Feynman integrals for a class of exponentially growing potentials*

Tobias Kuna
Universität Bielefeld, D 33615 Bielefeld, Germany

Ludwig Streit
Universität Bielefeld, D 33615 Bielefeld, Germany
CCM, Universidade da Madeira, P 9000 Funchal, Portugal

Werner Westerkamp
Universität Bielefeld, D 33615 Bielefeld, Germany

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Abstract

We construct the Feynman integrands for a class of exponentially growing time-dependent potentials as white noise functionals. We show that they solve the Schrödinger equation. The Morse potential is considered as a special case.

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1 Introduction

As an alternative approach to quantum mechanics Feynman introduced the concept of path integrals, see [FeHi65], which developed into an extremely useful tool in many branches of theoretical physics.

Unfortunately Feynman's intuitive idea of averaging over some set of paths is mathematically meaningful only for the heat equation where the underlying structure for the free motion is based on the Brownian paths with Wiener measure. This is stated by the famous Feynman-Kac formula. To write down solutions of the Schrödinger equation as path integrals is much more involved and often less direct. A measure does not exist for the Feynman integral as in the Euclidean case, see [Ca60]. We do not give a full list of references, but we would like to refer to [AlHK76], their method using infinite dimensional Fresnel integrals and their extensive list of references to further approaches to Feynman integrals. Further we want to mention one of the most widely used methods, analytic continuation. The calculation itself takes place on the Euclidean side, where we have the whole machinery of probabilistic theory at our disposal. The final step is to perform an analytic continuation in some parameter such as time or mass to get solutions of the Schrödinger equation. In Subsection 4.4 we will prove that continuation in the mass is incompatible with perturbation theory in the case we are considering. Additionally we want to mention the method of Doss, e.g. [Do80], using the Feynman-Kac formula and complex scaling. The method of Doss can also be used in the framework of white noise analysis, see [HKPS93] and [W95].

White noise analysis [HKPS93], [Ob94] and [Kuo96], is a framework which offers various generalizations of concepts known from finite dimensional analysis to the infinite dimensional case, among them are differential operators and Fourier transform.

The underlying random variable is not Brownian motion but rather its velocity, white noise. Being independent at each time, white noise provides a suitable infinite dimensional coordinate system. The "integral" is understood as the dual pairing of a distribution with a test function, so that the Feynman integrand itself has meaning as a distribution. This allows us to calculate not only the propagator but, more generally, time ordered expectation values. Important for the usefulness of any approach to Feynman integrals is the class of potentials we are able to handle. In white noise analysis Feynman integrals have been constructed for different classes of potentials. The first were proposed by [FPS91] and by Khandekar and Streit [KS92]. This latter construction was generalized in [LLSW93] and [GKSS96] to a wider class, allowing also time-dependent interactions. Potentials there were given as superpositions of δ-functions. Unfortunately this is restricted to one space dimension. In [W95] another set was considered, the so called Albeverio-Høegh-Krohn class [AlHK76] of potentials that are Fourier transforms of measures. Here the space dimension is arbitrary, on the other hand the potentials are smooth and bounded. We shall instead consider Laplace transforms of measures, again for arbitrary finite space dimension. In Section 4 we construct the Feynman integrand as a white noise distribution and show that the corresponding propagator solves the Schrödinger equation. The potentials are smooth but they grow in general exponentially at $\pm\infty$. They are too singular to be handled by Kato-Rellich perturbation theory. Nevertheless we show that the propagator is analytic in the coupling constant and we write it as a Dyson series. In Section 5 we consider the special case of Morse potentials $V(x) = g(e^{2ax} - be^{ax})$ for illustration and for more explicit calculations. This problem is solvable in closed form [Kl90], [PaSo84], [CaInWi83] and [FiLeMu92]. There the authors derive the Green function, the spectrum and the eigenfunctions. These quantities are in general not analytic in g. If we change from positive to negative g we also lose the essential self-adjointness of the corresponding Hamilton operator. This dramatic change however does not destroy the analyticity of the propagator.

2 White noise analysis

In this section we give a brief overview of concepts and theorems of white noise analysis which we use, see [HKPS93], [Kuo96] and [Ob94].

The starting point of d-dimensional white noise analysis is the real separable Hilbert space

$$L_d^2 := L^2(\mathbb{R}) \otimes \mathbb{R}^d, \quad d \in \mathbb{N}, \tag{1}$$

which is unitary isomorphic to a direct sum of d identical copies of $L^2(\mathbb{R})$ the space of real valued square-integrable functions with respect to Lebesgue measure. The norm in L_d^2 is given by

$$|f|_0 := \sum_{j=1}^{d} \int_{\mathbb{R}} f_j^2(s)\, \mathrm{d}s, \quad f \in L_d^2 \tag{2}$$

In this space we choose the densely imbedded nuclear space

$$S_d := S(\mathbb{R}) \otimes \mathbb{R}^d. \tag{3}$$

A typical element $\xi \in S_d$ is a d-dimensional vector where each component is a Schwartz test function. By $|\cdot|_p$ we denote a family of Hilbert norms topologizing S_d. Together with the dual space

$$S'_d := S'(\mathbb{R}) \otimes \mathbb{R}^d \tag{4}$$

we obtain the basic nuclear triple

$$S_d \subset L^2_d \subset S'_d. \tag{5}$$

Let $\mathcal{B}$ be the σ-algebra generated by cylinder sets on S'_d. Using Minlos' theorem we construct a measure space on $(S'_d, \mathcal{B})$ by fixing the characteristic function.

$$\int_{S'_d} \exp\left(i\left\langle \omega, \xi\right\rangle\right) \mathrm{d}\mu\left(\omega\right) = \exp\left(-\frac{1}{2}\left|\xi\right|_0^2\right), \quad \xi \in S_d, \tag{6}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between S'_d and S_d. The space $(S'_d, \mathcal{B}, \mu)$ is called the vector valued white noise space. Within this formalism a version of d-dimensional Wiener's Brownian motion is given by

$$B(\tau) := \left\langle \omega, 1\!\!1_{[0,\tau)} \right\rangle := \left(\left\langle \omega, 1\!\!1_{[0,\tau)} \otimes e_1 \right\rangle, \ldots, \left\langle \omega, 1\!\!1_{[0,\tau)} \otimes e_d \right\rangle\right), \quad \omega \in S'_d, \tag{7}$$

where $\{e_1, \ldots, e_d\}$ denotes the canonical basis of $\mathbb{R}^d$. We shall construct a Gel'fand triple with smooth and generalized functions of white noise around the complex Hilbert space

$$L^2(\mu) := L^2(S'_d, \mathcal{B}, \mu) \tag{8}$$

and we denote the scalar product in this space by

$$((f, g)) = \int_{S'_d} \bar{f}(\omega)\, g(\omega)\, \mathrm{d}\mu(\omega), \qquad f, g \in L^2(\mu). \tag{9}$$

We proceed by choosing first a special subspace $(S_d)^1$ of test functionals. Then we construct the Gel'fand triple

$$(S_d)^1 \subset L^2(\mu) \subset (S_d)^{-1}. \tag{10}$$

Elements of the space $(S_d)^{-1}$ are called Kondratiev distributions, the well known Hida distributions form a subspace. Instead of reproducing the explicit construction here, see e.g. [Ko78], [KLS96], we shall characterize the distributions by their T-transforms in Theorem 2 below. Let $\Phi \in (S_d)^{-1}$ then there exist $p, q \in \mathbb{N}_0$ such that we can define for every

$$\xi \in U_{p,q} := \left\{ \xi \in S_d \,\middle|\, |\xi|_p^2 < 2^{-q} \right\} \tag{11}$$

the T-transform by

$$T\Phi(\xi) := \langle\!\langle \Phi, \exp(i \langle \cdot, \xi \rangle) \rangle\!\rangle \tag{12}$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the bilinear extension of the scalar product of $L^2(\mu)$. The definition of the T-transform can be extended via analytic continuation to the complexification of S_d which we denote by $S_{d,\mathbb{C}}$. Further we need the definition of holomorphy in a nuclear space, see also [Ba85].

Definition 1 *A function $F : U \to \mathbb{C}$ is holomorphic on an open set $U \subseteq S_{d,\mathbb{C}}$ iff for all $\theta_0 \in U$*

1. *for any $\theta \in S_{d,\mathbb{C}}$ the mapping $\lambda \mapsto F(\theta_0 + \lambda\theta)$ is holomorphic in some neighborhood of 0 in $\mathbb{C}$,*
2. *there exists an open neighborhood U' of θ_0 such that F is bounded on U'.*

F is holomorphic at 0 iff F is holomorphic in a neighborhood of 0.

Now we can give the above mentioned characterization theorem, which is in the case of Hida distributions due to [PS91], [KLPSW96], and for Kondratiev distributions to [KLS96].

Theorem 2 *Let $U \subseteq S_{d,\mathbb{C}}$ be open and $F : U \to \mathbb{C}$ be holomorphic at zero, then there exists a unique $\Phi \in (S_d)^{-1}$ such that $T\Phi = F$. Conversely, let $\Phi \in (S_d)^{-1}$ then $T\Phi$ is holomorphic at zero. The correspondence between F and Φ is a bijection if we identify holomorphic functions which coincide on an open neighborhood of zero.*

As a consequence of the characterization we have also a criterion for sequences and integrals with respect to an additional parameter.

Theorem 3 *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(S_d)^{-1}$, such that there exists $U_{p,q}$, $p,q \in \mathbb{N}_0$, so that*

1. *all $T\Phi_n$ are holomorphic on $U_{p,q}$;*
2. *there exists $C > 0$ such that $|T\Phi_n(\theta)| \le C$ for all $\theta \in U_{p,q}$ and all $n \in \mathbb{N}$,*
3. *$(T\Phi_n(\theta))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for all $\theta \in U_{p,q}$.*

Then $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly in $(S_d)^{-1}$.

Theorem 4 *Let $(\Lambda, \mathcal{A}, \nu)$ be a measure space and $\lambda \mapsto \Phi_\lambda$ a mapping from Λ to $(S_d)^{-1}$. We assume that there exists $U_{p,q}$, $p,q \in \mathbb{N}_0$, such that*

1. *$T\Phi_\lambda$, is holomorphic on $U_{p,q}$ for every $\lambda \in \Lambda$;*
2. *the mapping $\lambda \mapsto T\Phi_\lambda(\theta)$ is measurable for every $\theta \in U_{p,q}$.*
3. *there exists $C \in L^1(\Lambda, \nu)$ such that*
$$|T\Phi_\lambda(\theta)| \le C(\lambda) \tag{13}$$
for all $\theta \in U_{p,q}$ and for ν-almost all $\lambda \in \Lambda$.

Then there exist p', $q' \in \mathbb{N}_0$, which only depend on p, q, such that Φ_λ is Bochner integrable.
In particular,
$$\int_\Lambda \Phi_\lambda \mathrm{d}\nu(\lambda) \in (S_d)^{-1} \tag{14}$$
and $T\left[\int_\Lambda \Phi_\lambda \mathrm{d}\nu(\lambda)\right]$ is holomorphic on $U_{p',q'}$. We may interchange dual pairing and integration
$$\left\langle\!\!\left\langle \int_\Lambda \Phi_\lambda \mathrm{d}\nu(\lambda), \varphi \right\rangle\!\!\right\rangle = \int_\Lambda \left\langle\!\!\left\langle \Phi_\lambda, \varphi \right\rangle\!\!\right\rangle \mathrm{d}\nu(\lambda), \quad \varphi \in (S_d)^1. \tag{15}$$

At the end of this section we want to give some examples of distributions, see e.g. [HKPS93].

Example 5 To define the kinetic energy factor in the path integrals one would like to give a meaning to the formal expression

$$\exp\left(c\int_{\mathbb{R}}\omega\left(t\right)^{2}\mathrm{d}t\right),$$

where c is a complex constant. We define the normalized exponential

$$\mathrm{Nexp}\left(c\int_{\mathbb{R}}\omega\left(t\right)^{2}\mathrm{d}t\right)$$

as a distribution via the following T-transform

$$T\,\mathrm{Nexp}\left(c\int_{\mathbb{R}}\omega\left(t\right)^{2}\mathrm{d}t\right)(\xi)=\exp\left(\frac{1}{4c-2}\int_{\mathbb{R}}\xi^{2}\left(\tau\right)\mathrm{d}\tau\right),\quad c\neq\frac{1}{2}.$$

Example 6 Donsker's delta function. In order to 'pin' Brownian motion at a point $a\in\mathbb{R}^{d}$ we want to consider the formal composition of the Dirac delta distribution with Brownian motion: $\delta\left(B\left(t\right)-a\right).$ This can be given a precise meaning as a Hida distribution, see e.g. [Kuo92]. Its T-transform is given by

$$T\left[\delta\left(B\left(t\right)-a\right)\right](\xi)=\frac{1}{\left(2\pi t\right)^{\frac{d}{2}}}\exp\left(-\frac{1}{2t}\left(\int_{0}^{t}i\xi\left(s\right)ds+x_{0}-a\right)^{2}-\frac{1}{2}\left|\xi\right|_{0}^{2}\right)$$

3 The free Feynman integrand

We follow [FPS91] and [HS83] in viewing the Feynman integral as a weighted average over Brownian paths. We use a slight change in the definition of the paths, which are here modeled by

$$x\left(\tau\right)=x-\sqrt{\frac{\hbar}{m}}\int_{\tau}^{t}\omega\left(\sigma\right)\mathrm{d}\sigma:=x-\sqrt{\frac{\hbar}{m}}\langle\omega,1\!\!1_{(\tau,t]}\rangle\tag{16}$$

In the sequel we set $\hbar=m=1$ unless otherwise stated. Correspondingly we define the Feynman integrand for the free motion by

$$I_{0}\left(x,t\mid x_{0},t_{0}\right)=\mathrm{Nexp}\left(\frac{i+1}{2}\int_{\mathbb{R}}\omega^{2}\left(\tau\right)\mathrm{d}\tau\right)\delta\left(x\left(t_{0}\right)-x_{0}\right)\tag{17}$$

see e.g. [LLSW93]. We recall that the delta distribution $\delta\left(x\left(t_0\right)-x_0\right)$ is used to fix the starting point and plays the role of an initial distribution. In the sequel instead of $I_0\left(x, t \mid x_0, t_0\right)$ we will often use the shorthand I_0. Thus we get for the T-transform

$$\begin{aligned} TI_0\left(\xi\right) &= \frac{1}{\left(2\pi i\left|t-t_0\right|\right)^{\frac{d}{2}}}\exp\left[-\frac{i}{2}\int_{\mathbb{R}}\xi^2\left(\tau\right)\mathrm{d}\tau\right. \\ &\quad \left.-\frac{1}{2i\left|t-t_0\right|}\left(\int_{t_0}^{t}\xi\left(\tau\right)\,\mathrm{d}\tau+x-x_0\right)^2\right], \end{aligned} \tag{18}$$

Not only the expectation but also the T-transform has a physical meaning. Integrating formally by parts we find

$$\begin{aligned} TI_0\left(\xi\right) &= \int_{\mathcal{S}_d'} I_0\left(\omega\right)\exp\left(-i\int_{t_0}^{t}x\left(\tau\right)\cdot\dot{\xi}\left(\tau\right)\mathrm{d}\tau\right)\mathrm{d}\mu\left(\omega\right) \\ &\quad \cdot\exp\left(-\frac{i}{2}\int_{\left[t_0,t\right]^c}\xi^2\left(\tau\right)\mathrm{d}\tau\right)\exp\left[ix\cdot\xi\left(t\right)-ix_0\cdot\xi\left(t_0\right)\right]. \end{aligned} \tag{19}$$

The multiplication denoted by dot is just the scalar product in $\mathbb{R}^d$. Indeed it is straightforward to verify that

$$\begin{aligned} K_0^{(\xi)}\left(x,t|x_0,t_0\right) &= TI_0\left(\xi\right) \\ &\quad \cdot\exp\left(\frac{i}{2}\int_{\left[t_0,t\right]^c}\xi^2\left(\tau\right)\mathrm{d}\tau\right)\exp\left[ix_0\cdot\xi\left(t_0\right)-ix\cdot\xi\left(t\right)\right] \end{aligned} \tag{20}$$

is the Green function corresponding to the potential $W=\dot{\xi}\left(t\right)\cdot x$, i.e., it obeys the Schrödinger equation

$$\left(i\partial_t+\frac{1}{2}\triangle_d-\dot{\xi}\left(t\right)\cdot x\right)K_0^{(\xi)}\left(x,t|x_0,t_0\right)=0 \tag{21}$$

with the initial condition

$$\lim_{t\searrow t_0}K_0^{(\xi)}\left(x,t|x_0,t_0\right)=\delta\left(x-x_0\right).$$

4 The Feynman integrand for a new class of unbounded potentials

Now we construct the Feynman integrand for a new class of potentials and calculate the propagators. In Subsection 4.3 we show that the propagators solve the corresponding Schrödinger equation and in Subsection 4.5 we generalize to time-dependent potentials.

4.1 The interactions

Definition 7 *Let m be a complex measure on the Borel sets on $\mathbb{R}^d$, $d \geq 1$ fulfilling the following condition*

$$\int_{\mathbb{R}^d} e^{C|\alpha|} \, \mathrm{d}|m|(\alpha) < \infty, \quad \forall C > 0. \tag{22}$$

We define a potential V on $\mathbb{R}^d$ by

$$V(x) = \int_{\mathbb{R}^d} e^{\alpha \cdot x} \mathrm{d}m(\alpha) \,. \tag{23}$$

Remark 8 A consequence of the above condition (22) is that the measure m is finite. By Lebesgue's dominated convergence theorem we obtain that the potentials are restrictions to the real line of entire functions. In particular they are locally bounded and without singularities. However they are in general unbounded at $\pm\infty$.

Remark 9 The time-dependent case will be considered in Subsection 4.5.

Example 10 Every finite measure with compact support fulfills the above condition (22).

Example 11 The simplest example is the Dirac measure in one dimension $m(\alpha) := g\, \delta_a(\alpha)$ for $a > 0$ and $g \in \mathbb{R}$. The associated potential is $V(x) = g\, e^{ax}$. Obviously all polynomials of exponential functions of the above kind are also in our class, too, e.g. $\sinh(ax)$, $\cosh(ax)$.

Example 12 In particular the well known Morse potential $V(x) := g(e^{-2ax} - 2\gamma e^{-ax})$ with $g, a, x \in \mathbb{R}$ and $\gamma > 0$ is included in our class. We will discuss this potential in Section 5 in more detail.

Example 13 If we choose a Gaussian density, we get potentials of the form $V(x) = g e^{bx^2}$ with $b, x \in \mathbb{R}$.

Example 14 Further entire functions of arbitrary high order of growth are inside of our class. More explicitly, the measures $m(\alpha) := \Theta(\alpha) \exp\left(-k\alpha^{1+b}\right)$ with b, $k > 0$ and $x \in \mathbb{R}$ fulfill the condition (22). The corresponding potentials are entire functions of order $1 + \frac{1}{b}$, see [Lu70] Lemma 7.2.1.

4.2 The Feynman integrand as a generalized white noise functional

In order to handle potentials of the form given above

$$V(x) = \int_{\mathbb{R}^d} e^{\alpha \cdot x} \mathrm{d}m(\alpha)$$

within our approach we must give a meaning to the following pointwise multiplication

$$I = I_0 \cdot \exp\left(-i \int_{t_0}^{t} V(x(\tau))\, \mathrm{d}\tau\right), \tag{24}$$

where

$$x(\tau) = x - \int_{\tau}^{t} \omega(s)\ \mathrm{d}s$$

is a path, as in Section 3. For convenience we assume $t_0 < t$ in the sequel. As a first step we formally expand the exponential

$$I = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(\tau_j)} \prod_{j=1}^{n} \mathrm{d}m(\alpha_j)\, \mathrm{d}^n\tau \tag{25}$$

into a perturbation series. In Theorem 18 we will show the existence of the integrals and the series . But first we have to give a definition for the product

$$I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(\tau_j)} \tag{26}$$

Products of this type have already been considered, see e.g. Example 6. In view of the Characterization Theorem 2 it is enough to define the product via its T-transform. Arguing formally we obtain

$$
\begin{aligned}
& T\left(I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(\tau_j)}\right)(\xi) && (27)\\
&= \int_{\mathcal{S}_d'} I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(\tau_j)} \exp\left(i \langle \omega, \xi \rangle\right) \mathrm{d}\mu(\omega) \\
&= TI_0\left(\xi + i \sum_{j=1}^{n} \alpha_j \mathbb{1}_{[\tau_j, t)}\right) \exp\left(\sum_{j=1}^{n} \alpha_j \cdot x\right).
\end{aligned}
$$

We only need to verify that TI_0 is extendable to $\xi + i \sum_{j=1}^{n} \alpha_j \mathbb{1}_{(\tau_j, t]}$. This is clearly fulfilled, since the explicit formula (18) extends continuously to all $\xi \in L_d^2$.

Hence we may define the product in this way:

Proposition 15 *Let $\tau_j \in [t_0, t]$ for $j = 1, \ldots, n$; $t_0 < t$ and $\alpha_j \in \mathbb{R}^d$. Then the pointwise product*

$$\Phi_n = I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(\tau_j)} \tag{28}$$

defined by

$$\begin{aligned} T\Phi_n(\xi) &= TI_0\left(\xi + i\sum_{j=1}^{n} \alpha_j \mathbb{1}_{(\tau_j,t]}\right) \exp\left(\sum_{j=1}^{n} \alpha_j \cdot x\right) \\ &= (2\pi i (t-t_0))^{-d/2} \exp\left[-\frac{i}{2}\int_{\mathbb{R}} \left(\xi(s) + i\sum_{j=1}^{n} \alpha_j \mathbb{1}_{(\tau_j,t]}(s)\right)^2 \mathrm{d}s\right] \\ &\quad \cdot \exp\left\{-\frac{1}{2i(t-t_0)}\left[\int_{t_0}^{t} \xi(s)\,\mathrm{d}s + i\sum_{j=1}^{n} \alpha_j (t-\tau_j) + (x-x_0)\right]^2\right\} \\ &\quad \cdot \exp\left(\sum_{j=1}^{n} \alpha_j \cdot x\right) \end{aligned}$$

is a Kondratiev distribution.

Proof. Obviously this has an extension in $\xi \in \mathcal{S}_d(\mathbb{R})$ to all $\theta \in \mathcal{S}_{d,\mathbb{C}}(\mathbb{R})$ and fulfills the first part of Definition 1. In order to prove that $\Phi_n \in (\mathcal{S}_d)^{-1}$ by applying Theorem 2, we need a bound

$$\begin{aligned} &|T\Phi_n(\theta)| \\ &\leq (2\pi(t-t_0))^{-d/2} \exp\left\{\frac{1}{2}|\theta|_0^2 + \int_{\mathbb{R}} \left|\theta(s)\sum_{j=1}^{n} \alpha_j \mathbb{1}_{(\tau_j,t]}(s)\right| \mathrm{d}s\right\} \\ &\quad \cdot \exp\left\{\frac{1}{2(t-t_0)}\left[2\left|\left((x-x_0) + \sum_{j=1}^{n} \alpha_j (t-\tau_j)\right)\int_{t_0}^{t} \theta(s)\,\mathrm{d}s\right|\right.\right. \\ &\qquad \left.\left. + (t-t_0)|\theta|_0^2 + 2\left|(x-x_0)\cdot\sum_{j=1}^{n} \alpha_j (t-\tau_j)\right|\right]\right\} \exp\left(\sum_{j=1}^{n} \alpha_j \cdot x\right) \end{aligned}$$

$$
\begin{aligned}
&\leq \quad (2\pi (t-t_0))^{-d/2} \exp\left(\sum_{j=1}^{n} |\alpha_j|\,|x-x_0| + \sum_{j=1}^{n} |\alpha_j|\,|x_0|\right) \qquad (29)\\
&\qquad \cdot \exp\left[|\theta|_0^2 + \left(2\sqrt{t-t_0}\sum_{j=1}^{n}|\alpha_j| + \frac{|x-x_0|}{\sqrt{t-t_0}}\right)|\theta|_0\right]\\
&=: \quad C_n(\alpha_1,\ldots,\alpha_j,\theta).
\end{aligned}
$$

Thus Φ_n is a Kondratiev distribution, in fact by the above bound it is also a Hida distribution, see e.g. [KLPSW96]. ■

Now we are able to prove the existence of the integrand.

Theorem 16 *Let V be as in Definition 7. Then*

$$
I := \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(\tau_j)} \prod_{j=1}^{n} \mathrm{d}m(\alpha_j)\,\mathrm{d}^n\tau \qquad (30)
$$

exists as a generalized white noise functional. The series converges in the strong topology of $(\mathcal{S}_d)^{-1}$. The integrals exist in the sense of Bochner integrals. Therefore we can express the T-transform by

$$
TI(\theta) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} T\left(I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(\tau_j)}\right)(\theta) \prod_{j=1}^{n} \mathrm{d}m(\alpha_j)\,\mathrm{d}^n\tau \qquad (31)
$$

for all θ in a neighborhood of zero

$$
U_{p,q} := \left\{\theta \in S_{d,\mathbb{C}} \,\middle|\, 2^q |\theta|_p < 1\right\} \qquad (32)
$$

for some $p, q \in \mathbb{N}_0$.

Proof. We have already shown in Proposition 15 that the product

$$
\Phi_n := I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(\tau_j)}
$$

is a Kondratiev distribution, moreover, we derived the estimate (29). In order to see that the integrals exist in the sense of Bochner we want to apply Theorem 4. As the T-transform of Φ_n is entire in $\xi \in \mathcal{S}_{d,\mathbb{C}}(\mathbb{R})$ and measurable, it remains only to derive a suitable bound

$$
\begin{aligned}
&\int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} C_n(\alpha_1,\ldots,\alpha_j,\theta) \prod_{j=1}^{n} \mathrm{d}|m|(\alpha_j)\,\mathrm{d}^n\tau \qquad (33)\\
&\leq \left(2\pi\,(t-t_0)\right)^{-d/2} (t-t_0)^n \exp\left(|\theta|_0^2 + \frac{|x-x_0|}{\sqrt{t-t_0}}\,|\theta|_0\right)\\
&\quad \cdot \left\{\int_{\mathbb{R}^d} \exp\left(\left[|x-x_0| + |x_0| + 2\sqrt{t-t_0}\,|\theta|_0\right]|\alpha|\right)\,\mathrm{d}|m|\,(\alpha)\right\}^n
\end{aligned}
$$

which is finite since the measure satisfies condition (22). Due to Theorem 4 there exists an open neighborhood U independent of n and

$$
I_n := \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} \Phi_n \prod_{j=1}^{n} \mathrm{d}m(\alpha_j)\,\mathrm{d}^n\tau \in (\mathcal{S}_d)^{-1}, \quad \forall n \in \mathbb{N}.
$$

with TI_n is holomorphic on U. To finish the proof we must show that the series converges in $(\mathcal{S}_d)^{-1}$ in the strong sense. For that we apply Theorem 3. We know that TI_n is holomorphic on U and we can bound it by

$$
\begin{aligned}
&|TI\,(\theta)| \qquad (34)\\
&\leq \sum_{n=0}^{\infty} \frac{1}{n!}\,|TI_n\,(\theta)|\\
&\leq \left(2\pi\,(t-t_0)\right)^{-d/2} \exp\left(|\theta|_0^2 + \frac{|x-x_0|}{\sqrt{t-t_0}}\,|\theta|_0\right)\\
&\quad \cdot \exp\left\{(t-t_0)\int_{\mathbb{R}^d} \exp\left[\left(|x| + 2\,|x_0| + 2\sqrt{t-t_0}\,|\theta|_0\right)|\alpha|\right]\,\mathrm{d}|m|(\alpha)\right\}\\
&< \infty
\end{aligned}
$$

for $\theta \in U$, so that we prove $I \in (\mathcal{S}_d)^{-1}$. ■

Remark 17 The bound established in the proof above has a trivial, but rather surprising consequence. For the forthcoming discussion it is convenient to show the dependence on the coupling constant explicitly, so that we get

$$
TI\,(\theta) = \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} T\left(I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(\tau_j)}\right)(\theta) \prod_{j=1}^{n} \mathrm{d}m(\alpha_j)\,\mathrm{d}^n\tau \qquad (35)
$$

which is a perturbation series in the coupling constant. Recall the bound we have already calculated; we obtain that

$$
\begin{aligned}
&|TI(\theta)| &(36)\\
&\leq \sum_{n=0}^{\infty} \frac{|g|^n}{n!} |TI_n(\theta)| \leq (2\pi(t-t_0))^{-d/2} \exp\left(|\theta|_0^2 + \frac{|x-x_0|}{\sqrt{t-t_0}} |\theta|_0\right) \\
&\quad \cdot \exp\left\{ |g|(t-t_0) \int_{\mathbb{R}^d} \exp\left(\left[|x-x_0| + |x_0| + 2\sqrt{t-t_0}\,|\theta|_0\right] |\alpha|\right) \mathrm{d}|m|(\alpha) \right\}
\end{aligned}
$$

and hence $TI(\theta)$ is entire in the coupling constant g for all fixed $x, x_0, t_0 < t$ and $\theta \in U_{p,q}$. This is surprising, since the corresponding Hamilton operators, even if they are essentially self-adjoint for $g > 0$, lose this property for $g < 0$ in general. Quantities such as eigenvalues and eigenvectors will not be analytic in the coupling constant. (On the other hand under a stronger condition than (22) Albeverio et al. [AlBrHa96] have shown that the solution of the Schrödinger equation $\Psi_t(x)$ is analytic in the coupling constant if the initial wave function Ψ_0 as a function of x is from a certain class of analytic functions.)

4.3 Schrödinger equation

Our aim is to prove that the propagator constructed above indeed solves the Schrödinger equation. We are able to show that for $t_0 < t$ the propagator does not only solve it in the sense of distributions, but also in the sense of ordinary functions. Similar to the free case, see equation (20), we can also give the test function in the T-transform a physical meaning corresponding to a time dependent homogeneous external small force in the sense of (32). To compensate the extra factors appearing in the formal integration by parts, see again (20), we consider

$$
TI(\theta) \cdot \exp\left[\frac{i}{2} \int_{[t_0,t]^c} \theta^2(s)\, \mathrm{d}s + ix_0 \cdot \theta(t_0) - ix \cdot \theta(t)\right]. \tag{37}
$$

This then produces the Schrödinger propagator as follows

Theorem 18 *Let V be as in Definition 7. Then*

$$\begin{aligned} & K^{(\theta)}(x,t \mid x_0,t_0) \qquad (38) \\ = & \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} (2\pi i (t-t_0))^{-d/2} \\ & \cdot \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} \exp\left\{ -\frac{i}{2} \int_{t_0}^{t} \left(\theta(s) + i \sum_{j=1}^{n} \alpha_j \mathbb{1}_{(\tau_j,t]}(s) \right)^2 \mathrm{d}s \right\} \\ & \cdot \exp\left\{ -\frac{1}{2i(t-t_0)} \left[\int_{t_0}^{t} \theta(s)\,\mathrm{d}s + i \sum_{j=1}^{n} \alpha_j (t-\tau_j) + (x-x_0) \right]^2 \right\} \\ & \cdot \exp\left(\sum_{j=1}^{n} \alpha_j \cdot x \right) \exp\left(i x_0 \cdot \theta(t_0) - i x \cdot \theta(t) \right) \prod_{j=1}^{n} \mathrm{d}m(\alpha_j)\, \mathrm{d}^n \tau \end{aligned}$$

solves the Schrödinger equation for all $x, x_0, t_0 < t$

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2}\triangle_d - gV(x) - x\cdot \dot{\theta}(t) \right) K^{(\theta)}(x,t \mid x_0,t_0) = 0. \qquad (39)$$

with initial condition

$$\lim_{t \searrow t_0} K^{(\theta)}(x,t \mid x_0,t_0) = \delta(x-x_0) \qquad (40)$$

Remark 19 *We may also write the propagator as a product of the free propagator with a perturbation series*

$$\begin{aligned} & K^{(\theta)}(x,t \mid x_0,t_0) \qquad (41) \\ = & K_0^{(\theta)}(x,t \mid x_0,t_0) \cdot \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} (t-t_0)^n \int_{[0,1]^n} \int_{\mathbb{R}^{dn}} \prod_{j=1}^{n} \mathrm{d}m(\alpha_j)\, \mathrm{d}^n \sigma \\ & \cdot \exp\left\{ -\frac{i}{2}(t-t_0) \left[\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \cdot \alpha_k (\sigma_j \sigma_k - \sigma_j \wedge \sigma_k) \right] \right\} \\ & \cdot \exp\left\{ \sum_{j=1}^{n} \alpha_j \cdot \left[\sigma_j x + (1-\sigma_j) x_0 + \sigma_j \int_{(t-t_0)\sigma_j + t_0}^{t} \theta(s)\,\mathrm{d}s \right.\right. \\ & \qquad \left.\left. -(1-\sigma_j) \int_{t_0}^{(t-t_0)\sigma_j+t_0} \theta(s)\,\mathrm{d}s \right] \right\}. \end{aligned}$$

Remark 20 *The bounds in the following proof also yield that for fixed $\theta \in U_{p,q}$ the above series as a function of x, t, x_0, t_0 is C^∞ and that we are allowed to interchange integration and summation with differentiation.*

Proof. It is easy to see that $T\Phi_n(\theta)$ is C^∞ in the variables x, x_0, t_0, t if $t < t_0$. For simplification we introduce the following abbreviation

$$\begin{aligned} &K^{(\theta)}(x, t \mid x_0, t_0) \\ &= \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} K_n^{(\theta)}(x, t \mid x_0, t_0) \end{aligned} \tag{42}$$

where

$$\begin{aligned} &K_n^{(\theta)}(x, t \mid x_0, t_0) \\ &= \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} T\Phi_n(\theta) \exp\left(\frac{i}{2}\int_{[t_0,t]^c} \theta^2(s)\, \mathrm{d}s + ix_0 \cdot \theta(t_0) - ix \cdot \theta(t)\right) \\ &\quad \cdot \prod_{j=1}^{n} \mathrm{d}m(\alpha_j)\, \mathrm{d}^n\tau. \end{aligned} \tag{43}$$

By direct computation we obtain

$$\begin{aligned} &\left(i\frac{\partial}{\partial t} + \frac{1}{2}\triangle_d\right)\left[T\Phi_n(\theta) \exp\left(\frac{i}{2}\int_{[t_0,t]^c} \theta^2(s)\, \mathrm{d}s + ix_0 \cdot \theta(t_0) - ix \cdot \theta(t)\right)\right] \\ &\quad = x \cdot \dot{\theta}(t) \left[T\Phi_n(\theta) \exp\left(\frac{i}{2}\int_{[t_0,t]^c} \theta^2(s)\, \mathrm{d}s + ix_0 \cdot \theta(t_0) - ix \cdot \theta(t)\right)\right] \end{aligned} \tag{44}$$

Thus the above product alone solves already the free Schrödinger equation with an extra small external force. The next step is to calculate the derivatives of $K_n^{(\theta)}$. We interchange them with the integrals. Since the domain of the τ-integrals depends on t, we get an extra term compared to (44) if we differentiate with respect to t. Using the theorem of Fubini we recognize that this term corresponds to the potential $V(x)$.

$$\begin{aligned} &\left(i\frac{\partial}{\partial t} + \frac{1}{2}\triangle_d\right) K_n^{(\theta)}(x, t \mid x_0, t_0) \\ &= x \cdot \dot{\theta}(t)\, K_n^{(\theta)}(x, t \mid x_0, t_0) + i\frac{n}{g} V(x)\, K_{n-1}^{(\theta)}(x, t \mid x_0, t_0) \end{aligned} \tag{45}$$

which is a typical recursion formula driven by the potential. By summing up we see that $K^{(\theta)}(x,t \mid x_0,t_0)$ solves formally the Schrödinger equation. It remains to justify the operations above. This can be done similarly to the proof of Theorem 16 if we bound the derivatives of $T\Phi_n(\theta)$ as follows. The derivatives of $T\Phi_n(\theta)$ have the form

$$\left\{ a_0(\tau_1,\ldots,\tau_n,\theta) + a_1(\tau_1,\ldots,\tau_n,\theta) \cdot \left(\sum_{j=1}^n \alpha_j\right) \right. \qquad (46)$$
$$\left. + a_2(\tau_1,\ldots,\tau_n,\theta) \cdot \left(\sum_{j=1}^n \alpha_j\right)^2 \right\} \cdot T\Phi_n(\theta)$$

where a_i are continuous in the τ_j. As the τ_j varies only in a compact domain we can bound the derivatives by

$$\{b_0(\theta) + b_1(\theta) + 2b_2(\theta)\} \cdot \exp\left(\sum_{j=1}^n |\alpha_j|\right) \cdot C_n(\alpha_1,\ldots,\alpha_j,\theta), \qquad (47)$$

where C_n is the bound in the proof of Proposition 15 and

$$b_i(\theta) := \sup_{\tau_j \in [t_0,t]} |a_i(\tau_1,\ldots,\tau_n,\theta)|. \qquad (48)$$

The rest of the proof can be done as before. ■

4.4 Continuation to imaginary mass

By continuation to imaginary mass we obtain a formal perturbation series for the propagator of the heat equation. If the measure in Definition 7 is positive we show that this perturbation series diverges. In the following discussion we need the explicit dependence of the propagator on $\hbar$ and m. For simplification we put $\theta = 0$. $K^{(0)}$ is abbreviated as K. By (41) we obtain

$$K(x,t \mid x_0,t_0) \qquad (49)$$
$$= K_0(x,t \mid x_0,t_0) \cdot \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-ig(t-t_0)}{\hbar}\right)^n \int_{[0,1]^n} \int_{\mathbb{R}^{dn}} \prod_{j=1}^n \mathrm{d}m(\alpha_j)\, \mathrm{d}^n\sigma$$
$$\cdot \exp\left\{-\frac{i\hbar}{2m}(t-t_0)\left[\sum_{j=1}^n \sum_{k=1}^n \alpha_j \cdot \alpha_k (\sigma_j\sigma_k - \sigma_j \wedge \sigma_k)\right]\right\}$$

$$\cdot \exp\left\{\sum_{j=1}^{n} \alpha_j \cdot [\sigma_j x + (1-\sigma_j)\, x_0]\right\}.$$

If we perform a formal analytic continuation in the mass from m to im we get

$$\begin{aligned} &K^H(x,t \mid x_0,t_0) \qquad (50)\\ &= K_0^H(x,t \mid x_0,t_0) \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-ig\,(t-t_0)}{\hbar}\right)^n \int_{[0,1]^n} \int_{\mathbb{R}^{dn}} \prod_{j=1}^{n} \mathrm{d}m(\alpha_j)\, \mathrm{d}^n\sigma \\ &\cdot \exp\left\{-\frac{\hbar}{2m}(t-t_0) \left[\sum_{j=1}^{n}\sum_{k=1}^{n} \alpha_j \cdot \alpha_k\, (\sigma_j\sigma_k - \sigma_j \wedge \sigma_k)\right]\right\} \\ &\cdot \exp\left\{\sum_{j=1}^{n} \alpha_j \cdot [\sigma_j x + (1-\sigma_j)\, x_0]\right\} \end{aligned}$$

This solves formally the heat equation with potential $-igV(x)$. Usually, convergence properties are easier to handle for the heat equation than for the Schrödinger equation. However, in our case it is the other way around.

Theorem 21 *Let $d = 1$ and let m be a positive measure on the Borel sets of $\mathbb{R}$ fulfilling condition (22). If $m(\mathbb{R}\setminus\{0\}) > 0$ then the power series in g*

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-ig\,(t-t_0)}{\hbar}\right)^n \int_{[0,1]^n} \int_{\mathbb{R}^{n}} \prod_{j=1}^{n} \mathrm{d}m(\alpha_j)\, \mathrm{d}^n\sigma \qquad (51)\\ &\cdot \exp\left\{-\frac{\hbar}{2m}(t-t_0) \left[\sum_{j=1}^{n}\sum_{k=1}^{n} \alpha_j \cdot \alpha_k\, (\sigma_j\sigma_k - \sigma_j \wedge \sigma_k)\right]\right\} \\ &\cdot \exp\left\{\sum_{j=1}^{n} \alpha_j \cdot [\sigma_j x + (1-\sigma_j)\, x_0]\right\} \end{aligned}$$

diverges for every $g \neq 0$ for any fixed $x_0, x, t_0 < t$.

Proof. Either there exists $a_0 > 0$ with $m([a_0,\infty)) > 0$ or $a_0 < 0$ with $m((-\infty,a_0]) > 0$. Without lost of generality we assume $a_0 > 0$. We use the shorthand

$$F(\alpha_1,\ldots,\alpha_n,\sigma_1,\ldots,\sigma_n) \qquad (52)$$

$$
\begin{aligned}
:= \; & \exp\left\{-\frac{\hbar}{2m}\left(t-t_0\right)\left[\sum_{j=1}^{n}\sum_{k=1}^{n}\alpha_j\cdot\alpha_k\left(\sigma_j\sigma_k-\sigma_j\wedge\sigma_k\right)\right]\right\} \\
& \cdot\exp\left\{\sum_{j=1}^{n}\alpha_j\cdot\left[\sigma_j x+\left(1-\sigma_j\right)x_0\right]\right\}.
\end{aligned}
$$

For $\frac{3}{16}\leq\sigma_j,\sigma_k\leq\frac{4}{16}$ we have

$$
\begin{aligned}
& \sigma_j\sigma_k-\sigma_j\wedge\sigma_k \\
\leq \; & -\frac{1}{16}.
\end{aligned}
\tag{53}
$$

Since the integrand is positive we get

$$
\begin{aligned}
& \left|\int_{[0,1]^n}\int_{\mathbb{R}^n}F\left(\alpha_1,\ldots,\alpha_n,\sigma_1,\ldots,\sigma_n\right)\prod_{j=1}^{n}\mathrm{d}m(\alpha_j)\,\mathrm{d}^n\sigma\right| \\
\geq \; & \int_{\left[\frac{3}{16},\frac{4}{16}\right]^n}\int_{[a_0,\infty)^n}F\left(\alpha_1,\ldots,\alpha_n,\sigma_1,\ldots,\sigma_n\right)\prod_{j=1}^{n}\mathrm{d}m(\alpha_j)\,\mathrm{d}^n\sigma \\
\geq \; & \int_{\left[\frac{3}{16},\frac{4}{16}\right]^n}\int_{[a_0,\infty)^n}\exp\left\{\frac{\hbar}{2m}\left(t-t_0\right)\frac{1}{16}n^2a_0^2\right\} \\
& \cdot\exp\left\{-\sum_{j=1}^{n}\left|\alpha_j\right|\left(\left|x_0\right|+\left|x\right|\right)\right\}\prod_{j=1}^{n}\mathrm{d}m(\alpha_j)\,\mathrm{d}^n\sigma \\
\geq \; & \left(\frac{1}{16}\right)^n\exp\left\{\frac{\hbar}{32m}\left(t-t_0\right)a_0^2n^2\right\}\left(\int_{a_0}^{\infty}\exp\left\{-\left|\alpha\right|\left(\left|x_0\right|+\left|x\right|\right)\right\}\mathrm{d}m\left(\alpha\right)\right)^n.
\end{aligned}
\tag{54}
$$

By the above assumption for a_0 the last factor does not vanish. Thus we get for the series

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\left|\frac{1}{n!}\left(\frac{-ig\left(t-t_0\right)}{\hbar}\right)^n\int_{[0,1]^n}\int_{\mathbb{R}^n}F\left(\alpha_1,\ldots,\alpha_n,\sigma_1,\ldots,\sigma_n\right)\prod_{j=1}^{n}\mathrm{d}m(\alpha_j)\,\mathrm{d}^n\sigma\right| \\
\geq \; & \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\left|g\right|\left(t-t_0\right)}{16\hbar}\int_{a_0}^{\infty}\exp\left\{-\left|\alpha\right|\left(\left|x_0\right|+\left|x\right|\right)\right\}\mathrm{d}m(\alpha)\right)^n \\
& \cdot\left(\exp\left\{\frac{\hbar}{32m}\left(t-t_0\right)a_0^2\right\}\right)^{n^2} \\
= \; & \infty.
\end{aligned}
$$

■

4.5 Time-dependent potentials

One of the advantages of the Feynman integral is that it can be easily extended to time-dependent potentials.

Theorem 22 *Let m denote a complex measure on the Borel sets of $\mathbb{R}^d \times [t_0', t']$; $d \geq 1$, such that*

$$\int_{\mathbb{R}^d} \int_{t_0'}^{t'} e^{C|\alpha|} \, \mathrm{d}|m|(\alpha, \tau) < \infty, \qquad \forall\, C > 0 \tag{55}$$

Then for $t_0' \leq t_0 < t \leq t'$

$$I := \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(\tau_j)} \prod_{j=1}^{n} \mathrm{d}m(\alpha_j, \tau_j) \tag{56}$$

exists as a generalized white noise functional in $(\mathcal{S}_d)^{-1}$. The T-transform fulfills the following equation

$$TI\,(\theta) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} T\left(I_0 \cdot \prod_{j=1}^{n} e^{\alpha_j \cdot x(\tau_j)} \right)(\theta) \prod_{j=1}^{n} \mathrm{d}m(\alpha_j, \tau_j) \tag{57}$$

for all θ in a neighborhood of zero.

Remark 23 If we consider the Schrödinger equation for the whole class defined in the above theorem, the potential becomes a distribution in the time variable. This causes technical difficulties; we only use the special forms given below.

Theorem 24 *Let m be as in Theorem 22. If additionally m has either the special form*

$$\mathrm{d}m(\alpha, \tau) = \sum_{j=1}^{k} \mathrm{d}m_j\,(\alpha)\, \rho_j\,(\tau)\, \mathrm{d}\tau \tag{58}$$

with $k \in \mathbb{N}$, m_j complex measures on the Borel sets of $\mathbb{R}^d$ and $\rho_j \in C^0\,(\mathbb{R}, \mathbb{C})$ for all $j = 1, \ldots, k$;
or the special form

$$\mathrm{d}m(\alpha, \tau) = \rho\,(\alpha, \tau)\, \mathrm{d}^d\alpha \; \mathrm{d}\tau \tag{59}$$

where $\rho : \mathbb{R}^d \times [t_0', t'] \to \mathbb{C}$ *with* $\rho(\alpha, \cdot)$ *continuous on* $[t_o', t']$ *for all* $\alpha \in \mathbb{R}^d$*and* $\sup_{\tau \in [t_0', t']} |\rho(\alpha, \tau)|$ *in* $L^1(\mathbb{R}, \mathrm{d}^d\alpha)$
then the propagator

$$\begin{aligned} & K^{(\theta)}(x, t \mid x_0, t_0) \qquad (60) \\ = & \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} (2\pi i (t - t_0))^{-d/2} \exp(ix_0 \cdot \theta(t_0) - ix \cdot \theta(t)) \\ & \cdot \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} \exp\left\{ -\frac{i}{2} \int_{t_0}^{t} \left(\theta(s) + i \sum_{j=1}^{n} \alpha_j \mathbb{1}_{(\tau_j, t]}(s) \right)^2 \mathrm{d}s \right\} \\ & \cdot \exp\left\{ -\frac{1}{2i(t - t_0)} \left[\int_{t_0}^{t} \theta(s)\,\mathrm{d}s + i \sum_{j=1}^{n} \alpha_j (t - \tau_j) + (x - x_0) \right]^2 \right\} \\ & \cdot \exp\left(\sum_{j=1}^{n} \alpha_j \cdot x \right) \prod_{j=1}^{n} \mathrm{d}m(\alpha_j, \tau_j) \end{aligned}$$

solves the Schrödinger equation with potential $x \cdot \overset{\bullet}{\theta}(t) + V(x,t)$ *for all* $t_0' < t_0 < t < t'$*, where*

$$V(x,t) := \sum_{j=1}^{k} \int_{\mathbb{R}^d} e^{\alpha \cdot x} \mathrm{d}m_j(\alpha) \rho_j(t) \qquad (61)$$

or respectively

$$V(x,t) := \int_{\mathbb{R}^d} e^{\alpha \cdot x} \rho(\alpha, t)\,\mathrm{d}^d\alpha. \qquad (62)$$

Proof of Theorem 22 and 24. The proof can be done in a similar way as in the previous subsection. Only the measure $\mathrm{d}m(\alpha_j)\,\mathrm{d}\tau_j$ has to be replaced by $\mathrm{d}m(\alpha_j, \tau_j)$. More explicitly we then get instead of bound (33) the following

$$\begin{aligned} & \int_{[t_0,t]^n} \int_{\mathbb{R}^{dn}} C_n(\alpha_1, \ldots, \alpha_j, \theta) \prod_{j=1}^{n} \mathrm{d}|m|(\alpha_j, \tau_j) \qquad (63) \\ \leq & \; (2\pi (t - t_0))^{-d/2} \exp\left(|\theta|_0^2 + \frac{|x - x_0|}{\sqrt{t - t_0}} |\theta|_0 \right) \\ & \cdot \left\{ \int_{\mathbb{R}^d} \int_{t_0}^{t} \exp\left(\left[|x - x_0| + |x_0| + 2\sqrt{t - t_0}\, |\theta|_0 \right] |\alpha| \right) \; \mathrm{d}|m|(\alpha, \tau) \right\}^n. \end{aligned}$$

To complete the proof of Theorem 24 we have to justify the interchange of the time derivative with the α-integrals. For this we use an extra condition as e.g. in the second special form the integrability of $\sup_{\tau\in[t'_0,t']} |\rho(\alpha,\tau)|$. ■

5 A special case: The Morse Potential

In order to illustrate the remarkable fact that the propagator is analytic in the coupling constant we discuss the Morse potential as a special case of the class of potentials we studied in Section 4. This potential has been very useful in molecular and nuclear physics, see for example [He50].

Definition 25 *In $L^2(\mathbb{R},\mathbb{C})$ we consider the Hamilton operator*

$$H := -\frac{1}{2}\triangle + g\left(e^{-2ax} - 2\gamma e^{-ax}\right) \tag{64}$$

with $\gamma, g, a \in \mathbb{R}$. As domain we choose

$$D(H) := C_0^\infty(\mathbb{R},\mathbb{C}) \tag{65}$$

the set of infinite differentiable functions with compact support. For $g > 0$ and $\gamma > 0$ this is called the Morse potential.

We now collect some well known results from operator theory, see [DuSc63] and [ReSi75].

Proposition 26 *H is symmetric and*

$$D(H^*) = \left\{ f \in L^2(\mathbb{R},\mathbb{C}) \cap C^1(\mathbb{R},\mathbb{C}) \;\middle|\; \begin{array}{c} f' \text{ is absolutely continous} \\ \text{and } Hf \in L^2(\mathbb{R},\mathbb{C}) \end{array} \right\}$$

Since we have a real potential the deficiency indices are equal and thus there exists a self-adjoint extension, which is not necessarily unique.

¿From Theorem X.8, X.9 in [ReSi75] we can derive the following proposition

Proposition 27 *H is essentially self-adjoint for $g \geq 0$ and it is not essentially self-adjoint for $g < 0$.*

For the Morse potential ($g > 0$) there exist treatments by algebraic, see [ChGuHa92] and by operator methods, see [NiSi79]. Furthermore path integral techniques have been applied, see e.g. [CaInWi83], [PaSo84] and [Kl90]. One uses path-dependent space-time transformation to convert the path integral for the Morse potential into the radial path integral for the harmonic oscillator in three dimensions which is well known. Further refinements were done in [FiLeMu92] and [Pe96]. By latter technique one calculates the Green function, the kernel of the resolvent, moreover one derives the spectrum and the eigenfunctions. Originally the method of path-dependent space-time transformation was applied to calculate the Feynman integral of the hydrogen atom, see e.g. [DuKl79] and [BlSi81]. The following formulas are taken from [FiLeMu92]. First we will have a look at the Green function.

$$\begin{aligned} G\left(x', x; E\right) &:= \left\langle x' \left| (H-E)^{-1} \right| x \right\rangle \qquad (66) \\ &= \frac{\Gamma\left(\frac{1}{2} + \frac{\sqrt{-2E}}{|a|} - \frac{\gamma\sqrt{2g}}{|a|}\right)}{\sqrt{2g}\,\Gamma\left(\frac{2}{|a|}\sqrt{-2E}+1\right)} e^{\frac{a}{2}(x+x')} \\ &\cdot \left\{ \Theta(a(x-x'))\, W_{\frac{\gamma}{|a|}\sqrt{2g}, \frac{\sqrt{-2E}}{|a|}} \left(\frac{2}{|a|}\sqrt{2g}\, e^{-ax'} \right) M_{\frac{\gamma}{|a|}\sqrt{2g}, \frac{\sqrt{-2E}}{|a|}} \left(\frac{2}{|a|}\sqrt{2g}\, e^{-ax} \right) \right. \\ &\left. + \Theta(a(x'-x))\, M_{\frac{\gamma}{|a|}\sqrt{2g}, \frac{\sqrt{-2E}}{|a|}} \left(\frac{2}{|a|}\sqrt{2g}\, e^{-ax'} \right) W_{\frac{\gamma}{|a|}\sqrt{2g}, \frac{\sqrt{-2E}}{|a|}} \left(\frac{2}{|a|}\sqrt{2g}\, e^{-ax} \right) \right\} \end{aligned}$$

with $x', x \in \mathbb{R}$. Θ denotes the Heaviside function with the convention $\Theta(0) = \frac{1}{2}$, Γ is Euler's gamma function, $M_{\varkappa,\frac{\mu}{2}}$ and $W_{\varkappa,\frac{\mu}{2}}$ Whittaker's functions, see [Er53]. The $\sqrt{\cdot}$ denotes the principal branch of the square root with the cut along the negative real half line. Then we get for the spectrum

$$\begin{aligned} \sigma(H) &= [0, \infty) \qquad (67) \\ & \bigcup \left\{ -\frac{1}{8}a^2 \left(\frac{2\gamma}{|a|}\sqrt{2g} - 2n - 1 \right)^2 \middle| n \in \mathbb{N}_0 \text{ with } \frac{2\gamma}{|a|}\sqrt{2g} - 2n - 1 > 0 \right\} \end{aligned}$$

and for the eigenvectors of the discrete eigenvalues

$$\begin{aligned} \Psi_n(x) &= \sqrt{\frac{|a|\left(\frac{2\gamma}{|a|}\sqrt{2g} - 2n - 1\right)\Gamma(n+1)}{\Gamma\left(\frac{2\gamma}{|a|}\sqrt{2g} - n\right)}} \left(\frac{2\sqrt{2g}}{|a|}\right)^{\frac{\gamma}{|a|}\sqrt{2g} - \frac{2n+1}{2}} \qquad (68) \\ & e^{-a\left(\frac{\gamma}{|a|}\sqrt{2g} - \frac{2n-1}{2}\right)x} e^{-\frac{1}{|a|}\sqrt{2g}e^{-ax}} L_n^{\left(2\frac{\gamma}{|a|}\sqrt{2g} - 2n - 1\right)} \left(\frac{2}{|a|}\sqrt{2g}e^{-ax} \right) \end{aligned}$$

where $L_n^{(\mu)}$ is a generalized Laguerre polynomial, see [Er53].

Proposition 28 *The Green function (66), the eigenvectors and the discrete eigenvalues are not analytic in g.*

Proof. For the discrete eigenvalues the above statement is obvious. The Green function is not even analytic in $\omega := \frac{2}{|a|}\sqrt{2g}$. Using [Bu69] we can rewrite (66) as

$$
\begin{aligned}
G\left(x', x; E\right) = &\frac{2\pi}{|a| \sin\left(\nu\pi\right)} e^{-\frac{\omega}{2}\left(e^{-ax'}+e^{-ax}\right)} \\
&\cdot \left\{ \Theta(a(x-x'))\, \frac{{}_1F_1\left(\frac{1}{2}\left(1+\nu-\gamma\omega\right); 1+\nu, \omega e^{-ax}\right)}{\Gamma(1+\nu)} \right. \\
&\quad \cdot \left[-\omega^{\nu} e^{-\nu\frac{a}{2}(x+x')} \frac{\Gamma\left(\frac{1}{2}\left(1+\nu-\gamma\omega\right)\right)}{\Gamma\left(\frac{1}{2}\left(1-\nu-\gamma\omega\right)\right)} \frac{{}_1F_1\left(\frac{1}{2}\left(1+\nu-\gamma\omega\right); 1+\nu, \omega e^{-ax'}\right)}{\Gamma(1+\nu)} \right. \\
&\qquad \left. + e^{\nu\frac{a}{2}(x'-x)} \frac{{}_1F_1\left(\frac{1}{2}\left(1-\nu-\gamma\omega\right); 1-\nu, \omega e^{-ax'}\right)}{\Gamma(1-\nu)} \right] \\
&\quad + \Theta(a(x'-x))\, \frac{{}_1F_1\left(\frac{1}{2}\left(1+\nu-\gamma\omega\right); 1+\nu, \omega e^{-ax'}\right)}{\Gamma(1+\nu)} \\
&\quad \cdot \left[-\omega^{\nu} e^{-\nu\frac{a}{2}(x+x')} \frac{\Gamma\left(\frac{1}{2}\left(1+\nu-\gamma\omega\right)\right)}{\Gamma\left(\frac{1}{2}\left(1-\nu-\gamma\omega\right)\right)} \frac{{}_1F_1\left(\frac{1}{2}\left(1+\nu-\gamma\omega\right); 1+\nu, \omega e^{-ax}\right)}{\Gamma(1+\nu)} \right. \\
&\qquad \left. \left. + e^{\nu\frac{a}{2}(x-x')} \frac{{}_1F_1\left(\frac{1}{2}\left(1-\nu-\gamma\omega\right); 1-\nu, \omega e^{-ax}\right)}{\Gamma(1-\nu)} \right] \right\}
\end{aligned}
$$

with $\nu := \frac{2}{|a|}\sqrt{-2E}$ and ${}_1F_1$ denotes a generalized hypergeometric function, see [Er53]. The function

$$
\frac{{}_1F_1\left(a; b, x\right)}{\Gamma\left(b\right)} \tag{69}
$$

is entire in a, b, x. Thus we only have to investigate $\Gamma(\frac{1}{2}(1+\nu-\gamma\omega))\omega^{\nu}$. This is obviously not analytic in ω near $\omega = 0$. For the eigenvectors we can proceed along the same line. ■

Although the Green function, the discrete eigenvalues and the eigenfunctions are not analytic, the propagator has a perturbation series which is uniformly absolutely convergent in the coupling constant for every compact set in the variables x, t, x_0, t_0. For the Morse potential we obtain an expansion for the propagator from (41). Putting $\theta = 0$ and doing the $\mathrm{d}m\left(\alpha_j\right)$

integrations we get

$$
\begin{aligned}
&K\left(x,t \mid x_0,t_0\right) && (70)\\
&= \quad K_0\left(x,t \mid x_0,t_0\right)\cdot\sum_{n=0}^{\infty}\frac{(-ig)^n}{n!}\left(t-t_0\right)^n \\
&\quad\cdot\sum_{j_1,\ldots,j_n=1}^{2}(-2\gamma)^{2n-\sum_{k=1}^{n}j_k}\int_{[0,1]^n}\exp\left\{-a\sum_{l=1}^{n}j_l\left(\sigma_l x+\left(1-\sigma_l\right)x_0\right)\right\} \\
&\quad\cdot\exp\left\{-\frac{i}{2}\left(t-t_0\right)a^2\sum_{l=1}^{n}\sum_{k=1}^{n}j_k j_l\left[\sigma_j\sigma_k-\sigma_j\wedge\sigma_k\right]\right\}\,\mathrm{d}^n\sigma.
\end{aligned}
$$

In the sum over n the coefficient of g^n is bounded by

$$\frac{1}{n!}\left|t-t_0\right|^n\left(1+2\left|\gamma\right|\right)^n e^{2n|a|\left(\left|x-x_0\right|+\left|x_0\right|\right)}.$$

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