Nonexistence of SBR Measures for some Diffeomorphisms that are "Almost Anosov"

Huyi Hu and *Lai-Sang Young**

Department of Mathematics, University of Arizona,
Tucson, AZ 85721

Abstract. The purpose of the paper is to present some simple examples that are hyperbolic everywhere except at one point, but which do not admit SBR measures. Each example has a fixed point at which the larger eigenvalue is equal to one and the smaller eigenvalue is less than one.

§0 Introduction

Let $f : M \to M$ be a C^2 Anosov diffeomorphism of a compact connected Riemannian manifold, and let m denote the Riemannian measure on M. A result of Sinai (see e.g. [S]) says that f admits a unique invariant Borel probability measure μ with the property that μ has absolutely continuous conditional measures on unstable manifolds. This is the invariant measure that is observed physically, for if $\phi : M \to I\!R$ is a continuous function, then for m-a.e. $x \in M$,

$$\frac{1}{n}\sum_{i=0}^{n-1}\phi(f^i x) \to \int \phi d\mu$$

as $n \to \infty$. The dynamical system (f, μ) is "chaotic" in the following sense: it has positive Lyapunov exponents; its metric entropy is equal to the sum of its positive Lyapunov exponents; (f, μ) is measure-theoretically isomorphic to a Bernoulli shift; and it has exponential decay of correlations for Hölder continuous test functions. These results have been extended to Axiom A attractors by Bowen, Ruelle, etc. (See e.g. [B].)

In this article we will refer to an invariant measure having absolutely continuous conditional measures on unstable manifolds as a *Sinai-Bowen-Ruelle measure* or an *SBR measure*. The work of Oseledec, Pesin and others allows us to extend this notion to a nonuniform setting. (See [P] and [LS].) While some of the properties of SBR measures carry over (see e.g. [LY], Part I), the question of *existence* of SBR measures in this broader context remains poorly understood. We formulate this question more precisely: given a diffeomorphism which appears to be hyperbolic in a large part of phase space, can one decide whether or not it admits an SBR measure? So far there are very few results outside of Axiom A, and these results involve delicate estimates. See e.g. [BC], [BY] for results on the Hénon attractors.

The purpose of the paper is to present some very simple examples that are hyperbolic everywhere except at one point, but which do not admit SBR measures. Precise statements of our results are given in §1. For now imagine slowly deforming a hyperbolic toral automorphism near the origin O until its derivative has one eigenvalue equal to 1 and the other eigenvalue less than 1. Our theorem says that for the resulting diffeomorphism,

* This research is partially supported by the National Science Foundation.

$\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i x} \to \delta_O$ for almost every x with respect to Lebesgue measure. (δ_x denotes Dirac measure at x.) This example can be made to be topologically conjugate to the original toral automorphism, and so it is topologically "chaotic". ¿From the statistical point of view, however, it is totally deterministic in the sense that for almost every initial condition, the trajectory spends nearly one hundred percent of its time arbitrarily near the origin O.

Our result can be thought of as a two dimensional version of the following result. Let $f : [0,1] \to [0,1]$ be a piecewise C^2, piecewise expanding map of the unit interval with $f' = 1$ at a fixed point. Then f cannot admit a finite absolutely continuous invariant measure. (See [PI].) The two dimensional situation is, however, not entirely identical to that in one dimension, for clearly there exist area preserving diffeomorphisms with positive Lyapunov exponents and nonhyperbolic fixed points. A more detailed analysis of whether or not systems that are "almost Anosov" can admit SBR measures will be given in a forthcoming paper by the first named author.

§1 Assumptions and Statements of Results

Let M be a C^∞ two dimensional compact manifold without boundary, let m denote the Riemannian measure on M, and let $f \in \mathrm{Diff}^2(M)$. We assume throughout this paper that f satisfies the following two conditions.

Assumption I.
1. f has a fixed point p, i.e. $fp = p$.
2. There exist a constant $\kappa^s < 1$, a continuous function κ^u with

$$\kappa^u(x)\begin{cases} = 1, & \text{at } x = p\ , \\ > 1, & \text{elsewhere,}\end{cases}$$

and a decomposition of the tangent space T_xM at every $x \in M$ into

$$T_xM = E^u_x \oplus E^s_x$$

such that

$$\|Df_x v\| \le \kappa^s \|v\|, \qquad \forall v \in E^s_x,$$

$$\|Df_x v\| \ge \kappa^u(x) \|v\|, \qquad \forall v \in E^u_x,$$

and

$$\|Df_p v\| = \|v\|, \qquad \forall v \in E^u_p.$$

Assumption II. f is topologically transitive on M.

Definition 1.1. *An $f-$invariant Borel probability measure μ on M is called an SBR measure for $f : M \to M$ if*

i) (f,μ) has positive Lyapunov exponents almost everywhere;

ii) μ has absolutely continuous conditional measures on unstable manifolds.

We give the precise meaning of the second condition above.

Let ξ be a measurable partition of a measure space X, and let ν be a probability measure on X. Then there is a family of probability measures $\{\nu_x^\xi : x \in X\}$ with $\nu_x^\xi\big(\xi(x)\big) = 1$ such that for every measurable set $B \subset X$, $x \to \nu_x^\xi(B)$ is measurable and

$$\nu B = \int_X \nu_x^\xi(B) d\nu(x). \tag{1.1}$$

The family $\{\nu_x^\xi\}$ is called a *canonical system of conditional measures* for ν and ξ. (For a reference, see e.g. [R].)

Suppose now that $f : (M, \mu) \to (M, \mu)$ has positive Lyapunov exponents almost everywhere. Then for a.e. x, the unstable manifold $W^u(x)$ exists and is an immersed submanifold of M (see [P]). A measurable partition ξ of M is said to be *subordinate to unstable manifolds* if for μ-a.e. x, $\xi(x) \subset W^u(x)$ and contains an open neighborhood of x in $W^u(x)$. Let m_x^u denote the Riemannian measure induced on $W^u(x)$. We say that μ has *absolutely continuous conditional measures on unstable manifolds* if for every measurable partition ξ that is subordinate to unstable manifolds, μ_x^ξ is absolutely continuous with respect to m_x^u (written $\mu_x^\xi << m_x^u$) for μ-a.e. $x \in M$. (For more details, see e.g. [LS]).

It is easy to verify that if $\mu_x^\xi << m_x^u$ for one measurable partition ξ subordinate to unstable manifolds, then the same property holds for all other measurable partitions subordinate to unstable manifolds.

We now state our results. Let f and M be as in the beginning of this section.

Theorem A. *f does not admit SBR measures.*

Theorem B. *For m-a.e. $x \in M$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x} = \delta_p$$

where δ_z is the Dirac measure at z, and the above convergence is in the weak * topology.

As a by-product of our proofs for Theorem A and Theorem B, we obtain the following.

Theorem C. *f has an infinite invariant measure $\bar{\mu}$ with the following properties:*

(i) if U is any open neighborhood of p in M, then $\bar{\mu}(M\backslash U) < \infty$;

(ii) $\bar{\mu}$ has absolutely continuous conditional measures on weak unstable manifolds.

Remark 1.2. Weak unstable manifolds are defined in Proposition 2.2 (2). Note that the definition of absolutely continuous conditional measures on unstable manifolds makes sense even though $\bar{\mu}$ is a $\sigma-$finite measure.

One could think of $\bar{\mu}$ as an infinite SBR measure. In this paper, however, the term "SBR measure" without any qualifications will always be reserved for probability measures.

§2. Preliminaries

Lemma 2.1. *The maps $x \to \{E_x^u\}$ and $x \to \{E_x^s\}$ are continuous.*

This is an easy consequence of the "gap" between κ^s and $1 = \inf\{\kappa^u(x) : \ x \in M\}$. The proof is left to the reader.

We will use the following notation: for $\beta > 0$, $E^u_x(\beta) = \{v \in E^u_x : \ |v| \le \beta\}$, $E^s_x(\beta) = \{v \in E^s_x : \ |v| \le \beta\}$, and $E_x(\beta) = E^u_x(\beta) \times E^s_x(\beta)$.

Proposition 2.2. *There exist two continuous foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ on M tangent to E^u and E^s respectively for which the following hold.*

(1) The leaf of $\mathcal{F}^s$ through x, denoted by $\mathcal{F}^s(x)$, is the stable manifold at x, i.e.

$$\mathcal{F}^s(x) = W^s(x) = \{y \in M : \ \exists C = C_y, \ \text{s.t. } d(f^n x, f^n y) \le C(\kappa^s)^n \ \forall n \ge 0\}.$$

(2) The leaf of $\mathcal{F}^u$ through x, denoted by $\mathcal{F}^u(x)$, is the unstable or "weak unstable" manifold at x, i.e.

$$\mathcal{F}^u(x) = \{y \in M : \ \lim_{n\to\infty} d(f^{-n}x, f^{-n}y) = 0\}.$$

(3) There exist constants $\beta > 0$ and $D > 0$ such that for all $x \in M$, if $\mathcal{F}^u_\beta(x)$ is the component of $\mathcal{F}^u(x) \cap \exp_x E_x(\beta)$ containing x, then $\exp_x^{-1} \mathcal{F}^u_\beta(x)$ is the graph of a function $\phi^u_x : \ E^u_x(\beta) \to E^s_x(\beta)$ with $\phi^u_x(0) = 0$ and $\|\phi^u_x\|_{C^2} \le D$. The analogous statement holds for $\mathcal{F}^s_\beta(x)$.

Proof: These results follow from Theorem 5.5 and Theorem 5A.1 in [HPS]. We indicate how $\mathcal{F}^u$ is obtained.

Let $x \in M$ be fixed. If $\phi : E^u_x(\beta) \to E^s_x(\beta)$ is a function with $\phi(0) = 0$ and $\mathrm{Lip}(\phi) \le 1$, we let $\mathcal{G}_x\phi : E^u_{fx}(\beta) \to E^s_{fx}(\beta)$ be the function defined by

$$\mathrm{graph}(\mathcal{G}_x\phi) = \big(\exp_{fx}^{-1} \circ f \circ \exp_x(\mathrm{graph}\,\phi)\big) \cap E_{fx}(\beta).$$

Note that $\mathcal{G}_x\phi$ is always well defined, even though f is not uniformly hyperbolic. The function ϕ^u_x in assertion (3) is then obtained as the pointwise limit as $n \to \infty$ of

$$\mathcal{G}_{f^{-1}x} \circ \cdots \circ \mathcal{G}_{f^{-n+1}x} \circ \mathcal{G}_{f^{-n}x}(0),$$

where 0 represents the 0 function from $E^u_{f^{-n}x}(\beta)$ to $E^s_{f^{-n}x}(\beta)$, and $\mathcal{F}^u(x)$ is defined to be

$$\bigcup_{n\ge 0} f^n\big(\mathcal{F}^u_\beta(f^{-n}x)\big). \qquad \blacksquare$$

Remark 2.3. For convenience we will write $W^u(x) = \mathcal{F}^u(x)$, $W^u_\beta(x) = \mathcal{F}^u_\beta(x)$ etc. and refer to $W^u(x)$ and $W^u_\beta(x)$ as the "unstable manifold" and "local unstable manifold" respectively at x, even though points on these manifolds may not be contracted exponentially in backwards time.

The Lipschitzness of the W^s-foliation will be very important for us later on. We give the form of the definition that will be used.

Definition 2.4. *Let Σ_1 and Σ_2 be two W^u-leaves, and let $\theta : \Sigma_1 \to \Sigma_2$ be a continuous map defined by sliding along the W^s-leaves, i.e. for $x \in \Sigma_1$, $\theta(x) \in \Sigma_2 \cap W^s(x)$. We say W^s is Lipschitz if θ is Lipschitz for every $(\Sigma_1, \Sigma_2; \theta)$.*

For $y \in W^s(x)$, let $d^s(x,y)$ denote the distance between x and y measured along $W^s(x)$, and for $z \in W^u(x)$, let $d^u(x,z)$ be defined similarly.

Proposition 2.5. *The W^s-foliation is Lipschitz. In fact, given $D_1 > 0$, there exists $L_1 > 0$ such that for every $(\Sigma_1, \Sigma_2; \theta)$ with $d^s(x, \theta(x)) < D_1$ $\forall x \in \Sigma_1$, the Lipschitz constant of θ is less than or equal to L_1.*

Proof: This result follows from the stronger statement that the map $x \to E^s_x$ is C^1, which can be obtained using the same ideas in the proof of Theorem 6.3 in [HP]. We sketch a more direct proof here for the convenience of the reader.

Let $x_1 \in \Sigma_1$, and let γ be an arbitrarily short segment in Σ_1 containing x_1. We will argue that $l(\gamma) \approx l(\theta\gamma)$, where l denotes length and "$\approx$" means "up to a constant".

By taking a suitably large iterate of f, we may assume that $f^n\gamma$ and $f^n(\theta\gamma)$ are very near each other, and $l(f^n\gamma) \approx d^s(f^n x, f^n(\theta x))$ for $x \in \gamma$. Notice that $\forall x \in \gamma$, $d^s(f^n(x), f^n(\theta x)) \le D_1(\kappa^s)^n$ $\forall n > 0$. Also, $\left|Df^n_x|_{E^u_x}\right| \ge 1$ $\forall n \ge 0$. Observe the following:

(1) $l(f^n\gamma) \approx l(\theta(f^n\gamma))$;

(2) $\left|Df^n_x|_{E^u_x}\right| \approx \left|Df^n_{\theta x}|_{E^u_{\theta x}}\right|$ $\forall x \in \gamma$;

(3) $\forall y_1, y_2 \in \gamma$,

$$\begin{aligned}\frac{\left|Df^n_{y_1}|_{E^u_{y_1}}\right|}{\left|Df^n_{y_2}|_{E^u_{y_2}}\right|} &\approx \prod_{i=0}^{n-1}\big(1 \pm \mathrm{const}\cdot d(f^i y_1, f^i y_2)\big) \le \big(1 \pm \mathrm{const}\cdot l(f^n\gamma)\big)^n \\ &\approx \big(1 \pm \mathrm{const}\cdot D_1(\kappa^s)^n\big)^n \approx \mathrm{const}.\end{aligned}$$

(The proof of (3) uses the boundness of the C^2 norms of ϕ^u_x. See Proposition 2.2.3).) Combining (1)-(3), we get $l(\gamma) \approx l(\theta\gamma)$. ∎

Lemma 2.1 and Proposition 2.2 imply that f has a local product structure, i.e. there exist constants $0 < \epsilon < \beta$ such that $\forall y, z \in M$ with $d(y,z) < \epsilon$, $[y,z] := W^u_\beta(y) \cap W^s_\beta(z)$ and $[z,y] := W^u_\beta(z) \cap W^s_\beta(y)$ each contains exactly one point.

A rectangle R is a set in M such that $y, z \in R$ implies $[y,z], [z,y] \in R$. If γ^u, γ^s are segments of W^u- and W^s-leaves respectively, then $[\gamma^u, \gamma^s]$ denotes the rectangle $\{[y,z] : y \in \gamma^u, z \in \gamma^s\}$ provided that everything makes sense. If R is a rectangle and $x \in R$, we let $W^u(x,R) = W^u_\beta(x) \cap R$ and $W^s(x,R) = W^s_\beta(x) \cap R$. If Q and R are two rectangles, we say that f^nQ *$u-$crosses* R if $\forall x \in Q$ with $f^n x \in R$, $f^n W^u(x,Q) \cap R = W^u(f^n x, R)$.

We record a simple fact that will be used in §4.

Proposition 2.6. *$W^u(p)$ and $W^s(p)$ are both dense in M.*

Proof: We only prove the proposition for $W^s(p)$.

Let P be a rectangle containing p and let R be any other rectangle, both with nonempty interiors. Let $\hat{R}$ be a strictly smaller rectangle lying in the interior of R. By the topological transitivity of f, $\exists n > 0$ such that $f^n\hat{R} \cap P \ne \emptyset$. For n sufficiently large, f^nR is considerably longer than $f^n\hat{R}$ in the $u-$direction. We may therefore assume that f^nR $u-$crosses P. This implies that $f^{-n}W^s(p,P) \cap R \ne \emptyset$. ∎

§3. Distortion Estimates

The goal of this section is to prove the following.

Proposition 3.1. *Given any small rectangle P containing p in its interior, there exist constants $\delta > 0$ and $J > 1$ such that if γ is a W^u-segment with $l(\gamma) \leq \delta$ and $\gamma \cap P = \emptyset$, then $\forall y, z \in \gamma$ and $n > 0$,*

$$J^{-1} \leq \frac{|Df_z^{-n}|_{E_z^u}|}{|Df_y^{-n}|_{E_y^u}|} \leq J.$$

We write $y_i = f^{-i}y$, $z_i = f^{-i}z$ and $\gamma_i = f^{-i}\gamma$. If Γ is a W^u-segment in P, let $\tilde{\Gamma} = \{[p, x] : x \in \Gamma\}$. In what follows, the letter C will be used to denote a generic constant, which is allowed to depend only on f.

Lemma 3.2. *Let P^+ be one of the components of $fP \backslash P$, and let $\gamma = W^u(x, P^+)$ for some $x \in P^+$. Assume that $\gamma_i \subset P$ for $i = 1, \cdots n-1$. Then for any $y, z \in \gamma$,*

$$\log \frac{|Df_z^{-n}|_{E_z^u}|}{|Df_y^{-n}|_{E_y^u}|} \leq C \frac{d^u(y, z)}{l(\gamma)}.$$

Proof: First, we have for $j \leq n$,

$$\log \frac{|Df_z^{-j}|_{E_z^u}|}{|Df_y^{-j}|_{E_y^u}|} = \log \prod_{i=0}^{j-1} \Big(1 + \frac{|Df_{z_i}^{-1}|_{E_{z_i}^u} - Df_{y_i}^{-1}|_{E_{y_i}^u}|}{|Df_{y_i}^{-1}|_{E_{y_i}^u}|}\Big) \leq C \sum_{i=0}^{j-1} |Df_{z_i}^{-1}|_{E_{z_i}^u} - Df_{y_i}^{-1}|_{E_{y_i}^u}|.$$

Using Proposition 2.2 (3) and the Lipschitzness of W^s (Proposition 2.5), we see that

$$|Df_{z_i}^{-1}|_{E_{z_i}^u} - Df_{y_i}^{-1}|_{E_{y_i}^u}| \leq C d^u(z_i, y_i) \leq C l(\gamma_i) \leq C l(\tilde{\gamma}_i).$$

Since the $\tilde{\gamma}_i$ are pairwise disjoint, we have

$$\sum_{i=1}^{j-1} l(\tilde{\gamma}_i) \leq l(W^u(p, P)).$$

The arguments above tell us that $\forall j \leq n$,

$$\frac{d^u(z_j, y_j)}{l(\gamma_j)} \leq C \frac{d^u(z, y)}{l(\gamma)}.$$

We conclude that

$$\log \frac{|Df_z^{-n}|_{E_z^u}|}{|Df_y^{-n}|_{E_y^u}|} \leq C \sum_{j=0}^{n-1} d^u(z_j, y_j) \leq C \frac{d^u(z, y)}{l(\gamma)}. \qquad \blacksquare$$

Proof of Proposition 3.1:
Let $0 = n_0 = n_0 + k_0 \leq n_1 < n_1 + k_1 < \cdots < n_t < n_t + k_t \leq n_{t+1} = n$ be such that

$$\gamma_j \cap P \neq \emptyset \qquad \forall n_i < j < n_i + k_i \quad 1 \leq i \leq t,$$

and

$$\gamma_j \cap P = \emptyset, \qquad \text{otherwise.}$$

Then

$$\log \frac{|Df_z^{-n}|_{E_z^u}|}{|Df_y^{-n}|_{E_y^u}|} = \sum_{i=1}^{t} \log \frac{|Df_{z_{n_i}}^{-k_i}|_{E_{z_{n_i}}^u}|}{|Df_{y_{n_i}}^{-k_i}|_{E_{y_{n_i}}^u}|} + \sum_{i=0}^{t} \sum_{j=n_i+k_i}^{n_{i+1}-1} \log \frac{|Df_{z_j}^{-1}|_{E_{z_j}^u}|}{|Df_{y_j}^{-1}|_{E_{y_j}^u}|}.$$

Lemma 3.2 applied to the terms in the first series gives each a contribution of $C' d^u(y_{n_i}, z_{n_i})$, where C' is a constant depending on P. This is summable since $d^u(y_{n_i}, z_{n_i})$ decreases exponentially in i. Each term in the second series is less than or equal to $Cd^u(y_j, z_j)$, so we again have a geometric series. ∎

§4 Proof of Theorem A

The following one-dimensional fact plays a key role.

Lemma 4.1. *Let $h : [-1,1] \to \mathbb{R}$ be a C^2 map with $h(0) = 0$, $h'(0) = 1$ and $h'(x) \geq 1$ $\forall x \in [-1,1]$. Let $a_0 \in [0,1]$, and let $a_i = h^{-i} a_0$ for $i \geq 1$. Then $\sum\limits_{i=0}^{\infty} a_i = \infty$.*

Proof: ¿From the Taylor expansion of h, we know that for $x > 0$,

$$hx \leq x + Lx^2 \tag{$*$}$$

for some L. Increasing L if necessary, we may assume that $a_1 \geq \frac{1}{L}$. We will show that $a_i \geq \frac{1}{Li}$ implies that $a_{i+1} \geq \frac{1}{L(i+1)}$. Suppose this is not true. Then

$$a_{i+1} + La_{i+1}^2 < \frac{1}{L(i+1)}\Big(1 + \frac{1}{i+1}\Big) = \frac{1}{Li}\Big(1 - \frac{1}{i+1}\Big)\Big(1 + \frac{1}{i+1}\Big) < \frac{1}{Li} \leq a_i,$$

contradicting $(*)$ with $x = a_{i+1}$.

With $a_i \geq \frac{1}{Li}$ $\forall i \geq 1$, the desired conclusion follows. ∎

Before giving the proof of Theorem A, we recall some facts from general nonuniform hyperbolic theory. Let f be an arbitrary $C^{1+\alpha}$ diffeomorphism (not having anything to do with the situation in this paper), and suppose that f preserves an SBR measure μ. Let ξ be a partition subordinate to W^u. Then it is proved in [L] that for μ-a.e. x, the density ρ_x of μ_x^ξ with respect to m_x satisfies

$$\frac{\rho_x(z)}{\rho_x(y)} = \frac{\prod\limits_{i=0}^{\infty} |Df_{z_i}^{-1}|_{E_{z_i}^u}|}{\prod\limits_{i=0}^{\infty} |Df_{y_i}^{-1}|_{E_{y_i}^u}|}$$

for all $y, z \in \xi(x)$. (In particular the quotient on the right makes sense.) The following is also known to be true. Let $\{\Lambda_l\}$ be the "Pesin sets", i.e. sets on which f^n has uniform estimates. Then for every l, $\exists \delta_l > 0$ such that $W^u_{\delta_l}(x)$ exists for every $x \in \Lambda_l$. Also, $\exists C_l > 0$ such that $\forall x \in \Lambda_l$,

$$C_l^{-1} \leq \frac{\rho_x(z)}{\rho_x(y)} \leq C_l \qquad \forall y, z \in W^u_{\delta_l}(x).$$

(See e.g. [P].)

We now return to the situation considered in this paper, i.e. f is again assumed to satisfy Assumptions I and II.

Proof of Theorem A:

Suppose, to derive a contradiction, that f admits an SBR measure μ. Then there is a rectangle $R \subset M$ such that $\mu(R \cap \Lambda_l) > 0$ for some l, and $W^u(x, R) \subset W^u_{\delta_l}(x)\ \forall x \in R \cap \Lambda_l$.

We fix a rectangle of the form $P = [W^u_\delta(p), W^s_\delta(p)]$ and let $Q = f^{-1}P \backslash P$. Let ξ be the partition of Q given by $\xi(x) = W^u(x, Q)$. An argument similar to that in Proposition 2.6 shows that $f^n R$ $u-$crosses Q for some $n > 0$. It follows from our discussion above that there is a set $\hat{Q} \subset Q \cap f^n R$ with $\mu\hat{Q} > 0$ such that

(i) $x \in \hat{Q} \Longrightarrow \xi(x) \subset \hat{Q}$; and

(ii) $\exists C_0 > 0$ such that $\forall x \in \hat{Q}$,

$$C_0^{-1} \leq \frac{\rho_x(z)}{\rho_x(y)} \leq C_0 \qquad \forall y, z \in W^u(x, Q).$$

Let

$$Q^{(i)} = \{y \in \hat{Q} : \ f^j y \in P \text{ for } j = 1, \cdots, i\},$$

and let $\tilde{\gamma}^{(i)}$ be the projection of $Q^{(i)}$ onto $W^u(p, P)$ by sliding along W^s. Then the density estimate above together with the Lipschitzness of the W^s-foliation gives

$$\mu Q^{(i)} \geq C l(\tilde{\gamma}^{(i)}).$$

Using the facts that $f^i Q^{(i)}$, $i = 1, 2, \cdots$, are pairwise disjoint subsets of P, and μ is an invariant measure, we have

$$\mu P \geq \sum_{i=1}^{\infty} \mu(f^i Q^{(i)}) = \sum_{i=1}^{\infty} \mu Q^{(i)} \geq C \sum_{i=1}^{\infty} l(\tilde{\gamma}^{(i)}).$$

Lemma 4.1 applied to $f|_{W^u_\beta(p)}$ tells us that this sum diverges, contradicting $\mu(M) = 1$. ▮

§5 Proofs of Theorems B and C

We first construct some neighborhoods of p that are convenient to work with. For a closed rectangle R, let $\partial^s R = \{x \in R : x \notin \text{int } W^u(x, R)\}$.

Lemma 5.1. *There exist rectangles P of the form $P = [\gamma^u, \gamma^s]$ satisfying the follows:*

(1) γ^u and γ^s are segments of $W^u(p)$ and $W^s(p)$ respectively such that p lies in the interiors of γ^u and γ^s;

(2) there is a compact segment $\hat{W}^s \subset W^s(p)$ with $f\hat{W}^s \subset \hat{W}^s$ such that $\partial^s P \subset \hat{W}^s$ and $\big(\hat{W}^s \backslash W^s(p, P)\big) \bigcap \operatorname{int} P = \emptyset$;

(3) there is a compact segment $\hat{W}^u \subset W^u(p)$ with analogous properties.

Moreover, the diameter of P can be chosen arbitrarily small.

Proof: Use the fact that $W^u(p)$ and $W^s(p)$ are dense in M. See Lemma 2.6. ∎

We fix P as above, and consider the first return map $g : M\backslash P \to M\backslash P$. That is, for $x \in M\backslash P$, if $\tau(x)$ is the smallest positive integer with $f^{\tau(x)}x \in M\backslash P$, then $gx = f^{\tau(x)}x$. Note that g is not defined on a set of Lebesgue measure 0 on $M\backslash P$, but this will not concern us.

Lemma 5.2. *There exists a $g-$invariant Borel probability measure μ with the property that μ has absolutely continuous conditional measures on the unstable manifolds of f.*

Proof: Let $L = W^u(x, P^+)$, where P^+ is one of the components of $fP\backslash P$ and $x \in P^+$. Let m_L be the Lebesgue measure on L and let $g_*^n m_L$ be the push-forward of m_L, i.e., $(g_*^n m_L)(E) = m_L(g^{-n}E)$. We may take μ to be any accumulation point in the weak * topology of $\frac{1}{n}\sum\limits_{i=0}^{n-1} g_*^i m_L$.

The $g-$invariance of μ is clear. To show that it has absolutely continuous conditional measures, it suffices to consider rectangles R in $M\backslash P$ with small diameters. We assume also that $\operatorname{int} R \cap \hat{W}^s = \emptyset$, where $\hat{W}^s$ is as in Lemma 5.1. The significance of the second condition is as follows: $g^i L$ is the disjoint union of W^u-segments, each one of which begins and ends at some point in $g^j\hat{W}^s$ for some $j \geq 0$. Since $f\hat{W}^s \subset \hat{W}^s$, each component of $g^i L$ that intersects R $u-$crosses R. Let ρ_i denote the density of $g_*^i m_L$ with respect to Lebesgue measure on $g^i L$. Then by Proposition 3.1, $\exists J > 0$ (independent of i) such that for all x, y in the same component of $g^i L \cap R$,

$$J^{-1} \leq \frac{\rho_i(x)}{\rho_i(y)} \leq J.$$

This bound on densities is passed on to the limit measure μ. ∎

Proof of Theorem C:

Let $Q_i = \{x \in M\backslash P : \ \tau(x) = i\}$, where τ is the return time to $M\backslash P$. Define

$$\bar{\mu} = \sum_{i=1}^{\infty}\sum_{j=0}^{i-1} f_*^j(\mu|_{Q_i}).$$

Then $\bar{\mu}$ is clearly $f-$invariant.

Let U be a neighborhood of p. Then $M\backslash U$ is contained in $M\backslash\big(\bigcap\limits_{i=-n}^{n} f^i P\big)$ for some large n, and this latter set clearly has finite $\bar{\mu}-$measure. This proves that $\bar{\mu}$ is at most

$\sigma-$finite. It cannot be finite because it has absolutely continuous conditional measures on unstable manifolds, and Theorem A says that f does not admit SBR measures. ∎

Recall that m denotes the Lebesgue measure on M. We now study the asymptotic behavior of trajectories starting at x for m-a.e. x.

Lemma 5.3. *Let g and μ be as in Lemma 5.2. Then (g,μ) is ergodic.*

Proof: We will follow the two standard steps in the proof of ergodicity of SBR measures for hyperbolic systems without discontinuities. The first step is to use Hopf's argument to show that given a rectangle R, m-a.e. $x \in R$ is "future-generic" with respect to some ergodic measure μ_R, with μ_R possibly depending on R. ("Future-genericity" means that $\frac{1}{n}\sum_{i=0}^{n-1} \phi \circ g^i(x) \to \int \phi d\mu_R$ as $n \to +\infty$ for every continuous function $\phi : M \to R$. "Past-genericity" is defined similarly.) The second step is to show that $\mu_R = \mu$ for all R.

Let $R \subset M \backslash P$ be a rectangle. Note that when we use the word "rectangle" or the symbol "$W^u(x,R)$" in this paper, we are always referring to the stable and unstable manifolds of f — which are not necessarily stable and unstable manifolds of g! First we need to argue that for suitable R, $W^u(x,R)$ *is* indeed a local unstable manifold of g, in the sense that $\forall y \in W^u(x,R)$, $d(g^{-n}x, g^{-n}y) \to 0$ as $n \to \infty$. This is true if $\operatorname{int} R \cap \hat{W}^s = \emptyset$, for this condition will guarantee that $\forall n \geq 0$, $f^{-n}W^u(x,R)$ is either entirely in P or it does not intersect P. Similarly, $W^s(x,R)$ is a local stable manifold of g if $R \cap \hat{W}^u = \emptyset$.

We recall Hopf's argument (see, e.g. [A]) for a rectangle R with the properties in the last paragraph. Since the conditional measures of μ are absolutely continuous on unstable manifolds (Lemma 5.2), there exists $L = W^u(x,R)$ such that m^u_x-a.e. $y \in L$ is generic (both future and past) with respect to some ergodic measure ν_y. All the ν_y's are in fact identical because as $n \to +\infty$, $d(g^{-n}y, g^{-n}z) \to 0 \quad \forall y,z \in L$. We call this common measure μ_R. Now if y is future generic with respect to μ_R, then z is future generic with respect to μ_R $\forall z \in W^s(y,R)$. It then follows from the Lipschitzness of the W^s-foliation (Proposition 2.5) that m-a.e. $z \in R$ is future generic with respect to μ_R.

To carry out the second step, it suffices to observe that if R_1 and R_2 are rectangles with the properties above, then $\exists n > 0$ such that $g^n R_1 \cap R_2 \neq \emptyset$. ∎

Proof of Theorem B:

We will show that given arbitrarily small numbers $\alpha > 0$ and $\epsilon > 0$, there exist neighborhoods $P_2 \subset P_1$ of p with $\operatorname{diam} P_1 \leq \alpha$ such that for m-a.e. $x \in M\backslash P_2$,

$$\frac{\#\{0 \leq k \leq n : \ f^n x \in M\backslash P_1\}}{\#\{0 \leq k \leq n : \ f^n x \in P_1\backslash P_2\}} < \epsilon$$

for all sufficient large n.

To see this, let P_1 be a rectangle of the type in Lemma 5.1. Let $g_1 : M\backslash P_1 \to M\backslash P_1$ and μ_1 be as in Lemma 5.2, and let $\bar{\mu}$ be the infinite measure in the proof of Theorem C. Let P_2 be chosen small enough that $\bar{\mu}(M\backslash P_1) \leq \epsilon\bar{\mu}(P_1\backslash P_2)$. Then $\mu_2 := \bar{\mu}|_{M\backslash P_2}$ is invariant under the first return map $g_2 : M\backslash P_2 \to M\backslash P_2$, and (g_2,μ_2) is ergodic. The Birkhoff Ergodic Theorem applied to (g_2,μ_2) completes the proof. ∎

References

[A] D. Anosov. Geodesic flows on closed Riemannian manifolds with negative curvature. *Proc. Steklov Inst. of Math.* No. 90(1970) 1-235.

[B] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Lecture Notes in Math. 470, Springer-Verlag (1975).

[BC] M. Benedicks & L. Carleson. The dynamics of the Hénon map. *Annals of Math.* 133(1991), 73-169.

[BY] M. Benedicks & L.-S. Young. Sinai-Bowen-Ruelle measure for certain Hénon maps. to appear in *Inventiones.*

[HP] M. Hirsch & C. Pugh. Stable manifolds and hyperbolic sets. *AMS Proc. Symp. Pure Math.* 14, (1970), 133-164.

[HPS] M. Hirsch, C. Pugh & M Shub. *Invariant Manifolds.* Lecture Notes in Math. 583, Springer-Verlag (1977).

[L] F. Ledrappier. Propriétés ergodiques des measure de Sinai. *Publ. Math. I.H.E.S.* 59(1984) 163-188.

[LS] F. Ledrappier & J.-M. Strelcyn. Estimation from below in Pesin's entropy formula. *Ergod. Th. & Dynam. Sys.* 2(1982), 203-219.

[LY] F. Ledrappier & L.-S. Young. The metric entropy of diffeomorphisms. *Annals of Math.* 122(1985), 509-574.

[P] Ya. B. Pesin. Families of invariant manifolds corresponding to non-zero characteristics exponents. *Math. of the USSR Izvestija* 10 (1978), 1261-1305.

[PI] G. Pianigiani. First return map and invariant measures. *Isreal J. Math.* 35(1980), 32-48.

[R] V. A. Rohlin. Lectures on the theory of entropy of transformations with invariant measures. *Russ. Math. Surveys* 22:5 (1967), 1-54.

[S] Ya. G. Sinai. Gibbs measure in ergodic theory. *Russ. Math. Surveys* 27:4 (1972), 21-69.