AMPLITUDE EQUATION FOR LATTICE MAPS,

A RENORMALIZATION GROUP APPROACH

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Abstract: We consider the development of instabilities of homogeneous stationary solutions of discrete time lattice maps. Under some generic hypothesis we derive an amplitude equation which is the space-time continuous Ginzburg-Landau equation. Using dynamical renormalization group methods we control the accuracy of this approximation in a large ball of its basin of attraction.

Keywords: multi scales, Ginzburg Landau, bifurcations, instabilities, continuous spectrum.

I.Introduction.

Instabilities of extended systems lead to a large number of interesting phenomena and in particular to the appearance of well defined structures[5,12]. We will consider in this paper the case of discrete time iteration of lattice systems. These are dynamical systems defined on the phase $l^\infty(\mathbf{Z}^d, \mathbf{R})$ which will be abbreviated below by l^∞. The time evolution is given by a map Φ of l^∞ into itself for which we will make the following basic hypothesis which also includes the dependence on a real parameter.

H1 $\Phi \, : \, \mathbf{R} \times l^\infty \longrightarrow l^\infty$. *$\Phi$ is C^2 in its first argument and C^4 in the second one on some neighborhood of the origin. Moreover Φ commutes with the translations on $\mathbf{Z}^d$.*

The time evolution for a fixed parameter η is given as follows. If $\underline{u}^t \in l^\infty$ $(t \in \mathbf{N})$ is the state of the system at time t, the state of the system at time $t+1$ is given by

$$\underline{u}^{t+1} = \Phi(\eta, \underline{u}^t) \ .$$

There are many examples of such dynamical systems. We will just mention the class of coupled lattice maps and also notice that many algorithms used for solving numerically partial differential equations are of this form (for example finite difference schemes).

One would like of course to understand the properties of this dynamical system and in particular the large time dynamics. In finite dimensional dynamical systems, varying a parameter is a very efficient way to discuss the increase of complexity of the time asymptotic dynamics. This increase of complexity can occur at well defined threshold values of the parameter called bifurcations for which detailed results are available in particular for the bifurcation of stationary solutions[8]. An important tool in bifurcation theory of stationary solutions is the normal form equation which provides a model for the dynamics. Moreover for generic systems these normal forms are in some sense universal.

For spatially extended continuous systems which depend on a parameter, amplitude equations have been derived in various situations[5,12], and rigorous derivations of the Ginzburg-Landau equation in one space dimension have been obtained[3,15,11,13,14]. The result is similar to what occurs for normal forms in finite dimensional dynamical systems, namely the space-time function reconstructed using the solution of the amplitude equation stays close to the solution of the true evolution (with the same initial condition) for a large time. However this method usually provides only an approximation of the true solutions and for very large times, neglected small errors usually grow due to instabilities[10]. Nevertheless, just before the approximation becomes too inaccurate one can restart the process with a new initial condition which is the present true solution. This provides a shadowing of the true solution by a sequence of chunks of solutions of the amplitude equation. We refer to the previously mentioned references for more detailed discussions of these results.

The shadowing of a solution of the true equation by a reconstructed function from a solution of the amplitude equation is only valid for initial conditions of a rather limited form, namely those which can be reconstructed from an initial condition of the amplitude equation. Eckhaus[6] has shown that this condition can be somewhat relaxed and that after a relatively short time any small enough initial condition leads to a solution having the right form.

We will consider below all the previously mentioned results for the case of discrete dynamics of one dimensional lattice systems. We will recover all the results obtained in the continuous case and extend them in various directions. In order to do this we will use a renormalization group approach[1,7]. Besides its systematic simplicity this method will also have the advantage of providing a rigorous basis for the universality of the amplitude equations. In the finite dimensional situation transversality arguments can be used, however in the infinite dimensional case the equivalent results are not known, and universality provided by the renormalization group method is a good replacement.

Another advantage of the renormalization group argument is that it will allow us to deal with initial conditions which are small but of order unity, independently of the smallness of the parameter. As we will see below the renormalization group flow draws an initial data of size of order one to a configuration of size $\sqrt{\eta}$. At the same time the renormalization group flow puts the solution in a resonant form providing a natural proof of the previously mentioned Eckhaus result. Finally when the renormalization group stops improving the size and form of the solution we get the amplitude equation together with a rigorous proof of the associated shadowing properties. The same ideas can probably be applied in the continuous space and time case and will eventually provide some improvements in term of simplicity and naturalness of proof as well as better bounds. Different approaches to amplitude equations using renormalization have been discussed by Goldenfeld et al.[4]. However since there is already a large literature for the continuous case we will not consider this situation. Note also that we will treat a rather large class of systems whose continuous equivalent would include pseudo differential operators.

We will mostly deal below with one dimensional systems ($d = 1$) although the same method works in higher dimensions, at least when the amplitude equation is not too involved. Systems of coupled equations (corresponding to vector valued fields) can be tee treated similarly.

Since we will be interested in instabilities of stationary solutions, we will now assume for simplicity that the zero element of l^∞ is a stationary solution of the dynamical system. In many cases this can be realized by a translation in function space.

H2 *$\Phi(\eta, \underline{0}) = \underline{0}$ for any $\eta \in \mathbf{R}$.*

We will make some assumptions on the linearized evolution around this fixed point.

H3 *We will assume that the operator $L_0 = D_2\Phi(0,0)$ which is bounded in l^∞ and commutes with translations can be represented as the convolution with a (real) sequence $l \in l^1$. Let $l(\theta)$ be the Fourier transform of l*

$$l(\theta) = \sum_{n\in\mathbf{Z}} l_n e^{-in\theta} \;.$$

We will also assume that

$$\sum_{n\in\mathbf{Z}} |n|^{10} |l_n| < \infty \;,$$

and that $|l(\theta)| \le 1$ with $|l(\omega)| = 1$ for a unique $\omega \in]0, \pi[$. We will also assume that $l(\omega) \ne -1$ and $\omega \notin \{\pi/2, 3\pi/3\}$. Moreover, we will make the non degeneracy assumption

$$D = -\Re\left[\frac{l''(\omega)}{l(\omega)} - \frac{l'(\omega)^2}{l(\omega)^2}\right] \ne 0 \;.$$

As we will see later on, this last number is always non-negative. Notice that $l(\omega)=\overline{l(-\omega)}$. The particular cases which are excluded in the above hypothesis correspond to some strong resonances. We will not consider them further below although some interesting results can probably be derived in these situations using the ideas of the present paper.

We now need some hypothesis on the higher order derivatives of Φ.

H4 *The translation invariant bilinear and trilinear operators* $Q=D_2^2\Phi(0,0)/2$ *and* $C=D_2^3\Phi(0,0)/6$ *are given by a double (respectively triple) real sequence* $(q_{n,m})\in l^1(\mathbf{Z}^2)$ *(respectively* $(c_{n,m,p})\in l^1(\mathbf{Z}^3)$*), namely*

$$Q(\underline{u})_r=\sum_{(n,m)\in\mathbf{Z}} q_{r-n,r-m}u_n u_m \qquad \text{and} \qquad C(\underline{u})_r=\sum_{(n,m,p)\in\mathbf{Z}} c_{r-n,r-m,r-p}u_n u_m u_p \ .$$

We will assume that these sequences satisfy

$$\sum_{(n,m)\in\mathbf{Z}} |q_{n,m}|(1+|n|+|m|)^6+\sum_{(n,m,p)\in\mathbf{Z}} |c_{n,m,p}|(1+|n|+|m|+|p|)^4<\infty \ .$$

We will also make the non linear stability assumption

$$g=-\Re\left(\frac{c(\omega,\omega,-\omega)}{l(\omega)}+\frac{4q(\omega,0)q(\omega,-\omega)}{3l(\omega)(|l(\omega)|^2-l(0))}+\frac{2q(-\omega,2\omega)q(\omega,\omega)}{3l(\omega)(l(\omega)^2-l(2\omega))}\right)>0\ ,$$

where

$$c(\phi_1,\phi_2,\phi_3)=\sum_{p,q,r} c_{p,q,r}e^{-i(p\phi_1+q\phi_2+r\phi_3)} \qquad \text{and} \qquad q(\phi_1,\phi_2)=\sum_{p,r} q_{p,r}e^{-i(p\phi_1+r\phi_2)} \ .$$

Finally we will also require that the dependence on the parameter allows the instability to occur.

H5. *We will assume that the operator* $L_1=D_1D_2\Phi(0,0)$ *which is bounded in* l^∞ *and commutes with translations can be represented as the convolution with a (real) sequence* $m\in l^1$*. We will assume that*

$$\sum_{n\in\mathbf{Z}} |n|^3|m_n|<\infty \ .$$

Let $m(\theta)$ *be the Fourier transform of* m*, we will also assume that*

$$\rho=\Re\frac{m(\omega)}{l(\omega)}>0 \ .$$

Note that as for the function l we have $m(-\theta)=\overline{m(\theta)}$.

A simple example satisfying the above hypothesis can be satisfied as follows. Let l be a bi-infinite sequence of real numbers satisfying **H3**, we define the map Φ by

$$\Phi(\eta,\underline{u})_n=\eta u_n+\sum_q l_{n-q}u_q-(u_n)^3 \ . \qquad (I.1)$$

In this particular situation, **H1** and **H2** are easily verified.

This is a natural generalization of the discrete Swift-Hohenberg evolution which is given by

$$u_n^{t+1} = (1+\eta)u_n^t - \rho\big((1+\gamma\Delta)^2\underline{u}^t\big)_n - (u_n^t)^3 \tag{DSH}$$

where Δ is the discrete Laplacian

$$(\Delta\underline{u})_n = u_{n+1} + u_{n-1} - 2u_n \ .$$

In order to explain some of the arguments in a simpler situation we will also sometimes make the following hypothesis.

H3' *l is even and satisfies* **H3**.

Note that in this case $l(\theta)$ is even and real, $l(\pm\omega) = 1$ and $l'(\pm\omega) = 0$.

For the particular case of (DSH) one has

$$l(\phi) = 1 - \rho(1 - 2\gamma + 2\gamma\cos\phi)^2 \ ,$$

and the hypothesis **H1-H5** and **H3'** are satisfied if one assumes

$$\gamma > 1/4 \qquad \text{and} \qquad 2/\rho > \max(1, (1-4\gamma)^2) \ .$$

We now formulate the main result of this paper for the evolution (I.1).

Theorem I.1. *Assume* **H3'**, *then there are constants $R > 0$, $1 > \eta_0 > 0$, $c_1 > 0, \cdots, c_4 > 0$ such that for any $\eta \in]0, \eta_0[$, for any $\underline{u} \in l^\infty$ such that $\|\underline{u}\|_{l^\infty} < R$, if $(\underline{u}^t)_{t\in\mathbf{N}}$ denotes the orbit of $\underline{u}$ under the discrete time evolution (I.1), then*

i) there is a positive number $T = T(\underline{u}, \eta)$ such that for any integer $t > T$

$$\|\underline{u}^t\|_{l^\infty} \le c_1\eta^{1/2} \ .$$

ii) For any integer $t > T$ there is a function $A(\tau, x)$ from $\mathbf{R}^+ \times \mathbf{R}$ to $\mathbf{C}$ satisfying

$$\partial_\tau A = \partial_x^2 A + A - A|A|^2 \tag{GL}$$

such that for any integer s with $0 \le s \le c_2\eta^{-1}\log\eta^{-1}$ we have

$$\|\underline{u}^{t+s} - \underline{v}^s\|_{l^\infty} \le c_3\eta^{1/2+c_4} \ .$$

where

$$v_n^s = 3^{-1/2}e^{i\omega n}\eta^{1/2}A(\eta s, \eta^{1/2}n\sqrt{2/D}) + c.c.$$

Theorem (I.1) can be understood in the light of the basic principles of bifurcation theory. First of all we have an "unstable" direction in phase space which is the sequence $(e^{i\omega n})$ (and its complex conjugate). As in finite dimensional bifurcation theory, the non trivial solutions are sought in the form $u_n = A_n e^{i\omega n}$, where A is a slowly varying function of space and time (it is a non trivial consequence of our theorem that locally there are no other long time solutions). The equation for A should ensure that in the evolution equation for $\underline{u}$ all resonant terms (i.e. terms containing $e^{\pm i\omega n}$) disappear. These resonant terms come of course from the linear part of the evolution equation but also from the non linear part. If we examine more carefully the second contribution, we see that quadratic terms in $\underline{u}$ (and all even higher order terms) should not contribute, whereas cubic terms (and higher odd powers) generate resonant terms. If the solution is small, cubic terms dominate. If we assume that the dispersion relation is non degenerate at the critical wave number (here $D > 0$), one is naturally lead by scaling arguments to the equation

$$\partial_t A = A'' - A|A|^2 \ . \tag{F.P.}$$

Note that this equation has the one parameter family of scale invariance $x \to \lambda x$, $t \to \lambda^2 t$, $A \to \lambda A$. This equation can be considered as a fixed point of a renormalization operation. An important property of (F.P.) is that all solutions with bounded initial conditions tend to zero when t tends to infinity. A term σA added to (F.P.) corresponds to a relevant direction$^{(1)}$. This is how the complete Ginzburg-Landau equation appears. The proof of Theorem (I.1) is essentially a control of the R.G. flow in the vicinity of (F.P.) done from the point of view of the solutions i.e. proving that the unstable manifold is transversally stable.

We implement these ideas in section II. We first give estimates on various operators which are needed to control the time evolution of the remainder. The recursive renormalization argument is given at the end of the section.

In section III we will give the equivalent result under the more general hypothesis **H1-H5**. Technically the proof is not much different (except for the treatment of the quadratic term). However one has to deal simultaneously with several details which somewhat obscure the main ideas, this is why we have chosen to explain first in section II the simple case, and explain in section III the necessary modifications for the general case.

II. Renormalization proof of Theorem II.1.

In this section we will prove Theorem I.1. We will therefore always assume that η is a positive number.

We start by fixing a renormalization factor S which is a positive number larger than one. Although the final result will be essentially independent of S, it is convenient in the present discrete space-time setting to take for S and integer (for example $S = 2$). This avoids using integer parts of various time scales. We can now define a sequence of scales $(\delta_n)_{n\in\mathbf{N}}$ for the wave numbers by

$$\delta_n = S^{-n} \ .$$

In other words we have a sequence of space scales given by $(\delta_n^{-1})_{n\in\mathbf{N}}$. As we will see below, since we have a "trivial" scaling, this induces a a sequence of time scales $(\delta_n^{-2})_{n\in\mathbf{N}}$, and a sequence of field scales given by $(\delta_n)_{n\in\mathbf{N}}$. We will also need below a sequence of scales $(\theta_n)_{n\in\mathbf{N}}$ for a window of unstable wave numbers. It is convenient to choose this sequence of the form

$$\theta_n = (\delta_n^{-1})^{\varsigma}$$

where ς is a positive number smaller than one and small enough. We will for definiteness choose $\varsigma = 1/5$.

As usual with renormalization group arguments, all the steps are similar except for a change of scales. Therefore we will start by assuming that a scale has been fixed (i.e. an n in the previous sequences). Hence we have a scale $\delta = \delta_n$ for the solution, namely we will assume from now on that the initial condition $\underline{u}$ satisfies

$$S^{-1}\delta < \|\underline{u}\|_{l^\infty} \le \delta \ .$$

We also have a scale θ for the normalized width of the window of wave numbers. One should think of δ as a small number (but possibly of order one, i.e. a negative power of S), and θ as a large number such that $\delta\theta^4$ is small (i.e. $\theta = \delta^{1/5}$). Explicit constraints on δ and θ will be given later on. In particular we will always assume that $\delta\theta < 1/8$ (see appendix B).

We now introduce three operators whose sum is the identity and which are essentially projections on sequences with wave numbers near ω. They are constructed as follows. First we consider a cut-off function ψ which is infinitely differentiable, non negative with support in $[-2, 2]$ and which is equal to one on the interval $[-1, 1]$. We then define a sequence $K_{\delta,\theta}$ by

$$K_{\delta,\theta}(n) = \delta\theta\mathcal{F}\psi\,(n\delta\theta) \ ,$$

where $\mathcal{F}$ denotes Fourier transform on the real line. Convolution by $K_{\delta,\theta}$ localizes the wave numbers in an interval of length $4\delta\theta$ centered at the origin. In particular, for $\delta\theta < \pi/2$, since that ψ has compact support we have for $-\pi \le \phi \le \pi$

$$\sum_n K_{\delta,\theta}(n)e^{in\phi} = \psi(\theta^{-1}\delta^{-1}\phi) \ .$$

We now define three operators $P^+_{\delta,\theta}$, $P^-_{\delta,\theta}$ and $R_{\delta,\theta}$ by their kernels, namely

$$P^+_{\delta,\theta}(n,m) = e^{i\omega n}K_{\delta,\theta}(n-m)e^{-i\omega m} \ ,$$

$$P^-_{\delta,\theta}(n,m) = e^{-i\omega n} K_{\delta,\theta}(n-m) e^{i\omega m} \ ,$$

and

$$R_{\delta,\theta} = I - P^+_{\delta,\theta} - P^-_{\delta,\theta} \ .$$

From the above remark about the action of the convolution by $K_{\delta,\theta}$, we see that $P^+_{\delta,\theta}$ localizes the wave numbers around ω and $P^-_{\delta,\theta}$ around $-\omega$. In particular, if δ is small enough, we obtain two disjoint intervals of wave numbers.

We now define a constant C_1 by

$$C_1 = 2\left(1 + \sup_{0\le j\le 5} \|\partial^j \mathcal{F}\psi\|_1\right) \ .$$

This constant will be a convenient bound for several quantities below.

Lemma II.1. *For any $0 < \delta\theta < \pi/2$, the operators $P^+_{\delta,\theta}$, $P^-_{\delta,\theta}$ and $R_{\delta,\theta}$ have a norm bounded by C_1 in l^∞.*

The proof is an immediate consequence of the fact that the function ψ belongs to the Schwarz space $\mathcal{S}$.

We now decompose a sequence $\underline{u} \in l^\infty$ using the above three operators.

Lemma II.2. *For any real sequence $\underline{u} \in l^\infty$, there is a function $B \in C^4(\mathbf{R},\mathbf{C})$ such that*

$$e^{-i\omega n} P^+_{\delta,\theta}(\underline{u})_n = B(\delta n) \quad \forall n \in \mathbf{Z} \ ,$$

and satisfying

$$\max\left\{\theta^{-j}\|\partial^j B\|_{l^\infty} \ ; j = 0,\cdots,4\right\} \le C_1\|u\|_{l^\infty} \ .$$

We have also

$$e^{i\omega n} P^-_{\delta,\theta}(\underline{u})_n = \overline{B}(\delta n) \quad \forall n \in \mathbf{Z} \ .$$

This is an immediate consequence of the definition of $P^\pm_{\delta,\theta}$ and interpolation. In order to fix entirely B one can use the following explicit choice

$$B(x) = \delta\theta \sum_m \mathcal{F}\psi\left(x\theta - m\delta\theta\right) e^{-im\omega} u_m \ .$$

We also define the remainder $\underline{r}^0$ by

$$\underline{r}^0 = R_{\delta,\theta}\underline{u} \ .$$

We now denote by A the solution of the Ginzburg Landau equation

$$\partial_\tau A = A'' + \eta\delta^{-2}A - A|A|^2 \ , \tag{$II.1$}$$

with initial data

$$A(0,x) = \sqrt{3}\,\delta^{-1} B(x\sqrt{D/2}) \ . \tag{$II.1'$}$$

It will be convenient in the sequel to denote by $\underline{\mathcal{A}}^t$ the sequence

$$\mathcal{A}_n^t = 3^{-1/2} e^{in\omega} A(\delta^2 t, n\delta\sqrt{2/D}) + c.c. \tag{II.2}$$

Finally, the equation for the evolution of the remainder term

$$\underline{r}^t = \underline{u}^t - \delta\underline{\mathcal{A}}^t$$

is given by

$$r_n^{t+1} = (L\underline{r}^t)_n - 3\delta^2 r_n^t (\mathcal{A}_n^t)^2 - 3(r_n^t)^2 \mathcal{A}_n^t - (r_n^t)^3 + \mathcal{R}_n^t \ , \tag{II.3}$$

where we have denoted by L the operator

$$L\underline{v} = l * \underline{v} + \eta\underline{v} \ ,$$

and the sequence $\underline{\mathcal{R}}^t$ is given by

$$\mathcal{R}_n^t = \delta(L\underline{\mathcal{A}}^t)_n - \delta^3(\mathcal{A}_n^t)^3 - \delta\mathcal{A}_n^{t+1} \ . \tag{II.4}$$

Using equation (II.1), we have immediately the following estimate for the forcing term $\underline{\mathcal{R}}^t$.

Lemma II.3. *There is a constant $C_2 > C_1$ such that for any $\delta\theta^4 < 1$, and $\delta > \eta^{1/2}$, if B is a function of x satisfying*

$$\max\left\{\theta^{-j}\|\partial^j B\|_{l^\infty} \ ; j = 0, \cdots, 4\right\} \le \delta C_1 \ ,$$

if A denotes the solution of (II.1) with initial condition (II.1'), and if $\underline{\mathcal{A}}^t$ is constructed from A as in (II.2), then for any $t \in \mathbf{N}$ we have

$$\|\underline{\mathcal{A}}^t\|_{l^\infty} \le C_2 \ , \ \|\underline{\mathcal{R}}^t\|_{l^\infty} \le C_2\delta^3 \ , \ \ \|\underline{\mathcal{R}}^t + \underline{\mathcal{C}}^t\|_{l^\infty} \le C_2\delta^4\theta^4 \ ,$$

where the sequence $\underline{\mathcal{C}}^t$ is given by

$$\mathcal{C}_n^t = 3^{-3/2}\delta^3 e^{3in\omega} A^3(\delta^2 t, n\delta\sqrt{2/D}) + c.c.$$

Proof. It follows easily from (II.1') that

$$\|A(0, \cdot)\|_{l^\infty} \le \sqrt{3}C_1 \ .$$

The Lemma follows then at once from Lemmas A.3 and A.4 of appendix A using $\delta\theta^4 < 1$.

We now start estimating various operators needed below in the proof of Theorem I.1.

Lemma II.4. *There is a constant $C_3 > e$ such that for any $0 < \eta \le 1$ and for any $t \in \mathbf{N}$ we have*

$$\|L^t\|_{l^\infty} \le C_3 e^{\eta(t-\eta^{-1})} \ .$$

The proof is given in appendix B.

Lemma II.5. *For any $t \in \mathbf{N}$, $0 < \eta < 1$ and $1 > \delta^2\theta^2 > 4D\eta$ we have*

$$\|L^t R_{\delta,\theta}\| \le C_4 e^{-tD\delta^2\theta^2/8} .$$

The proof is given in appendix B.

It will be convenient to denote by L_t the linear operator given by

$$(L_t \underline{v})_n = -3\delta^2 (\mathcal{A}_n^t)^2 v_n - 3\delta r_n^t \mathcal{A}_n^t v_n - (r_n^t)^2 v_n .$$

Note that

$$r_n^{t+1} = (L\underline{r}^t)_n + (L_t \underline{r}^t)_n + \mathcal{R}_n^t . \tag{II.5}$$

Lemma II.6. *We have for any $t > 0$*

$$\|L_t\|_{l^\infty} \le 5(\delta^2 \|\underline{\mathcal{A}}^t\|_{l^\infty}^2 + \|\underline{r}^t\|_{l^\infty}^2) .$$

This is obvious from the definition of L_t.

Lemma II.7. *If $0 \le t_0 < t \le \eta^{-1}$ we have*

$$\|(L + L_t)\cdots(L + L_{t_0})\|_{l^\infty} \le C_3 \left(1 + \sup_{t_0 \le s \le t} 5C_3(\delta^2 \|\underline{\mathcal{A}}^s\|_{l^\infty}^2 + \|\underline{r}^s\|_{l^\infty}^2)\right)^{t-t_0+1} .$$

Proof. We first expand the product $(L + L_t)\cdots(L + L_{t_0})$. It is a sum of 2^{t-t_0+1} terms of the form

$$L^{1-\zeta_t} L_t^{\zeta_t} \cdots L^{1-\zeta_{t_0}} L_{t_0}^{\zeta_{t_0}} ,$$

where the numbers $\zeta_.$ are equal to zero or one. Using Lemmas II.4 and II.6 it easily follows that

$$\|L^{1-\zeta_t} L_t^{\zeta_t} \cdots L^{1-\zeta_{t_0}} L_{t_0}^{\zeta_{t_0}}\|_{l^\infty} \le C_3^{k+1} 5^k \left(\sup_{t_0 \le s \le t} (\delta^2 \|\underline{\mathcal{A}}^s\|_{l^\infty}^2 + \|\underline{r}^s\|_{l^\infty}^2)\right)^k$$

where k is the number of $\zeta_.$ equal to one. Therefore

$$\|(L + L_t)\cdots(L + L_{t_0})\|_{l^\infty} \le \sum_{k=0}^{t-t_0+1} \binom{t - t_0 + 1}{k} C_3^{k+1} 5^k \left(\sup_{t_0 \le s \le t} (\delta^2 \|\underline{\mathcal{A}}^s\|_{l^\infty}^2 + \|\underline{r}^s\|_{l^\infty}^2)\right)^k ,$$

and the result follows.

If $t - t_0 > \eta^{-1}$, we can split this time interval in various sub-intervals of length $[\eta^{-1}]$ plus a remainder. We immediately obtain the following corollary.

Corollary II.8. *For any* $0 < t_0 < t$ *and* $\eta < 1/2$ *we have*

$$\|(L+L_t)\cdots(L+L_{t_0})\|_{l^\infty} \le C_3^{1+2\eta(t-t_0)} e^{5C_3(t-t_0+1)\sup_{t_0\le s\le t}(\delta^2\|\underline{A}^s\|_{l^\infty}^2+\|\underline{r}^s\|_{l^\infty}^2)} \ .$$

We now start estimating the remainder $\underline{r}^t$. This will be done in two steps. Namely, during a time of order δ^{-2} we do not loose too much on the estimate of the remainder. After this time has elapsed and up to a time of order $\mathcal{O}(1)\delta^{-2}\log\delta^{-1}$ the estimate improves because the remainder is in a contracting subspace of the linear evolution. When we reach this time (after which we may somewhat loose again on the estimate), we perform a renormalization step and start the whole process again.

Lemma II.9. *There is a constant* $1 > C_4 > 0$ *such that if* $C_4 > \delta^2 > \eta$*, if* $t < C_4\delta^{-2}$ *and* $\theta < \delta^{-1/4}/(2C_2)$ *then*

$$\|\underline{r}^t\|_{l^\infty} < 4C_2C_3\delta \ .$$

Proof. Let the non decreasing sequence of positive numbers σ_t be defined by

$$\sigma_t = \delta^{-1}\sup_{0\le s\le t}\|\underline{r}^s\|_{l^\infty} \ .$$

Note that by definition of C_1 we have $\sigma_0 \le C_1 < 4C_2C_3$. We have from equation (II.5) the identity

$$\underline{r}^{t+1} = (L+L_t)\cdots(L+L_0)\underline{r}^0 + \sum_{j=0}^{t}(L+L_t)\cdots(L+L_{j+1})\underline{\mathcal{R}}^j \ .$$

Using Lemma II.3 and Lemma II.7 we obtain

$$\begin{aligned}
&\|\underline{r}^{t+1}\|_{l^\infty}\\
&\le C_3\left(1+5C_3\delta^2(C_2^2+\sigma_t^2)\right)^{t+1}\|\underline{r}^0\|_{l^\infty} + \sum_{j=0}^{t} C_3\left(1+5C_3\delta^2(C_2^2+\sigma_t^2)\right)^{t-j}\|\underline{\mathcal{R}}^j\|_{l^\infty}\\
&\le C_3(C_1+C_2)\delta e^{(t+1)\delta^2 5C_3(C_2^2+\sigma_t^2)}
\end{aligned}$$

since $t\delta^2 < 1$. The result now follows recursively from the choice of the constant

$$C_4 = \frac{\log 2}{170C_3^3C_2^2} \ .$$

We now improve the previous result using the support properties of the spectrum of $\underline{r}^0$ and of $\underline{\mathcal{R}}^t$. We first derive an equivalent of Corollary II.7.

Lemma II.10. *There is a constant $C_5 > 1$ such that if $\eta < 1/2$, $\delta\theta < 1$, $\delta^2 > \eta$, and t is such that for $0 \le s \le t$ we have $\|\underline{r}^s\| \le 4C_2C_3\delta$, then*

$$\|(L+L_t)\cdots(L+L_0)R_{\delta,\theta}\|_{l^\infty} \le C_5\theta^{-2}e^{C_5 t\delta^2} + C_5 e^{-t\delta^2\theta^2/C_5} \ .$$

Proof. We first observe that

$$(L+L_t)\cdots(L+L_0) = L^{t+1} + \sum_{s=0}^{t}(L+L_t)\cdots(L+L_{s+1})L_sL^s \ ,$$

where as usual, the elements with indices out of range are equal to identity. We now apply Lemmas II.5-II.7 and Corollary II.8 to get

$$\|(L+L_t)\cdots(L+L_0)R_{\delta,\theta}\|_{l^\infty} \le C_4 e^{-tD\delta^2\theta^2/8} + 5(C_2^2+16C_2^2C_3^2)\delta^2$$

$$\sum_{s=0}^{t-1} e^{-sD\delta^2\theta^2/8}C_3^{1+2\eta(t-s)}e^{5C_3(t-s)\delta^2(C_2^2+16C_2^2C_3^2)} \ ,$$

from which the result follows immediately.

Corollary II.11. *There are constants $C_6 > C_5$ and $C_7 > 0$ such that if δ is small enough so that the numbers $\pm 3\omega \pmod{2\pi}$ do not belong to $[\omega - 2\theta\delta, \omega + 2\theta\delta] \cup [-\omega - 2\theta\delta, -\omega + 2\theta\delta]$,then if t is such that for $0 \le s \le t$ we have $\|\underline{r}^s\| \le 4C_2C_3\delta$, and if $C_7\delta^{-2}\log\delta^{-1} > t \ge C_4\delta^{-2}$ we have*

$$\|\underline{r}^t\| \le C_6\delta\theta^{-1/2} \ .$$

The proof is similar to the proof of Lemma II.9 using Lemma II.10, Corollary C1, and the relation

$$\sum_{j=0}^{t}(L+L_t)\cdots(L+L_{j+1})\underline{\mathcal{R}}^j =$$

$$\sum_{j=0}^{t} L^{t-j}\underline{\mathcal{R}}^j + \sum_{j=0}^{t}\sum_{s=j+1}^{t}(L+L_t)\cdots(L+L_{s+1})L_sL^{s-j-1}\underline{\mathcal{R}}^j \ .$$

We can now give the recursive proof of Theorem I.1. We start by selecting an integer n_0 large enough so that all the constraints on $\delta = \delta_n$ and $\theta = \theta_n$ in the previous Lemmas are satisfied for $n > n_0$. We also assume n_0 large enough so that $\log\delta_0^{-1} > (S+1)^2 + 1$. We define $R = \delta_0$. Note that the choice of n_0 is independent of η. We now define η_0 by $\eta_0 = S^{-4}\delta_0^2/8$.

Since the proof is recursive we will therefore assume that we have reached a situation corresponding to scale δ_n $(n < n_0)$. In other words we assume that $\underline{u}^0$ satisfies

$$\|\underline{u}^0\|_{l^\infty} \le \delta_n \ .$$

We will also assume that

$$\|\underline{u}^0\|_{l^\infty} > S^{-1}\delta_n ,$$

otherwise the choice of n is not optimal. Finally since we are for the moment interested in the first part of the Theorem, we will also assume that $\delta_n^2 > \eta$.

We now consider the time evolution starting from this initial data $\underline{u}^0$, and we will apply the previous Lemmas with $\delta = \delta_n$ and $\theta = \theta_n = \delta_n^{-\varsigma}$. We first apply Lemma II.2 to get a function B with the corresponding estimates on the derivatives, and we define the function $A(\tau, x)$ solution of (II.1) with initial data (II.1'). We also define the remainder $\underline{r}^0$.

The time evolution will now be split into two parts. We define a first time

$$t_n = [C_4\delta_n^{-2}] .$$

On the time interval $[0, t_n]$, we have from Lemma II.9

$$\|\underline{r}^t\|_{l^\infty} \leq 4C_2C_3\delta_n .$$

We now define a second time by

$$T_n = [C_8\delta_n^{-2}\log\delta_n^{-1}] ,$$

where $C_7 > C_8 > 0$ and C_8 is small enough so that for $t < C_8\delta^{-2}\log\delta^{-1}$, all the exponentially growing factors which appear in all the previous Lemmas are smaller than $\theta^{1/4}$.

On the time interval $[t_n, T_n]$, we can now apply Lemma II.11. In particular, there is a time $\mathcal{T}_n$ which satisfies

$$t_n \leq \mathcal{T}_n < T_n$$

where we have

$$\|\underline{r}^{\mathcal{T}_n}\|_{l^\infty} \leq S^{-1}\delta_n/2 ,$$

and also using Lemma A.2

$$\|3^{-1/2}A(\delta_n^2\mathcal{T}_n, \cdot)\|_{L^\infty} \leq S^{-1}/2 .$$

Therefore

$$\|\underline{u}^{\mathcal{T}_n}\|_{l^\infty} \leq \delta_{n+1} ,$$

and we can now repeat the argument with $n+1$.

The proof of the second part of Theorem I.1 is essentially similar. We first define an integer N as the smallest integer such that $\delta_N \leq S\eta^{1/2}$. We can now repeat the above proof and the result follows since T_N is of the right order.

III. The General Case.

In this section we will prove the following theorem which is a generalization of Theorem I.1.

Theorem III.1. *Assume hypothesis* **H1-H5**. *Then there exists constants $R>0$, $\eta_0>0$, $c_1>0,\cdots,c_4>0$ such that for any $\eta\in]0,\eta_0[$, for any $\underline{u}\in l^\infty$ such that $\|\underline{u}\|_{l^\infty}<R$, if $(\underline{u}^t)_{t\in\mathbf{N}}$ denotes the orbit of $\underline{u}$ under the discrete time evolution Φ, then*

i) there is a positive number $T=T(\underline{u},\eta)$ such that for any integer $t>T$

$$\|\underline{u}^t\|_{l^\infty}\le c_1\eta^{1/2}\ .$$

ii) Define the constants α, β, σ, γ and V by

$$\alpha=\Im\frac{l''(\omega)}{Dl(\omega)}\ ,\qquad e^{i\sigma}=l(\omega)\ ,\qquad \gamma=\Im\frac{m(\omega)}{\rho l(\omega)}\ ,\qquad V=-i\frac{l'(\omega)}{l(\omega)}\ ,$$

and

$$\beta=-\Im\left(\frac{c(\omega,\omega,-\omega)}{gl(\omega)}+\frac{4q(\omega,0)q(\omega,-\omega)}{3gl(\omega)(|l(\omega)|^2-l(0))}+\frac{2q(-\omega,2\omega)q(\omega,\omega)}{3gl(\omega)(l(\omega)^2-l(2\omega))}\right)\ .$$

For any integer $t>T$ there is a function $A(\tau,x)$ from $\mathbf{R}^+\times\mathbf{R}$ to $\mathbf{C}$ satisfying

$$\partial_\tau A=(1+i\alpha)\partial_x^2A+A-(1+i\beta)A|A|^2 \tag{CGL}$$

such that for any integer s with $0\le s\le c_2\eta^{-1}\log\eta^{-1}$ we have

$$\|\underline{u}^{t+s}-\underline{v}^s\|_{l^\infty}\le c_3\eta^{1/2+c_4}\ .$$

where

$$v_n^s=e^{i(\sigma t+\eta\gamma t+\omega n)}\eta^{1/2}\sqrt{\frac{\rho}{3g}}A\big(\rho\eta t,\eta^{1/2}\sqrt{2\rho/D}(n+Vt)\big)+c.c.\ .$$

We recall that the constants D and ρ have been defined in hypothesis **H3** and **H5** respectively.

Lemma III.2. *The number D is non negative, and the number V is real.*

Proof. Let $\zeta(\theta)=l(\theta)/l(\omega)$, this function is equal to one in $\theta=\omega$ and this is a point where its modulus is maximum. Let ζ_R and ζ_I denote the real and imaginary parts of ζ. Writing that the first derivative of the square of the modulus of ζ in ω is equal to zero and the second derivative is non-positive, one gets immediately (using $\zeta_I(\omega)=0$)

$$\zeta_R'(\omega)=0\ ,\qquad\text{and}\qquad \zeta_R''(\omega)+\big(\zeta_I'(\omega)\big)^2\le 0\ ,$$

and the lemma follows.

From the differentiability of Φ we can write

$$\Phi(\eta,\underline{u})=L'\underline{u}+\eta L''\underline{u}+Q(\underline{u},\underline{u})+C(\underline{u},\underline{u},\underline{u})+G(\eta,u)\ , \tag{III.1}$$

where G is a nonlinear map, differentiable which is at least of degree 4 in $\underline{u}$, at least of degree 2 in η and at least of degree one in $\eta(\underline{u})^2$.

Our first goal is to eliminate the quadratic term Q. As we will see, this is not entirely possible, but it will be enough to eliminate the dominant part of this map. The main idea is of course that Q should not give rise to a resonant term, and therefore it should be possible to eliminate the quadratic terms by a nonlinear change of variables. An implementation of this idea is given by the following result.

Theorem III.3. *There is a (differentiable) quadratic operator S in l^∞ such that if $\mathcal{S}$ denotes the operator*

$$\mathcal{S}(\underline{u}) = \underline{u} + S(\underline{u}) \ ,$$

then on some neighborhood of zero $\mathcal{S}$ is invertible (with differentiable inverse), and the maps

$$\tilde{\Phi}(\eta, \cdot) = \mathcal{S}^{-1} \circ \Phi(\eta, \cdot) \circ \mathcal{S}$$

satisfy the hypothesis **H1-H5** *with a quadratic part $\tilde{Q} = D_2^2\tilde{\Phi}(0,0)/2$ such that its Fourier transform $\tilde{q}(\phi_1, \phi_2)$ is identically zero on a neighborhood of the points $(\pm\omega, \pm\omega)$ and of the lines $\phi_1 + \phi_2 = \pm\omega$. $\tilde{q}$ satisfies also*

$$\sum_{n,m} (1 + |n| + |m|)^6 |\tilde{q}_{n,m}| < \infty \ .$$

Moreover, we have

$$D_2\tilde{\Phi}(0,0) = L' \ , \qquad D_1 D_2 \tilde{\Phi}(0,0) = L'' \ ,$$

and

$$D_2^3\tilde{\Phi}(0,0)(\underline{v}, \underline{v}, \underline{v})/6 = C(\underline{v}, \underline{v}, \underline{v}) + 2Q(\underline{v}, S(\underline{v})) \ ,$$

and the Fourier coefficients of this form decay also polynomially fast.

Proof. By continuity, it follows from **H3** that we can find a number $h < \omega/4$ such that if ϕ_1 and ϕ_2 belong to the intervals $[\pm\omega - 2h, \pm\omega + 2h]$ then $l(\phi_1 + \phi_2) - l(\phi_1)l(\phi_2) \neq 0$. Moreover, we can choose h small enough such that if $|\phi_1 + \phi_2 \pm \omega| < 2h$ then either $|\phi_1 \pm \omega| > 2h$ or $|\phi_2 \pm \omega| > 2h$ and $l(\phi_1 + \phi_2) - l(\phi_1)l(\phi_2) \neq 0$. Let

$$\psi_0(\theta) = \psi((\theta + \omega)h^{-1}/2) + \psi((\theta - \omega)h^{-1}/2) \ ,$$

where ψ is the cut-off function defined at the beginning of section 2. We now define a double sequence $(\Gamma_{n,m})$ by

$$\Gamma_{n,m} = \frac{1}{4\pi^2} \int\int \frac{\psi_0(\phi_1)\psi_0(\phi_2) + \psi_0(\phi_1 + \phi_2)}{l(\phi_1 + \phi_2) - l(\phi_1)l(\phi_2)} e^{-i(\phi_1 n + \phi_2 m)} d\phi_1 d\phi_2 \ .$$

Since the function ψ defined in section 2 is regular with compact support, it follows that the sequence $(\Gamma_{n,m})$ belongs to $l^1(\mathbf{Z}^2)$. Note also that this sequence is real. Moreover, we have

$$\sum_{n,m} |\Gamma_{n,m}| (1 + |n| + |m|)^k < \infty$$

for any integer k.

We now define the double sequence s by

$$s = \Gamma * q \ ,$$

and the quadratic map S by

$$S(\underline{v})_n = \sum_{p,q} s_{n-p,n-q} v_p v_q \ .$$

The proof follows by easy computations.

We will now drop the tilde on the various transformations, i.e. we will assume that the function $q(\phi_1,\phi_2)$ satisfies the conclusions of the previous Theorem. We can recover the general result by applying the above Theorem.

The proof of Theorem III.1 is now very analogous to the proof of Theorem I.1. It goes along the same steps of the renormalizaztion group using estimates analogous to those of Lemmas II.3-II.11. We will only indicate the non-elementary changes in the proofs without keeping a detailed account of the various constants, i.e. the notation $\mathcal{O}(1)$ will be used for any constant independent of δ, η and $\underline{u}$.

We first decompose $\underline{u}^0$ as in section 2, namely we first apply Lemma II.2 and we obtain a function B. As in section II, we set

$$\underline{u}^t = \delta\underline{\mathcal{A}}^t + \underline{r}^t$$

where now

$$\mathcal{A}_n^t = e^{i(\sigma t+\omega n)}(3g)^{-1/2}A(\delta^2 t, \delta\sqrt{2/D}(n+Vt)) \ , \tag{III.2}$$

with A a solution of the complex Ginzburg Landau equation

$$\partial_\tau A = (1+i\alpha)A'' + (\rho+i\gamma)\eta\delta^{-2}A - (1+i\beta)A|A|^2 \ , \tag{CGL}$$

with initial condition

$$A(0,x) = \sqrt{3g}\,\delta^{-1}B(x\sqrt{D/2}) \ . \tag{CGL'}$$

The remainder $\underline{r}^t$ satisfies the equation

$$\underline{r}^{t+1} = (\mathcal{L}_t + L_t)\underline{r}^t + \underline{\mathcal{R}}^t \ , \tag{III.3}$$

where

$$\underline{\mathcal{R}}^t = \delta L'\underline{\mathcal{A}}^t + \eta\delta L''\underline{\mathcal{A}}^t + \delta^2 Q(\underline{\mathcal{A}}^t,\underline{\mathcal{A}}^t) + \delta^3 C(\underline{\mathcal{A}}^t,\underline{\mathcal{A}}^t,\underline{\mathcal{A}}^t) + G(\eta,\delta\underline{\mathcal{A}}^t) - \delta\underline{\mathcal{A}}^{t+1} \ ,$$

and the linear operators L_t and $\mathcal{L}_t$ are defined by

$$\mathcal{L}_t\underline{v} = L'\underline{v} + \eta L''\underline{v} + Q(\underline{v}, 2\delta\underline{\mathcal{A}}^t + \underline{r}^t)$$

and

$$L_t\underline{v} = 3C(\underline{v},\delta\underline{\mathcal{A}}^t,\delta\underline{\mathcal{A}}^t+\underline{r}^t) + C(\underline{v},\underline{r}^t,\underline{r}^t) + \int_0^1 d\xi D_2G(\eta,\delta\underline{\mathcal{A}}^t+\xi\underline{r}^t)\underline{v} \ .$$

We first describe the estimations on the linear operators. Note that Lemma II.7, Corollary II.8 and Lemma II.10 follow as before if we have Lemmas II.4-II.6.

The following equivalent of Lemma II.6 is immediate from the above formulas.

Lemma III.4. *If* $\|\delta\underline{\mathcal{A}}^t\|_{l^\infty} + \|\underline{r}^t\|_{l^\infty} < 1$, *then*

$$\|L_t\|_{l^\infty} \le \mathcal{O}(1)(\delta^2\|\underline{\mathcal{A}}^t\|_{l^\infty}^2 + \|\underline{r}^t\|_{l^\infty}^2) \ .$$

We now come to the equivalent of Lemma II.4 and Lemma II.5.

Lemma III.5. *There are two constants $c>1$ and $c'>0$ such that for any $\eta<\delta^2\theta^2\in\,]0,b/2]$ (b is the constant of Lemma C3) and for any $t\ge t_0\ge 0$ such that*

$$\sup_{t_0\le s\le t}\left(\delta\|\underline{\mathcal{A}}^t\|_{l^\infty}+\|\underline{r}^t\|_{l^\infty}\right)\le c'\delta\ ,$$

we have

$$\|\mathcal{L}_t\cdots\mathcal{L}_{t_0}\|_{l^\infty}\le ce^{c\eta(t-t_0)}$$

and

$$\|\mathcal{L}_t\cdots\mathcal{L}_{t_0}R_{\delta,\theta}\|_{l^\infty}\le ce^{-\delta^2\theta^2(t-t_0)/c}\ .$$

Proof. We first observe that $\mathcal{L}_t=L+\mathcal{L}'_t$ where

$$L\underline{v}=L'\underline{v}+\eta L''\underline{v}\qquad\text{and}\qquad \mathcal{L}'_t=Q(\underline{v},2\delta\underline{\mathcal{A}}^t+\underline{r}^t)\ .$$

With this notation we have

$$\mathcal{L}_t\cdots\mathcal{L}_{t_0}=L^{t-t_0+1}+\sum_{s=t_0+1}^{t}L^{t-s}\mathcal{L}'_{s-1}\mathcal{L}_{s-2}\cdots\mathcal{L}_{t_0}\ .$$

Using Lemma C3 and Lemma II.4 we get

$$\|\mathcal{L}_t\cdots\mathcal{L}_{t_0}\|_{l^\infty}\le$$

$$\mathcal{O}(1)e^{c\eta(t-t_0+1)}+\sum_{s=t_0+1}^{t}\mathcal{O}(1)e^{-b(t-s)}(\delta\|\underline{\mathcal{A}}^{s-1}\|_{l^\infty}+\|\underline{r}^{s-1}\|_{l^\infty})\|\mathcal{L}_{s-2}\cdots\mathcal{L}_{t_0}\|_{l^\infty}\ .$$

The proof of the first part of the Lemma follows now recursively.

To get the second part, we use the following formula

$$\mathcal{L}_t\cdots\mathcal{L}_{t_0}=L^{t-t_0+1}$$

$$+\sum_{k=1}^{t-t_0+1}\ \sum_{\substack{n_1+\cdots+n_{k+1}=t-t_0+1-k\\ n_1\ge0,\ \cdots,\ n_{k+1}\ge0}}L^{n_1}\mathcal{L}'_{t-n_1}L^{n_2}\mathcal{L}'_{t-n_1-n_2}L^{n_3}\cdots L^{n_k}\mathcal{L}'_{t-n_1-\cdots-n_k-k}\,L^{n_{k+1}}\ .$$

The second part of the Lemma follows now easily using Lemma II.5 and Lemma C3.

As explained before, Lemmas II.7, Corollary II.8 and Lemma II.10 follow as before.

We now come to the estimates concerning the forcing term $\underline{\mathcal{R}}^t$. From Lemma C2, the term $\delta^2Q(\underline{\mathcal{A}}^t,\underline{\mathcal{A}}^t)$ is at least of order $\delta^4\theta^2$. For the term $\underline{\mathcal{R}}^t-\delta^2Q(\underline{\mathcal{A}}^t,\underline{\mathcal{A}}^t)$we have the same estimates as in Lemma II.3 and therefore we get bounds analogous to those of Lemmas II.9 and II.11.

The proof of Theorem III.1 then follows exactly the same steps as the proof of Theorem I.1 and we will not repeat them here. Finally one has to apply the inverse of the map $\mathcal{S}$ to conclude the proof.

Appendix A. Some estimates on the solutions of CGL.

In this appendix we will derive estimates on the solution of equation (CGL). General estimates were derived before[2], however the present situation is slightly better in the sense that the initial conditions will never be larger than some predefined constant. We will also restrict ourselves to dimensions one and two where the complex Ginzburg-Landau equation is known to have well behaved solutions for any values of the parameters α and β. We will in fact obtain a stronger result than what we need in sections II and III.

We consider the equation

$$\partial_t A = (1+i\alpha)\Delta A + \sigma(\rho + i\gamma)A - (1+i\beta)A|A|^2 \ , \tag{A.1}$$

where σ is a positive parameter smaller than a given positive constant S, and α, β, γ and ρ are fixed constants. Note that we could have as well rescaled σ and fixed for example $\rho = 1$. We will not do this in order to have an immediate application of the result to section III.

Lemma A.1. *Let $A(t,x)$ be a solution of (A.1). Assume that there is a number $k \geq 2$, a constant $K > 1$ and a constant $\theta > 1$ such that*

$$\sup_{0\le j\le k} \theta^{-j} \|\partial_2^j A(0,\,\cdot\,)\| \le K \ .$$

Then there is a constant M which depends only on K, S, α, β, γ, ρ, such that for any $t > 0$

$$\sup_{0\le j\le k} \theta^{-j} \|\partial_2^j A(t,\,\cdot\,)\|_{L^\infty} \le M \ .$$

Moreover, for any integer $p \le k/2$ we have

$$\sup_{0\le j\le p} \theta^{-2j} \|\partial_1^j A(t,\,\cdot\,)\|_{L^\infty} \le M \ .$$

There is a number $U > 0$ which is independent of θ and A such that for $t > U$ the same estimates hold without the term θ^{-j}. We can take U as small as needed eventually increasing M.

Proof. Let G_t denote the function

$$G_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-x^2/4t} \ .$$

We then consider the integral equation

$$A(t,\,\cdot\,) = e^{i\alpha t} G_t * A(0,\,\cdot\,)$$

$$+ \int_0^t ds G_{t-s} * \big(\sigma(\rho+i\gamma)A(s,\,\cdot\,) - (1+i\beta)|A(s,\,\cdot\,)|^2 A(s,\,\cdot\,)\big) \ . \tag{A.2}$$

It is easy to verify that there is a number $1 > T > 0$ which depends only on k and K, S, α, β, γ, ρ, such that this equation can be solved by the contraction mapping method on the time interval $[0, T]$ with a solution in the ball of radius $2K$ in L^∞. Moreover, the solution is also a solution of (A.1) with initial data $A(0,\,\cdot\,)$. Taking the successive derivatives in x of (A.2) we get

$$\partial^j A(t,\,\cdot\,) = e^{i\alpha t} G_t * \partial^j A(0,\,\cdot\,) + \int_0^t G_{t-s} * \big(\sigma(\rho + i\gamma)\partial^j A(s,\,\cdot\,)$$

$$-2(1 + i\beta)|A(s,\,\cdot\,)|^2 \partial^j A(s,\,\cdot\,) - (1 + i\beta) A(s,\,\cdot\,)^2 \partial^j \overline{A}(s,\,\cdot\,)\big)\, ds + F_j \ ,$$

where F_j is a combination of derivatives of A of lower order. Eventually taking a smaller T (independently of j) we can again solve this equation for $\partial^j A$ by contraction and obtain recursively estimates on the derivatives up to order k with the right dependence in θ. The first part of the Lemma is then proven on the time interval $[0, T]$.

We now observe that at time T the estimate is in fact better. Indeed what we said before is also true for the time interval $[0, T/2]$. We can now write another integral equation, namely for $T/2 < \tau \le T$

$$A(\tau,\,\cdot\,) = e^{i\alpha(\tau - T/2)} G_{(\tau - T/2)} * A(T/2,\,\cdot\,)$$

$$+ \int_{T/2}^{\tau} ds\ G_{\tau - s} * \big(\sigma(\rho + i\gamma) A(s,\,\cdot\,) - (1 + i\beta)|A(s,\,\cdot\,)|^2 A(s,\,\cdot\,)\big) \ ,$$

and for the gradient

$$\nabla A(\tau,\,\cdot\,) = e^{i\alpha(\tau - T/2)} \nabla G_{(\tau - T/2)} * A(T/2,\,\cdot\,) + \int_{T/2}^{\tau} \nabla G_{\tau - s} * \big(\sigma(\rho + i\gamma)\partial^{j-1} A(s,\,\cdot\,)$$

$$-(1 + i\beta)|A(s,\,\cdot\,)|^2 A(s,\,\cdot\,)\big)\, ds \ ,$$

We then deduce from the integrability in $t = 0$ of the L^1 norm (in x) of ∇G_t an estimate on the L^∞ norm of the gradient of A on the time interval $]T/2, T]$ which is independent of θ. Similarly we can control recursively higher order derivatives by writing down integral equations starting at various increasing times between $T/2$ and T.

In other words, even if we started at time zero with a number θ which was quite large, after a time of order at most one we get an estimate where the function and all its derivatives up to order k are of order unity. The result for all times then follows immediately from previous results[2]. Finally the result on the time derivatives follows at once from the result on the space derivatives using equation (A.1).

We now need to prove that when σ is small, if at the initial time A is of order unity, then it will decrease. Note that if $\alpha = \beta = 0$ this follows easily from the maximum principle applied to the square of the modulus of A. In the general case, we will use a method of local energy estimate[3,2].

Lemma A.2. *Given the numbers α, β, ρ, S, and K as before, there is a number $\Theta > 0$ and a number $\sigma_0 > 0$ such that if $\sigma \in [0, \sigma_0]$ and $t > \Theta$, we have*

$$\|A(t, \cdot)\|_{L^\infty} \leq S^{-1}/2 \ .$$

Note that we could get a bound of order $\sqrt{\sigma}$ but after some larger time which depends on σ. One can also derive similar estimates for the derivatives.

Proof. The proof is essentially based on a local energy estimate[(2)]. First of all we can get rid of the term $i\gamma\sigma A$ by considering the function $e^{-i\gamma\sigma t}A(t, x)$. We introduce then a cut-off function depending on a positive parameter ϵ

$$\varphi(x) = \frac{\epsilon^d}{(1 + \epsilon^2|x|^2)^d}$$

where $d = 1$ or 2 is the dimension. One then introduces the function of time

$$\Psi(t) = \int \varphi(x)|A(t, x)|^2 \ dx \ .$$

It is easy to verify using integration by parts that since A is bounded, this function satisfies

$$\frac{d}{dt}\Psi(t) \leq -2 \int \varphi(x)|\nabla A(t, x)|^2 + 2\sigma\rho \int \varphi(x)|A(t, x)|^2$$

$$+2(1 + |\alpha|) \int |\nabla\varphi(x)||\nabla A(t, x)||A(t, x| - 2 \int \varphi(x)|A(t, x)|^4 \ .$$

Completing the squares and using that $|\nabla\phi|/\phi \leq \mathcal{O}(1)\epsilon$ we get for some number $a > 1$ independent of ϵ

$$\frac{d}{dt}\Psi(t) \leq +2(\sigma\rho + a\epsilon^2) \int \varphi(x)|A(t, x)|^2 - 2 \int \varphi(x)|A(t, x)|^4 \ .$$

Using the last term on the right hand side to control the first one, we finally get

$$\frac{d}{dt}\Psi(t) \leq -(\sigma\rho + a\epsilon^2) \int \varphi(x)|A(t, x)|^2 + \mathcal{O}(1)(\sigma\rho + a\epsilon^2)^2 \ .$$

This implies since $\Psi(0) = \mathcal{O}(1)$ that beyond a time of order $(\sigma\rho + a\epsilon^2)^{-1}$ we have

$$\Psi(t) \leq \mathcal{O}(1)(\sigma\rho + a\epsilon^2) \ .$$

Let now t be larger than the number U of the previous Lemma. Then we have

$$\|\nabla A(t, \cdot)\|_{L^\infty} \leq \mathcal{O}(1) \ .$$

From the previous estimate we also know that the average of $|A|^2$ on any square (segment) of size ϵ^{-1} is bounded by $\mathcal{O}(1)(\sigma\rho + a\epsilon^2)$. Given S, this implies that in dimension one we can choose σ_0 and ϵ small enough such that the estimate holds. In dimension 2, one needs a similar energy estimate for the gradient to apply an adequate Sobolev inequality. We refer to the literature for analogous estimates.

Appendix B: Proof of Lemmas II.4 and II.5.

In this appendix we will always assume that the parameter η is positive and smaller than one. We will first give an integral representation of the kernels of the operators L^t and $L^t R_{\delta,\theta}$. These operators are convolution operators with a sequence and we will estimate the l^1 norm of this sequence. The operator L^t is the convolution with the sequence $\underline{\mathcal{P}}^t$ given by

$$\mathcal{P}_n^t = (2\pi)^{-1} \int_0^{2\pi} e^{-in\varphi}(\eta + l(\varphi))^t d\varphi \tag{B1}$$

and the operator $L^t R_{\delta,\theta}$ is the convolution with the sequence $\underline{\mathcal{M}}^t$

$$\mathcal{M}_n^t = (2\pi)^{-1} \int_0^{2\pi} e^{-in\varphi}(\eta + l(\varphi))^t \left(1 - \psi\left(\frac{\omega - \varphi}{\delta\theta}\right) - \psi\left(\frac{\omega + \varphi}{\delta\theta}\right)\right) d\varphi \tag{B2}$$

since $2\pi/\delta\theta$ is larger than twice the diameter of the support of ψ.

First of all, since l is C^2, we can find a positive number $a < \min(\omega, \pi - \omega)$ such that on the intervals $[\pm\omega - a, \pm\omega + a]$ we have $|l''(\varphi) + D| < D/2$. Moreover, there is a positive constant $B < 1$ such that

$$\sup_{\varphi \notin]\pm\omega - a, \pm\omega + a[} |l(\varphi)| \le B \ .$$

In the integral (B1) we separate the domain of integration in the union of the two intervals $[\pm\omega - a, \pm\omega + a]$ and their complements. Using integration by parts, one checks easily that the contribution of the complement to the integral is bounded in l^1 norm by $\mathcal{O}(1)(1+t)^2(\eta + B)^t$. For the contribution of each interval we obtain by standard arguments[9]

$$\left|(2\pi)^{-1} \int_{\pm\omega - a}^{\pm\omega + a} e^{-in\varphi}(\eta + l(\varphi))^t d\varphi\right| \le \mathcal{O}(1) e^{t\eta} t^{-1/2} \ .$$

On the other hand, integrating twice by parts we get for $t \ge 2$ and $n \ne 0$

$$n^2 \mathcal{P}_n^t = -(2\pi)^{-1} \int_0^{2\pi} e^{-in\varphi} \left(t l''(\varphi)(\eta + l(\varphi)) + t(t-1) l'(\varphi)^2\right) (\eta + l(\varphi))^{t-2} d\varphi$$

and by similar estimates as before, for $n \ne 0$ and $t \ge 2$

$$|\mathcal{P}_n^t| \le \mathcal{O}(1) n^{-2} t^{1/2} e^{\eta t} \ .$$

Combining the two estimates we get

$$|\mathcal{P}_n^t| \le \mathcal{O}(1)(1 + n^2/t)^{-1} t^{-1/2} e^{\eta t} \ .$$

which implies immediately

$$\|\mathcal{P}^t\|_{l^1} \le \mathcal{O}(1) e^{\eta t} \ ,$$

and this proves Lemma II.4.

We now come to the proof of Lemma II.5. First of all, if $t < (\delta\theta)^{-2}$, the estimate follows at once from Lemma II.4 and Lemma II.1 (as long as $\delta^2\theta^2 > \eta$ which is always assumed). From now on we will assume $t \ge (\delta\theta)^{-2}$.

It follows by estimates similar to the previous ones that

$$|\mathcal{M}_n^t| \le \mathcal{O}(1)(1 + n^2/t)^{-1} t^{1/2} e^{-t(D\delta^2\theta^2/2 - \eta)} \ .$$

from which Lemma II.5 follows.

Appendix C. Estimation of oscillating sums.

Lemma C.1. *There is a constant $C_9 > 0$, a constant $a > 0$ and a constant $1 > \eta_0 > 0$ such that if $\eta \in [0, \eta_0]$, if δ, θ, B and A satisfy the hypothesis of Lemma II.3, if the sequence $\underline{f}^t$ is defined by*

$$f_n^t = A^3(\delta^2 t, n\delta\sqrt{2/D}) ,$$

then for any integers s and t we have

$$\|L^s \underline{e^{(3i\,\cdot\,)} f^t}\|_{l^\infty} \leq C_9 \left(\delta\theta e^{\eta s} + e^{-as}\right) .$$

Proof. From hypothesis **H3** it follows that there is a number $\Upsilon > 0$ such that $|4\omega \pmod{2\pi}| > \Upsilon$ and $|2\omega \pmod{2\pi}| > \Upsilon$. We can now write for the kernel of the operator L^s

$$L^s(n,m) = \frac{1}{2\pi}\int e^{i\phi(n-m)}(\eta + l(\phi))^s d\phi = \frac{1}{2\pi}\int_{|\phi\pm\omega|>\Upsilon} e^{i\phi(n-m)}(\eta + l(\phi))^s d\phi$$

$$+\frac{1}{2\pi}\int_{|\phi-\omega|\leq\Upsilon} e^{i\phi(n-m)}(\eta + l(\phi))^s d\phi + \frac{1}{2\pi}\int_{|\phi+\omega|<\Upsilon} e^{i\phi(n-m)}(\eta + l(\phi))^s d\phi .$$

Using techniques analogous to those of the previous appendix, it is easy to show that the first integral is the kernel of an operator with norm $\mathcal{O}(1)e^{-as}$ in l^∞ for some constant $a > 0$ which can be chosen independent of η if $\eta_0 > 0$ is small enough. The second and third integrals are estimated by similar methods of "summation by parts" and we will only treat the second one.

We observe that

$$\sum_m f_m^t \frac{1}{2\pi}\int_{|\phi-\omega|\leq\Upsilon} e^{i\phi(n-m)+3i\omega m}(\eta + l(\phi))^s d\phi$$

$$= \sum_m (f_m^t - f_{m+1}^t)\frac{1}{2\pi}\int_{|\phi-\omega|\leq\Upsilon} \frac{e^{i\phi(n-m)+3i\omega m}(\eta + l(\phi))^s}{1 - e^{i(\phi-3\omega)}} d\phi .$$

From the properties of the function A (see Lemma A.1) we have

$$|f_m^t - f_{m+1}^t| \leq \mathcal{O}(1)\delta\theta .$$

Using again stationary phase methods, it follows as in the previous section that

$$\left|\frac{1}{2\pi}\int_{|\phi-\omega|\leq\Upsilon} \frac{e^{i\phi(n-m)+3i\omega m}(\eta + l(\phi))^s}{1 - e^{-i(\phi-3\omega)}} d\phi\right| \leq \frac{\mathcal{O}(1)s^{-1/2}e^{\eta s}}{1 + (n-m)^2/s} ,$$

and the Lemma follows.

Lemma C.2. *Let $(q_{n,m})$ be such that*

$$\sum_{n,m}(1+|n|+|m|)^{k+2}|q_{n,m}|<\infty\ ,$$

and $q(\phi_1,\phi_2)=0$ on a neighborhood of the four points $(\pm\omega,\pm\omega)$. Then there is a number $\delta_0>0$ such that if $\underline{\mathcal{A}}$ is constructed as in III.2 with an $A(x)$ such that there is a constant M and a number $k>0$ such that

$$\sup_{0\le j\le k}\ \theta^j\,\|\partial^j A\|\le M\ ,$$

then

$$\sup_r\left|\sum_{n,m}q_{r-n,r-m}\mathcal{A}_n\mathcal{A}_m\right|\le\mathcal{O}(1)\delta^k\theta^k$$

Proof. The proof again essentially mimics the proof of the stationary phase estimates, namely it relies on an integration by parts. Let $F(x)=(3g)^{-1/2}A(\delta^2 t,\sqrt{2/D}(x+\delta Vt))$, we have to consider four terms which are essentially of the same form. We will only treat one of them in details, the others are estimated similarly. We have

$$\sum_{n,m}q_{r-n,r-m}e^{i(\omega(n+m))}F(\delta n)F(\delta m)=$$

$$e^{2i\omega r}\sum_{n,m}F(\delta n)F(\delta m)\frac{1}{4\pi^2}\int q(\phi_1,\phi_2)e^{i((\phi_1-\omega)(r-n)+(\phi_2-\omega)(r-m))}d\phi_1,d\phi_2\ .$$

Let ϖ be a positive number such that $q(\phi_1,\phi_2)=0$ on the four sets $[\epsilon_1\omega-\varpi,\epsilon_1\omega+\varpi]\times[\epsilon_2\omega-\varpi,\epsilon_2\omega+\varpi]$ where $\epsilon_1=\pm1$ and $\epsilon_2=\pm1$. We now decompose the integration domain into a finite union of domains, in each of which ϕ_1 or ϕ_2 (but may-be not both) is at a distance at least ϖ of $\pm\omega$. Let us call $\mathcal{D}$ such a domain, and assume for definiteness that on $\mathcal{D}$ we have

$$\inf_{(\phi_1,\phi_2)\in\mathcal{D}}|\phi_1\pm\omega|>\varpi\ .$$

It is then easy to verify that

$$\sum_{n,m}F(\delta n)F(\delta m)\frac{1}{4\pi^2}\int_{\mathcal{D}}q(\phi_1,\phi_2)e^{i((\phi_1-\omega)(r-n)+(\phi_2-\omega)(r-m))}d\phi_1 d\phi_2=$$

$$\sum_{n,m}F(\delta(r-n))F(\delta(r-m))\frac{1}{4\pi^2}\int_{\mathcal{D}}q(\phi_1,\phi_2)e^{i((\phi_1-\omega)n+(\phi_2-\omega)m)}d\phi_1 d\phi_2=$$

$$\sum_{n,m}(F(\delta(r-n))-F(\delta(r-n+1))F(\delta m)\frac{1}{4\pi^2}\int_{\mathcal{D}}\frac{q(\phi_1,\phi_2)}{1-e^{i(\phi_1-\omega)}}e^{i((\phi_1-\omega)n+(\phi_2-\omega)m)}d\phi_1,d\phi_2\ .$$

From our hypothesis, we have

$$|F(\delta(r-n))-F(\delta(r-n+1))|\le\mathcal{O}(1)\delta\theta\ .$$

The result follows by several applications of the argument, and controlling the convergent of the ensuing sum by the summability properties of $(q_{n,m})$.

Lemma C.3. *There is a number $b>0$ such that if $0\le\eta<1$, if the sequence $(q_{n,m})$ is such that*

$$\sum_{n,m}(1+|n|+|m|)^2|q_{n,m}|<\infty\ ,$$

and $q(\phi_1,\phi_2)=0$ on a neighborhood of $\phi_1+\phi_2=\pm\omega$, then for any $t\ge 0$ we have

$$\|\mathcal{L}^t\underline{q}(\underline{v}\,\underline{w})\|_{l^\infty}\le\mathcal{O}(1)e^{-bt}\|\underline{v}\|_{l^\infty}\|\underline{w}\|_{l^\infty}\ .$$

Proof. We have easily the formula

$$\big(\mathcal{L}^t\underline{q}(\underline{v}\,\underline{w})\big)_r=\sum_{n,p}v_nw_p\frac{1}{4\pi^2}\int\big(l(\phi_1+\phi_2)+\eta m(\phi_1+\phi_2)\big)^tq(\phi_1,\phi_2)e^{i(\phi_1(r-n)+\phi_2(r-p))}d\phi_1d\phi_2\ .$$

The result follows at once from the fact that on the support of q we have $\phi_1+\phi_2\ne\pm\omega$. The necessary summability results are obtained by integration by parts as in the previous Lemma.

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