying the above assumptions such that the perturbed system (2.1) does not have an invariant curve and moreover there are trajectories going to infinity.*

Proof. Consider the following perturbed family of maps, depending on the parameter $\varepsilon \geq 0$

$$\begin{cases} \varphi \to \varphi + \alpha + \varepsilon h \,(\mathrm{mod}\ 1) \\ r \to r + \varepsilon g(\varphi) \end{cases} \tag{2.2}$$

where $h \in [0, 1)$ is a constant and $g(\varphi)$ is a C^∞ 1-periodic function. The system (2.2) is a skew product of two maps, and the first (angle) coordinate does not depend on the second (radial) one. Clearly we can choose the constant h in such a way that the number $\alpha + \varepsilon h$ will be rational. In this case the first map has a periodic trajectory $\varphi_1, \varphi_2, \ldots, \varphi_k$. Now setting $g \equiv 0$ we immediately see that all the trajectories are periodic (with the same period), and thus do not fill densely corresponding invariant circles.

To show that at least some trajectories may go to infinity we need to define the function g in a bit more delicate way. We define the function g at the points φ_i as $+1$ and extend it as a C^∞-function to the unit circle, such that it rapidly goes to -1 in small neighborhoods of the points φ_i and is equal to -1 otherwise. This definition clearly implies the intersection property, while the trajectory starting at the point $(\varphi_1, 1)$ goes to infinity with the linear rate εn. ■

3 Twist maps with discontinuous perturbations

Definition 3.1 By an *ε-discretization*, $\varepsilon > 0$, of a set $X \subset \mathbb{R}^d$ we mean a choice of an ordered lattice (a collection of points) X_ε in the set X such that the distances between its neighboring elements do not exceed ε. By an *operator of ε-discretization* we mean a map $D_\varepsilon : X \to X_\varepsilon$ that associates to each point $x \in X$ its nearest point on the lattice $x_\varepsilon = D_\varepsilon(x) \in X_\varepsilon$ (if there are several such points we choose the point with the minimal index). Recall that the lattice

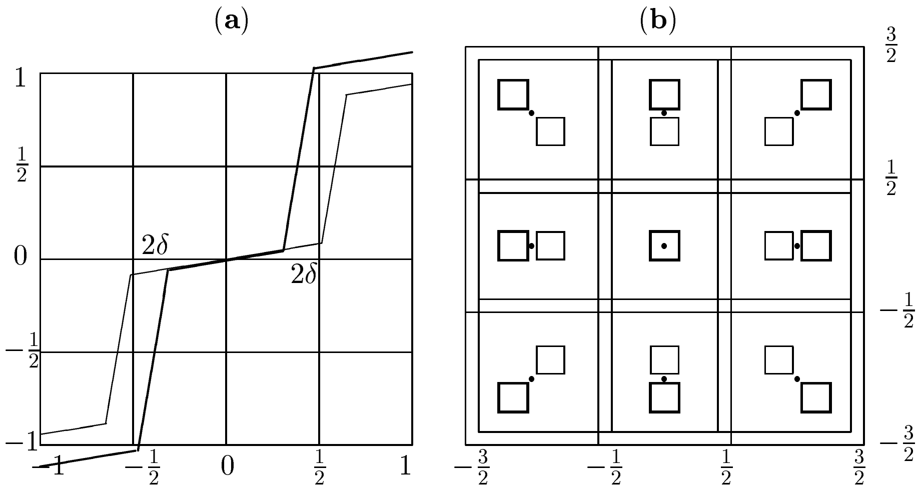

Figure 3: Approximation for the operator D_1. (a) One-dimensional approximation. (b) Two-dimensional approximation. By thick (thin) lines we indicate the approximation from above (below). On figure (b) by thick black dots we indicate integer points, thick-lined small squares correspond to the images under the approximation from above of unit rectangles centered at integer points, without 2δ-narrow strips near their boundaries, indicated by thin lines. Thin small squares correspond to the images under the approximation from below.

X_ε is assumed to be ordered. The value of the parameter $\varepsilon > 0$ is called the *diameter* of the ε-discretization or the magnitude of the corresponding perturbation.

Definition 3.2 By an ε-*discretized* (perturbed) system for the map f with the phase space X we mean a pair $(f_\varepsilon,\ X_\varepsilon)$, where $f_\varepsilon = D_\varepsilon \circ f$.

Note that contrary to usual perturbation schemes, the discretized (perturbed) system here is given not on the original phase space X but on the discrete lattice X_ε.

In the case of round-off errors in computer modeling we deal with a uniform ε-discretization, i.e., for $\varepsilon = 1/N$, $N \in \mathbb{Z}_+$, the lattice X_ε consists of all points $x \in X$ with rational coordinates with the denominator N.

Denoting the twist map (1.1) by F we may represent the ε-discretized twist map $D_\varepsilon \circ F$ as follows:

$$\begin{cases} \varphi \to \varphi + \Phi(r) + \frac{\varepsilon}{r} h(\varphi, r, \varepsilon) \,(\mathrm{mod}\ 1) \\ r \to r + \varepsilon g(\varphi, r, \varepsilon) \end{cases} \tag{3.1}$$

The perturbations due to round-off errors that we consider are discontinuous, but there is an additional small parameter $1/r$ (for $r \gg 1$) in the first relation, which arises due to the fact that the perturbation of the angle is inversely proportional to the distance from the origin.

Our aim is to construct C^∞ smooth approximations $D_\varepsilon^\pm$ for the discretization operator D_ε. The first approximation D_ε^+ is the approximation from above in the sense that for any point $x \in \mathbb{R}^2$ we have

$$|D_\varepsilon^+(x)| \geq |D_\varepsilon(x)|,$$

while for the second approximation D_ε^- – the approximation from below the opposite inequality

$$|D_\varepsilon^-(x)| \leq |D_\varepsilon(x)|$$

holds. Both of these approximations should satisfy also the property that their graphs lie in the $3\delta\varepsilon$ neighborhood of the graph of the operator D_ε. Here $0 < \delta \ll 1$ is the parameter of the approximation. Notice that $D_\varepsilon(x) = \varepsilon D_1(x/\varepsilon)$ for any $\varepsilon > 0$ and therefore it is enough to construct the approximations for $\varepsilon = 1$.

We define the the approximation from above D_1^+ in three steps. On the first step we define the following one-dimensional piecewise linear function, depending on a small parameter $0 < \delta \ll 1$:

$$f_1^+(t) := \begin{cases} \frac{2\delta}{1/2-2\delta} t & \text{if } 0 \le t \le 1/2 - 2\delta \\ \frac{1-\delta}{\delta}\left(t - D_1(t) - \frac{1}{2} + 2\delta\right) + D_1(t) + 2\delta & \text{if } -2\delta < t - D_1(t) - \frac{1}{2} \\ & \quad \le -\delta,\ t > 0 \\ \frac{\delta}{1-\delta}\left(t - D_1(t) - \frac{1}{2} + \delta\right) + D_1(t) + 1 + \delta & \text{if } -\frac{1}{2} - \delta < t - D_1(t) \\ & \quad \le \frac{1}{2} - 2\delta,\ t > \frac{1}{2} \\ -f_1^+(-t) & \text{if } t < 0 \end{cases}$$

The graph of this function is shown by thick lines on Figure 3(a). By the construction we have

$$\begin{aligned} f_1^+(t) &\le D_1(t) \text{ if } t < 0 \\ f_1^+(t) &\ge D_1(t) \text{ if } t \ge 0. \end{aligned}$$

Thus $|f_1^+(t)| \ge |D_1(t)|$ for any $t \in \mathbb{R}^1$. Straightforward estimates also show that the distance between the graphs of D_1 and f_1^+ is equal to 2δ.

From the first sight it seems that the one-dimensional approximation from above for the discretization operator can be done in a much more simple way, by the function:

$$\hat{f}_1(t) := \begin{cases} 2x & \text{if } |x| < 1/2 \\ x - 1/2 & \text{if } x < -1/2 \\ x + 1/2 & \text{if } x > 1/2 \end{cases}.$$

However for any twist map with the perturbation $\varepsilon\hat{f}_1(t/\varepsilon)$ on each coordinate almost all trajectories go to infinity, and thus there is no invariant curves.

On the second step we smooth out this piecewise linear function to obtain a new function having a given number of derivatives. The simplest way to do it is to apply the following integral smoothing operator:

$$\mathcal{S}_\sigma h(t) := \int_{|s|<\sigma} \sigma^{-1} S_\sigma(s) h(t-s)\, ds,$$

where $0 < \sigma \ll 1$, and the kernel $S_\sigma \in C^\infty$, $S_\sigma(t) = 0$ for $|t| > \sigma$ and

$$\int_{-\infty}^{\infty} t^k S_\sigma(t)\, dt = \begin{cases} \sigma & \text{for } k = 0 \\ 0 & \text{for } 0 < k < l \end{cases},$$

where l is a fixed number. It is shown (see details, for example, in [9]) that for any continuous function h the function $\mathcal{S}_\sigma h \in C^\infty$ and its derivatives up to the order l are effectively estimated. Moreover we have a good control over the difference $h - \mathcal{S}_\sigma h$, which enables to show that if the parameter σ is sufficiently small, for example, $\sigma = \delta^2$, then applying this operator to f_1^+ we obtain a function $\tilde{f}_1^+ \in C^\infty$, being the approximation of the discretization operator from above. The distance between the graphs will be somewhat enlarged, but is still less than 3δ.

The multidimensional discretization operator D_1 acts on each coordinate independently of others. Thus on the last step of the approximation procedure we apply the function $\tilde{f}_1^+$ on each coordinate independently, which defines the smooth define approximation D_1^+ for the two-dimensional discretization operator D_1. The action of this two-dimensional transformation, shown on Figure 3(b), is the following. First, it maps almost linearly unit squares centered at integer points (except 2δ-narrow strips near their boundaries, indicated by thin lines on the figure) to small squares of size δ, located near the corresponding integer points and shifted outward with respect to the origin. Second, the mentioned above strips are expanded to the complement of these small squares. Since f_1^+ is the approximation from above for the one-dimensional discretization operator, similar property holds also for D_1^+:

$$|D_1^+(x)| \geq |D_1(x)|$$

for any $x \in \mathbb{R}^2$. Thus, according to our definition, D_1^+ is the approximation from above for D_1. Straightforward estimates also show that the distance between the graphs D_1 and D_1^+ is not more than to 3δ.

In a similar way one can construct the approximation from below D_1^-. The corresponding one-dimensional version is defined as

$$f_1^-(t) := \begin{cases} \frac{\delta}{1/2+\delta} t & \text{if } 0 \leq t \leq 1/2+\delta \\ \frac{1-3\delta}{\delta}\left(t - \frac{1}{2} - \delta\right) + \delta & \text{if } \delta \leq t - \frac{1}{2} < 2\delta \\ \frac{\delta}{1-\delta}\left(t - D_1(t) - \frac{1}{2} + 2\delta\right) + D_1(t) - 2\delta & \text{if } -\frac{1}{2} + 2\delta < t - D_1(t) \\ & \quad \leq \frac{1}{2} + \delta,\ t > \frac{1}{2} \\ \frac{1-\delta}{\delta}\left(t - D_1(t) - \frac{1}{2} + \delta\right) + D_1(t) - \delta & \text{if } \delta < t - D_1(t) + \frac{1}{2} \leq 2\delta, \\ & \quad t > \frac{1}{2} \\ -f_1^-(-t) & \text{if } t < 0 \end{cases}$$

The graph of this function is shown by thin lines on Figure 3(a). By the construction we have

$$f_1^-(t) \geq D_1(t) \text{ if } t < 0$$
$$f_1^-(t) \leq D_1(t) \text{ if } t \geq 0$$

for any point $|t| > 1$. Thus $|f_1^-(t)| \leq |D_1(t)|$ for $|t| > 1$, which makes f_1^- the approximation from below everywhere except the central interval $[-1, 1]$. Notice that there is no continuous one-dimensional approximation from below with nondegenerate derivatives for D_1 for all $t \in \mathbb{R}^1$.

The second and the third steps of the procedure are exactly the same as in the construction of the approximation from above. The scheme of the action of the approximation from below is shown by thin-lined small squares on Figure 3(b). It is worth noting that the operator D_1^- does not satisfy our requirement to be the approximation from below in the $1 + 2\delta$-square centered at the origin. However we shall apply our approximation scheme only for the annulus A with nonzero smaller radius. Therefore we do not need to define the discretization operator near the origin. Another problem is to verify the approximation from below property for unit squares intersecting with the coordinate axes. The operator D_1^- restricted to such squares enlarges one of the coordinates and decreases another one. However, take an integer point $(0, n)$ and notice that

$$(n - \delta)^2 + \delta^2 = n^2 - 2\delta(1 - \delta) < n^2.$$

Therefore the emphasized small square lies entirely inside of the circle of radius n centered at the origin. Thus the effect of the decreasing is stronger, which proves the desired property.

The proof of the intersection property for the approximation from above is as follows. Consider a circle centered at the origin with a sufficiently large radius. Then locally this circle is arbitrary close the the straight line. Take the segment of the circle with the "slope" -1. Clearly there is a point (φ, r) in the lattice of "half-integer points" $\varepsilon(\mathbb{Z}^2 + (1/2, 1/2))$ arbitrary close to this segment in the same "half-plane" as the origin. Then by the construction of our approximation there is an intersection of the image of the circle with the circle itself near the image of the point (φ, r). Indeed in the $2\varepsilon\delta$-neighborhood of the point (φ, r) the function D_ε^+

takes both values $(\varphi_-, r - \varepsilon/2)$ and $(\varphi_+, r + \varepsilon/2)$, which yields the intersection. In the same way we can deal with a curve sufficiently C^1-close to the circle. In the same way one can prove the intersection property for the approximation from below D_ε^-.

Therefore the both maps $D_\varepsilon^\pm \circ F$ satisfy the conditions of Theorem 2.1, and thus they have invariant curves, defined by the relations $r = \Gamma_\pm(\varphi)$. In spite of the fact that our approximation is the approximation from above (from below), the angles corresponding to the points $D_\varepsilon^\pm \circ F(x)$ and $D_\varepsilon \circ F(x)$ are different. Therefore we cannot apply the idea that every time the trajectory of the approximating map is further from the origin or at least not closer (on closer in the case of the approximation from below) than the discretized one, being at the same time uniformly bounded (by the invariant curves).

To prove the boundedness we need an additional trick. For any point $x \in \mathbb{Z}^2$ consider its "shadow" point x^s on the invariant curve (the point with the same angle). Now comparing the images of these points $D_\varepsilon^\pm \circ F(x^s)$ and $D_\varepsilon \circ F(x)$ we obtain the boundedness of the trajectory of the discretized map.

This completes the proof of the first part of Theorem 1.1.

To prove the second part we restrict our twist map F to the annulus $\mathcal{A}(r_-, r_+)$ such that $1 \ll R \le r_- < 2R \le r_+ \le 3R$ with a sufficiently large R. Then, after the following change of variables: $(\varphi, r) \to (\varphi, r_- r)$, we obtain the twist map of the new annulus $\mathcal{A}(1, r_+/r_-)$ with the perturbation of order ε/r_-, which can be made arbitrary small by the choise of the constant r_-. Therefore the argument in the first part of the proof yields the boundedness of any trajectory of the discretized map with arbitrary $\varepsilon < \infty$. On the other hand, in a bounded region there is only a finite number of points of the discretized lattice, which yields the eventual periodicity. Theorem 1.1 is proven. ■

The similar construction can be applied for the "integer part" discretization, when each point is mapped to its integer part. In spite of of some differences, this type of discretizations leads to the same eventually periodic behavior. On the other hand, the truncation operator for arbitrary small truncation error leads to quite different behavior: almost any trajectory of the truncated system goes to zero with time [6].

4 Discretization of the twist map in Lobachevski's metric

In this section we discuss some our findings about the properties of discretizations of the twist map in Lobachevski's metric. The Lobachevski plane $\mathbb{L}_2$ can be realized as the upper semi-plane $y \ge 0$ of the Eucledian two-dimensional plane $\mathbb{R}^2$ with the metric $ds^2 = ((dx)^2 + (dy)^2)/y^2$, which we denote by $\rho(\cdot,\cdot)$. We shall consider two different discretizations of $\mathbb{L}_2$: one with respect to the usual integer lattice $\mathbb{Z}^2$, and another one with respect to $\mathbb{Z}_L = \{(x,y), x \in \mathbb{Z},\ y = 2^k,\ k \in \mathbb{Z}\}$. The reason for the introduction of the second lattice is that it is uniform with respect to the Lobachevski's metric, which is not the case for the lattice $\mathbb{Z}^2$. Notice that the lattice $\mathbb{Z}_L$ is not locally finite.

Observe that the distance in the Lobachevski's metric between any two horizontal straight lines vanishes as $x \to \pm\infty$. Therefore any vertical straight line plays a role of the radius from the infinity $y = \infty$, while any horizontal straight line is a circle centered at the infinity. Consider a map $F : \mathbb{L}_2 \to \mathbb{L}_2$ defined as $F(x,y) = (x + \Phi(y), y)$. The discretization operator $D : \mathbb{L}_2 \to \mathbb{Z}^2 \cup \mathbb{Z}_L$ we define as the closest point map in the Lobachevski's metric.

Proposition 4.1 *The intersection of $\mathbb{Z}^2$ with any horizontal line with the integer coordinate $y \in \mathbb{Z}$ is invariant with respect to the $(\mathbb{Z}^2 \cup \mathbb{Z}_L)$-discretized map $D \circ F$.*

Proof. It suffices to show that $D(x,n)=(m,n)$ for any numbers $x\in\mathbb{R}$ and $n\in\mathbb{Z}$ and some integer $m\in\mathbb{Z}$. Let $x=m+\delta$, where $m\in\mathbb{Z}$ and $|\delta|\le 1/2$. Then for $D(x,n)=(x',y')$ we have:

$$\rho((x,n),(x',y'))\ge\left|\int_n^{y'}\frac{dy}{y}\right|=|\ln(y'/n)|.$$

Therefore if $y'\ne n$ and $n>1$, then

$$\rho((x,n),(x',y'))\ge|\ln((n+1)/n)|>\frac{1}{n}-\frac{1}{2n^2}>\frac{|\delta|}{n}.$$

On the other hand, $\rho((x,n),(x',y'))<|\delta|/n$ for $(x',y')=(m,n)$.

It remains to check the case $n=1$ and $y'=2^{-k}$ with $k\ge 1, k\in\mathbb{Z}$. In this case

$$\rho((x,1),(x',y'))\ge\left|\int_{\frac{1}{2}}^{1}\frac{dy}{y}\right|=\ln 2>|\delta|>\rho((x,1),(m,1)).$$

∎

It is of interest that the intersection of the second lattice $\mathbb{Z}_L$ with horizontal lines is not invariant with respect to the discretized map. The proof of this statement and the study of the discretization of finite radius rotation maps in Lobachevski's metric is technically sufficiently complex and will be published elsewhere.

References

[1] M. Blank, *Ergodic properties of discretizations of dynamical systems*, Dokl. Russian Ac. of Sci, **278**:4 (1984), 779-782.

[2] M. Blank, *Discreteness and continuity in problems of chaotic dynamics*, Monograph, Amer. Math. Soc., 1997.

[3] M. Blank, *Marginal singularities, almost invariant sets, and small perturbations*, Chaos, **1**:3(1991), 347-356.

[4] M. Blank, *Pathologies generated by round-off in dynamical systems*, Physica D **78**:1-2 (1994), 93-114.

[5] M. Blank, *Round-off induced instabilities and some unusual arithmetics*, Comptes Rendus CRAS, II, **321** (1995), 139-145.

[6] P. Diamond, P. Kloeden, V. Kozyakin, A. Pokrovskii, *Boundedness and dissipativity of truncated rotations on uniform lattices*, Technical Report M95/06, Deakin University, Australia, June 1995.

[7] V. Kozyakin, N. Kuznetsov, A. Pokrovskii, I. Vladimirov, *Some problems in analysis of discretizations of continuous dynamical systems*, preprint Deakin University. Centre for Applied Dynamical Systems & Environmental Modeling. CADSEM Reports, 96–019 (1996), 1-19.

[8] T. Krüger, L. Pustylnikov, S. Troubetzkoy, *Acceleration of bouncing balls in external fields*, Nonlinearity, **8** (1995), 397-410.

[9] J. K. Moser, *On invariant curves of area-preserving mappings of an annulus*, Nachr. Akad. Wiss. Gottingen II, Math.-Phys. Kl., **1** (1962), 1-20.