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Regularity of Interacting Nonspherical Fermi Surfaces: The Exact Self–Energy

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Abstract

Regularity of the deformation of the Fermi surface under short-range interactions is established to all orders in perturbation theory. The proofs are based on a new classification of all graphs that are not doubly overlapping. They turn out to be generalized RPA graphs. This provides a simple extension to all orders of the regularity theorem of the Fermi surface movement proven in [FST2]. Models in which S is not symmetric under the reflection $\mathbf{p} \to -\mathbf{p}$ are included.

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1. Introduction

This paper is a continuation of [FST1,FST2], which we refer to as I and II in the following. It completes the proof of the regularity of the counterterm function K that describes the deformation of the Fermi surface under the interaction. In II, we proved regularity of the random–phase–aproximation (RPA) self–energy by a detailed, and delicate, analysis of singularities arising from tangential intersections of the Fermi surface with its translates. The reason we treated the RPA separately is not that the RPA is popular or time–honoured, but that, as we prove in this paper, it emerges as the leading contribution in a natural way: the least regular contributions to the self–energy are from generalized RPA graphs. The contributions from all other graphs have a higher degree of differentiability. We define in Section 3.6 what we mean by generalized RPA graphs, but let us say right away that these graphs occur as natural generalizations of RPA graphs when the renormalization group scale flow is studied. Traditional RPA graphs have interaction vertices at scale zero only, generalized RPA graphs have interaction vertices that are effective vertex functions of the theory, and lines that carry scales.

We give a very brief outline of our model, and state our hypotheses and the two main theorems in this Introduction. We refer to I and II for the detailed motivation.

Our model is a nonrelativistic fermion quantum field theory, defined by a band structure $e : \mathcal{B} \to \mathbb{R}, \mathbf{p} \mapsto e(\mathbf{p})$ and an interaction $\hat{v} : \mathbb{R} \times \mathcal{B} \to \mathbb{C}, (p_0, \mathbf{p}) \to \hat{v}(p_0, \mathbf{p})$ between the fermions, which has a small coupling constant λ in front. The set $\mathcal{B} \subset \mathbb{R}^d$ is a bounded region of momentum space (for example, the first Brillouin zone) and imposes an ultraviolet cutoff on $\mathbf{p}$. The function $e(\mathbf{p})$ is the energy of a fermion with momentum $\mathbf{p}$ if there is no interaction between the fermions ($\lambda = 0$), measured relative to the chemical potential. In other words, in the system of independent fermions, and at zero temperature, all states with $e(\mathbf{p}) < 0$ are filled, and all states with $e(\mathbf{p}) > 0$ are empty. The covariance $1/(ip_0 - e(\mathbf{p}))$, related to the heat kernel of the free Hamiltonian, determines the propagation of free fermions in the system. The feature that makes the physical behaviour of the system interesting and the mathematical treatment difficult is the singularity of this covariance on the Fermi surface $S = \{\mathbf{p} : e(\mathbf{p}) = 0\}$. The problem addressed in this work is the regularity of the movement of the singular set S as the interaction is turned on.

To state our hypotheses, we need the following standard norms. Let $|\cdot|_k$ be the norm

$$|f|_k = \sup_{p \in \mathbb{R} \times \mathcal{B}} \sum_{|\alpha| \le k} |D^\alpha f(p)| \tag{1.1}$$

where $\alpha = (\alpha_0, \ldots, \alpha_d) \in \mathbb{Z}^{d+1}$, $\alpha_i \ge 0$ for all i, is a multiindex, $|\alpha| = \sum_{i=0}^{d} \alpha_i$, and $\partial^\alpha = \left(\frac{\partial}{\partial p_0}\right)^{\alpha_0} \ldots \left(\frac{\partial}{\partial p_d}\right)^{\alpha_d}$. Let $0 < h \le 1$. We denote the space of C^k functions on a set Ω whose k^{th} derivative is h–Hölder continuous by $C^{k,h}(\Omega)$, and use the norm

$$|f|_{k,h} = \sup_{p \in \Omega} \sum_{|\alpha| \le k} |D^\alpha f(p)| + \max_{\alpha : |\alpha| = k} \sup_{\substack{x, y \in \Omega \\ x \ne y}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^h}. \tag{1.2}$$

For $h = 0$, we define $C^{k,0}(\Omega) = C^k(\Omega)$.

We use the following assumptions on e, $\hat{v}$, and S (not all assumptions are needed in all parts of the proof; the details are stated in the Lemmas and Theorems). For some $h \geq 0$,

(H1)$_{k,h}$ *$\hat{v} \in C^{k,h}(\mathbb{R} \times \mathcal{B}, \mathbb{C})$ with all derivatives of order at most k uniformly bounded on $\mathbb{R} \times \mathcal{B}$, and $\hat{v}$ satisfies $\hat{v}(-p_o, \mathbf{p}) = \overline{\hat{v}(p_o, \mathbf{p})}$. There is a bounded real–valued function $\tilde{v} \in C^{k,h}(\mathcal{B}, \mathbb{R})$ such that $\lim_{p_o \to \infty} \hat{v}(p_o, \mathbf{p}) = \tilde{v}(\mathbf{p})$. The convergence is at rate $|p_o|^{-\alpha}$ uniformly in $\mathbf{p}$ for some $\alpha > 0$.*

(H2)$_{k,h}$ *$e \in C^{k,h}(\mathcal{B}, \mathbb{R})$, and $\nabla e(\mathbf{p}) \neq 0$ for all $\mathbf{p} \in S$.*

(H3)   *The curvature of S is strictly positive everywhere.*

(H5)   *The Fermi surface S is such that $\{u\mathbf{p} + v\mathbf{q} : \mathbf{p}, \mathbf{q} \in S, u, v = \pm 1\} \subset \mathring{\mathcal{F}}$, where $\mathcal{F}$ is the fundamental domain of the action of the group Γ of the Fourier space lattice.*

The meaning of these assumptions is discussed in detail in I and II. We state here only their consequences as regards the constants that will appear in the statements of our theorems (for details, see Lemma I.2.1 and Sections II.2.1 and II.2.2). *(H1)* is trivially satisfied if $\hat{v}$ is a real-valued $C^{2,h}$–function that is independent of p_o, in other words, if the interaction is instantaneous and decays fast enough. The assumptions *(H2)*$_{2,h}$ and *(H3)* on e and S imply that there is a constant $r_o > 0$ such that in the neighbourhood of S given by the condition $U_{r_o} = \{\mathbf{p} \in \mathcal{B} : |e(\mathbf{p})| < r_o\}$, the following statements hold.

(1) There is $g_o > 0$ such that for all $\mathbf{p} \in U_{r_o}$, $|\nabla e(\mathbf{p})| > g_o$.

(2) There is a C^∞ vector field u transversal to S, i.e. satisfying $|u(\mathbf{p})| \leq 1$ and such that for all $\mathbf{p} \in U_{r_o}$, $|u(\mathbf{p}) \cdot \nabla e(\mathbf{p})| \geq u_o = \frac{g_o}{2}$, and there is a $C^{2,h}$–diffeomorphism $\phi : ((-r_o, r_o) \times S^{d-1} \to U_{r_o} \subset \mathcal{B}$ such that $e(\phi(\rho, \theta)) = \rho$. Furthermore, when $d = 2$, $\partial_\theta \phi(0, \theta)$ is of constant (nonzero) length. Throughout, we denote the function ϕ by $\mathbf{p}$, i.e., we write $\mathbf{p}(\rho, \theta) = \phi(\rho, \theta)$. ρ and θ are our standard radial and tangential coordinates around S.

(3) There is a constant $w_o > 0$, related to the minimal curvature κ_o in *(H3)*, such that, e.g. in $d = 2$, for all $\mathbf{p} \in U_{r_o}$, $(\partial_\theta \mathbf{p}, e''(\mathbf{p}) \partial_\theta \mathbf{p}) \geq w_o$ (here e'' is the matrix of second derivatives of e).

For the full regularity proofs, we need another hypothesis. We call a band structure e *symmetric* if

(Sy)   *For all $\mathbf{p} \in \mathcal{B}$, $e(-\mathbf{p}) = e(\mathbf{p})$.*

and *asymmetric* otherwise. We require that either e is symmetric, or, if e is asymmetric, we assume *(H4)* and *(H4')* (stated in Chapter 2 of II). We do not repeat these assumptions in all detail here, since their main consequences were already derived in II. They concern the curvature of S at a point $\mathbf{p} \in S$ and at its antipode $\mathbf{a}(\mathbf{p}) \in S$ (defined by the requirement that the unit normal $\mathbf{n}$ at $\mathbf{a}(\mathbf{p})$ should be the opposite to that at $\mathbf{p}$, $\mathbf{n}(\mathbf{a}(\mathbf{p})) = -\mathbf{n}(\mathbf{p})$). *(H4)* requires that these two curvature must not differ by too much (if *(Sy)* holds, they are equal), where 'too much' means a fixed number. *(H4')* requires, on the other hand, that for asymmetric e, the curvature at a point and its antipode have to differ at all but a finite number of points. These geometrical conditions on the Fermi surface lead to rather delicate bounds which imply, for the models with an asymmetric e, the regularity of the RPA self–energy and counterterm, as well as the suppression of the Cooper instability (see II, Appendix C and Theorem II.4.9). As discussed in II. the regularity proof is easier for symmetric e, partly because the antipode function is, by definition, in general only C^1 if e is C^2, and partly because certain singular points arise only in the asymmetric case (see Lemma II.4.6).

The connected, amputated Green functions of the interacting system are generated by the effective action. Because of the singularity, we regularize the propagator by a cutoff $\varepsilon = M^I$ (here $M > 1$ and $I \in \mathbb{Z}, I < 0$ (this will be recalled in detail in Chapter 3) and study the limit $\varepsilon \to 0$. At positive ε, the effective action

$$e^{\mathcal{G}(\chi,\bar\chi)} = \int d\mu_{C(\varepsilon)}(\psi,\bar\psi) e^{\lambda V^{(0)}(\psi+\chi,\bar\psi+\bar\chi)}, \tag{1.3}$$

with $C(\varepsilon)$ the regularized covariance and $V^{(0)}$ the initial interaction given by $\hat v$ (see I.1.23), is well-defined. For $\varepsilon > 0$, an expansion of the exponential in λ, the polymer expansion for the logarithm, and a determinant bound, imply that $\mathcal{G}$ is analytic in λ in a disk whose radius is ε–dependent. As $\varepsilon \to 0$, the radius of convergence shrinks to zero, and even the coefficients of λ^r diverge for all $r \ge 3$ ('infrared divergences').

In I,II, this paper, and a forthcoming paper (IV), we prove the existence of the limit $\varepsilon \to 0$ of the effective action in any finite order in perturbation theory in the coupling λ. As explained in I and II, this is a rather nontrivial problem even at the perturbative level, because in the absence of spherical symmetry (which is always broken by the crystal lattice) one has to do renormalization with momentum–dependent counterterms. Essentially, one has to prove regularity of the Fermi surface deformation under the interaction. It is the purpose of the present series of papers to show that this is indeed possible. The counterterm function K depends on e, λV, and $\mathbf{p}$, $K = K(e,\lambda V,\mathbf{p})$. When the counterterm $(\bar\psi, K\psi)$ is added to the action, the Fermi surface stays fixed, independent of λ, because the counterterm removes those parts of the self-energy that lead to the movement and deformation of the Fermi surface. Consequently, as shown in I, the infrared divergences of the unrenormalized expansion disappear. In reality, however, the counterterm is not there at the beginning, and the Fermi surface responds to the interaction by a deformation. Merely adding a counterterm changes the model. It is explained in detail in I and II that to renormalize without changing the model, one has to solve the equation

$$E = e + K(e,\lambda V) \tag{1.4}$$

in a suitable space $\mathcal{E}$ of functions from $\mathcal{B}$ to $\mathbb{R}$. The results of II and the present paper imply, very roughly speaking, that $\mathcal{E}$ can be chosen as a ball of fixed radius in $(C^2(\mathcal{B},\mathbb{R}), |\cdot|_2)$ around a given E which gives rise to a C^2 Fermi surface with strictly positive curvature. More precisely, we have the following theorem about the formal power series for K,

$$K(e,\lambda V,\mathbf{p}) = \sum_{r=1}^{\infty} \lambda^r\, K_r(e,V,\mathbf{p}), \tag{1.5}$$

and, for $d \ge 3$, the self energy

$$\Sigma(p) = \sum_{r=1}^{\infty} \lambda^r\, \Sigma_r(p) \tag{1.6}$$

Theorem 1.1

(*i*) *Let $d = 2$. For all $h \in [0,\frac12)$: if (H1)$_{2,h}$, (H2)$_{2,h}$, (H3), (Sy), and (H5) hold, then there is a constant $\mathcal{C}_1$ such that for all $r \ge 1$, $K_r \in C^{2,h}(\mathcal{B},\mathbb{R})$, and*

$$|K_r|_{2,h} \le \mathcal{C}_1{}^r\, r! \tag{1.7}$$

The constant $\mathcal{C}_1$ depends only on h, $|e|_{2,h}$, $|\hat{v}|_{2,h}$, g_o, r_o, and w_o. $h = 0$ is allowed.

(ii) *Let $d = 2$. For all $h \in (0, \frac{1}{3})$: if (H1)$_{2,h}$, (H2)$_{2,h}$, (H3), (H4), (H4'), and (H5) hold, then there is a constant $\mathcal{C}_2$ such that for all $r \geq 1$, $K_r \in C^{2,h}(\mathcal{B}, \mathbb{R})$, and*

$$|K_r|_{2,h} \leq \mathcal{C}_2{}^r\, r! \tag{1.8}$$

The constant $\mathcal{C}_2$ depends only on h, $|e|_{2,h}$, $|\hat{v}|_{2,h}$, g_o, r_o, K_a and w_o. $h = 0$ is not allowed. (K_a is defined in (H4')).

(ii) *Let $d \geq 3$. There is $h_o > 0$ such that for all $h \in [0, h_o]$: if (H1)$_{2,h}$, (H2)$_{2,h}$, (H3), and (H4) hold, then there is a constant $\mathcal{C}_3$ such that for all $r \geq 1$, $K_r \in C^{2,h}(\mathcal{B}, \mathbb{R})$, and*

$$|K_r|_{2,h} \leq \mathcal{C}_3{}^r\, r! \tag{1.9}$$

The constant $\mathcal{C}_3$ depends only on d, h, $|e|_{2,h}$, $|\hat{v}|_{2,h}$, g_o, r_o, and w_o. $h = 0$ is allowed. Moreover, there is a constant $\mathcal{C}_4$ such that for all $r \geq 1$, $\Sigma_r \in C^{2,h}(\mathcal{B}, \mathbb{R})$, and

$$|\Sigma_r|_{2,h} \leq \mathcal{C}_4{}^r\, r!. \tag{1.10}$$

For an iteration of the map to get a solution, as done in IV, we need the constants to be uniform on the set $\mathcal{E}$ where we want to do the iteration. The quantities g_o, r_o and w_o are all uniform even for $e \in C^{2,0}$, i.e., with $h = 0$. The set $\mathcal{E}$ will be defined, and the iteration will be done, in IV.

To prove Theorem 1.1, we use the Feynman graph representation, which is a rewriting of every K_r (and every Σ_r) as a sum over values of graphs with r vertices. A restriction of Theorem 1.1 to the contributions from RPA graphs was proven in II. Here we extend it to the full K. We also prove the following statement about the self–energy, announced in II, which is much simpler since it requires only an analysis of simply overlapping graphs. It holds without the filling restriction *(H5)*, and, for asymmetric e, without the condition *(H4')*.

Theorem 1.2 *Let $d \geq 2$ and $0 \leq \gamma < 1$. If (H1)$_{1,\gamma}$, (H2)$_{2,0}$, (H3), and (H4) hold, then $\Sigma_r \in C^{1,\gamma}(\mathbb{R}\times\mathcal{B}, \mathbb{C})$ for all $r \geq 1$. In particular, if (H1)$_{2,0}$, (H2)$_{2,0}$, (H3), and (H4) hold, then $\Sigma_r \in C^{1,\gamma}(\mathbb{R} \times \mathcal{B}, \mathbb{C})$ for all $\gamma < 1$ and all $r \geq 1$.*

As noted in II, the calculations in [F] indicate that in two dimensions, Σ_2 is not twice differentiable because of a logarithmic singularity in the second derivative, even for an $e \in C^\infty$ and $\hat{v} = 1$. As discussed in detail in Section 1.3 of II, this is the main reason why we take the function K, and not Σ itself, to do renormalization. Note also that while $\Sigma_r \in C^{1,\gamma}$ requires only $\hat{v} \in C^{1,\gamma}$, we have to require e to be at least $C^{2,0}$ because our proof of the volume bound Theorem II.1.1, on which Theorem 1.2 is based, requires $e \in C^2$.

As stated in Theorem 1.1, in three or more dimensions, not only K, but also Σ is a C^2 function of its arguments. Although a superficial analysis would suggest that the power counting behaviour is essentially independent of dimension because the codimension of the Fermi surface S is one in any dimension d, the truth is that there is a nontrivial dimension-dependence even in perturbation theory. In our case, the main reason for the better behaviour in $d \geq 3$ is that the singularities of the Jacobian analyzed in II are point singularities on the strictly convex Fermi surface, i.e. point singularities located on a $(d-1)$–dimensional submanifold. The codimension of these point singularities on S is $d-1$ and thus grows with d, and, as shown in `II`, Section 3.5, this implies the better regularity properties of Σ.

The analysis done in `II` was rather intricate even for the very small class of RPA graphs, and a generalization of that analysis to all graphs contributing to K looks very complicated. The observation which allows us to avoid having to extend the analysis of `II` to all graphs is the following. Each volume gain from overlapping loops increases the degree of smoothness by almost one derivative. If there were no volume gains, graphs contributing to the proper selfenergy would be $C^{0,\gamma}$ for every $\gamma < 1$. Thus the value of any graph that provides two volume gains is $C^{2,\gamma}$, hence almost three times differentiable. We call such graphs doubly overlapping. The heart of this paper is the classification of all graphs that are not doubly overlapping. They turn out to be the generalized RPA graphs mentioned above. The analysis of `II` applies to these graphs, so a combination of the results of `II` with the bounds for values of doubly overlapping graphs (along the lines of the tree decomposition done in `I`) implies Theorem 1.1.

It is a natural question whether the statement that e is C^2 implies that K is C^2 generalizes to a statement $e \in C^k$ implies $K \in C^k$, at least if e is symmetric. In `II`, we proved this for the class of RPA graphs. As mentioned, an extension of this proof looks rather difficult, even when combined with further volume techniques.

We end with an overview of the rest of the paper. Chapter 2 contains the main new idea of the proofs given here. It is graph theoretical and can therefore be understood without any knowledge of the scale decomposition. All nontrivial graph theoretical notions required are defined there, so this part is self–contained. In Chapter 3 we prove Theorems 1.1 and 1.2. We outline the strategy at the beginning of Chapter 3. Although the idea of the proof will be clear from the developments in Chapter 2, familiarity with the results of `I` and `II` is required for an understanding of the details. It is a good idea to keep a copy of `I` and `II` in reach when reading Chapter 3 because we shall refer frequently to these papers, and use the notation and results thereof.

2. Classification of Skeleton Graphs

We denote the vertex set of a graph G by $V(G)$ and its set of internal lines by $L(G)$. A line is internal (resp. external) if both (resp. only one) of its ends are hooked to vertices (resp. is hooked to a vertex). If a vertex $v \in V(G)$ has incidence number n, we call it an n-legged vertex. A vertex is called external if it is hooked to an external leg of the graph. If all vertices of G have even incidence number, then G has an even number of external legs. We assume throughout that all graphs have vertices with even incidence number. This is the case for the effective vertices of our model. Thus all graphs (and subgraphs) appearing in our analysis have an even number of external legs.

We define a skeleton graph to be a connected graph with at least two vertices, that has no two–legged vertices and no proper two–legged subgraphs. If G is a two–legged skeleton graph, G is one-particle irreducible (1PI). That is, G cannot be disconnected by cutting one internal line. In fact, we show in Section 2.3 the stronger statement that, if all vertices of the two–legged graph G have even incidence number then G is a skeleton graph if and only if G is two–particle–irreducible.

2.1 Doubly overlapping graphs

We briefly recall the notion of overlapping loops defined in I. Let T be a spanning tree for G, i.e. a subgraph of G that is a connected tree and contains all vertices of G, and $\ell \notin L(T)$ a line of G. We associate to ℓ a loop in G as follows. Denote the vertices at the ends of ℓ by v and w. If $v = w$, the loop generated by ℓ contains only the line ℓ. If $v \neq w$, there is a unique path P_ℓ in T from v to w. The loop generated by ℓ (and T) contains ℓ and all lines of P_ℓ. A graph is overlapping if for some choice T^* of the spanning tree there are lines $\theta \in L(T^*)$ and $\ell_1 \neq \ell_2 \in L(G) \setminus L(T^*)$ such that the loops generated by ℓ_i both contain θ (this statement is equivalent to the definition given in I, see Remark I.2.18 (iv)). It was shown in Lemma I.2.34, that if G is overlapping, then not only T^*, but every spanning tree of G has this property. It is straightforward to verify that the 'sunset' graph shown in Figure 1 is overlapping according to the above criterion. More generally, any graph that contains a sunset subgraph is overlapping. We showed in Section 2.4 of I that the only nonoverlapping graphs with two external legs are generalized Hartree-Fock graphs and that the only nonoverlapping graphs with four external legs are the dressed bubble chains which contribute to a generalized RPA resummation.

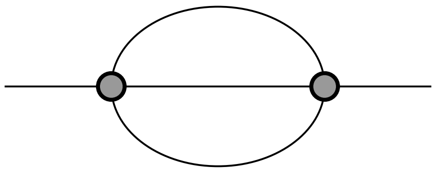


Figure 1: The sunset graph

Definition 2.1 We say that a spanning tree T^* gives rise to two separate overlaps if there are lines $\theta \neq \zeta \in T^*$ and $\ell_1, \ell_2, k_1, k_2 \in L(G) \setminus L(T^*)$, all distinct, such that the loops generated by ℓ_1 and ℓ_2 both contain θ, and the loops generated by k_1 and k_2 both contain ζ, but not θ. We call a graph doubly overlapping (DOL) if it has a spanning tree T^* that gives rise to two separate overlaps.

Since by this definition, a DOL graph has to have at least four loops, the sunset graph of Figure 1 is not DOL. Moreover, the graph has to have at least three vertices, because otherwise there can be at most one line in the spanning tree. By the usual counting arguments, a connected graph with $E = 2$ external legs and only four-legged vertices has $|V(G)|$ loops. Thus any such graph with at least four vertices has at least four loops. For any such graph to be DOL, it must have at least four vertices. Replacing a four-legged vertex by one with more than four legs increases the number of loops. There exists a two-legged graph with three vertices, one of which is six-legged, that is DOL (see Figure 3).

The significance of the notion of overlapping graphs for our analysis is that the value of any overlapping graph, all of whose lines are restricted to carrying momenta p obeying $|e(\mathbf{p})| \leq \varepsilon$, contains a subintegral bounded by

$$\mathcal{W}(\varepsilon) = \sup_{\mathbf{q}\in\mathcal{B}} \max_{v_i \in \{\pm 1\}} \int\limits_{S^{d-1}\times S^{d-1}} d\theta_1\, d\theta_2\ \mathbb{1}\left(|e(v_1\mathbf{p}(0,\theta_1) + v_2\mathbf{p}(0,\theta_2) + \mathbf{q})| \leq \varepsilon\right) . \tag{2.1}$$

Here $\mathbf{p}(\rho, \theta)$ denotes a parametrization of a neighbourhood of the Fermi surface S with $\rho = 0$ corresponding to the Fermi surface itself (see Section 2.2 of II). By Theorem II.1.1, there is a constant Q_V, depending only on $|e|_{2,0}$, r_0, g_0, and w_0, such that

$$\mathcal{W}(\varepsilon) \le Q_V \varepsilon |\log \varepsilon|. \tag{2.2}$$

This *volume improvement bound* leads to an improvement in power counting, as proven for all overlapping graphs in Lemma I.2.35 (and as discussed once more in detail in second order in II), which implies differentability of the self-energy and the counterterm function.

We show below that the spatial momentum integrals for any DOL graph with lines of energy scale ε contain an improvement factor $\mathcal{W}(\varepsilon)^2$, where $\mathcal{W}$ is the volume improvement function defined in (2.1), and that this implies that the value of all DOL graphs is C^2.

Remark 2.2 If G has a connected subgraph H such that H and G/H are overlapping, then G is DOL.

Proof: Recall that G/H is the quotient graph of G obtained by replacing H by a vertex, so that in particular $L(G/H) = L(G) \setminus L(H)$. We shall show that under the given hypotheses, any choice of the spanning tree for G that remains a tree in G/H gives rise to two separate overlaps. Let T be such a spanning tree for G. Then its restriction to H, T_1, is a spanning tree for H, and its quotient T_2 is a spanning tree for G/H. T_1 only consists of lines internal to H. H is overlapping, so there are two lines $k_1 \ne k_2 \in L(H) \setminus L(T_1)$ such that the loops generated by k_1 and k_2 contain a common line $\zeta \in L(T_1)$. Both loops never get out of H, so they contain no line of $L(G) \setminus L(H) = L(G/H)$, and therefore no line of T_2. Since G/H is overlapping, there are two lines $\ell_1 \ne \ell_2$ in $L(G/H) \setminus L(T_2)$ whose loops contain a line $\theta \in L(T_2)$. We now turn to the situation in G itself. The loops generated by ℓ_1, ℓ_2 in G still contain θ. If these loops contain lines of the subgraph H, they may also contain the line ζ. But the loops generated by k_1 and k_2 remain unchanged, so they contain ζ, but not θ. Thus G is DOL. ■

Remark 2.3 Not every spanning tree for a DOL graph G gives rise to two separate overlaps. An example is given in Figure 2. The heavy lines are those in the spanning tree.

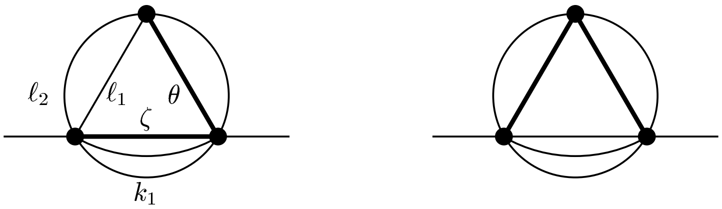


Figure 2

If G is two-legged, 1PI, and has two external vertices, then G is overlapping by Lemma I.2.22. There are paths of minimal length between these two vertices. Let θ be one of them, identify the map θ with the subtree of G that it defines, let $|L(\theta)| = t$ and number the vertices of θ in the order of the walk from the first to the second external vertex as $v_0, \ldots, v_t$. The integer t is the minimal number of steps required to walk from one to the other external vertex.

Theorem 2.4 *All two–legged skeleton graphs with $t \geq 2$ are doubly overlapping.*

This theorem is a direct consequence of the following, more detailed, lemma.

Lemma 2.5 *Let G be a two–legged skeleton graph and $t \geq 2$. Then G is DOL. More precisely, for all r, s with $0 \leq r < s \leq t-1$, G takes the form shown in Figure 3. The subgraphs G_1 and G_2 are connected, and each of them connects to G' by at least three lines. If it is not possible to choose G_1, G_2 and G' all connected, then G is as indicated in Figure 5, with C' and C_1 connected and $\overline{m}_1 \geq 2$ ($\overline{m}_1$ is the number of lines joining G_1 to C') and $n_1 \geq 3$, or as in Figure 6, with C_1 connected, and with $\overline{m} \geq 2$ and $\overline{n} \geq 2$.*

Proof: For $v \in V(G)$, let $s(v)$ be the minimal number of steps required to walk from v_0 to v over lines of G. Thus $s(v_0) = 0$, and $s(v_r) = r$ for all $r \in \{1, \ldots, t\}$ by minimality of θ (of course, there may also be $v \in V(G)$ for which $s(v) > t$). Let S_k be the subgraph of G defined as follows. The set of vertices is $V(S_k) = \{v \in V(G) : s(v) \leq k\}$. The lines of S_k are all those lines $l \in L(G)$ that join vertices in S_k. Obviously, S_k is a connected subgraph of G that contains v_0. If $k = 0$, S_k is the vertex v_0. Also, $v_k \in S_k$ and $v_{k+1} \notin S_k$. Let T_m be the analogous graph constructed from v_t, i.e. if $s'(v)$ is the minimal number of steps from v_t to v, going over lines of G, $V(T_m) = \{v \in V(G) : s'(v) \leq m\}$. T_m is a connected subgraph of G that contains v_t, and $v_{t-m} \in T_m$, but $v_{t-m-1} \notin T_m$.

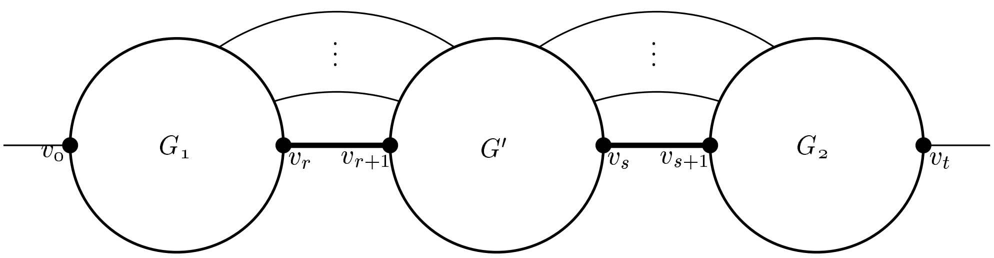


Figure 3

For r and s as given in the statement of the Lemma, let $G_1 = S_r$ and $G_2 = T_{t-s-1}$; both of these graphs are nonempty for all $t \geq 2$ (if $t = 2$, $G_1 = v_0$ and $G_2 = v_2$). Moreover, they are disjoint by minimality of θ and because $r + t - s - 1 < t - 1$. The vertices v_{r+1} and v_s are in neither of the two graphs ($v_{r+1} = v_s$ is possible). Let G' be the subgraph of G with vertex set $V(G') = V(G) \setminus (V(G_1) \cup V(G_2))$ and with those lines of G that join vertices in $V(G')$.

The remaining lines $l \in L(G) \setminus (L(G') \cup L(G_1) \cup L(G_2)$ are external lines of the three subgraphs, and they connect G', G_1 and G_2 to form G. However, by minimality of θ, there can be no line that connects a vertex in G_1 to a vertex in G_2 (if there were such a line, it could be used to make a path of length strictly less than t between v_0 and v_t).

All vertices of G have an even number of legs, so G_1, G_2 and G' must all have an even number of legs. Thus, the number of lines k_1 between G_1 and G' is odd, and the number of lines k_2 from G' to G_2 is odd. G is 1PI, so $k_1 \geq 3$ and $k_2 \geq 3$, hence G is of the form shown in Figure 3.

If G' is disconnected, we decompose it into its connected components C_α. Let C be one of these components. If all external lines of C connect to G_1, an even number of external lines of G_1 is bound. We absorb all these components in G_1. Similarly, we absorb in G_2 all C_α's connected directly only to G_2. There is at least one component of G' that connects to both G_1 and G_2. If there is only one such component, then G is as in Figure 3. Otherwise, G is as shown in Figure 4, where $a \geq 2$, and all subgraphs G_1, G_2 and $C_1, \ldots, C_a$ are connected. Since all the subgraphs have an even number of legs, and because G has no two–legged subdiagrams,

$$m_b \geq 1, \quad n_b \geq 1, \quad m_b + n_b \text{ is even and } m_b + n_b \geq 4 \tag{2.3}$$

holds for all $b \in \{1, \ldots, a\}$. Moreover,

$$\sum_{b=1}^{a} m_b \quad \text{and} \quad \sum_{b=1}^{a} n_b \quad \text{are odd.} \tag{2.4}$$

Note that the m_b and n_b do not both have to be ≥ 2; the dots in Figure 4 should be interpreted that way.

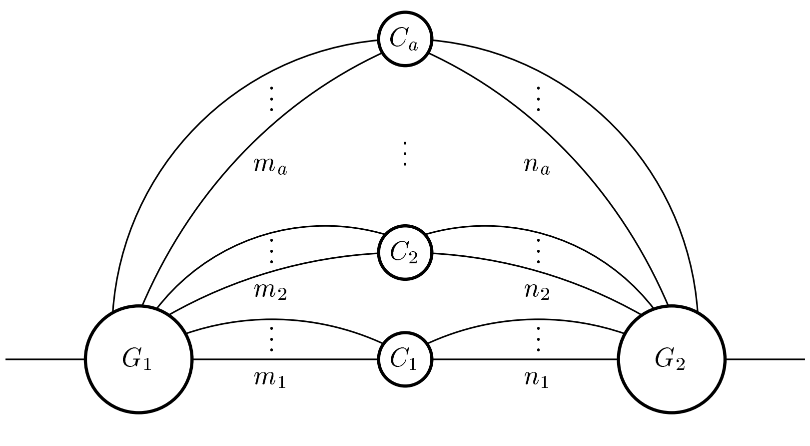


Figure 4

Assume that there is k such that $m_k = 1$ or $n_k = 1$. By the symmetry of the graph and by renumbering, we may assume that $m_1 = 1$. Then $n_1 \geq 3$ by (2.3). We redraw G by collecting $C_2, \ldots, C_a$ and G_2 in a subgraph C'. C' is connected, and G takes the form shown in Figure 5. $\overline{m}_1 = \sum_{b=1}^{a} m_b - 1$ is even by (2.4) and nonzero because G is 1PI, hence $\overline{m}_1 \geq 2$.

If $m_b > 1$ and $n_b > 1$ for all $b \in \{1, \ldots, a\}$, there is a k such that m_k is odd and hence at least 3. Without loss of generality, we may assume that $k = 1$. By (2.3), n_1 must also be odd, hence $n_1 \geq 3$ as well. We collect $C_2, \ldots, C_a$ into a subgraph D. D is disconnected if $a > 2$, but always nonempty. This brings G into the form shown in Figure 6. By construction of G_1 and G_2, and because D is nonempty, $\overline{m} > 0$ and $\overline{n} > 0$. By (2.4), they must be even, so $\overline{m} \geq 2$ and $\overline{n} \geq 2$.

In Figures 3, 5, and 6, the lines drawn fat are those in a possible spanning tree, where one can see directly from the definition that the graph is DOL (in all cases, one can choose a quotient graph H such that Remark 2.2 holds, e.g. in Figure 5, H can be taken to consist of C_1 and C' and the lines joining them). ■

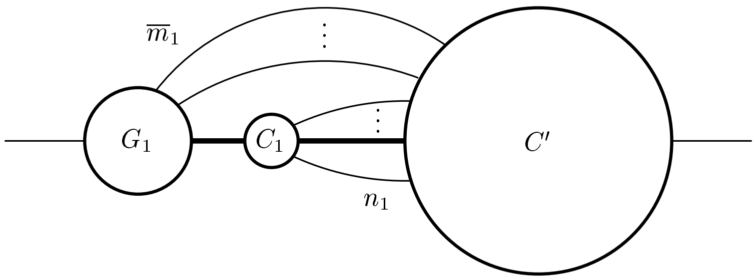


Figure 5

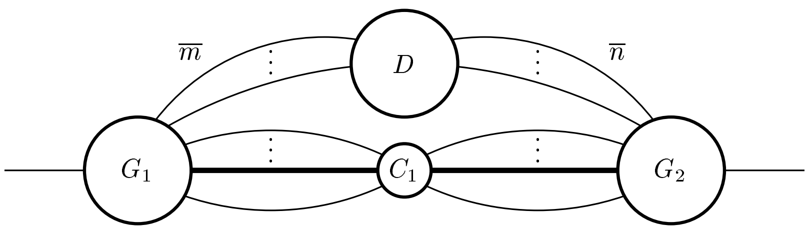


Figure 6

2.2 Graphs with $t = 1$

To complete the classification of doubly overlapping graphs, we turn to the case $t = 1$. We call the external vertices v_0 and v_1. If G has only these two vertices, then G is the sunset diagram shown in Figure 1, or

G is the multiple sunset shown in Figure 7. Otherwise, let D be the subgraph of G consisting of all those vertices of G other than the external vertices, and of all the lines between these vertices. The connected components of D may be joined only to v_0, only to v_1, or to v_0 and v_1. All components that connect only to v_0 are absorbed in a connected subgraph G_1 of G, and similarly we construct G_2. If there is no component of D that connects both to v_0 and v_1, then G is as shown in Figure 1 or 7, with the disks now representing the subgraphs G_1 and G_2. If D is nonempty and G_1 has at least one vertex in addition to v_0, then, by the definition of skeleton graphs, G_1 is overlapping and hence is G is DOL.

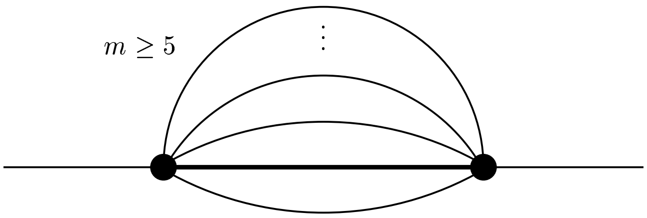


Figure 7: The multiple sunset

Otherwise, G is as in Figure 4, except that now there are $m \geq 1$ additional lines that join G_1 directly to G_2. We first show that for $a \geq 2$ components C_b between G_1 and G_2, the graph is DOL. Since adding lines to a DOL graph keeps it DOL, it suffices to consider the 'minimal' case $a = 2$, $m = 1$, and $m_1+n_1 = m_2+n_2 = 4$. Up to exchanges of G_1 and G_2, or of C_1 and C_2, there are only the four cases shown in Figure 8. They are all DOL by direct inspection.

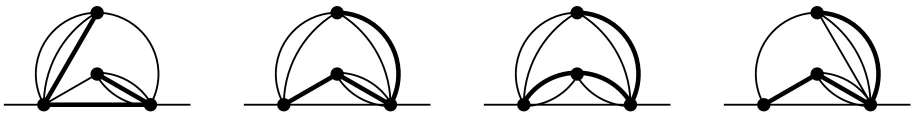
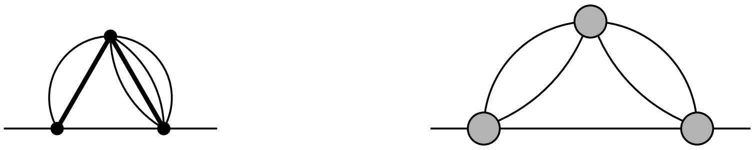


Figure 8

If $a = 1$, we consider $m \geq 2$ first. Again, we may reduce to the minimal possible number of lines, $m_1 + n_1 = 4$, $m = 2$ (reduced from general m even), or $m = 3$ (reduced from general m odd). These graphs are drawn in Figure 9; they are DOL.

Figure 9

There remains the case $m = 1$. If $m_1 + n_1 \geq 6$, we may reduce to $m_1 + n_1 = 6$. Then G looks as in Figure 10 (a), so it is DOL. If $m_1 + n_1 = 4$, G is as in Figure 10 (b).



(a) (b)

Figure 10

If any of the subgraphs drawn as shaded disks is overlapping, G is DOL. If the subgraphs are nonoverlapping, they are dressed bubble chains (see Definition 2.24 and Figure 7 in I) by Lemma 2.26 of I. Since G has $t = 1$, G must then look as in Figure 11, with the disks representing connected subgraphs, or as in Figure 12, with the disks representing either four-legged vertices or $2m$–legged vertices with $m-2$ self–contractions.

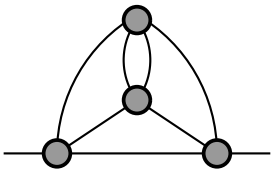


Figure 11

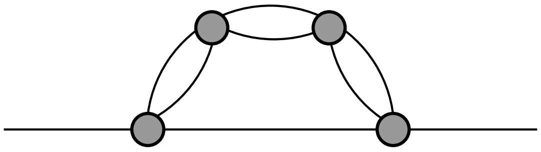


Figure 12*: The wicked ladder. There may be an arbitrary number of bubbles*

The graph in Figure 11 is DOL, but the one in Figure 12, the wicked ladder, is not. In summary, we have proven the following theorem.

Theorem 2.6 *Let G be a non–DOL two–legged skeleton graph. Then G is a sunset, or a multiple sunset, or a wicked ladder, with the vertices possibly being $2m$–legged vertices with $m-2$ self–contractions.*

2.3 Two–particle irreducible graphs

A connected graph is called two–particle irreducible (2PI) if cutting any two particle lines does not disconnect the graph and two–particle reducible (2PR) otherwise. In this section we give a characterization of 2PI graphs with two and four external legs. The statement that G has a two–legged subgraph H includes the requirement that H is a proper subgraph of G.

Proposition 2.7 *Let G be a 1PI graph with vertices that all have even incidence number.*

(*i*) *Let G be two–legged. Then*

$$G \text{ is 2PR} \iff G \text{ has a two–legged subgraph} \tag{2.5}$$

In other words, all the two–legged 2PI graphs are skeleton graphs, and vice versa.

(*ii*) *Let G be four–legged. Then*

$$G \text{ is 2PR} \iff \begin{array}{l} G \text{ has a two–legged subgraph or} \\ G \text{ is as shown in Figure 13,} \\ \text{where } C_1 \text{ and } C_2 \text{ are 1PI} \end{array} \tag{2.6}$$

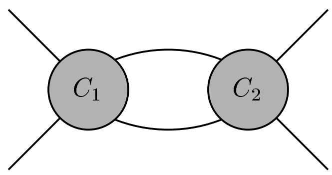


Figure 13: Two–particle reducible four–legged graphs

Proof: For a graph G and line l of G, $G - l$ denotes the graph where l is cut. $E(G)$ denotes the number of external legs of the graph G.

It is obvious that if a graph G contains a two–legged subgraph H, G is 2PR since cutting the lines that connect H to the other vertices of G makes the remaining graph disconnected. It is also obvious that the graph shown in Figure 13 is 2PR. So $\Leftarrow$ holds, and it remains to prove $\Rightarrow$.

Let G be 2PR. Let l_1 and l_2 be any two lines such that $G - l_1 - l_2$ is disconnected. Since G is 1PI, $G - l_1$ and $G - l_2$ are both connected, so $G - l_1 - l_2$ can fall into at most two connected components, C_1 and C_2, which are joined by l_1 and l_2 only, since otherwise $G - l_1 - l_2$ would be connected. Cutting a line gives two external legs, so

$$E(C_1) + E(C_2) = 4 + E(G) \tag{2.7}$$

must hold.

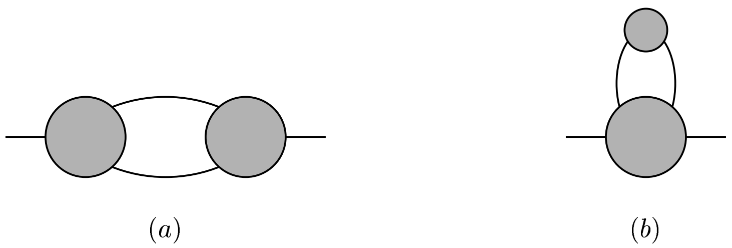


Figure 14*: Two–particle reducible two–legged graphs*

Let $E(G)$ =2. There are two cases: (a) If C_1 and C_2 both contain an external leg of G, G is as shown in Figure 14 (a). (b) If only one of them connects to an external leg, G is as shown in Figure 14 (b), and thus has a two–legged subgraph. But case (a) is impossible: all vertices of G have even incidence number, so all subgraphs of G have an even number of external legs.

Let $E(G) = 4$. If $E(C_1) = 2$ and $E(C_2) = 6$ C_1 is a two–legged subgraph of G. If $E(C_1) = E(C_2) = 4$, G is as in Figure 13. If C_1 or C_2 were 1PR, G would have a two–legged subgraph, because by Theorem I.2.23, a four–legged 1PR graph always has a two–legged subgraph. ■

3. Regularity to All Orders

In this chapter, we prove regularity of the counterterm function K to all orders in perturbation theory. In II, we proved regularity properties of all non–DOL graphs. We now turn to the problem of larger graphs, where the two derivatives applied to $Val(G)$ may act on at least two lines. We first show that the volume bounds alone are sufficient to control the second derivative for all two–legged skeleton graphs with $t \geq 2$. We then show the same for all graphs with at least three vertices and $t \geq 1$, except for those already treated in II. The proof of Theorems 1.1 and 1.2 follows from a combination of these results with an extension of the methods developed in I. At the beginning of this Chapter, we recall briefly some conventions and results of I and II.

3.1 Some Definitions and Results from I and II

We briefly recall the equivalence of the theory with the full propagator and the initial four-fermion interaction to that with the (physically relevant) infrared part of the propagator and a bounded and regular *scale zero effective action*, defined in II.2.50, and give the most basic properties of the single scale propagators.

Integrate out the ultraviolet (that is, large k_0) part of the model specified in the Introduction. By Lemma II.2.3, all vertex functions of the resulting scale zero effective action are $C^{2,h}$, with all derivatives uniformly bounded, if *(H1)*$_{2,h}$ and *(H2)*$_{2,h}$ hold. We may therefore assume that the initial interaction is the scale zero effective action. Thus our graphs have vertices with even incidence number (not necessarily four), and vertex functions that are $C^{2,h}$ with bounds uniform in the scales, and the propagators associated to the lines are the infrared propagators

$$C_{<0}(p_0, E) = \sum_{j<0} C_j(p_0, E) \tag{3.1}$$

(see Section I.2.1). The single scale propagators C_j are defined in Section I.2.1 (see also (II.2.53)) and they satisfy for all $s \le 2$

$$\max_{|\alpha|=s} |D^\alpha C_j(p_0, e(\mathbf{p}))| \le W_s M^{-(s+1)j} \, \mathbb{1}\left(|ip_0 - e(\mathbf{p})| \le M^j\right) \tag{3.2}$$

where D^α is a derivative with respect to p (α is a multiindex with $|\alpha| = s$, $0 \le s \le k$) of order s. The indicator functions take the value $\mathbb{1}(X) = 1$ if X is true and $\mathbb{1}(X) = 0$ otherwise. Here we choose the constant M as in I and II, namely $M \ge \max\{4^3, \frac{1}{r_0}\}$.

The estimates on scale sums for general labelled graphs, will be stated in terms of the function

$$\lambda_n(j, \epsilon) = \sum_{p=1}^{\infty} (|j| + p + 1)^n M^{-\epsilon p}. \tag{3.3}$$

It satisfies $(|h|+1)^m \lambda_n(h,\epsilon) \le \lambda_{m+n}(h,\epsilon)$ and

$$\lambda_m(h,\epsilon)\lambda_n(h,\epsilon) \le \lambda_{m+n}(h,\epsilon). \tag{3.4}$$

For a proof of the second relation, and further properties of λ_n, see Lemma I.2.44.

3.2 Outline of the strategy; the string lemma

To make the proof easier to read, we split it into several lemmas and give an outline of the procedure now.

We do an induction on the order of perturbation theory, with the inductive hypothesis that the root scale behaviour of the second derivative of the value of any two–legged graph is bounded by a power of $|j|$ and that the root scale behaviour of the second derivative of the localized value of a graph is summable. All subgraphs, in particular all two–legged proper subgraphs, of an order r graph are at most of order $r-1$ in the coupling constant λ, so the induction hypothesis applies to these subgraphs. We show as a first step that strings of two–legged subdiagrams have the same scale behaviour as single propagators.

To every labelled graph G, we can associate a skeleton graph G' in a natural way by replacing strings of two–legged subdiagrams by single lines. By the above, the propagators associated to these lines obey bounds that lead to exactly the same power counting as the free propagator. To do the induction step, one then has to show that the value of G', as given by the assignment of propagators and effective vertices, is C^2.

We shall first consider the simplified problem in which all vertices are of scale zero and all lines are of the same scale. We show that all DOL graphs have double volume gain, so that the root scale behaviour of the second derivative is still summable as $j \to -\infty$. By Theorem 2.6, this implies the convergence of the second derivative for the contribution from all two–legged skeleton graphs except for those shown in Figure 1, 12, and 7. The multiple sunset graph in Figure 7 is easily treated in the simplified problem because the scale zero vertices and their derivatives are all uniformly bounded. The values of the graphs in Figure 1 and 12 were shown to be C^2 in II.

After that, we shall put in the full scale structure. Let $\Gamma = (G')\tilde{}(\phi)$ be the root scale quotient of G' (that is, the graph obtained by collapsing all lines that are not of root scale – see Definition 2.27 and Remark 2.28 of I). If Γ is DOL, there are two volume gains on root scale, which suffices to take two derivatives and to show Hölder continuity of degree $\gamma < 1$ of the second derivative. Otherwise, Γ is as in Figure 1, 12, and 7. For the graphs as in Figure 1 or 12, we proceed as in II to prevent the second derivative from acting on root scale. However, it may now act on the effective vertex function, which now replaces the scale zero vertex function as the function P in (II.3.4), and whose second derivative does not have a uniform bound in the scales. Whenever the derivative acts on an effective vertex like this, we analyze the graph at the higher scale where the derivative acts on a fermion propagator. If the graph is still not DOL at that scale, it must still be as in Figure 1 or 12, and we keep going until a second volume gain arises or until we end up at scale zero, in which case the results of II apply.

The graph in Figure 7 requires a separate argument which uses that scale sums over effective vertices with more than four legs grow and that therefore a gain at higher scale amounts to a gain at root scale. This will be discussed in more detail below.

We now come to the details. We start with the Lemma about strings. Let

$$|g|_s^{(j)} = \sum_{\alpha:|\alpha|\le s} \sup\{|\partial^\alpha g(p)| : |ip_0 - e(\mathbf{p})| \le M^j\}, \tag{3.5}$$

and for a differentiable function T defined on the r_0–neighbourhood of S, let $\frac{\partial}{\partial\theta}T$ be defined as

$$\frac{\partial}{\partial\theta}T(p) = \nabla T(p)\cdot\partial_\theta \mathbf{p}(\rho,\theta) \tag{3.6}$$

(since p is near to S, $p = (p_0, \mathbf{p}(\rho,\theta))$).

Lemma 3.1 *Assume (H2)$_{2,0}$. Let $\varepsilon > 0$, $j < 0$, $j_0, \ldots, j_n \in \{j, j+1\}$ with $\min\{j_0,\ldots,j_n\} = j$, and let*

$$S_j(p) = C_{j_0}(p_0, e(\mathbf{p})) \prod_{k=1}^n T_k^{(j)}(p) C_{j_k}(p_0, e(\mathbf{p})) \tag{3.7}$$

Then $\operatorname{supp} S_j \subset \{p : |ip_0 - e(\mathbf{p})| \le M^j\}$. *If the $T_k^{(j)}$ are C^2 and if there are $\tau_k > 0$ and $n_k \in \mathbb{N}$ such that*

$$\left|T_k^{(j)}\right|_s^{(j)} \le \tau_k \lambda_{n_k}(j,\varepsilon) \begin{cases} M^j & s=0 \\ 1 & s=1 \\ |j|^2 & s=2 \end{cases} \tag{3.8}$$

and

$$\begin{aligned} \left|\frac{\partial}{\partial\theta}T_k^{(j)}\right|_0^{(j)} &\le \tau_k |j|^2\, \lambda_{n_k}(j,\varepsilon)\, M^j \\ \left|\frac{\partial}{\partial\theta}\frac{\partial T_k^{(j)}}{\partial p_\alpha}\right|_0^{(j)} &\le \tau_k |j|^2\, \lambda_{n_k}(j,\varepsilon) \end{aligned} \tag{3.9}$$

then

$$\sum_{\alpha:|\alpha|\le s} |\partial^\alpha S_j(p)| \le (n+1)^s\, M^{-j(1+s)}\, \mathbb{1}\left(|ip_0 - e(\mathbf{p})| \le M^j\right)\, B_{s,n} \prod_{k=1}^n \tau_k \lambda_{n_k}(j,\varepsilon) \tag{3.10}$$

with $B_{0,n} = {W_0}^{n+1}$, $B_{1,n} = {W_0}^n(2W_0 + W_1)$ *and* $B_{2,n} = {W_0}^{n-1}({W_0}^2 + {W_1}^2 + W_0(2W_1 + W_2))$. *The* W_r *are as in* (II.2.49). *Moreover*

$$\left|\frac{\partial}{\partial\theta} S_j(p_0, \mathbf{p}(\rho,\theta))\right| \le n{W_0}^{n+1}\ |j|^2\ M^{-j}\ \mathbb{1}\left(|ip_0 - \rho| \le M^j\right) \prod_{k=1}^n \tau_k \lambda_{n_k}(j,\varepsilon)$$
$$\left|\frac{\partial}{\partial\theta}\frac{\partial}{\partial p_\alpha} S_j(p_0, \mathbf{p}(\rho,\theta))\right| \le \tilde{W}_n |j|^2\ M^{-2j}\ \mathbb{1}\left(|ip_0 - \rho| \le M^j\right) \prod_{k=1}^n \tau_k \lambda_{n_k}(j,\varepsilon), \tag{3.11}$$

with $\tilde{W}_n = (2n+1)^2(W_0^n W_1 + W_0^{n+1})$.

This Lemma implies that strings of two–legged subdiagrams behave like single propagators because the bound on the T_i will be proven for two–legged insertions (r– or c–forks or single scale insertions). (3.11) means that derivatives with respect to θ do not affect the exponential scaling behaviour, i.e. they produce only a factor $|j|$, not an M^{-j}. For the scale analysis, this is as good as the behaviour of the free propagator C_j, which satisfies $\frac{\partial}{\partial\theta} C_j(p_0, \mathbf{p}(\rho,\theta)) = 0$. More precisely, as proven in Theorem II.3.5, the critical point analysis of II is unchanged under the replacement of C_j by a string S_j satisfying (3.10) and (3.11).

Proof: The support property follows directly from that of C_j. Let $s = 0$. By the support properties of C_j,

$$|S_j|_0 \le \prod_{k=0}^n |C_{j_k}|_0^{(j)} \prod_{k=1}^n \left|T_k^{(j)}\right|_0^{(j)}. \tag{3.12}$$

By (II.2.49) and (3.8), this is

$$\le {W_0}^{n+1}\ M^{-(n+1)j} \prod_{k=1}^n (\tau_k \lambda_{n_k}(j,\varepsilon) M^j) = {W_0}^{n+1}\ M^{-j} \prod_{k=1}^n \tau_k \lambda_{n_k}(j,\varepsilon). \tag{3.13}$$

Let $s = 1$. The derivative in $\frac{\partial}{\partial p_\alpha} S_j$ can act on a C_{j_k} or on a $T_k^{(j)}$. The contribution where the derivative acts on the product of propagators is bounded by $(n+1){W_0}^n W_1 M^{-2j} \prod\limits_{k=1}^n \tau_k \lambda_{n_k}(j,\varepsilon)$. By (3.8), the contribution from the derivative acting on the product of $T_k^{(j)}$ is bounded by

$$n\, {W_0}^{n+1}\ M^{-(n+1)j} \left(\prod_{k=1}^n \tau_k \lambda_{n_k}(j,\varepsilon)\right) M^{(n-1)j} \le n{W_0}^{n+1}\ M^{-2j} \prod_{k=1}^n \tau_k \lambda_{n_k}(j,\varepsilon). \tag{3.14}$$

so

$$\left|D^1 S_j\right|_0 \le {W_0}^n (W_0 + W_1)(n+1)\ M^{-2j} \prod_{k=1}^n \tau_k \lambda_{n_k}(j,\varepsilon). \tag{3.15}$$

Adding $|S_j|_0$, we get the desired bound. $s = 2$ is similar, since $M > e$, and therefore $|j| \le M^{-j}$. (3.11) follows from $\frac{\partial}{\partial\theta} C_j = 0$ and (3.8). ■

3.3 Volume estimates for DOL graphs

We continue with the volume improvement bounds for overlapping and doubly overlapping graphs. Recall that $r_o > 0$ determines the size of the neighbourhood of the Fermi surface where radial and angular variables are introduced (see Section 2.2 of II). Without loss of generality, we may assume $r_o < 1$.

Lemma 3.2 *Assume (H2)$_{2,0}$–(H4). Let $\varepsilon \in (0, r_o)$ with $|\log \varepsilon| > 1$ and G be a skeleton graph with oriented lines. Let*

$$g_{OL}(G) = \begin{cases} 0 & \text{if } G \text{ is nonoverlapping} \\ 1 & \text{if } G \text{ is overlapping but not DOL} \\ 2 & \text{if } G \text{ is DOL} \end{cases} \tag{3.16}$$

and let T be a spanning tree of G; if G is DOL let T give rise to two separate overlaps. To every external leg ℓ of G we associate a momentum $\mathbf{r}_\ell \in \mathcal{B}$. To every line $\ell \in L(G) \setminus L(T)$ we associate an integration variable θ_ℓ running over the unit sphere S^{d-1} and a momentum $\mathbf{p}_\ell = \mathbf{p}(0, \theta_\ell)$. To every line $b \in L(T)$ we associate a momentum $\mathbf{q}_b$ in the usual way, i.e. we choose an endpoint of b (it does not matter which one), and set $\mathbf{q}_b = \mathbf{p}_{b,in} - \mathbf{p}_{b,out}$, where $\mathbf{p}_{b,in}$ is the sum of $\mathbf{p}_\ell$ and $\mathbf{r}_{\ell'}$ from the incoming lines except b, and $\mathbf{p}_{b,out}$ is the sum of $\mathbf{p}_\ell$ and $\mathbf{r}_{\ell'}$ from the outgoing lines except b. Then the integration volume

$$\mathcal{V} = \sup_{\mathbf{r}_\ell} \int \prod_{\ell \in L(G)\setminus L(T)} \frac{d\theta_\ell}{|S^{d-1}|} \prod_{b \in L(T)} \mathbb{1}(|e(\mathbf{q}_b)| \le \varepsilon) \tag{3.17}$$

satisfies

$$\mathcal{V} \le \mathcal{W}(\varepsilon)^{g_{OL}(G)} \le \begin{cases} (Q_V\ \varepsilon |\log \varepsilon|)^{g_{OL}(G)} & d = 2 \\ (Q_V\ \varepsilon)^{g_{OL}(G)} & d \ge 3 \end{cases}, \tag{3.18}$$

and

$$\begin{aligned} \tilde{\mathcal{V}} &= \sup_{\mathbf{r}_\ell} \int \prod_{\ell \in L(G)\setminus L(T)} (d\mathbf{p}_\ell\ \mathbb{1}(|e(\mathbf{p}_\ell)| \le \varepsilon)\ ) \prod_{b \in L(T)} \mathbb{1}(|e(\mathbf{q}_b)| \le \varepsilon) \\ &\le (\nu_o \varepsilon)^{|L(G)|-|L(T)|} \begin{cases} (\nu_1 \varepsilon |\log \varepsilon|)^{g_{OL}(G)} & d = 2 \\ (\nu_1 \varepsilon)^{g_{OL}(G)} & d \ge 3 \end{cases} \end{aligned} \tag{3.19}$$

with

$$\begin{aligned} \nu_o &= 2|S^{d-1}||J|_o \\ \nu_1 &= 2Q_V(1 + 2\tfrac{|e|_1}{u_o})^2. \end{aligned} \tag{3.20}$$

Here J is the Jacobian of the change of variables to (ρ, θ) and Q_V is the constant in Theorem II.1.1.

Proof: The proof is a simple extension of that of Lemma I.2.35. If G is nonoverlapping, the right hand side of (3.18) is one. The bound is obviously obtained by dropping the product over $b \in L(T)$ in (3.17). If G is overlapping, but not DOL, let ℓ_1 and ℓ_2 be two loop lines whose loops overlap on $b_o \in L(T)$. Then $\mathbf{q}_{b_o} = v_1 \mathbf{p}(0, \theta_{\ell_1}) + v_2 \mathbf{p}(0, \theta_{\ell_2}) + \mathbf{Q}$ with $v_i \in \{1, -1\}$ and $\mathbf{Q}$ possibly depending on the θ_ℓ, ℓ different from ℓ_1 and ℓ_2, and on the external momenta. So

$$\begin{aligned} \mathcal{V} &\le \sup_{\mathbf{r}} \int \prod_{\ell \notin \{\ell_1, \ell_2\}} \frac{d\theta_\ell}{|S^{d-1}|} \int d\theta_{\ell_1}\ d\theta_{\ell_2}\ \mathbb{1}(|e(\mathbf{q}_{b_o})| \le \varepsilon) \\ &\le \sup_{\mathbf{r}} \sup_{\theta_\ell, \ell \notin \{\ell_1, \ell_2\}} \int d\theta_{\ell_1}\ d\theta_{\ell_2}\ \mathbb{1}(|e(\mathbf{q}_{b_o})| \le \varepsilon)\ . \end{aligned} \tag{3.21}$$

The last integral is bounded, uniformly in $\mathbf{Q}$, by $\mathcal{W}(\varepsilon)$, so $\mathcal{V} \le \mathcal{W}(\varepsilon)$, and (3.18) follows from Theorem II.1.1.

If G is DOL, let k_1, k_2, ℓ_1, ℓ_2 and $a, b \in L(T)$ such that the loops of k_1 and k_2 both contain a, and such that the loops of ℓ_1 and ℓ_2 both contain b but not a. We use

$$\prod_{l \in L(T)} \mathbb{1}\left(|e(\mathbf{q}_l)| \le \varepsilon\right) \ \le\ \mathbb{1}\left(|e(\mathbf{q}_a)| \le \varepsilon\right) \quad \mathbb{1}\left(|e(\mathbf{q}_b)| \le \varepsilon\right) \tag{3.22}$$

and proceed as in the previous case, to bound

$$\mathcal{V} \le \sup_{\mathbf{r}} \sup_{\theta_\ell, \ell \notin \{k_1, k_2, \ell_1, \ell_2\}} S_{ab}(\varepsilon) \tag{3.23}$$

with

$$S_{ab}(\varepsilon) = \int d\theta_{k_1}\ d\theta_{k_2}\ d\theta_{\ell_1}\ d\theta_{\ell_2}\ \ \mathbb{1}\left(|e(\mathbf{q}_a)| \le \varepsilon\right) \quad \mathbb{1}\left(|e(\mathbf{q}_b)| \le \varepsilon\right) \tag{3.24}$$

and

$$\begin{aligned} \mathbf{q}_a &= v_1 \mathbf{p}(0, \theta_{k_1}) + v_2 \mathbf{p}(0, \theta_{k_2}) + \mathbf{Q}_a \\ \mathbf{q}_b &= v_1 \mathbf{p}(0, \theta_{\ell_1}) + v_2 \mathbf{p}(0, \theta_{\ell_2}) + \mathbf{Q}_b. \end{aligned} \tag{3.25}$$

As before, $\mathbf{Q}_a$ and $\mathbf{Q}_b$ may depend on many other momenta. However, $\mathbf{Q}_a$ does not depend on θ_{ℓ_1} or θ_{ℓ_2} because the overlaps are separate. Thus

$$\begin{aligned} S_{ab}(\varepsilon) &= \int d\theta_{k_1}\ d\theta_{k_2}\ \ \mathbb{1}\left(|e(\mathbf{q}_a)| \le \varepsilon\right) \ \int d\theta_{\ell_1}\ d\theta_{\ell_2}\ \ \mathbb{1}\left(|e(\mathbf{q}_b)| \le \varepsilon\right) \\ &\le \int d\theta_{k_1}\ d\theta_{k_2}\ \ \mathbb{1}\left(|e(\mathbf{q}_a)| \le \varepsilon\right) \ \ \mathcal{W}(\varepsilon) \le \mathcal{W}(\varepsilon)^2 \end{aligned} \tag{3.26}$$

which proves the first inequality of (3.18). The second inequality follows by Theorem II.1.1.

The proof of (3.19) is trivial in the nonoverlapping case and similar in the two others; we do the DOL case. We first change variables to (ρ_ℓ, θ_ℓ) for all $\ell \in L(G) \setminus L(T)$. This is possible because $|\varepsilon_l| < r_0$ for all l, so the support of every loop integral is contained in the neighbourhood of S where ρ and θ are defined. Using (3.22),

$$\tilde{\mathcal{V}} \le |J|_0^{|L(G)|-|L(T)|} \int \prod_{\ell \in L(G) \setminus L(T)} d\rho_\ell\ \ \mathbb{1}\left(|\rho_\ell| \le \varepsilon\right) \quad \prod_{\ell \in L(G) \setminus L(T)} d\theta_\ell\ \ \mathbb{1}\left(|e(\mathbf{q}_a)| \le \varepsilon\right) \quad \mathbb{1}\left(|e(\mathbf{q}_b)| \le \varepsilon\right)\ . \tag{3.27}$$

Now $\mathbf{q}_a$ depends also on ρ_{k_1} and ρ_{k_2}, and $\mathbf{q}_b$ depends also on ρ_{ℓ_1} and ρ_{ℓ_2}, and possibly also on ρ_{k_1} and ρ_{k_2}. We use $|\rho_\ell| \le \varepsilon$ and Taylor expansion to get

$$\begin{aligned} \mathbb{1}\left(|e(\mathbf{q}_a)| \le \varepsilon\right) \ &=\ \mathbb{1}\left(|e(\pm \mathbf{p}(\rho_{k_1}, \theta_{k_1}) \pm \mathbf{p}(\rho_{k_2}, \theta_{k_2}) + \mathbf{Q}_a)| \le \varepsilon\right) \\ &\le\ \mathbb{1}\left(|e(\pm \mathbf{p}(0, \theta_{k_1}) \pm \mathbf{p}(0, \theta_{k_2}) + \mathbf{Q}_a)| \le \varepsilon(1 + 2\tfrac{|e|_1}{u_0})\right) \end{aligned} \tag{3.28}$$

and similarly,

$$\mathbb{1}\left(|e(\mathbf{q}_b)| \le \varepsilon\right) \ \le\ \mathbb{1}\left(|e(\pm \mathbf{p}(0, \theta_{k_1}) \pm \mathbf{p}(0, \theta_{k_2}) + \mathbf{Q}_b)| \le \varepsilon(1 + 2\tfrac{|e|_1}{u_0})\right)\ . \tag{3.29}$$

We integrate over $\theta_{k_1}, \theta_{k_2}, \theta_{\ell_1}, \theta_{\ell_1}$ first, to get a subintegral similar to S_{ab}, and take a sup over all other θ–variables. Using (3.26) and Theorem II.1.1, and collecting the constants, we get (3.19). ■

We state a generalization of this Lemma where the support condition may depend on the line, i.e. ε gets replaced by ε_ℓ on every line.

Lemma 3.3 *Assume (H2)$_{2,0}$–(H4). Let G be a skeleton graph with oriented lines and let T be a spanning tree of G; if G is DOL let T give rise to two separate overlaps. Let the momentum assignments be as in Lemma 3.2, associate an $\varepsilon_\ell \in (0, r_0)$ to every line $\ell \in L(G)$, and let $\underline{\varepsilon} = \min_{\ell\in L(G)} \varepsilon_\ell$. Then*

$$\bar{\mathcal{V}}(G,T) = \int \prod_{\ell\in L(G)\setminus L(T)} (d\mathbf{p}_\ell \;\; \mathbb{1}(|e(\mathbf{p}_\ell)| \le \varepsilon_\ell)\,) \prod_{\beta\in L(T)} \mathbb{1}(|e(\mathbf{q}_\beta)| \le \varepsilon_\beta) \tag{3.30}$$

satisfies

$$\bar{\mathcal{V}}(G,T) \le (\nu_1|\log\underline{\varepsilon}|)^{g_{OL}(G)} \left(\prod_{\ell\in L(G)\setminus L(T)} \nu_0\varepsilon_\ell\right) \begin{cases} 1 & \text{if } G \text{ is nonoverlapping} \\ \varepsilon_a & \text{if } G \text{ is overlapping, but not DOL} \\ \varepsilon_a\ \varepsilon_b & \text{if } G \text{ is DOL.} \end{cases} \tag{3.31}$$

a and b are lines of T common to the two (separate) pairs of overlapping loops. For $d \ge 3$ the factor $|\log\underline{\varepsilon}|^{g_{OL}(G)}$ is absent.

Proof: The only difference to the previous proof is that one has to be careful in the Taylor expansion steps (3.28) and (3.29) because ε_l now depends on l. We only discuss the modifications because of this. The point is that it may happen that the line in the tree is on a lower scale than those in the loops, i.e. that in the integral

$$\mathcal{V}_2 = \int d\mathbf{p}_1\ d\mathbf{p}_2\ \ \mathbb{1}(|e(\mathbf{p}_1)| \le \varepsilon_1)\ \ \mathbb{1}(|e(\mathbf{p}_2)| \le \varepsilon_2)\ \ \ \mathbb{1}(|e(v_1\mathbf{p}_1 + v_2\mathbf{p}_2 + \mathbf{Q})| \le \varepsilon_3) \tag{3.32}$$

$\varepsilon_3 < \varepsilon_1$, or $\varepsilon_3 < \varepsilon_2$, in which case the Taylor expansion does not give (3.28). Assume, without loss of generality, that $\varepsilon_2 = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. Change variables from $\mathbf{p}_2$ to $\mathbf{p}_3 = v_1\mathbf{p}_1 + v_2\mathbf{p}_2 + \mathbf{Q}$, which is then also integrated over $\mathcal{B}$, so that

$$\mathcal{V}_2 = \int d\mathbf{p}_1\ d\mathbf{p}_3\ \mathbb{1}(|e(\mathbf{p}_1)| \le \varepsilon_1)\ \ \mathbb{1}(|e(\mathbf{p}_3)| \le \varepsilon_3)\ \ \mathbb{1}(|e(v_2\mathbf{p}_3 - v_1v_2\mathbf{p}_1 - v_2\mathbf{Q})| \le \varepsilon_2)\ . \tag{3.33}$$

Changing to variables ρ and θ, the Taylor expansion argument in ρ_1 and ρ_3 now works since $\varepsilon_2 \ge \max\{\varepsilon_1, \varepsilon_3\}$. Therefore

$$\mathcal{V}_2 \le (2|J|_0)^2 \varepsilon_1\ \varepsilon_3 \mathcal{W}\left((1 + 2\tfrac{|e|_1}{u_0})\varepsilon_2\right) \le (2|J|_0)^2\varepsilon_1\ \varepsilon_3\ \nu_1\varepsilon_2\ |\log\varepsilon_2| \tag{3.34}$$

The factor $\log\varepsilon_2$ is bounded by $|\log\underline{\varepsilon}|$. In $d \ge 3$ this factor is absent. Note that in the DOL case, the momentum $\mathbf{Q}_a$ is independent of the integration momenta $\mathbf{p}_1 = \mathbf{p}_{\ell_1}$ and $\mathbf{p}_2 = \mathbf{p}_{\ell_2}$ of the inner loop and so is not affected by the change of variables to $\mathbf{p}_3$. ■

In the following, we assume that r_0 is chosen so small that all bounds from `II` and the volume bounds apply, and such that

$$\begin{aligned} |\log r_0| &\ge 1 \\ \nu_1 r_0 |\log r_0| \log M &< 1. \end{aligned} \tag{3.35}$$

3.4 Doubly volume–improved power counting for skeleton graphs

Let $m \geq 1$ and G be a skeleton graph with $E(G) = 2m$ external legs and n vertices $v_1, \ldots, v_n$. Denote the incidence number of vertex v by $2m_v$ (where $m_v \geq 2$). To every vertex we associate a C^2 vertex function

$$\mathcal{U}_v : (\mathbb{R} \times \mathcal{B})^{2m_v - 1} \times \{\uparrow, \downarrow\}^{2m_v} \to \mathbb{C} \tag{3.36}$$

that is totally antisymmetric under simultaneous exchange of momenta and spins (see Definition 2.10 *(ii)* and Definition 2.8 *(ii)* and *(iii)* of I) and define the norm

$$|\mathcal{U}_v|_s = \sum_{\alpha : |\alpha| \leq s} \sup_{p_1, \ldots p_{2m_v - 1}} \max_{A \in \{\uparrow, \downarrow\}^{2m_v}} |D^\alpha \mathcal{U}_v(p_1, \ldots, p_{2m-1}, A)|. \tag{3.37}$$

We start with an estimate where all lines of the graph have the same scale.

Lemma 3.4 *Assume (H2)$_{2,0}$–(H4). Let $j < 0$ and G be a skeleton graph, with g_{OL} defined as in* (3.16). *To every line ℓ, associate a propagator $S_\ell(p_\ell)\, \delta_{\alpha\alpha'}$, and define the value of G in the standard way (see Definition* 2.10 *(ii) of* I*). If for all $\ell \in L(G)$ and $s \leq 2$ there are constants $P_{\ell,s} \leq 1$ such that for any $\sigma = (\sigma_0, \ldots \sigma_d)$ with $|\sigma| = \sigma_0 + \ldots + \sigma_d = s$,*

$$|D^\sigma S_\ell(p)| \leq P_{\ell,s} M^{-j(1+s)}\, \mathbb{1}\left(|ip_0 - e(\mathbf{p})| \in [M^{j-2}, M^j]\right)\ , \tag{3.38}$$

then

$$|Val(G)|_s \leq 2(4\nu_0)^{|L(G)|}\ M^{j(2-m-s)}\ \left(\nu_1\ |j| M^j\ \log M\right)^{g_{OL}(G)}\ \mu_s(G) \tag{3.39}$$

with

$$\mu_s(G) = \max_T \sum_{\sigma \in Z(s,T,G)} \prod_{l \in L(G)} P_{l, |\sigma_l|} \prod_{v \in V(G)} \left(|\mathcal{U}_v|_{|\sigma_v|}\, M^{j(m_v - 2 + |\sigma_v|)} \right) \tag{3.40}$$

The maximum in (3.39) *is over all spanning trees T of G, and $Z(s,T,G)$ is the set of multiindices*

$$\sigma :\ (L(G) \times \{0, \ldots, d\}) \times \prod_{v \in V(G)} (\{1, \ldots, 2m_v\} \times \{0, \ldots, d\}) \quad \to \{0, 1, 2\} \tag{3.41}$$

such that

$$\sum_{\ell \in L(G)} |\sigma_\ell| + \sum_{v \in V(G)} |\sigma_v| \leq s, \tag{3.42}$$

$\sigma_\ell = 0$ if $\ell \notin L(T)$, and $\sigma_{v,k,r} = 0$ if the leg k of v is not hooked to a line in $L(T)$. Here $|\sigma_\ell| = \sum\limits_{i=0}^{d} \sigma_{\ell,i}$ and $|\sigma_v| = \sum\limits_{v,k,i} \sigma_{v,k,i}$. The constants are as in Lemma 3.2.

Remark 3.5 Up to powers of $|j|$, $M^{-j(m_v + \sigma_v - 2)}$ is precisely the standard power counting behaviour of a $\mathcal{U}_v$ given by an effective vertex of the model on scale j, proven in Theorem 2.45 of I for $\sigma_v \leq 1$ for arbitrary (not necessarily skeleton) graphs. Inspection of the proof of that theorem shows that for skeleton graphs, it also holds for $\sigma_v = 2$. We shall show below that this scaling behaviour holds for $\sigma_v = 2$ and arbitrary graphs, so there will be no need to go through the proof of Theorem 2.45 in I.

Proof: Let T be a spanning tree for G. If G is DOL then let T give rise to two separate overlaps. Fixing the momenta on the lines of T, we obtain (denoting all spin indices by A and $A_v = (\alpha_1^{(v)}, \ldots, \alpha_{2m_v}^{(v)})$) for a multiindex β with $|\beta| \le 2$

$$D^\beta Val(G^J)(q_1, \ldots, q_{2m-1}, A) = \sum_\sigma m(\beta,\sigma) \sum_{A_1,\ldots,A_v} \int \prod_{l\in L(G)\setminus L(T)} đ^{d+1}p_l \prod_{l \in L(G)} D^{\sigma_l} S_l(p_l) \delta_{\alpha_l \alpha_l'} \prod_{v\in V(G)} D^{\sigma_v} \mathcal{U}_v(p_1^{(v)}, \ldots, p_{2m_v-1}^{(v)}, A_v) \tag{3.43}$$

where the sum is over multiindices σ and the possible values of the assignment of $\sigma_{l,r}$ and $\sigma_{v,r}$ depend on the multiindex β, but the restrictions stated in the Lemma apply because only the momenta on lines in the tree can depend on the external momenta. The spin indices α_l and α_l' are fixed by the corresponding indices of the vertex functions at the endpoints of the line. $m(\beta,\sigma)$ is a multinomial factor, and for each $t \in L(T)$, p_t is a linear combination of the loop momenta $(p_l)_{l\in L(G)\setminus L(T)}$ and, possibly, of the external momentum q.

The Kronecker delta for the spins in the propagator implies that there is only one spin sum per line. We bound this sum by 2 times the supremum over all spin values, and use (3.38) to bound the propagators. We also use that $m(\beta,\sigma) \le 2$. This gives

$$\left| D^\beta Val(G^J) \right|_{\text{o}} \le 2\mathcal{V}(G,T)\, 2^{|L(G)|} Y(\beta,T) \tag{3.44}$$

with

$$Y(\beta,T) = \sum_\sigma \prod_{l\in L(G)} P_{l,|\sigma_{l,r}|} M^{-j(1+|\sigma_{l,r}|)} \prod_{v\in V(G)} |\mathcal{U}_v|_{|\sigma_{v,r}|} \tag{3.45}$$

and the integration volume

$$\mathcal{V}(G,T) = \int \prod_{l\in L(G)\setminus L(T)} đ^{d+1}p_l \prod_{l\in L(G)} 1\!\!1\left(|i(p_l)_{\text{o}} - e(\mathbf{p}_l)| \le M^j \right) . \tag{3.46}$$

Collecting the exponent of M^j and noting that $\sum_{v,r} \sigma_{v,r} + \sum_{l,r} \sigma_{l,r} = |\beta|$, we get

$$Y(\beta,T) = M^{-j(|L(G)|+|\beta|)}\, M^{j\sum_v (2-m_v)} \sum_\sigma \prod_{l\in L(G)} P_{l,|\sigma_l|} \prod_{v\in V(G)} \left(|\mathcal{U}_v|_{|\sigma_v|} M^{j(m_v - 2 + |\sigma_v|)} \right). \tag{3.47}$$

We sum over all β with $|\beta| \le s$ so that the sum over σ now runs over the set $Z(s,T,G)$, and bound $M^{-j|\beta|} \le M^{-js}$. Taking the maximum over all spanning trees T of G, we get

$$Y(\beta,T) \le M^{-j(|L(G)|+s)}\, M^{j\sum_v (2-m_v)}\, \mu_s(G). \tag{3.48}$$

After all this combinatorics, we turn to the essential part of the estimate, the bound for $\mathcal{V}(G,T)$. Because $|ip_{\text{o}} - e(\mathbf{p})| \le M^j$ implies $|p_{\text{o}}| \le M^j$ and $|e(\mathbf{p})| \le M^j$,

$$\mathcal{V}(G,T) \le \tilde{\mathcal{V}} \int \prod_{l\in L(G)\setminus L(T)} d(p_l)_{\text{o}}\ 1\!\!1\left(|(p_l)_{\text{o}}| \le M^j \right) \tag{3.49}$$

with $\tilde{\mathcal{V}}$ given in (3.19). By (3.19),

$$\mathcal{V}(G,T) \le \left(2\nu_0 M^{2j}\right)^{|L(G)|-|L(T)|} \; (\nu_1\; |j| M^j\; \log M)^{g_{OL}(G)}. \tag{3.50}$$

Rearranging the various factors, using that $|L(G)| = \sum_v m_v - m$ and $|L(T)| = |V(T)| - 1 = \sum_v 1\; - 1$, we obtain the result. ∎

3.5 The multiple sunset

Before considering the full scale structure, we show how the estimates for the multiple sunset graph shown in Figure 7 fit into the picture. The point is here that because there are five or more lines connecting the two vertices, the strategy from the second–order case cannot be used directly, because there are now at least five fermion momenta whose sum is to be near to the Fermi surface instead of three. Also, on root scale, there is only one volume gain from any of the two–loop subintegrals that one can choose, and one volume gain alone is not sufficient to cancel the large factors arising from two derivatives acting on root scale. However, the values of these graphs are C^2 again by volume effects only, by the following argument. The scale zero vertices of the model have at most four legs, so the vertex functions $\mathcal{U}_v$ associated to the two effective vertices on root scale must be values of subdiagrams with $2m \ge 6$ external legs. There are two possibilites:

- (i) both effective vertices are scale zero effective vertices
- (ii) at least one of them is not a scale zero effective vertex, and thus corresponds to a subgraph with lines carrying scales $j_l \le -1$.

In case (i), the value of G is (up to an unimportant sign factor)

$$Val(G) = \int \left(\prod_{l=2}^{i} d\!\!\!{}^-p_l\; S_{l,j}(p_l) \right) S_{1,j}(q + \sum_{l=2}^{i} v_l p_l)\, \mathcal{U}_1(q,\underline{p}) \mathcal{U}_2(q,\underline{p}) \tag{3.51}$$

where i is the number of lines joining vertex 1 to vertex 2, $\mathcal{U}_k$ is the vertex function associated to vertex k, $v_l \in \{1,-1\}$, and the $S_{l,j}$ satisfy (3.38). Since G is a *multiple* sunset, $i \ge 5$. The derivatives of $\mathcal{U}_1$ and $\mathcal{U}_2$ are uniformly bounded, so the worst case is when all derivatives act on $S_{1,j}$. By (3.38), $\max\limits_{|\alpha|=s} |D^\alpha S_{1,j}|_0 \le P_{1,s} M^{-(1+s)j}$, so, using this estimate after taking at most two derivatives and estimating the integrals in the standard way (not even using any extra volume effect), we have

$$|Val(G)|_2 \le \text{ const } M^{-3j} M^{(i-1)j} \le \text{ const } M^{-3j} M^{4j} \le \text{ const } M^j \tag{3.52}$$

with the constant given in terms of the $P_{l,s}$ with $s \le 2$. Thus the sum over $j < 0$ is convergent. Moreover, the procedure of taking differences of Section II.3.4 implies Hölder continuity of degree h if *(H1)*$_{2,h}$ and *(H2)*$_{2,h}$ hold. In brief, since the scale zero effective vertex functions with $2m \ge 6$ external legs are bounded, i.e. behave as $1 = M^{0(2-m)}$ instead of growing like $M^{j(2-m)}$, the number of integrations alone already suffices to make two derivatives converge.

Case (ii): We shall show in the next lemma that the graph must be DOL at a higher scale $h < 0$. In general, a gain at a scale h does not result in a gain at a lower scale j, once h is summed down to j. But the subgraphs here have $m \geq 3$, and thus the root scale behaviour of their values is without improvement $M^{h(2-m)} \geq M^{-h}$, i.e. it grows because $m \geq 3$. *Therefore, a gain at the higher scale h slows down the rate of growth of the scale sum that gives $\mathcal{U}_v$.* Using

$$\sum_{h \geq j} M^{h(2-m)}\ M^{\epsilon h} \leq \text{ const } (\epsilon, M) M^{j(2-m)}\ M^{\epsilon j} \qquad \text{for all} \quad 2 + \epsilon - m < 0 \tag{3.53}$$

it is 'transported' down to a gain at scale j when h is summed down to j. The dominant term in $\sum\limits_{h \geq j} M^{h(2-m)} M^{\epsilon h}$ is that with $h = j$. On the other hand, for $m = 1$ the scale sum is already convergent, so that only the speed of convergence, but not the scale behaviour of the sum, is changed by the improvement factor $M^{\epsilon h}$. For $m = 2$, the same holds since the improvement factor $M^{\epsilon h}$ only removes a polynomial growth in $|j|$. Thus, for $m \leq 2$, the dominant term in (3.53) is that with $h = 0$. In the following Lemma, we prove that for the graphs of Figure 7, the gain arising at a higher scale can indeed be transported to root scale.

To formulate the lemmas that follow, we now need the definitions of the tree formalism (see [FT1,2] or I). We also assume familiarity with the results of Sections 2.4 – 2.6 of I, although we shall explain the most essential notions briefly.

Let $\mathcal{T}$ be a tree. The vertices of $\mathcal{T}$ with incidence number 1 are called leaves; the others are the forks of $\mathcal{T}$. Let $\mathcal{T}$ be rooted at a fork ϕ and have n leaves, and let π be the predecessor map that maps every vertex $f \neq \phi$ to the unique fork $\pi(f)$ of T whose distance from ϕ (measured in steps over lines of T) is one less than that of f. $\mathcal{T}$ is compatible to G, $\mathcal{T} \sim G$, if there is a family $(G_f)_{f \in \mathcal{T}}$ of connected subgraphs of G such that

$G_\phi = G$,
for all f and f', either $G_f \subset G_{f'}$ or $G_{f'} \subset G_f$ or $G_f \cap G_{f'} = \emptyset$.
G_f is a subgraph of $G_{f'}$ if and only if f' is between ϕ and f in the ordering of the tree.
if $\pi(f) = f'$ then G_f is a proper subgraph of $G_{f'}$

Let $j < 0$ and define

$$\mathcal{J}(\mathcal{T}, j) = \{(j_f)_{f \in \mathcal{T}} : j_\phi = j, \text{ and for all } f \in \mathcal{T} \setminus \{\phi\} \quad j_f \in \{j_{\pi(f)} + 1, \ldots, -1\}\} \tag{3.54}$$

This set is well–defined because of the recursive structure of $\mathcal{T}$ given by π. Note that scales are also associated to the the leaves of $\mathcal{T}$; we shall need this because these leaves may be forks of the original Gallavotti-Nicolò trees of our model, at which the latter have been trimmed. Every element of $\mathcal{J}(\mathcal{T}, j)$ defines a labelling $J : L(G) \to \{j, \ldots, -1\}$, $\ell \mapsto j_\ell$, where $j_\ell = j_f$ for all ℓ that are in $L(G_f)$ but not in $L(G_{f'})$ for any f' with $\pi(f') = f$. Equivalently,

$$j_\ell = \max\{j_f : \ell \in L(G_f)\}. \tag{3.55}$$

Given an assignment of propagators to the lines of G and vertex functions to the vertices of G, the value of the labelled graph G^J is defined in the standard way (see Definition 2.10 (ii) of I and the discussion in Lemma 3.4).

For a subtree $\mathcal{T}'$ of $\mathcal{T}$, the quotient graph $\tilde{G}(\mathcal{T}')$ is the quotient graph of $G_{\phi_{\mathcal{T}'}}$ in which, for all leaves b of $\mathcal{T}'$, the subgraph G_b of G is replaced by a vertex with the same incidence number (see Section 2.6 of I for details).

Lemma 3.6 *Let G be a skeleton graph with at least three vertices, $j < 0$, $\mathcal{T}$ be a tree, and J a labelling of G consistent with $\mathcal{T}$ and such that the root scale is j. Assume that $\tilde{G}(\phi)$ is a multiple sunset (see Figure 7). Let the forks of $\mathcal{T}$ corresponding to the two vertices v_1 and v_2 in Figure 7 be f_1 and f_2, and assume that G_{f_1} contains an additional vertex of G. There exists a subtree $\mathcal{T}'$ of $\mathcal{T}$, consisting of forks $\phi < f_1 < f_3 < \ldots < f_n < f'$ such that for all $k \in \{3, \ldots, n\}$, f_k has incidence number 2 on $\mathcal{T}$, and f' has incidence number at least 3 on $\mathcal{T}$. Moreover, $E(G_{f_1}) = 2m_{f_1} \geq 6$, and $E(G_{f_k}) = 2m_{f_k} \geq 2m_{f_{k-1}} + 2$ for all $k \geq 3$. The spanning tree for $\tilde{G}(\mathcal{T}')$ can be chosen such that one overlap takes place on scale j. The other gain arises on scale j', where $j' = j_{f'}$. j' is also the lowest scale on the tree $\mathcal{T}'$ where a derivative with respect to the external momentum can act on any line in G_{f_1}.*

Note that the statement is not that the subgraph G_{f_1} itself becomes overlapping at scale j'. It is the entire graph that becomes DOL at that scale. See Figure 15 (c) for an example how the graph $\tilde{G}(\mathcal{T}'_{f_1})$ associated to the effective vertex v_1 may look.

In Lemma 3.6, we do not assume that the vertices of G are four–legged. If they are, then G will always have at least four scale zero vertices if $\tilde{G}(\phi)$ is of the form of Figure 7.

Proof: If G were not DOL at any scale, then, by Theorem 2.6, it would have to be a wicked ladder (see Figure 12), since it has at least three vertices. But no quotient graph of a wicked ladder can be a multiple sunset, so G has to be DOL at some scale, and there a minimal scale $h \leq 0$ on which G is DOL. We now show the more detailed statements. As before, we denote by $t_G \geq 0$ the number of steps required to walk from one external vertex to the other over lines of G. By assumption, $t_{\tilde{G}(\phi)} = 1$, so any quotient graph $\bar{G}$ of G containing the lines of $\tilde{G}(\phi)$ has $t_{\bar{G}} \geq 1$. We grow the tree $\mathcal{T}'$ by adding forks and leaves to ϕ by the following algorithm (the various possibilities that can arise during the procedure are sketched in Figure 15). In the first step, we add the fork f_2 to get $\mathcal{T}' = \begin{array}{c} f_2 \\ | \\ \phi \end{array}$ and consider the graph $G' = \tilde{G}(\mathcal{T}')$. If G' has more than two vertices, $\mathcal{T}'$ is complete.

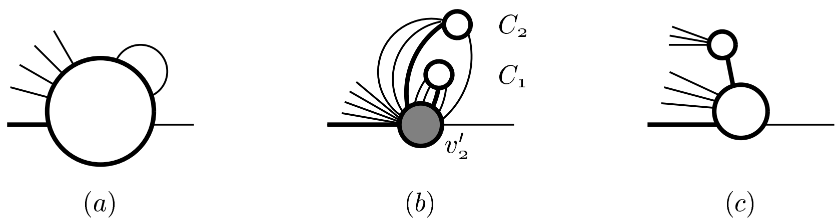


Figure 15

If G' still has only two vertices, then going from $\tilde{G}(\phi)$ to G' has only uncovered some self–contractions of a vertex v'_2 that replaces v_2 in G' (see Figure 15 (a)). If a graph H_1 is obtained from H_2 by a self–contraction (tadpole line), H_2 must have incidence number at least two more than H_1. Therefore the fork f'_2 with $\pi(f'_2) = f_2$ must have $E(G_{f'_2}) \geq 8$ (there is only one such fork, otherwise there would have been more than one effective vertex in G_{f_1}). In this case we go on by adding f'_2 to $\mathcal{T}$ as a leaf above f_2. If the number of vertices still does not increase, we repeat this procedure, and we stop when a new vertex appears, so that G' has more than two vertices. Then $\mathcal{T}'$ is complete. By construction, all forks $f > \phi$ of $\mathcal{T}'$ have $E(G_f) \geq 6$. If at no scale a new vertex appears, then the subgraph G_{f_2} consists only of a single vertex with self–contractions as lines. In that case, we start from the beginning by resolving G_{f_1}, i.e. we apply the same procedure to $\begin{array}{c} f_1 \\ | \\ \phi \end{array}$. Then a third vertex must appear at some scale because G has at least three vertices.

By the symmetry of the graph, we may assume without loss that the above procedure has produced a third vertex, and thus a suitable candidate $\mathcal{T}'$, by resolving G_{f_2}.

We split the proof that G' and $\mathcal{T}'$ have the desired properties into two cases. Recall that under our conventions, a path in G is a non-selfintersecting walk over the lines of G, i.e. no vertex is visited twice by a path (see Definition 2.17 of I). G' consists of v_1 and a subgraph $G'_2 = \tilde{G}_{f_2}(f_2)$, i.e. when G'_2 is collapsed to a point, G' becomes G.

(1) There is no path in G' between the external vertices that contains at least two lines. This implies that all the extra vertices of G' can only be connected to v'_2, so that G'_2 takes the form shown in Figure 15 (b) (not all the subgraphs drawn in that figure have to be there, but at least one of them must be there). All connected components C_α of $G'_2 - v'_2$ must have at least four external legs because all quotient graphs of G are skeleton graphs. This is also the reason why generalized self–contractions cannot appear. It is obvious that the subgraph G'_2 is overlapping, hence G is DOL. Choosing any spanning tree for the subgraphs and combining them with the fat lines in Figures 7 and 15 (b), the statement of the Lemma is obvious.

(2) There is a path in G' between the external vertices with at least two lines (even though the *shortest* path may still be of length one). If $t_{G'} \geq 2$, then G' is as shown in figure 5 or 8, so it is DOL. Taking the quotient graph of G' to get back $\tilde{G}(\phi)$, we see that one of the lines carrying the overlaps must be at scale j. That leaves $t_{G'} = 1$, i.e. the shortest path still has length one. By construction of G', one of the lines of $\tilde{G}(\phi)$, ℓ_1, is the first step in a walk of length at least two from v_1 to the other external vertex, v'_2, of G'. ℓ_1 carries scale $j_{\ell_1} = j$. Call w the other endpoint of ℓ_1. Collect all other vertices of G_2 into a graph D_2, and let Γ_2 be the connected component of D_2 that contains v'_2. Collect all other connected components together with w and the lines between them into a connected graph Γ. Then G' takes the form sketched in Figure 16 ((a) and (c) are the cases where m_1 and n_1 are even, (b) is the case where they are odd. The dots indicate possible additional lines, i.e. the lines drawn are the minimal number that has to be there. All k_1 lines crossing the dashed line are of scale j, the others are of higher scale. $k_1 \geq 5$ because G_1 has incidence number six or more.

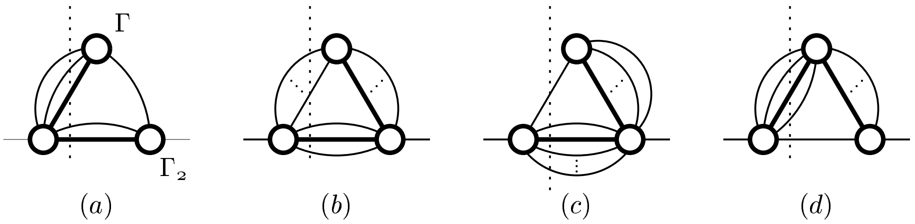


Figure 16

The lines of a possible spanning tree for G' are drawn fat in Figure 16. It is clear by inspection that the statement of the Lemma holds. Moreover, one of the lines ϑ_1 in the spanning tree is at scale j, and so are two lines generating loops that overlap on ϑ_1. The cases in Figure 16 are equivalent to those of Figure 9; we redrew the graphs to bring out the scale structure. ■

Lemma 3.7 *Assume (H2)$_{2,0}$–(H4). Let G be a skeleton graph with $2m$ external legs and vertices v with an incidence number $2m_v \geq 4$. Let $\mathcal{T}$ be a tree rooted at a fork ϕ and such that $\mathcal{T} \sim G$. Assume that there are $\epsilon > 0$, $n_v \geq 0$ $n_\ell \geq 0$, and that for all $s \in \{0, \ldots, 2\}$ there are $\xi_{v,s} > 0$ and $\mathbb{Q}_{l,s} > 0$ such that*

$$|\mathcal{U}_v|_s \leq \begin{cases} \xi_{v,s} M^{j_v(2-m_v-s)} \lambda_{n_v}(j_v, \epsilon) & \text{if } v \text{ is a vertex of scale } j_v < 0 \\ \xi_{v,s} & \text{if } v \text{ is a vertex of scale } j_v = 0 \end{cases} \tag{3.56}$$

(where λ_n is the function defined in (3.3)*) and that the propagators associated to the lines of G satisfy*

$$|D^\sigma S_\ell(p)| \leq \mathbb{Q}_{\ell,s} \lambda_{n_\ell}(j_\ell, \epsilon) M^{-j_\ell(1+s)} \;\; 1\!\!1\left(|ip_0 - e(\mathbf{p})| \in [M^{j_\ell - 2}, M^{j_\ell}]\right) . \tag{3.57}$$

For a fork $f \in \mathcal{T}$ define

$$\begin{aligned} n_f = &\sum_{\ell \in L(G_f)} n_\ell + \sum_{v \in V(G_f)} n_v + |\{v \in V(G_f) : 2m_v = 4, v \text{ not scale zero } \}| \\ &+ \left|\{f' \in \mathcal{T} : f' \text{ fork}, f' > f, G_{f'} \text{ four–legged}, \tilde{G}_{f'}(f') \text{ is nonoverlapping}\}\right|. \end{aligned} \tag{3.58}$$

Let $j < 0$ and

$$\mathcal{S}(j, \mathcal{T}, G) = \sum_{J \in \mathcal{J}(\mathcal{T}, j)} Val(G^J). \tag{3.59}$$

Then for $s \in \{0, \ldots, 2\}$

$$|\mathcal{S}(j, \mathcal{T}, G)|_s \leq \mathcal{M}(s, G) \mathbb{K}_0^{|L(G)|} \left(\nu_1 |j| M^j \log M\right)^{g_{OL}(\tilde{G}(\phi))} \; M^{j(2-m-s)} \lambda_{n_\phi}(j, \epsilon) \tag{3.60}$$

with

$$\mathbb{K}_0 = 8\nu_0 \left(\nu_1 \log M\right)^2 \tag{3.61}$$

and

$$\mathcal{M}(s, G) = \max_T \sum_{\sigma \in Z(s,T,G)} \prod_{\ell \in L(G)} \mathbb{Q}_{\ell, |\sigma_l|} \prod_{v \in V(G)} \xi_{v, \sigma_v}, \tag{3.62}$$

where $Z(s,T,G)$ is as in Lemma 3.4*.*

Proof: Recall that the value of G^J is given by

$$\begin{aligned} Val(G^J)(r_1, \ldots, r_{2m-1}) = \sum_{(A_v)_{v \in V(G)}} \int \prod_{\ell \in L(G) \setminus L(T)} \frac{dp_\ell}{(2\pi)^{d+1}} \prod_{\ell \in L(G)} (S_\ell(p_\ell))_{\alpha_\ell \alpha'_\ell} \\ \prod_{v \in V(G)} \mathcal{U}_v\left(p_1^{(v)}, \ldots, p_{2m_v - 1}^{(v)}, A_v\right) \end{aligned} \tag{3.63}$$

where the $p_k^{(v)}$ are fixed by the usual rules in terms of the p_ℓ and the external momenta $r_1, \ldots, r_{2m-1}$, and where $A_v \in \{\uparrow, \downarrow\}^{2m_v}$, and the α_ℓ and α'_ℓ are determined by the A_v of the adjacent vertices.

Let $\tilde{G} = \tilde{G}(\phi)$, $\tilde{V} = V(\tilde{G})$ and $\tilde{L} = L(\tilde{G})$. We may assume that the spanning tree of G is chosen such that the subgraph $\tilde{T}$ that it induces in $\tilde{G}$ is also a tree, and hence a spanning tree of $\tilde{G}$. If $\tilde{v} \in \tilde{V}$ is the image of G_f belonging to a fork f with $\pi(t) = \phi$ under the projection from G to $\tilde{G}$, then the vertex function associated to $\tilde{v}$ is

$$\tilde{\mathcal{U}}_{\tilde{v}} = \sum_{j'>j} \sum_{J \in \mathcal{J}(\mathcal{T},j')} Val(G_f^J). \tag{3.64}$$

If $\tilde{v}$ is a vertex $\tilde{v} = v \in V(G)$ with $\pi(v) = \phi$ (i.e. there is no fork $f > \phi$ such that $v \in G_f$) and if v carries scale $j_v < 0$, then

$$\tilde{\mathcal{U}}_{\tilde{v}} = \sum_{j_v > j} \mathcal{U}_v. \tag{3.65}$$

If $\tilde{v}$ is a scale zero vertex, then $\tilde{\mathcal{U}}_{\tilde{v}} = \mathcal{U}_v$. With these definitions,

$$\mathcal{S}(j, \mathcal{T}, G) = Val(\tilde{G}). \tag{3.66}$$

To prove the Lemma, we do an induction in the height of $\mathcal{T}$, defined as

$$h(\mathcal{T}) = \max\{k : \exists \text{ forks } f_1, \ldots, f_k \in \mathcal{T}, \text{ such that } f_k = \phi \text{ and } \pi(f_l) = f_{l+1} \text{ for all } l \in \{1, \ldots, k-1\}\}. \tag{3.67}$$

If $h(\mathcal{T}) = 0$, $G = \tilde{G}(\phi)$, and $j_\ell = j$ for all $\ell \in L(G)$. Then every vertex function $\tilde{\mathcal{U}}_v$ is a scale sum (3.65) or given by $\mathcal{U}_v$ for a scale zero vertex. In the latter case,

$$\left|\tilde{\mathcal{U}}_{\tilde{v}}\right|_s \le \xi_{v,s} \le \xi_{v,s} M^{j(2-m_v-s)} \tag{3.68}$$

since $2 - m_v - s \le 0$ for all $s \ge 0$ and all v. In the former case,

if $m_v = 2$ and $s = 0$,

$$\begin{aligned} \left|\tilde{\mathcal{U}}_{\tilde{v}}\right|_0 &\le \xi_{v,0} \sum_{j_v > j} \lambda_{n_v}(j_v, \epsilon) \le \xi_{v,0}(|j| + 1)\lambda_{n_v}(j, \epsilon) \\ &\le \xi_{v,0}\lambda_{n_v+1}(j_v, \epsilon) \end{aligned} \tag{3.69}$$

if $m_v \ge 3$ or $s > 0$, we have $2 - m_v - s < 0$, so

$$\begin{aligned} \left|\tilde{\mathcal{U}}_{\tilde{v}}\right|_s &\le \xi_{v,s} \sum_{j_v > j} \lambda_{n_v}(j_v, \epsilon) M^{j_v(2-m_v-s)} \\ &\le \xi_{v,s}\lambda_{n_v}(j, \epsilon) M^{j(2-m_v-s)} \sum_{k=1}^{\infty} M^{-k(2-m_v-s)} \\ &\le \frac{1}{M-1}\xi_{v,s}\lambda_{n_v}(j, \epsilon) M^{j(2-m_v-s)}. \end{aligned} \tag{3.70}$$

Thus the vertex functions of $\tilde{G}$ satisfy

$$\left|\tilde{\mathcal{U}}_{\tilde{v}}\right|_\sigma \le \xi_{v,s}\lambda_{\hat{n}_v}(j, \epsilon) M^{j(2-m_v-\sigma)} \tag{3.71}$$

with

$$\hat{n}_v = \begin{cases} n_v + 1 & \text{if } m_v = 2 \text{ and } v \text{ is not a scale zero vertex} \\ n_v & \text{otherwise.} \end{cases} \tag{3.72}$$

By Lemma 3.4, and since $j_\ell = j$ for all $\ell \in L(G)$,

$$
\begin{aligned}
\left| Val(\tilde G) \right|_s \le 2\,(4\nu_0)^{|L(G)|}\; M^{j(2-m-s)} \left(\nu_1 |j| M^j \log M\right)^{g_{OL}(\tilde G)}\; \mathcal{M}(s,G) \\
\prod_{\ell\in L(G)} \lambda_{n_\ell}(j,\epsilon) \prod_{v\in V(G)} \lambda_{\hat n_v}(j,\epsilon).
\end{aligned}
\tag{3.73}
$$

The product over the factors λ_n is bounded by $\lambda_N(j,\epsilon)$, where, by (3.4),

$$
N = |\{v\in V(G) : \pi(v)=\phi, m_v = 2, v \text{ not scale zero }\} + \sum_{\ell\in L(G)} n_\ell + \sum_{v\in V(G)} n_v. \tag{3.74}
$$

Since $h(\mathcal{T})=0$, there are no forks $f>\phi$ on $\mathcal{T}$, so there are in particular no four–legged forks. Therefore $N = n_\phi$, and the statement of the Lemma is proven for $h(\mathcal{T})=0$.

Let $h(\mathcal{T})>0$ and (3.60) be proven for all $\mathcal{T}'$ with $h(\mathcal{T}') \le h(\mathcal{T})-1$ and all skeletons G' with $\mathcal{T}' \sim G'$. Again, we first bound the vertex functions $\tilde{\mathcal{U}}_{\tilde v}$ associated to the vertices $\tilde v$ of $\tilde G$. If $\tilde v = v \in V(G)$ with $\pi(v)=\phi$, the bound (3.71) holds. If $\tilde v$ belongs to a fork f with $\pi(f)=\phi$, the subtree $\mathcal{T}_f$ of $\mathcal{T}$ rooted at f and G_f fulfil the induction hypothesis. Using (3.64) and bounding the scale sums as in the proof of (3.71), we get

$$
\left|\tilde{\mathcal{U}}_{\tilde v}\right|_\sigma \le \mathcal{M}(\sigma, G_f)\; (\nu_1 \log M)\; \mathbb{K}_0^{|L(G_f)|} \lambda_{\hat n_f}(j,\epsilon) M^{j(2-m_f-\sigma)} \tag{3.75}
$$

where $2m_f$ is the number of external legs of G_f and

$$
\hat n_f = \begin{cases} n_f + 1 & \text{if } 2m_f = 4 \text{ and } \tilde G_f(f) \text{ is nonoverlapping} \\ n_f & \text{otherwise.} \end{cases} \tag{3.76}
$$

All lines of $\tilde G$ have scale $j_\ell = j$. Lemma 3.4 applies to $\tilde G$. Thus, by (3.66), calling $R(M) = \nu_1 \log M$,

$$
|\mathcal{S}(j,\mathcal{T},G)|_s \le \; 2(4\nu_0)^{|\tilde L|}\; M^{j(2-m-s)} (|j| M^j \nu_1 \log M)^{g_{OL}(\tilde G)}\; \Lambda\,\Gamma\,\Sigma \tag{3.77}
$$

with

$$
\begin{aligned}
\Lambda &= \prod_{\substack{v\in V(G)\\ \pi(v)=\phi}} \lambda_{\hat n_v}(j,\epsilon) \prod_{\substack{f\in\mathcal{T}\\ \pi(f)=\phi}} \lambda_{\hat n_f}(j,\epsilon) \prod_{\ell\in\tilde L} \lambda_{n_\ell}(j,\epsilon) \le \lambda_{n_\phi}(j,\epsilon) \\
\Gamma &= \prod_{\substack{v\in V(G)\\ \pi(v)=\phi}} R(M) \prod_{\substack{f\in\mathcal{T}\\ \pi(f)=\phi}} R(M) \mathbb{K}_0^{|L(G_f)|} \\
\Sigma &= \sum_{\sigma\in Z(s,\tilde T,\tilde G} \prod_{\ell\in\tilde L} \mathbb{Q}_{\ell,|\sigma_\ell|} \prod_{\substack{v\in V(G)\\ \pi(v)=\phi}} \xi_{v,|\sigma_v|} \prod_{\substack{\tilde v = f\in\mathcal{T}\\ \pi(f)=\phi}} \mathcal{M}(|\sigma_{\tilde v}|, G_f)
\end{aligned}
\tag{3.78}
$$

To combine the constants, we use that

$$
L(G) = L(\tilde G) \,\dot\cup \bigcup_{f:\pi(f)=\phi}^{\cdot} L(G_f) \tag{3.79}
$$

and

$$
V(G) = \{v\in V(G) : \pi(v)=\phi\} \,\dot\cup \bigcup_{f:\pi(f)=\phi}^{\cdot} V(G_f). \tag{3.80}
$$

Thus

$$\prod_{\substack{v\in V(G)\\ \pi(v)=\phi}} R(M) \prod_{\substack{f\in\mathcal{T}\\ \pi(f)=\phi}} R(M) \le R(M)^{|\tilde V|} \le R(M)^{|\tilde L|+1} \le R(M)^{2|\tilde L|}$$
$$\prod_{\substack{f\in\mathcal{T}\\ \pi(f)=\phi}} \mathbb{K}_{\rm o}^{|L(G_f)|} = \mathbb{K}_{\rm o}^{|L(G)|-|\tilde L|}, \tag{3.81}$$

so

$$2(4\nu_{\rm o})^{|\tilde L|} \prod_{\substack{f\in\mathcal{T}\\ \pi(f)=\phi}} \mathbb{K}_{\rm o}^{|L(G_f)|} \prod_{\substack{v\in V(G)\\ \pi(v)=\phi}} R(M) \prod_{\substack{f\in\mathcal{T}\\ \pi(f)=\phi}} R(M) \le \mathbb{K}_{\rm o}^{|L(G)|}. \tag{3.82}$$

Finally, we bound Σ. Every $\mathcal{M}(|\sigma_f|, G_f)$ in the product is a maximum over spanning trees of G_f, which is attained at some T_f since the set of spanning trees is finite. The union of all such T_f and $\tilde T$ is a spanning tree T^* of G. The sum over the multiindices of the G_f and those of G combines to a sum over $\sigma \in Z(s, T^*, G)$. Collecting the $\mathbb{Q}_{\ell,|\sigma_\ell|}$ and $\xi_{v,|\sigma_v|}$, we obtain

$$\Sigma = \sum_{\sigma\in Z(s,T^*,G)} \prod_{\ell\in L(G)} \mathbb{Q}_{\ell,|\sigma_\ell|} \prod_{v\in V(G)} \xi_{v,|\sigma_v|}\ . \tag{3.83}$$

This sum is bounded by its maximum over all possible spanning trees of G, which is $\mathcal{M}(s,G)$. ∎

Theorem 3.8 *Assume (H2)$_{2,h}$–(H4) and let G be a two–legged skeleton graph obeying the hypotheses of Lemma 3.7. Let $t_{\tilde G(\phi)} \ge 1$, and $\tilde G(\phi)$ not be a sunset (Figure 1) or a wicked ladder (Figure 12). Let all vertices v have scale $j_v = 0$, and*

$$|\mathcal{U}_v|_s \le \xi_{v,s} \quad \left(= \xi_{v,s} M^{j_v(2-m_v-s)}\right). \tag{3.84}$$

Let

$$\mathcal{S}(j,\mathcal{T},G) = \sum_{J\in\mathcal{J}(\mathcal{T},j)} Val(G^J). \tag{3.85}$$

Then for $s\in\{0,1,2\}$,

$$\left|\mathcal{S}(j,\mathcal{T},G)\right|_s \le \mathbb{K}_{\rm o}^{|L(G)|} |j|^3 M^{j(3-s)} \mathcal{M}(s,G) \lambda_{n_\phi}(j,\epsilon), \tag{3.86}$$

with $\mathcal{M}$ and $\mathbb{K}_{\rm o}$ given as in Lemma 3.7.

Proof: Let $\tilde G = \tilde G(\phi)$. Then $\tilde G$ is a skeleton or it consists of one external vertex only, with self–contractions. $\tilde G$ has vertices $\tilde v$ that carry scales $j_{\tilde v}$, where $j_{\tilde v} = 0$ for scale zero vertices, or $j_{\tilde v} > j$ for a vertex belonging to a fork $f \in \mathcal{T}$ with $\pi(f) = \phi$. In the latter case, the vertex function is

$$\mathcal{U}_{\tilde v} = \sum_{J\in\mathcal{J}(\mathcal{T}_f, j_{\tilde v})} Val(G_f^J). \tag{3.87}$$

If $\tilde G$ is not a multiple sunset, then $\tilde G$ is DOL by Theorem 2.6. By Lemma 3.7,

$$\left|\mathcal{S}(j,\mathcal{T},G)\right|_s \le \mathcal{M}(s,G) \mathbb{K}_{\rm o}^{|L(G)|} (\nu_1 \log M)^2 |j|^2 M^{2j} M^{j(1-s)} \lambda_{n_\phi}(j,\epsilon) \tag{3.88}$$

which proves (3.86). So let $\tilde G$ be a multiple sunset. There are three cases.

1. One of the vertices, say v_1, is of scale zero. Let the number of lines between v_1 and v_2 be n, then $n \geq 5$, and G has $n-1$ loops. All $j_\ell = j$. $g_{OL}(\tilde{G}) = 1$. Since all vertices of the graph G are scale zero, the scale-dependent vertex v_2 of $\tilde{G}$ corresponds to a fork f with a subgraph G_f. Lemma 3.7 applies to G_f, so the vertices of $\tilde{G}$ satisfy

$$\begin{aligned} &|\mathcal{U}_{v_1}|_s \leq \xi_{v,s} \qquad s \in \{0,1,2\} \\ &\sum_{j_{\tilde{v}}>j} |\mathcal{U}_{v_2}|_s \leq R(M)\mathcal{M}(s,G_f)\mathbb{K}_0^{|L(G_f)|}\lambda_{n_f}(j,\epsilon)M^{j(2-m_f-s)} \qquad s \in \{0,1,2\}. \end{aligned} \tag{3.89}$$

Let $\ell_1 \in L(\tilde{G})$ be the line in the spanning tree, then

$$\begin{aligned} |\mathcal{S}(j,\mathcal{T},G)|_s = \left| Val(\tilde{G}) \right|_s \leq & \sum_{\sigma_{v_1}+\sigma_{v_2}+\sigma_{\ell_1} \leq s} |\mathcal{U}_{v_1}|_{|\sigma_1|} \left(\sum_{j_{\tilde{v}}>j} |\mathcal{U}_{v_2}|_{|\sigma_2|} \right) \mathbb{Q}_{\ell_1,|\sigma_{\ell_1}|} \; M^{-j(1+|\sigma_{\ell_1}|)} \\ & \left(\prod_{\ell \neq \ell_1} \mathbb{Q}_{\ell,0} \right) M^{-(n-1)j}(2\nu_0 M^{2j})^{n-1} \; \nu_1 M^j |j| \log M \\ \leq & \sum_{\sigma_1+\sigma_2+\sigma_3 \leq s} \xi_{v_1,|\sigma_1|} \mathcal{M}(|\sigma_2|,G_f)\mathbb{K}_0^{|L(G_f)|}\lambda_{n_f}(j,\epsilon)M^{j(2-m_f-|\sigma_2|)} \\ & (2\nu_0)^{n-1}(\nu_1|j|\log M)M^{j(n-2+1-|\sigma_3|)}\mathbb{Q}_{\ell_1,|\sigma_3|} \prod_{\ell \neq \ell_1} \mathbb{Q}_{\ell,0}. \end{aligned} \tag{3.90}$$

We use $m_f = \frac{n+1}{2}$ and $|\sigma_2| + |\sigma_3| \leq s$ to get

$$\begin{aligned} |\mathcal{S}(j,\mathcal{T},G)|_s \leq & M^{j(\frac{n+1}{2}-s)}|j| \sum_{\sigma_1+\sigma_2+\sigma_3 \leq s} \xi_{v_1,|\sigma_1|}\mathcal{M}(|\sigma_2|,G_f)\mathbb{K}_0^{|L(G_f)|} \\ & \lambda_{n_f}(j,\epsilon)(2\nu_0)^{n-1}(\nu_1 \log M)\mathbb{Q}_{\ell_1,|\sigma_3|} \prod_{\ell \neq \ell_1} \mathbb{Q}_{\ell,0} \\ \leq & M^{j(\frac{n+1}{2}-s)}|j|\mathbb{K}_0^{|L(G)|}\lambda_{n_\phi}(j,\epsilon)\mathcal{M}(s,G). \end{aligned} \tag{3.91}$$

Since $n \geq 5$, (3.86) follows. If v_1 comes from a summed vertex, the same bound holds.

2. Both vertices of $\tilde{G}$ are of scale zero. The bound in 1. improves by a factor $M^{j(\frac{n+1}{2})-2}$ (see the discussion around (3.51).

3. Both vertices $\tilde{v}_1$ and $\tilde{v}_2$ are scale-dependent. Since G has only scale zero vertices, the $\tilde{v}_i$ must belong to subgraphs of G. If all lines of these subgraphs are self–contractions of scale zero vertices, the bound is similar to that of case 2. So we may assume that G has at least three vertices. Therefore Lemma 3.6 applies. Let $\mathcal{T}'$ be as constructed in Lemma 3.6, and let $G' = \tilde{G}(\mathcal{T}')$, so that

$$Val(\tilde{G}) = \sum_{J \in \mathcal{J}(\mathcal{T}',j)} Val(G'^J). \tag{3.92}$$

The vertex functions for all $v' \in G'$ are given by scale sums $\mathcal{S}(j_f,\mathcal{T}_f,G_f)$ of skeleton graphs G_f with $\pi(f) = f'$, where f' is the highest fork of $\mathcal{T}$ that is in $\mathcal{T}'$. j_f is summed down to $j' = j_{f'}$. The $\sum_{j_f>j'} \mathcal{S}(j_f,\mathcal{T}_f,G_f)$ therefore obey (3.60), with $g_{OL} = 0$ and j replaced by j'. We choose a suitable spanning tree T^* of G', as in

Lemma 3.6. Derivatives of $Val(G'^J)$ are given by (3.43), and can be bounded as in (3.44) and (3.45), with $\mathcal{V}(G,T)$ replaced in (3.44) by the integration volume

$$\begin{aligned}\mathcal{V}'(G',T^*) = \int \prod_{\ell\in L(G')\setminus L(T^*)} & \left(d^{d+1}p_\ell \;\mathbb{1}\left(|i(p_\ell)_0 - e(\mathbf{p}_\ell)| \le M^{j_\ell}\right)\right) \\ & \prod_{\beta\in L(T^*)} \mathbb{1}\left(|i(p_\beta)_0 - e(\mathbf{p}_\beta)| \le M^{j_\beta}\right) .\end{aligned} \tag{3.93}$$

Doing the p_0–integrals, we get

$$\mathcal{V}'(G',T) \le \bar{\mathcal{V}}(G',T^*) \prod_{\ell\in L(G)\setminus L(T^*)} (2M^{j_\ell}), \tag{3.94}$$

with $\bar{\mathcal{V}}$ as in Lemma 3.3. G' is DOL, and by Lemma 3.6, one of the volume gains arises at scale j, and the other one at scale j', which is the lowest scale on which a derivative can act in G_{f_1}. By Lemma 3.3, and since $j_\ell \ge j$ for all $\ell \in L(G')$,

$$\mathcal{V}'(G',T^*) \le (\nu_1|j|)^2\; M^{j+j'} \prod_{\ell\in L(G)\setminus L(T^*)} (2\nu_0 M^{2j_\ell}). \tag{3.95}$$

Also,

$$\begin{aligned} Y(\beta,T^*) &= \sum_\sigma \left(\prod_{\ell\in L(G')} \mathbb{Q}_{\ell,|\sigma_{\ell,r}|}\lambda_{n_\ell}(j_\ell,\epsilon)M^{-j_\ell(1+|\sigma_{\ell,r}|)}\right) \prod_{v\in V(G')} |\mathcal{U}_v|_{|\sigma_{v,r}|} \\ &\le \sum_\sigma \left(\prod_{\ell\in L(G')} \mathbb{Q}_{\ell,|\sigma_{\ell,r}|}\lambda_{n_\ell}(j_\ell,\epsilon)M^{-j_\ell(1+|\sigma_{\ell,r}|)}\right) \\ &\prod_{v\in V(G')} \mathcal{M}(|\sigma_{v,r}|,G_v)\mathbb{K}_0^{|L(G_v)|}M^{j_v(2-m_v-|\sigma_{v,r}|)}\lambda_{n_v}(j_v,\epsilon). \end{aligned} \tag{3.96}$$

We bound $M^{-j_\ell|\sigma_{\ell,r}|} \le M^{-j|\sigma_{\ell,r}|}$ and $M^{-j_v|\sigma_{v,r}|} \le M^{-j|\sigma_{v,r}|}$, and use

$$\sum |\sigma_{\ell,r}| + \sum |\sigma_{v,r}| \le s. \tag{3.97}$$

The scale factors without the improvement factor $M^{j+j'}$ and the derivative factor M^{-sj} add up to

$$B = -\sum_{\ell\in L(G')} j_\ell + \sum_{v\in V(G')} j_v(2-m_v) + \sum_{\ell\in L(G')\setminus L(T^*)} 2j_\ell. \tag{3.98}$$

Since

$$j_\ell = j + \sum_{\substack{\phi<f\in T' \\ \ell\in G_{f'}}} (j_f - j_{\pi(f)}) \tag{3.99}$$

and (since the scale sum over the effective vertices is already done)

$$j_v = j + \sum_{\substack{\phi<f\in T' \\ v\in G_{f'}}} (j_f - j_{\pi(f)}) \tag{3.100}$$

so

$$\begin{aligned} B &= j\left(\sum_{v\in V(G')} (2-m_v) + |L(G')| - 2|L(T^*)|\right) \\ &+ \sum_{f>\phi} (j_f - j_{\pi(f)}) \left(\sum_{v\in V(G'_f)} (2-m_v) + |L(G'_f)| - 2|L(G'_f)\cap L(T^*)|\right) . \end{aligned} \tag{3.101}$$

Using $|L(G'_f)| = \sum_{v\in V(G'_f)} m_v \ - m$ and $|L(G'_f)\cap L(T^*)| = |V(G'_f)|-1$, we get

$$B = j(2-m) + \sum_{\phi < f\in\mathcal{T}'} (j_f - j_{\pi(f)})(2-m_f). \tag{3.102}$$

By the structure of $\mathcal{T}'$ given in Lemma 3.6,

$$B = j(2-m) + \sum_{f_1\le f\le f'} (j_f - j_{\pi(f)})(2-m_f), \tag{3.103}$$

so, adding j and $j' = j + \sum_{f_1\le f\le f'} (j_f - j_{\pi(f)})$, we get

$$j + j' + B = j(4-m) + \sum_{f_1\le f\in\mathcal{T}'} (j_f - j_{\pi(f)})(3-m_f). \tag{3.104}$$

Using $m=1$ and adding the scale effect $-sj$ from s derivatives, we have

$$j + j' - sj + B = j(3-s) + \sum_{f_1\le f\in\mathcal{T}'} (j_f - j_{\pi(f)})(3-m_f). \tag{3.105}$$

By Lemma 3.6, only f_1 can have $m_f = 3$, all higher forks of $\mathcal{T}'$, if existent, must have $3-m_f\le -1$. Thus, combining the constants,

$$\begin{aligned} \left| Val(\tilde G)\right|_s &\le M^{j(3-s)}\lambda_{n_\phi}(j,\epsilon)|j|^2\mathcal{M}(s,G')\mathrm{I\!K}_0^{|L(G')|} \\ &\qquad \sum_{(j_f)_{f\in\mathcal{T}'\setminus\{\phi\}}} \prod_{f\in\mathcal{T}'\setminus\{\phi\}} M^{(3-m_f)(j_f-j_{\pi(f)})} \\ &\le |j|^3 M^{j(3-s)}\lambda_{n_\phi}(j,\epsilon)\mathcal{M}(s,G)\mathrm{I\!K}_0^{|L(G)|}. \end{aligned} \tag{3.106}$$

■

Remark 3.9 None of the results of this section required *(H5)* because only volume gains were used in the bounds. *(H5)* is only necessary to control the contributions from RPA graphs.

3.6 RPA skeletons and general graphs in two dimensions

Definition 3.10 Let G be a two–legged graph and $\mathcal{T}$ be a tree compatible to G, rooted at ϕ. The graph G is a generalized RPA graph (for tree $\mathcal{T}$) if $\tilde G(\phi)$ is a sunset or a wicked ladder.

To bound the values of generalized RPA graphs (not covered in Theorem 3.8), we shall need *(H5)*. The proof of regularity is by reduction to DOL graphs at higher scales, or by reduction to the cases treated in `II`. In the proof of the following theorem, we shall go through a number of easy arguments to do this reduction.

Theorem 3.11 *Assume (H2)$_{2,h}$, (H3), and (H4). Let G be a two–legged graph without two–legged proper subgraphs and $\mathcal{T}$ be a tree compatible to G, rooted at ϕ, such that $\tilde{G} = \tilde{G}(\phi)$ is a sunset, or a wicked ladder, or has only one external vertex. Let G have vertices v with an incidence number $2m_v \ge 4$, all of scale zero, with C^2 vertex functions $\mathcal{U}_v$ obeying* (3.84)*, and with propagators satisfying* (3.10) *associated to the lines. Then $\mathcal{S}(j,\mathcal{T},G)$, defined in* (3.85)*, satisfies for $s=1$ and $s=2$*

$$|\mathcal{S}(j,\mathcal{T},G)|_s \le \mathbb{K}_5(G)|j|M^{(2-s)j}(\nu_1 \log M)\lambda_{n_\phi}(j,\epsilon). \tag{3.107}$$

Assume (H5) and let the propagators associated to lines be given by the strings S_{j_ℓ} of (3.7) *satisfying* (3.10) *and* (3.11)*. Then the projection ℓ onto the Fermi surface satisfies*

$$|\ell\mathcal{S}(j,\mathcal{T},G)|_2 \le \mathbb{K}_6(G)|j|M^{j/3}\lambda_{n_\phi}(j,\epsilon), \tag{3.108}$$

and

$$\left|\frac{\partial}{\partial\theta}\frac{\partial}{\partial p_\alpha}\mathcal{S}(j,\mathcal{T},G)\right|_0 \le \mathbb{K}_7(G)|j|M^{j/3}\lambda_{n_\phi}(j,\epsilon) \tag{3.109}$$

Proof: Let $\tilde{G} = \tilde{G}(\phi)$. Consider the case in which $t_{\tilde{G}} \ge 1$. The case when $t_{\tilde{G}} = 0$ will be considered at the end of the proof. Now $\tilde{G}$ is a skeleton. $\tilde{G}$ has vertices $\tilde{v}$ that carry scales $j_{\tilde{v}}$, where $j_{\tilde{v}} = 0$ for scale zero vertices, or $j_{\tilde{v}} > j$ for a vertex belonging to a fork $f \in \mathcal{T}$ with $\pi(f) = \phi$. In the latter case, the vertex function is as in (3.87). If $\tilde{G}$ is a sunset or a wicked ladder, then $g_{OL}(\tilde{G}) \ge 1$, so by Lemma 3.7

$$|\mathcal{S}(j,\mathcal{T},G)|_s \le \mathcal{M}(s,G)\mathbb{K}_0^{|L(G)|}(\nu_1 \log M)|j|M^j M^{j(1-s)}\lambda_{n_\phi}(j,\epsilon) \tag{3.110}$$

which proves (3.107) and the bounds on the 0^{th} and the first derivatives required for (3.108). It remains to bound the second derivative contributions to (3.108) and (3.109).

1. Let $\tilde{G}$ be a sunset. Then there are three possibilities for G itself: G can be a sunset, in which case $G = \tilde{G}$, or G is a wicked ladder, or G is DOL.

1.1 G a sunset. The vertices of G may have more than four legs and have self–contractions, but that is inessential for the following discussion since the external momentum cannot enter those 'tadpole lines', and because the scale behaviour of a loop containing exactly one fermion line is $M^{-j}\,M^{2j} = M^j$, which is summable and produces the same bound as if the self–contraction were not there. We may therefore assume that there are no self–contractions. The projection ℓ means evaluation on the Fermi surface, $(\ell F)(p) = F(0,\mathbf{p}(0,\theta))$, so the only derivatives we look at are those with respect to θ.

The proof is an application of Theorem II.3.5. Unlike $C_j(p_0, e(\mathbf{p}))$, the propagators $S_{\ell,j_\ell}(p)$ depend on θ because the $T_k^{(j)}$ from (3.7) do. However, (3.11) ensures that the same change of variables used in II will prevent the derivative from degrading the scale behaviour here too. We use the notation of II. The function P (see (II.3.5)) entering (II.3.13) still contains scale zero vertex functions only, hence is C^2 with bounds uniform in the scales. We now have instead of (II.3.14)

$$Y_{j,a}(p) = \int dp_1 \int dp_2\; S_{\ell_1,j}(p_1)\; S_{\ell_2,j}(p_2)\; S_{\ell_3,j}(L_a(p_1,p_2,p))\; P(p_1,p_2,p) \tag{3.111}$$

with ζ_a and e_a given by (II.3.15). All $S_{\ell_k,j}$ are $C^{2,h}$. We take one derivative with respect to p right away. In the integrand, it acts only on $S_{\ell_3,j}$ and on P. Thus $\frac{\partial}{\partial p} Y_{j,a}(p)$ consists of two terms, both of the form given in II.3.81. In the first term, where the derivative acts on P, $A = \frac{\partial}{\partial p} P$, and $\nu_1 = \nu_2 = \nu_3 = 1$. In the second term, where the derivative acts on $S_{\ell_3,j}$, $\nu_1 = \nu_2 = 1$, and $\nu_3 = 2$. Both terms fulfil the hypotheses of Theorem II.3.5. Thus, by Theorem II.3.5,

$$\left| \frac{\partial}{\partial \theta} \frac{\partial Y_{j,a}}{\partial p} \right|_0 \le \text{ const } |j_1| M^{j_1 + j_2 - \frac{3}{2} j_3} \tag{3.112}$$

with the constant given as in Theorem II.3.5. This implies convergence of the scale sums, as shown in all detail in II, Section 3.2.

1.2 $\quad G$ a wicked ladder.

1.2.1 $\quad$ *(Sy)* does not hold, and G is a particle–particle wicked ladder. Then we can use an argument as in the proof of Theorem II.4.8 to show that the extra volume gain from Lemma II.4.7 suffices to make the second derivative convergent. Note that although not all lines have the same scale, the spanning tree can always be chosen consistent with the scales such that both the gain of Lemma II.4.7 and the gain from a pair of overlapping loops can be extracted (the latter on root scale j). An example is the graph shown in Figure 17. The line connecting the two external vertices carries scale j because the $\tilde{G}(\phi)$ is a sunset. If, for instance, $j_2 \ge j_3$ and $j_5 \ge j_4$, one puts the lines with scales j_2 and j_5 into the spanning tree, as indicated by the heavy lines in the Figure. The details of the volume bound are given in Lemma II.4.8 and Theorem II.4.9. In this example, there are two volume gains from particle–particle bubbles and one gain from overlapping loops on scale j. The total improvement is $M^{j + \frac{j_5 + j_2}{3}}$ by Lemma II.4.8. This improvement is used to control derivatives that act strictly above root scale. Those that act at root scale are controlled using the change of variable arguments of Theorem II.3.5.

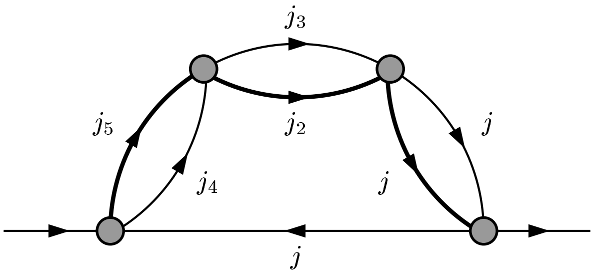


Figure 17: A particle–particle wicked ladder

1.2.2 $\quad$ If *(Sy)* holds or G is not a particle–particle wicked ladder, we choose the spanning tree for G as in Figure II.6. For the same reasons as in case 1.1 (namely (3.11)), we may apply the same change of variables procedure as for the wicked ladder with propagators C_j. It is no complication that the lines in different bubbles may be of different scale since the line with p_1 is always on scale j. However, even the two lines in a single bubble may now have different scales, and in particular, the line in the tree may be the one with the lower scale. In the particle–particle wicked ladder, we may simply change the spanning tree to contain the higher of the two lines since this amounts to a change of variables from $\mathbf{p}_3$ to $\mathbf{q}_3 = \mathbf{p} + \mathbf{p}_1 - \mathbf{p}_3$, thus $\mathbf{p}_3 = \mathbf{p} + \mathbf{p}_1 - \mathbf{q}_3$, which does not change any signs and hence also no critical points. In the particle–hole wicked ladder, the change of the spanning tree only exchanges the signs in front of $\mathbf{p}_1$ and $\mathbf{p}$, and the critical points do not change because of that. Thus the procedure of Section II.3.5 applies directly.

1.3 G DOL. We construct a maximal non–DOL quotient graph $\tilde{G}(\mathcal{T}')$ of G, associated to a tree $\mathcal{T}'$, which is grown as follows. For both effective vertices of $\tilde{G} = \tilde{G}(\phi)$, associated to forks f_1 and f_2, we first determine if $\tilde{G}_i = \tilde{G}_{f_i}(f_i)$ are overlapping. If $\tilde{G}_i$ is nonoverlapping, we add f_i to the tree $\mathcal{T}'$. We now attempt to add each fork f associated to effective four-legged vertices of the current quotient graph. After each trial addition we check to see if the resulting $\tilde{G}(\mathcal{T}')$ is DOL. This happens, for example, if the graph in Figure 10 (b) is expanded to the one in Figure 11. In this case, the fork f is rejected. Otherwise, $\tilde{G}(\mathcal{T}')$ must be a wicked ladder, and the fork is appended to $\mathcal{T}'$. Once all candidate forks are rejected, $\mathcal{T}'$ is complete. As a result, Γ is a wicked ladder, with vertices that are of scale zero or belong to overlapping subgraphs, or Γ becomes a DOL graph $\tilde{\Gamma}$ when one of its nonoverlapping four-legged effective vertices is expanded. In the former case, 1.2.1 and 1.2.2 apply, the only change being that there are now vertex functions that depend on the scales. However, they come from overlapping graphs, so whenever derivatives act on them, there is also one volume gain that improves the scaling behaviour by $M^j|j|$. In the latter case, we apply the procedure for the wicked ladders to Γ. Whenever a derivative hits a vertex of Γ , the corresponding subgraph is overlapping, or the derivative acts on a line of $\tilde{\Gamma}$ at the scale where the second overlap takes place. Therefore, its effect is controlled by the volume gain at that scale.

2. Let $\tilde{G}$ be a wicked ladder. The proof is the same as in case 1.2 and 1.3. One only has to construct $\mathcal{T}'$ before applying the results of Chapter II.4, to ensure that the effective vertex functions appearing in the wicked ladder $\tilde{G}(\mathcal{T}')$ are either scale zero vertices or values of overlapping graphs.

The remaining case is $t_{\tilde{G}} = 0$. There is the trivial case, where a scale zero vertex has only self–contractions. Then $\mathcal{S}(j, \mathcal{T}, G)$ obeys the desired bounds by $(H1)$ and the properties of the scale zero action. Otherwise, G is overlapping. We consider the minimal quotient graph Γ of G that has $t_\Gamma \geq 1$. All the considerations of the above case apply, the only difference being that the lowest scale on which the derivatives can act is a scale strictly above root scale, not root scale itself, which just improves the estimates.

■

Remark 3.12 Let $\gamma \in (0, 1)$ and G be a 1PI graph. If *(H1)*$_{2,\gamma}$, *(H2)*$_{2,0}$, *(H3)*, and either *(Sy)* or *(H4)* hold, and if the pair $(\mathcal{T}, G)$ is such that $\tilde{G}(\phi)$ is DOL, then the scale sum

$$\Sigma_{\mathrm{DOL}}(\mathcal{T}, G, p) = \sum_{j<0} \sum_{J \in \mathcal{J}(\mathcal{T},j)} Val(G^J)(p) \tag{3.113}$$

is $C^{2,\gamma}$ in p. In other words, $\Sigma_{\mathrm{DOL}}(\mathcal{T}, G)$ *is more regular than* e. The reason for this is that the double volume gain on root scale suffices to control almost three derivatives, and that for 1PI graphs, one can always use an integration by parts to remove a derivative from a propagator, so that one can distribute the derivatives such that at most one derivative and one difference operator can act on any given line (see Case 6 of the proof of Theorem I.2.46). Thus at most the second derivative of e appears in the bound.

Theorem 3.13 *Let $d = 2$, assume (H1)$_{2,h}$, (H2)$_{2,h}$, and (H3)–(H5), and let K^I_r be the r^{th} order coefficient of K^I i.e., $K^I = \sum\limits_{r=1}^{\infty} K^I_r \lambda^r$ as a formal power series in λ. Then*

$$\left|K^I_r\right|_2 \le Q\; |\lambda|^r. \tag{3.114}$$

The constant Q depends only on r, g_0, r_0, and w_0. As $I \to -\infty$, K^I_r converges in $|\cdot|_2$ to a C^2 function K_r.

Remark 3.14 The r–behaviour is

$$Q = \tilde{Q}^r\; r! \tag{3.115}$$

where $\tilde{Q}$ depends on g_0, r_0, and w_0.

Proof: Let G be a graph contributing to K^I_r. Then G is two–legged and 1PI (otherwise ℓ of its value would be zero). The contribution of G to K^I_r can be written as a scale sum $\sum_{j=I}^{-1} \sum_{\mathcal{T}} \mathcal{S}(j, \mathcal{T}, G)$, where $\mathcal{S}$ denotes the value of the graph (see I.2.76 for the detailed formula for K^I_r).

The statement of the Theorem holds for $r = 1$ and $r = 2$ by Theorem II.1.2. We do an induction in r. The inductive hypothesis consists of (3.107), (3.108), and (3.109). Let $r \ge 3$, assume the statement to be proven for all $r' < r$, and let G be a graph contributing to K^I_r. Let G_1 be the skeleton graph obtained from G by replacing strings of two–legged subdiagrams by single lines l, and associate the value S_{l,j_l} of the string to the line of G_1. Every subdiagram is of order at most $r - 1$, so the inductive hypothesis applies to it. We now show that the values of the c– and r–forks fulfil the hypotheses of Lemma 3.1.

We start with the c–forks. They are given by a scale sum

$$T^{(j)}(p) = \sum_{I \le h \le j} \ell \tilde{T}_h(p) \tag{3.116}$$

with $\tilde{T}(p)$ the value of a two–legged 1PI graph G_f corresponding to a fork f of $\mathcal{T}$. Since $\tilde{T}_h$ obeys (3.107), and since the highest scale in the sum appearing is $h = j$, (3.8) is obvious for $s = 0$ and $s = 1$ (for details, see (I.2.151-153)). Also, derivatives acting on $T^{(j)}(p)$ can only act as θ–derivatives. Similarly, (3.108) and a summation over $h \le j$ imply that $\left|T^{(j)}\right|_2 \le$ const $\lambda_{n_f}(j, \varepsilon) M^{j/2}$, which fulfils (and is actually much better than) (3.8) with $s = 2$.

If $T^{(j)}$ belongs to an r–fork, then

$$T^{(j)}(p_0, \mathbf{p}(\rho,\theta)) = \sum_{h>j} (1-\ell)\tilde{T}_h(p_0, \mathbf{p}(\rho,\theta)) = \sum_{h>j} \left(\tilde{T}_h(p_0, \mathbf{p}(\rho,\theta)) - \tilde{T}_h(0, \mathbf{p}(0,\theta))\right) \tag{3.117}$$

so (3.8) follows by the same scale summations and estimates as in (I.2.154–157). Moreover,

$$\begin{aligned}
\frac{\partial}{\partial\theta}(1-\ell)T^{(j)}(p_0, \mathbf{p}(\rho,\theta)) &= \sum_{h>j} \frac{\partial}{\partial\theta} \left(\tilde{T}_h(p_0, \mathbf{p}(\rho,\theta)) - \tilde{T}_h(0, \mathbf{p}(0,\theta))\right) \\
&= \sum_{h>j} \left((\nabla\tilde{T}_h)(p_0, \mathbf{p}(\rho,\theta)) \cdot \frac{\partial\mathbf{p}}{\partial\theta}(\rho,\theta) - (\nabla\tilde{T}_h)(0, \mathbf{p}(0,\theta)) \cdot \frac{\partial\mathbf{p}}{\partial\theta}(0,\theta) \right) \\
&= \left((1-\ell)(\nabla T^{(j)} \cdot \frac{\partial\mathbf{p}}{\partial\theta}) \right) (p_0, \mathbf{p}(\rho,\theta))
\end{aligned} \tag{3.118}$$

In other words, the θ derivative does not upset the renormalization cancellation. Taylor expanding, and inserting (3.107), we get

$$\begin{aligned}\left|\frac{\partial}{\partial\theta}(1-\ell)T^{(j)}\right|_0^{(j)} &\le M^j \sum_{h>j} \mathbb{K}_5(G_f)\ |h|\nu_1 \log M\ \lambda_{n_\phi}(h,\epsilon) \\ &\le \mathbb{K}_5(G_f)\nu_1 \log M\ \lambda_{n_\phi}(j,\epsilon)|j|^2 M^j.\end{aligned} \tag{3.119}$$

The second inequality in (3.9) is proven by summing (3.107).

Thus S_{l,j_l} fulfils the hypotheses of Lemma 3.1, and thus of Lemma 3.7, Theorem 3.8, and Theorem 3.11. By Theorem 3.11, the value of every graph with root scale j and GN tree $\mathcal{T}$ contributing to $K^I_{r,j}$ ($K^I_r = \sum\limits_{I\le j<0} K^I_{r,j}$) satisfies (3.107), (3.108), and (3.109). Thus the scale sum over j converges. The sum over trees gives a constant to the power r. The proof of the $r!$–bound is as in [FT1]. ■

Remark 3.15 Hölder continuity of the second derivative follows by the standard argument given in the Appendix and in Section II.3.4 (the hypotheses on the general propagators in Theorem II.3.8 are proven for strings in Lemma A.3).

3.7 Higher dimensions

The detailed analysis of the preceding section, as well as *(H5)*, were necessary only for $d=2$. For $d\ge 3$, the simpler argument of Section II.3.5 applies to the generalized RPA graphs. This is one point where one sees that the effective strength of the infrared singularity is the same for all $d\ge 2$ only on the level of naive power counting. For improved power counting, which is necessary for taking derivatives, increasing the dimension helps. The geometrical reason for this is that although the codimension of S always stays the same, the codimension of the critical points of the change-of-variable procedure of II increases with the dimension.

Theorem 3.16 *Let $d\ge 3$, and assume (H1)$_{2,h}$, (H2)$_{2,h}$, (H3), and, in the asymmetric case, (H4). Let G be a two–legged graph contributing to Σ, $j<0$, and $\mathcal{T}$ be a tree, rooted at a fork ϕ, and compatible to G. Then there is $\delta>0$ (independent of G) such that*

$$\sum_{J\in\mathcal{J}(\mathcal{T},j)} \left|Val(G^J)\right|_2 \le \text{ const } M^{\delta j}\lambda_{n_\phi}(j,\epsilon) \tag{3.120}$$

Proof: The proof is a combination of the methods of II, combined with Theorem 3.11, Lemma 3.7, and Theorem 3.8, exactly as was done in the proof of Theorem II.1.2. We therefore leave it to the reader. ■

Appendix : The Hölder argument and the proof of Theorem 1.2

In this appendix, we prove Theorem 1.2. The essence of the proof of the Hölder statements of Theorem 1.1 is the same as in the proof in this appendix: the extra scale decay leaves the freedom to extract Hölder continuity. We first illustrate the technique using the example of the second order graph. All other proofs are easy generalizations of that proof.

Lemma A.1 *Let $0 < \gamma < 1$ and assume (H1)$_{1,\gamma}$, (H2)$_{2,0}$, (H3), and (H4). Then $\Sigma_2 \in C^{1,\gamma}(\mathbb{R} \times \mathcal{B}, \mathbb{C})$, i.e.,*

$$|\Sigma_2|_{1,\gamma} \le K(\gamma) \tag{A.1}$$

where $K(\gamma)$ diverges in the limit $\gamma \to 1$.

Proof: As in II, we only have to deal with the sunset graph because all other contributions are $C^{1,\gamma}$ by the properties of the scale zero effective action (no derivative or difference affects the propagators on internal lines of these graphs).

We do the scale decomposition as in Section II.3.1. We then need to bound $|\cdot|_{1,\gamma}$ of

$$Y^\pi_{\underline{j}}(p) = \int đp_1 đp_2 \; C_{j_1}(z_1, e(\mathbf{p}_1)) \; C_{j_2}(z_2, e(\mathbf{p}_2)) \; C_{j_3}(\zeta_{\pi(3)}, e_{\pi(3)}) \; P_\pi(p_1, p_2, p) \tag{A.2}$$

where $j_1 < j_2 < j_3$,

$$\zeta_a = L_a(z_1, z_2, z), \qquad e_a = e(L_a(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p})), \tag{A.3}$$

L_a is given by

$$L_a(p_1, p_2, p) = \begin{cases} p + p_1 - p_2 & \text{if } a = 1 \\ p - p_1 + p_2 & \text{if } a = 2 \\ -p + p_1 + p_2 & \text{if } a = 3 \end{cases}, \tag{A.4}$$

π is a permutation of $\{1, 2, 3\}$, and P_π is given by a $C^{1,\gamma}$ function of all momenta with bounds uniform in j_1, j_2, j_3. It suffices to prove that the scale sum over $\{(j_1, j_2, j_3) : 0 > j_3 > j_2 > j_1 \ge I\}$ of $\left|Y^\pi_{\underline{j}}\right|_{1,\gamma}$ converges uniformly in I to prove the Lemma (see Section II.3.2). The convergence of the scale sum for $\left|Y^\pi_{\underline{j}}\right|_{1,0}$ was shown in Section II.3.2.

The derivative with respect to p can act only on the product $C_{j_3} P_\pi$, so it gives two terms

$$\frac{\partial}{\partial p} Y^\pi_{\underline{j}}(p) = U^\pi_{\underline{j}}(p) + V^\pi_{\underline{j}}(p) \tag{A.5}$$

with

$$\begin{aligned} U^\pi_{\underline{j}}(p) &= \int đp_1 đp_2 \; C_{j_1}(z_1, e(\mathbf{p}_1)) \; C_{j_2}(z_2, e(\mathbf{p}_2)) \; C_{j_3}(\zeta_{\pi(3)}, e_{\pi(3)}) \; \frac{\partial P_\pi}{\partial p}(p_1, p_2, p) \\ V^\pi_{\underline{j}}(p) &= \int đp_1 đp_2 \; C_{j_1}(z_1, e(\mathbf{p}_1)) \; C_{j_2}(z_2, e(\mathbf{p}_2)) \; \frac{\partial C_{j_3}}{\partial p}(\zeta_{\pi(3)}, e_{\pi(3)}) \; P_\pi(p_1, p_2, p) \end{aligned} \tag{A.6}$$

Both $U^\pi_{\underline{j}}$ and $V^\pi_{\underline{j}}$ are $C^{0,\gamma}$ functions of p. Since $U^\pi_{\underline{j}}$ has the same scaling behaviour as the undifferentiated function, the bound for $\left|U^\pi_{\underline{j}}\right|_{0,\gamma}$ is a power of M^{j_1} better than that for $\left|V^\pi_{\underline{j}}\right|_{0,\gamma}$ so we do only the latter bound in detail.

By II.3.31, with $s=1$,

$$\left|V_{\underline{j}}^{\pi}\right|_0 \le \text{ const } M^{j_1+j_2-j_3}|j_3|. \tag{A.7}$$

Summing over $j_3 > j_2$ and $j_2 > j_1$, we get

$$\sum_{j_2,j_3:j_3>j_2>j_1} \left|V_{\underline{j}}^{\pi}\right|_0 \le \text{ const } |j_1|^2 \, M^{j_1} \tag{A.8}$$

which is summable over $j_1 < 0$. The difference

$$\Delta_{\underline{j}}^{\pi}(p,p') = V_{\underline{j}}^{\pi}(p) - V_{\underline{j}}^{\pi}(p') \tag{A.9}$$

can be bounded trivially by

$$\left|\Delta_{\underline{j}}^{\pi}(p,p')\right| \le 2\left|V_{\underline{j}}^{\pi}\right|_0 \tag{A.10}$$

so

$$\sum_{j_2,j_3:j_3>j_2>j_1} \left|V_{\underline{j}}^{\pi}(p) - V_{\underline{j}}^{\pi}(p')\right| \le \text{ const } |j_1|^2 M^{j_1} \tag{A.11}$$

We now give a bound for $\left|\Delta_{\underline{j}}^{\pi}(p,p')\right|$ that depends nontrivially on p and p'. The difference in $\Delta_{\underline{j}}^{\pi}(p,p')$ acts on a product, so we use the discrete product rule (II.3.123) to convert it into a sum of terms in which the difference acts on only one factor of the product, and bound all these terms separately. In other words, we write

$$\begin{aligned} \frac{\partial C_{j_3}}{\partial p}(p)\; P_\pi(p_1,p_2,p) - \frac{\partial C_{j_3}}{\partial p}(p')\; P_\pi(p_1,p_2,p') &= \left(\frac{\partial C_{j_3}}{\partial p}(p) - \frac{\partial C_{j_3}}{\partial p}(p')\right) P_\pi(p_1,p_2,p) \\ &\quad + \frac{\partial C_{j_3}}{\partial p}(p')\left(P_\pi(p_1,p_2,p) - P_\pi(p_1,p_2,p')\right) \end{aligned} \tag{A.12}$$

Consequently, $\Delta_{\underline{j}}^{\pi}(p,p') = \Delta_{\underline{j},1}^{\pi}(p,p') + \Delta_{\underline{j},2}^{\pi}(p,p')$. By our hypotheses, P_π is $C^{1,\gamma}$ with bounds uniform in the scales, so

$$\left|\Delta_{\underline{j},2}^{\pi}(p,p')\right| \le \text{ const } |p-p'| M^{j_1+j_2-j_3}|j_3| \tag{A.13}$$

which gives a convergent scale sum, as above. To bound $\Delta_{\underline{j},1}^{\pi}(p,p')$, we use

$$\frac{\partial}{\partial p_\alpha} C_{j_3}(\zeta_{\pi(3)}, e_{\pi(3)}) = \pm \begin{cases} (\partial_1 C_{j_3})(\zeta_{\pi(3)}, e_{\pi(3)}) & \text{if } \alpha = 0 \\ (\partial_2 C_{j_3})(\zeta_{\pi(3)}, e_{\pi(3)})\; \dfrac{\partial e}{\partial p_\alpha}(L_{\pi(3)}(p_1,p_2,p)) & \text{if } \alpha \in \{1,\ldots,d\}, \end{cases} \tag{A.14}$$

where the sign $\pm$ depends on the sign factor in front of p in $L_{\pi(3)}$.

Taking the difference of C_{j_3} at p and p', we get again two terms if $\alpha \in \{1,\ldots,d\}$. The one involving $\frac{\partial e}{\partial p_\alpha}(L_{\pi(3)}(p_1,p_2,p)) - \frac{\partial e}{\partial p_\alpha}(L_{\pi(3)}(p_1,p_2,p'))$ is similar to $\Delta_{\underline{j},2}^{\pi}(p,p')$ and therefore leads to a convergent scale sum. In the remaining term, we Taylor expand, for $i \in \{1,2\}$, the difference (we abbreviate $\zeta_{\pi(3)}(p_1,p_2,p) = \zeta_{\pi(3)}(p)$ and $e_{\pi(3)}(p_1,p_2,p) = e_{\pi(3)}(p)$)

$$\begin{aligned} &(\partial_i C_{j_3})(\zeta_{\pi(3)}(p), e_{\pi(3)}(p)) - (\partial_i C_{j_3})(\zeta_{\pi(3)}(p'), e_{\pi(3)}(p')) \\ &\quad = \pm \sum_{k=1}^{2} \int_0^1 dt\; (\partial_k \partial_i C_{j_3})\left(\zeta_{\pi(3)}(p(t)), e_{\pi(3)}(p(t))\right)\; \left((p_0 - p_0')\delta_{k1} + (\mathbf{p}-\mathbf{p}')\cdot \nabla e(\mathbf{p}(t))\delta_{k2}\right) \end{aligned} \tag{A.15}$$

with $p(t) = (1-t)p' + tp$. This gives

$$\begin{aligned}
&\left|(\partial_i C_{j_3})(\zeta_{\pi(3)}(p), e_{\pi(3)}(p)) - (\partial_i C_{j_3})(\zeta_{\pi(3)}(p'), e_{\pi(3)}(p'))\right| \\
&\qquad \le (1+|e|_1)\,|p-p'| \int_0^1 dt \max_{i,k\in\{1,2\}} \left|\partial_k \partial_i C_{j_3}\left(\zeta_{\pi(3)}(p(t)), e_{\pi(3)}(p(t))\right)\right| \\
&\qquad \le W_2(1+|e|_1)\,|p-p'| \int_0^1 dt\, M^{-3j_3}\, \mathbb{1}\left(\left|e(L_{\pi(3)}(p_1,p_2,p(t))\right| \le M^{j_3}\right)
\end{aligned} \tag{A.16}$$

Inserting this, we get

$$\begin{aligned}
\left|\Delta^{\pi}_{\underline{j},1}(p,p')\right| &\le |P_\pi|_0 (1+|e|_1) W_0^2 W_2\, |p-p'| M^{j_1+j_2-3j_3} \\
&\int_0^1 dt \int d\theta_1 d\theta_2\, \mathbb{1}\left(\left|e(L_{\pi(3)}((0,\mathbf{p}(0,\theta_1)),(0,\mathbf{p}(0,\theta_2)),p(t)))\right| \le \left(1+2\tfrac{|e|_1}{u_0}\right) M^{j_3}\right) .
\end{aligned} \tag{A.17}$$

Since p_1 and p_2 are independent of t, we may use Theorem II.1.1 to bound the θ–integral by $\mathcal{W}((1+2\frac{|e|_1}{u_0}))M^{j_3}|j_3|$, which is independent of t. Thus

$$\left|\Delta^{\pi}_{\underline{j},1}(p,p')\right| \le |P_\pi|_0\, \mathbb{K}(Q_V,|e|_2,u_0)\, |p-p'|\, |j_1|\, M^{j_1+j_2-2j_3} \tag{A.18}$$

Summing over j_3 and j_2, we get

$$\sum_{j_2,j_3: j_3>j_2>j_1} \left|\Delta^{\pi}_{\underline{j},1}(p,p')\right| \le \text{ const } |p-p'|\, |j_1|. \tag{A.19}$$

This bound is linear in $p-p'$, but its sum over $j_1 < 0$ diverges. However, taking the weighted geometric mean with $(A.11)$, and adding the terms $\Delta^{\pi}_{\underline{j},2}(p,p')$ which we estimated before (and which have a convergent scale sum), we get for any $\gamma \in (0,1)$

$$\sum_{j_2,j_3: j_3>j_2>j_1} \left|V^{\pi}_{\underline{j}}(p) - V^{\pi}_{\underline{j}}(p')\right| \le \text{ const } |p-p'|^{\gamma}\, M^{j_1(1-\gamma)}|j_1|^{2-\gamma}. \tag{A.20}$$

The sum over j_1 now converges, which proves the statement of the Lemma. ∎

To show the same for Σ_r, for all $r \ge 3$, it suffices to bound the value of any order r two-legged 1PI graph. Thus Theorem 1.2 follows from the next Lemma.

Lemma A.2 *Let G be a 1PI two–legged graph and J a labelling of G. Let $0<\gamma<1$. Assume (H1)$_{1,\gamma}$, (H2)$_{2,0}$, (H3), and (H4). Let $\mathcal{T}$ be a tree consistent with G. Then for all $j<0$, and all labellings $J \in \mathcal{J}(\mathcal{T},j)$ of G ($\mathcal{J}$ is defined in* (I.2.73)*), $Val(G^J)$ is $C^{1,\gamma}$, and there is $\alpha > 0$ and a constant K such that for all $\sigma \in \{0,\ldots,d\}$*

$$\sum_{J\in\mathcal{J}(\mathcal{T},j)} \left|\frac{\partial}{\partial p_\sigma} Val(G^J)(p) - \frac{\partial}{\partial p_\sigma} Val(G^J)(p')\right| \le K|j|^{\alpha} M^{j(1-\gamma)}|p-p'|^{\gamma}. \tag{A.21}$$

K and α depend on G. For any two–legged 1PI graph G, the sum

$$\sum_{j<0}\sum_{\mathcal{T}\sim G}\sum_{J\in\mathcal{J}(\mathcal{T},j)}\left|Val(G^J)\right|_{1,\gamma} \tag{A.22}$$

converges.

Proof: It suffices to prove $(A.21)$ because $(A.22)$ then follows simply by summation over $\mathcal{T}$ and j. By *(H1)*$_{1,\gamma}$ and Lemma II.2.3, the vertex functions associated to the scale zero effective vertices are $C^{1,\gamma}$, with bounds uniform in the scales. The bounds depend only on $|\hat{v}|_{1,\gamma}$. We do an induction in the depth of $(\mathcal{T},G)$, defined as

$$P_2(G,\mathcal{T})=\max\{k:\exists f_1>f_2>\ldots>f_k>\phi:E(G_{f_l})=2 \text{ for all } l\in\{1,\ldots,k\}\} \tag{A.23}$$

with $(A.21)$ as the inductive hypothesis. If $P_2=0$, G is a skeleton graph (or G has at most one vertex, in which case the statement is obvious by the properties of the vertex functions).

Let $\tilde{G}(\phi)$ be the root scale quotient of G (see Definition I.2.27). Let $P_2(G,\mathcal{T})=0$.

Case 1: $\tilde{G}(\phi)$ is overlapping. We take the difference

$$\Delta_J(p,p')=\left(\frac{\partial}{\partial p_\sigma}Val(G^J)\right)(p)-\left(\frac{\partial}{\partial p_\sigma}Val(G^J)\right)(p'). \tag{A.24}$$

As in second order, the derivative can act on vertex functions or propagators. Again, we rewrite the result by the discrete product rule (II.3.123). This gives a sum of terms of the following types (denoted by $T_J^{(i)}$)

(1) Both the derivative and the difference operator act on a vertex function. This term has the same scale behaviour as the undifferentiated graph because the vertex function has uniform bounds. Since $\tilde{G}(\phi)$ is overlapping, Theorem I.2.46 (i) implies that

$$\sum_{J\in\mathcal{J}(\mathcal{T},j)}\left|T_J^{(1)}(p,p')\right|\le \text{ const } |j|^{\alpha_1}M^{2j}|p-p'|. \tag{A.25}$$

(2) A vertex function gets differentiated, but the difference operator acts on a propagator. Rewriting the difference of the propagator at p and p' by Taylor expansion, we see that this term behaves like the first derivative of the value of the graph. By Theorem I.2.46 (i),

$$\sum_{J\in\mathcal{J}(\mathcal{T},j)}\left|T_J^{(2)}(p,p')\right|\le \text{ const } |j|^{\alpha_2}M^{j}|p-p'| \tag{A.26}$$

(3) The derivative acts on the propagator on line ℓ_1 and the difference on line ℓ_2 ($\ell_1=\ell_2$ is possible). With the same Taylor expansion as above, the scale behaviour deteriorates by a factor const $M^{-j_{\ell_1}-j_{\ell_2}}\le M^{-2j}$. Thus this term is bounded by

$$\sum_{J\in\mathcal{J}(\mathcal{T},j)}\left|T_J^{(2)}(p,p')\right|\le \text{ const } |j|^{\alpha_3}|p-p'|. \tag{A.27}$$

Summing up all these terms, we obtain

$$\sum_{J\in\mathcal{J}(\mathcal{T},j)}|\Delta_J(p,p')|\le \text{ const } |j|^{\alpha}|p-p'|. \tag{A.28}$$

On the other hand, we also have the trivial estimate

$$\sum_{J \in \mathcal{J}(\mathcal{T},j)} |\Delta_J(p,p')| \le 2 \sum_{J \in \mathcal{J}(\mathcal{T},j)} \left| \frac{\partial}{\partial p_\sigma} Val(G^J) \right|_0 \le \text{ const } M^j |j|^\alpha \tag{A.29}$$

The geometric mean of $(A.28)$ and $(A.29)$ gives

$$\sum_{J \in \mathcal{J}(\mathcal{T},j)} |\Delta_J(p,p')| \le \text{ const } |p-p'|^\gamma \; M^{j(1-\gamma)} |j|^\alpha. \tag{A.30}$$

Case 2: $\tilde{G}(\phi)$ is nonoverlapping. Since $P_2(G,\mathcal{T}) = 0$, this means that $\tilde{G}(\phi)$ is an ST diagram (see Definition I.2.21). Thus all lines on scale j are self–contractions, and therefore the propagators associated to them do not depend on the external momentum p. Thus no derivative or difference operation can act on these propagators. Let τ_ϕ be the maximal subtree of t rooted at ϕ such that $\tilde{G}(\tau_\phi)$ is nonoverlapping. By Lemma I.2.31 (ii), τ_ϕ exists and is unique. Let $\tilde{G} = \tilde{G}(\tau_\phi)$. By definition of τ_ϕ, the lowest scale j^* on which the derivative or difference can have an effect is also the scale where a volume gain from overlapping loops occurs (if $j^* = 0$, there is no volume gain, but then the derivative acts only on a scale zero vertex function, and thus does not affect the scale behaviour at all). All considerations of the previous case apply, only with j replaced by j^*. Thus the derivatives and differences produce at worst a factor M^{-2j^*} and volume improvement produces a factor M^{j^*}. Summing the scales, we get again $(A.28)$ and $(A.29)$.

Let $P_2(G,\mathcal{T}) > 0$. Now G can have two–legged subgraphs, corresponding to r– and c–forks of t. They have smaller depth P_2. Therefore the inductive hypothesis applies to them. We construct the skeleton graph G' associated to G by replacing the strings of two–legged subdiagrams by new propagators. We now verify that these strings have the same behaviour as ordinary propagators when they are differentiated and when differences are taken.

The value of such a string is given by (I.2.93). When the derivative and the difference operator act on one of the propagators the scaling behaviour changes in the same way as for a single propagator. If they act on an r–fork, the renormalization cancellation is lost, but the inductive hypothesis implies that no other factor $M^{-\beta j}$, $\beta > 0$, occurs. If the derivative acts on a c–fork, by $(A.26)$, the net effect is the same as if the derivative had acted on a line of scale j_ϕ rather than internal to the c–fork. If the derivative and the difference operator act on the c–fork, the inductive hypothesis $(A.30)$ implies that the c–fork scale sum is still convergent. If derivatives and differences act on different factors in the value of the string, Taylor expansion implies that the same bounds hold. Thus strings behave like single propagators, which puts us back to the case $P_2 = 0$, which we have already done. ■

Finally, we state the Lemma that implies that the strings of two–legged subdiagrams S_{l,j_l} have the properties required of the general propagators appearing in Lemma II.3.8.

Lemma A.3 *Assume the hypotheses of Lemma* 3.1, *assume* $(H2)_{2,h}$, *and and assume that there are* $\tau_{k,h} > 0$ *and* $n_k \in \mathbb{N}$ *such that*

$$\left| D^\alpha T_k^{(j)}(p) - D^\alpha T_k^{(j)}(p') \right| \le \tau_{k,h} |p-p'|^h |j|^2 M^{-hj} \lambda_{n_k}(j,\varepsilon) \tag{A.31}$$

for all multiindices α with $|\alpha| = 2$, for some $\varepsilon > 0$. Then

$$\left| D^\alpha S^{(\nu_l)}_{l,j_l}(p) - D^\alpha S^{(\nu_l)}_{l,j_l}(p') \right| \le H_S |p-p'|^\beta \; M^{-j_l(\nu_l+k-1+\beta)} \;\; 1\!\!1\left(|ip_{\rm o} - e(\mathbf{p})| \le G_{\rm o} M^{j_l}\right) \tag{II.3.126}$$

and

$$\begin{aligned} &\left| \left(\tfrac{\partial}{\partial\theta}\right)^{k-1} S^{(\nu_l)}_{l,j_l}(p_{\rm o}, \mathbf{p}(\rho,\theta)) - \left(\tfrac{\partial}{\partial\theta}\right)^{k-1} S^{(\nu_l)}_{l,j_l}(p_{\rm o}, \mathbf{p}(\rho',\theta')) \right| \\ &\qquad \le H_S |\mathbf{p}(\rho,\theta) - \mathbf{p}(\rho',\theta')|^\beta \; M^{-j_l(\nu_l+\beta)} \;\; 1\!\!1\left(|ip_{\rm o} - \rho| \le 4M^{j_l}\right) \end{aligned} \tag{II.3.127}$$

hold, with $\nu_l = 1$, $\beta = h$ and $H_s =$ const $|j|^2 \prod_k \lambda_{n_k}(j,\varepsilon)$ where the constant depends only on $\tau_{1,h}, \ldots, \tau_{n,h}$ and $|e|_{2,h}$ and ε.

Proof: If $|p-p'| \ge M^j$, then $|p-p'|^{-\beta} \le M^{-j\beta}$ so the statement follows from $|S_j(p) - S_j(p')| \le |S_j(p)| + |S_j(p')|$, and Lemma 3.1. So let $|p-p'| < M^j$. Using the discrete product rule in the usual way, we get contributions involving the differences $D^{\alpha'} T^{(j)}_k(p) - D^{\alpha'} T^{(j)}_k(p')$ and $D^{\alpha'} C_j(p_{\rm o}, e(\mathbf{p})) - D^{\alpha'} C_j(p'_{\rm o}, e(\mathbf{p}'))$ with $\alpha' \le 2$. If $|\alpha'| < 2$, Taylor expansion and the inequality

$$|p-p'| \le |p-p'|^h M^{j(1-h)} \tag{A.32}$$

do the job. In particular, we have for propagators an obvious analogue of $(A.16)$ with, possibly, a more complicated linear combination of other momenta, and $(A.32)$ implies a bound involving $M^{-j(1+h+|\alpha'|)}$. Let $\alpha' = 2$. If the difference acts on $D^{\alpha'} T^{(j)}_k$, the result follows from $(A.31)$. $D^{\alpha'} C_j$ contains the summand $(\partial_2 C_j) D^{\alpha'} e$, which is the only summand containing the second derivative of e. In the difference, we use again the discrete product rule. The term involving the difference of e is bounded as

$$\left| D^{\alpha'} e(p) - D^{\alpha'} e(p') \right| \le |e|_{2,h} \; |p-p'|^h. \tag{A.33}$$

In the other terms, we are back to the case (with $\alpha' \le 1$). ■

Remark A.4 The polynomial dependence of H_S on j does not change the conclusion of Theorem II.3.8 because the convergence factor in the scale sum is $M^{\delta j}$ with $\delta > 0$.

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