# Large deviations in the van der Waals limit

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**Abstract.** In this paper we extend the analysis in [3] by proving a strong large deviation principle for the empirical distribution of Ising spins in $d \geq 2$ dimensions when the interaction is determined by a Kac potential and the temperature is below the critical value.

## 1. Introduction.

Large deviations theory is the natural setup for studying the structure and the geometry of the interfaces. At a phase transition the cost (i.e. the logarithm of the Gibbs probability) of having a deviation from equilibrium in some given region inside the system may be "only" proportional to its surface and not to its volume, as customary when there is no phase transition, because the deviation may involve just a change of the phase in the given region. The process then looks atypical only in a neighborhood of the interface, hence the cost is proportional to its surface. The rate function of the large deviations quantifies the cost of an interface and gives the probability of its appearence. Moreover with the help of a strong large deviation principle (LDP) we can, for instance, determine the interface which realizes a given constraint, by relating this to the solution of the corresponding variational problem with the rate function. The best known example is the Wulff problem about the optimal shape of the interface once the volume fraction of the two phases has been fixed.

The $d = 2$ dimensional, nearest neighbor, ferromagnetic Ising system is the most remarkable example where all this has been developed, see [10], [12], [13], [16], [17]. Unfortunately not many other models have been worked out. Here we study and solve the problem under a simplifying feature, namely we consider the $d \geq 2$ ferromagnetic Ising system with Kac potentials. As proposed by Kac, Uhlenbeck and Hemmer, [14], and Lebowitz and Penrose, [15], we consider after the thermodynamic limit ($L \to \infty$) also the scaling limit $\gamma \to 0$, where $\gamma > 0$ is the scaling parameter of the Kac potential (Kac parameter). In [14]

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and [15] it is shown that this procedure yields a rigorous derivation of the van der Waals theory. In [3] a weak LDP is proved, showing that the rate function is the perimeter of the interface times the van der Waals surface tension. Here we prove a strong LDP which allows for instance to characterize the optimal shape of the interface under a general class of constraints.

The order of the limits is very important, we emphasize that the scaling limit $\gamma \to 0$ is done here after the thermodynamic limit $L \to \infty$; the simultaneous limit with $L$ and $\gamma$ suitably related has been examined earlier in [2], [4], [1] where it is solved together with the proof that the non local van der Waals excess free energy functional $\Gamma$-converges to the perimeter functional (times the van der Waals surface tension). Our analysis is intermediate between this case and the other one with only $L \to \infty$ and $\gamma > 0$ maybe very small but fixed, like in [8], [6], [5], [7] where the goal was to prove phase transitions at fixed $\gamma > 0$. Unfortunately our techniques do not allow to extend the analysis to the large deviations at fixed $\gamma > 0$, but we hope they may provide a step forward in this direction.

## 2. Basic notation and main results.

We use the same notation as in [3] that we recall briefly here for the reader's convenience.

*Microscopic, mesoscopic and macroscopic representations of the Ising system*

We consider in this paper the Ising spin system with configuration space $\{-1,1\}^{\mathbb{Z}^d}$, $d \geq 2$, its elements being denoted by $\sigma = \{\sigma(i),\ i \in \mathbb{Z}^d\}$, $\sigma(i)$ the spin at the site $i$. As the spin configurations $\sigma$ give a complete description of the state of the system we will refer to this as to the "microscopic representation" of the system. We will actually restrict to tori $\Lambda$ of $\mathbb{Z}^d$ of side $L = 2^n$, $n \in \mathbb{N}$, and use the following notation: for any subset $\Delta$ of $\mathbb{Z}^d$, $\sigma_\Delta \in \{-1,1\}^\Delta$ denotes the restriction of $\sigma$ to $\Delta$.

The macroscopic state of the system is instead determined by an order parameter which specifies the phase of the system (we will be working at a fixed temperature for which there are just two pure equilibrium phases, i.e. two extremal, translationally invariant Gibbs states, see below). It is convenient to choose the order parameter $u$ in such a way that at the two equilibrium phases $u$ has the values $\pm 1$. The two pure phases are then represented by the two functions $u(r)$ constantly equal to 1 and to $-1$. We will suppose that the macroscopic region where our system is confined is the unit torus $\mathcal{T}$ in $\mathbb{R}^d$ with center the origin. Then $r \in \mathcal{T}$ and $u(r) = 1$ means that at $r$ there is the phase $+1$. Our goal is to investigate the structure of macroscopic states $u(\cdot)$ where the order parameter takes both the value $+1$ and $-1$, but we will also consider states where it takes non equilibrium values (not in $\{\pm 1\}$). As we will see, these states are much less probable than the others. Thus the order parameter ranges in some interval $[-A, A]$, $A > 1$ and the macroscopic configurations are elements of

$$\mathcal{X} = L^1(\mathcal{T}; [-A, A]) \tag{2.1}$$

with $\|u\|$ denoting the $L^1(\mathcal{T})$ norm of $u$. The $L^1$ norm reflects the choice that two macroscopic configurations will be considered close to each other if their difference is small except possibly for a small fraction of the volume. The macroscopic observables are then elements of $C(\mathcal{X})$.

The order parameter as a function of the spin configurations will be defined later via a limit procedure which involves empirical averages. To this end it is convenient to represent the Ising configurations as functions on $\mathbb{R}^d$. Let $\hat{e} \in \mathbb{R}^d$ be the point with coordinates all equal to $1/2$ and $\mathcal{D}$ the partition into unit cubes $C$ with centers the points $i+\hat{e}$, $i \in \mathbb{Z}^d$. A face in common to two cubes is attributed to the one with the largest center, so that the cube with center $i+\hat{e}$ contains $i$. We also use the notation $C(r)$ for the cube of $\mathcal{D}$ which contains $r$. Finally $\mathcal{D}^{(\ell)}$, $\ell \in \{2^n,\, n \in \mathbb{Z}\}$, denotes the partition into cubes $C^{(\ell)}$ of side $\ell$ obtained by scaling $\mathcal{D}$ by $\ell$ and given a bounded function $f$ on $\mathbb{R}^d$ we define the empirical averages (coarse graining) of $f$ as

$$f^{(\ell)}(r) = \frac{1}{\ell^d} \int_{C^{(\ell)}(r)} dr' f(r') \tag{2.2}$$

The macroscopic region corresponding to the tori $\Lambda$ of side $L$ is always the unit torus $\mathcal{T}$. The spin configurations are then represented by functions $s \in L^\infty(\mathcal{T}; \{\pm 1\})$ that are $\mathcal{D}^{(1/L)}$-measurable, i.e. constant on the cubes $C^{(1/L)}$ of $\mathcal{D}^{(1/L)}$. The relation with the microscopic representation is then given by

$$s(r) = \sigma(i), \qquad Lr \in C(i) \tag{2.3}$$

where $C(i)$ is the cube of $\mathcal{D}$ that contains $i$. In this way the thermodynamic limit $L \to \infty$ is represented as a continuum limit with the mesh $1/L$ of the coarse graining going to 0.

In many instances it is convenient to work on an intermediate scale, the mesoscopic scale, whose units are chosen so that the range of the interaction becomes 1. As we will see, in microscopic units the range is $\gamma^{-1}$, where $\gamma$, the Kac parameter, takes values in $\{2^{-n}\}$, we will always restrict to the case $L_\gamma := \gamma L > 1$. The mesoscopic space is then the torus $L_\gamma \mathcal{T}$ of $\mathbb{R}^d$ and the the mesoscopic spin configurations are the functions $S \in L^\infty(L_\gamma \mathcal{T}; \{\pm 1\})$ which are $\mathcal{D}^{(1/L_\gamma)}$-measurable, so that

$$S(x) = s(L_\gamma^{-1} x), \quad L_\gamma = \gamma L, \qquad S(x) = \sigma(i), \quad \gamma^{-1} x \in C(i) \tag{2.4}$$

To distinguish the points in the various spaces we write (when possible) $r$ for macroscopic, $x$ for mesoscopic and $i$ for microscopic.

*Kac interaction and Gibbs measures*

For any $\gamma > 0$ and any bounded set $\Delta$ in $\mathbb{Z}^d$, we define the energy of $\sigma_\Delta$ in interaction with $\sigma_{\Delta^c}$ as

$$H_\gamma(\sigma_\Delta | \sigma_{\Delta^c}) = -\frac{1}{2} \sum_{i \neq j \in \Delta} J_\gamma(i,j)\sigma(i)\sigma(j) - \sum_{i \in \Lambda, j \in \Delta^c} J_\gamma(i,j)\sigma(i)\sigma(j) \tag{2.5}$$

where

$$J_\gamma(i,j) := \gamma^d J(\gamma|i-j|), \qquad \forall\, i,j \in \mathbb{Z}^d \tag{2.6}$$

and $J$ is a nonnegative, smooth function supported by $[0,1]$ and normalized so that

$$\int_{\mathbb{R}^d} dr\, J(|r|) = 1 \tag{2.7}$$

The conditional Gibbs probability of $\sigma_\Delta$ given $\sigma_{\Delta^c}$ is

$$\mu_{\gamma,\Delta}(\sigma_\Lambda|\sigma_{\Delta^c}) = Z_{\gamma,\Delta}(\sigma_{\Delta^c})^{-1} \exp\big[-\beta H_\gamma(\sigma_\Delta|\sigma_{\Delta^c})\big] \tag{2.8}$$

$Z_{\gamma,\Delta}(\sigma_{\Delta^c})$ being the partition function. The Gibbs measure on the torus $\Lambda$ of side $L$ will be denoted by $\mu_{\gamma,L}$.

The infinite volume Gibbs measures $\mu_\gamma$ are the probabilities on $\{-1,1\}^{\mathbb{Z}^d}$ whose conditional probabilities satisfy (2.8). In [8] and [6] it is shown that if $\beta > 1$ there is $\gamma_\beta > 0$ so that for all $\gamma \le \gamma_\beta$ there are two distinct, translationally invariant Gibbs states $\mu_\gamma^\pm$, limits of the finite volume Gibbs states with all $+1$ and, respectively, all $-1$ boundary conditions. In [7] it is shown that these are the only extremal, translationally invariant Gibbs states. Moreover their magnetizations, $\pm m_{\beta,\gamma}$, converge when $\gamma \to 0^+$ to $\pm m_\beta$, where $m_\beta$ is the positive root of

$$m_\beta = \tanh\{\beta m_\beta\} \tag{2.9}$$

*Large deviations*

Our order parameter is the ratio of the magnetization density with its equilibrium value $m_{\beta,\gamma}$, and since the absolute value of the magnetization density cannot exceed 1, we take $A$ in (2.1) so that $A > m_{\beta,\gamma}^{-1}$, for all $\gamma \le \gamma_\beta$. We will define the order parameter as a function of the spin configurations by a limit procedure. Starting from a spin configuration $\sigma$, we first go to its macro representation $s$ and then, recalling the definition (2.2) of the empirical averages, we take as an approximation for the order parameter the (normalized) coarse grained configurations $s^{(\varepsilon)}/m_{\beta,\gamma}$. Our limit procedure is to first take the thermodynamic limit $L \to \infty$, then $\varepsilon \to 0$ and eventually $\gamma \to 0$ (it would be much nicer if we could avoid the last limit and keep $\gamma > 0$ fixed). In Theorem 1.2 of [3] it is proved that for all $\gamma$ small enough

$$\lim_{L\to\infty} \mu_{\gamma,L}\Big(\|s^{(\varepsilon)}/m_{\beta,\gamma} \mp 1\| \le \delta\Big) = \frac{1}{2}$$

for all $\delta > 0$ and all $\varepsilon > 0$. In the thermodynamic limit therefore the probability concentrates on the two pure phases where the order parameter is constantly equal to 1 or to $-1$. Regarding the coarse grained configurations $s^{(\varepsilon)}/m_{\beta,\gamma}$ as elements of $\mathcal{X}$, see (2.1), we will prove in the next theorem a LDP in $\mathcal{X}$ for $s^{(\varepsilon)}/m_{\beta,\gamma}$. However, as the LDP holds unchanged for $s^{(\varepsilon)}/m_\beta$, we will rather state it for the latter, for notational simplicity.

The rate function in the LDP is the following one. Setting $K = BV(\mathcal{T};\{\pm 1\})$ and denoting by $P(u)$, $u \in K$, the perimeter of the set $\{u = 1\}$ and by $\tau_\beta > 0$ the van der Waals surface tension, see (1.20) in [3], we define the functional $\mathcal{I}$ on $\mathcal{X}$ as

$$\mathcal{I}(v) = \begin{cases} \tau_\beta P(v) & \text{if } v \in K \\ +\infty & \text{else} \end{cases} \tag{2.10}$$

Notice that $\mathcal{I}$ is a good rate function in the sense that it is lower semicontinuous and its level sets are compacts, as the sets

$$K_a = \{u \in K : P(u) \le a\} \tag{2.11}$$

are compact in $\mathcal{X}$, see [9].

Now we can state the main result of the paper, that is a strong LDP for $s^{(\varepsilon)}/m_\beta$.

**Theorem 2.1** *For any closed subset $F$ of $\mathcal{X}$,*

$$\limsup_{\gamma\to 0}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}\big(m_\beta^{-1}s^{(\varepsilon)} \in F\big) \le -\inf_{u\in F}\mathcal{I}(u) \tag{2.12}$$

*and for any open subset $G$ of $\mathcal{X}$,*

$$\liminf_{\gamma\to 0}\liminf_{\varepsilon\to 0}\liminf_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}\big(m_\beta^{-1}s^{(\varepsilon)} \in G\big) \ge -\inf_{u\in G}\mathcal{I}(u) \tag{2.13}$$

Recalling that in our scheme the observables are elements $f$ of $C(\mathcal{X})$ the physically most interesting questions concern the events

$$\Big\{u \in \mathcal{X} : |f(u) - c| < \delta\Big\}$$

$\delta > 0$, namely the probability that a measurement of $f$ gives the value $c$ with tolerance $\delta$. By Theorem 2.1, using the lower semicontinuity and compactness of the rate function $\mathcal{I}(\cdot)$ we have, calling $g = |f - c|$,

$$\lim_{\delta\to 0}\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\lim_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}\Big(g(m_\beta^{-1}s^{(\varepsilon)}) < \delta\Big) = -\inf_{g(u)=0}\mathcal{I}(u) \tag{2.14}$$

(this is a shorthand for the statement that the right hand side is the limit both with all limsup and all liminf on the left hand side). (2.14) thus states that the probability of having $g = 0$ is reduced to the variational problem about the minimizer of the rate function under the contraint $\{g = 0\}$. Our proofs actually show that we can interchange the limits $\delta \to 0$ and $\gamma \to 0$, provided we change the normalization writing $m_{\beta,\gamma}^{-1}s^{(\varepsilon)}$ instead of $m_\beta^{-1}s^{(\varepsilon)}$. The special case where $g(v) = \|u - v\|$, $u \in K \equiv BV(\mathcal{T};\{\pm 1\})$, had already been worked out in [3]. The case

$$g(v) = \Big|\int_{\mathcal{T}} dr\, v(r) - s\Big|, \qquad |s| < m_\beta$$

corresponds to the Wulff problem.

The lower bound (2.13) is a straight consequence of the weak LDP proved in [3], its proof will be omitted. The upper bound follows from a proof that the coarse grained magnetization $s^{(\varepsilon)}$ is exponentially supported by neighborhoods of the compact sets $K_a$, see Proposition 3.1. This is the main technical point in the paper that will be proved in the next section, using contours and Peierls estimates. The upper bound (2.12) is proved in Section 4 using the exponential tightness and the upper bound of the weak LDP established in [3], the proof is classical and it is reported for the sake of completeness.

## 3. Exponential tightness

For any set $A$ in $L^1(\mathcal{T})$ and any $\delta > 0$ we denote by $A^\delta$ the $\delta$-neighborhood of $A$ in the $L^1$-norm, that is

$$A^\delta = \left\{ u \in L^1(\mathcal{T}) : \inf_{v \in A} \|u - v\| \le \delta \right\} \tag{3.1}$$

In this section we will prove "weak exponential tightness" in the sense that:

**Proposition 3.1** *There is a constant $c > 0$ such that for any $a > 0$ and $\delta > 0$*

$$\limsup_{\gamma \to 0} \limsup_{\varepsilon \to 0} \limsup_{L \to \infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma, L} \Big( m_\beta^{-1} s^{(\varepsilon)} \notin K_a^\delta \Big) \le -c\, a \tag{3.2}$$

*Outline of the proof.* After recalling from [3] the basic definitions of the block spin configurations $\eta$ and of the corresponding contours $\Gamma$, we will use these notions to construct $\pm 1$ valued, random variables $T(x)$, $x \in L_\gamma \mathcal{T}$, with the property that with large probability for $a$ large $P(T) \le a L_\gamma^{d-1}$ ($P(T)$ the perimeter of the boundary of the set $\{T = +1\}$). $T$ will be obtained from $\eta$ by "erasing the small contours" and by putting $T = \pm 1$ in the "large contours" in some careful way that will be specified below. In Lemma 3.2 we will then show that $P(T) \le a L_\gamma^{d-1}$ with large probability for $a$ large and in Lemma 3.3 that $\eta$ is super-exponentially close in $L^1$-norm to $T$. With these ingredients we will then prove Proposition 3.1 at the end of the section.

*Block spins and contours.*

We start from the coarse grained spin configuration $S^{(2^{-k})} \in L^\infty(L_\gamma \mathcal{T}; [-1,1])$, $k \in \mathbb{N}$, see (2.2). Given $k$ and $h$ in $\mathbb{N}$, $\zeta > 0$, we then define the block spin $\eta \in L^\infty(L_\gamma \mathcal{T}; \{0, \pm 1\})$ as

$$\eta(x) = \begin{cases} \pm 1 & \text{if } \left| S^{(2^{-k})}(y) \mp m_\beta \right| < \zeta \quad \text{for all } y \in C^{(2^h)}(x) \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

We also define the block spin $\eta$ induced by a function $m \in L^\infty(\Delta; [-1,1])$ using the analogous of (3.3)

The point $x$ is called correct, or, equivalently, $\eta(x)$ is correct, if $\eta(x) \neq 0$ and $\eta(y) = \eta(x)$ on the cubes $C^{(2^h)}$ that are $\star$-connected to $C^{(2^h)}(x)$. $x$ is incorrect if it is not correct.

Each maximal $\star$-connected component of the incorrect set is the support of a contour, the contour $\Gamma$ is defined by its support and by the values of the block spins on its support. When there is no risk of confusion, we may denote by $\Gamma$ only its support. We denote by $\#\Gamma$ the number of block cubes $C^{(2^h)}$ in the spatial support of $\Gamma$ and by $|\Gamma|$ its length ($|\Gamma| = 2^{hd}\#\Gamma$). $\mathrm{Ext}(\Gamma)$ is the largest connected component of $\Gamma^c$ and $\mathrm{Int}(\Gamma) = \mathrm{Ext}(\Gamma)^c$; finally $\mathrm{vol}(\Gamma)$ the number of block cubes $C^{(2^h)}$ in $\mathrm{Int}(\Gamma)$. If $\Gamma$ is a contour produced by a spin configuration $\sigma$, we write $\sigma \Rightarrow \Gamma$ and we say that $\{\Gamma_1, \dots, \Gamma_k\}$ is a collection of compatible contours if there is a spin configuration which produces all of them. In the same way, we write $m \Rightarrow \Gamma$ when the block spin $\eta$ is induced by $m \in L^\infty(\Delta; [-1,1])$.

*Non local excess free energy functional, Peierls estimates.*

Let $\Lambda$ be a measurable set in $\mathbb{R}^d$ (or in a torus) and $m \in L^\infty(\Lambda, [-1,1])$. The excess free energy $\mathcal{F}_\Lambda(m)$ of $m$ in $\Lambda$ is defined by the formula (A.1) in the appendix, we do not need its explicit expression here. As a consequence of the Peierls estimates it is shown in Lemma 6.5 of [3] that if $\underline{\Gamma} = \{\Gamma_1, \dots, \Gamma_k\}$ is a collection of compatible contours, then

$$\mu_{\gamma,L}\Big(\sigma \Rightarrow \underline{\Gamma}\Big) \leq \exp\left[-\beta\gamma^{-d}\sum_{i=1}^{\kappa}\left(\inf_{m\Rightarrow\Gamma_i}\mathcal{F}_{\Gamma_i}(m) - o_\gamma(1)|\Gamma_i|\right)\right] \tag{3.4}$$

where $o_\gamma(1)$ vanishes with $\gamma$. Moreover, by Theorem 6.2 of [3], there is a constant $\alpha > 0$ ($\alpha$ depends on $\zeta$ and $k$ and can be chosen as $\alpha = c\zeta^2 2^{-kd}$) such that for any contour $\Gamma$

$$\inf_{m\Rightarrow\Gamma}\mathcal{F}_\Gamma(m) \geq \alpha\#\Gamma \tag{3.5}$$

Therefore

$$\log\mu_{\gamma,L}\Big(\sigma \Rightarrow \underline{\Gamma}\Big) \leq -\beta\gamma^{-d}\sum_{i=1}^{\kappa}\left(\frac{1}{2}\inf_{m\Rightarrow\Gamma_i}\mathcal{F}_{\Gamma_i}(m) + \left(\alpha/2 - o_\gamma(1)2^{hd}\right)\#\Gamma_i\right) \tag{3.6}$$

We fix $\zeta' > 0$ and for $\Lambda \subset L_\gamma\mathcal{T}$, $m \in L^\infty(\Lambda; [-1,1])$ we consider

$$\Phi_m(x) = \begin{cases} -1 & \text{if } S^{(1)}(x) \leq -m_\beta + \zeta' \\ 1 & \text{otherwise} \end{cases} \tag{3.7}$$

We will prove in the Appendix that there is $c > 0$ dependent only on $\zeta'$, such that

$$N^{\pm}(m) \leq c\mathcal{F}_\Lambda(m) \tag{3.8}$$

where $N^{\pm}(m)$ is the number of pairs of cubes $C$ in $\Lambda$ which are connected and where $\Phi_m$ has opposite signs.

*The set of small contours and the random variable $T(x)$.*

We denote by $\Omega^b$, $b \in (0,1/d)$, the set of all the contours with length less than $L_\gamma^b$ and we define $T(x)$, $x \in L_\gamma \mathcal{T}$, as follows. If $x$ belongs to $\mathrm{Int}(\Gamma)$, where $\Gamma \in \Omega^b$ is maximal in $\Omega^b$ (it is not contained in the interior of any other contour of $\Omega^b$), then $T(x) = \pm 1$ according to the sign of the cubes in $\mathrm{Ext}(\Gamma)$ $\star$-connected to the boundary of $\Gamma$. If $x$ is not in a contour, we set $T(x) = \eta(x)$ and finally, if $x$ belongs to a contour $\Gamma \notin \Omega^b$, we consider a minimizer $m^\star$ of $\inf_{m \Rightarrow \Gamma} \mathcal{F}_\Gamma(m)$ and put $T(x) = \Phi_{m^\star}(x)$. We also define $t(r) = T(L_\gamma r)$.

**Lemma 3.2** *There is a constant $c > 0$ such that for any $a > 0$*

$$\limsup_{\gamma \to 0} \limsup_{L \to \infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,L}\big(P(T) > a L_\gamma^{d-1}\big) \le -c\, a \tag{3.9}$$

*Proof.* Let $\underline{\Gamma} = \{\Gamma_1, \dots, \Gamma_\kappa\}$ be the collection of long contours produced by a spin configuration. We first remark from the definition of the variable $T$ that we can bound its perimeter $P(T)$ proportionally to $\sum_{i=1}^{\kappa} N_i^\pm$. $N_i^\pm$ is the number of couples of connected cubes $C^{(1)}$ in the support of $\Gamma_i$ where $\Phi_{m_i^\star}$ has opposite signs, $m_i^\star$ being the minimizer of $\inf_{m \Rightarrow \Gamma_i} \mathcal{F}_{\Gamma_i}(m)$. So using (3.8), there is a constant $c' > 0$ (depending only on $\zeta'$) such that

$$\mu_{\gamma,L}\big(P(T) > a L_\gamma^{d-1}\big) \le \sum_{\underline{\Gamma} \in \mathcal{G}_{c'a}} \mu_{\gamma,L}\big(\sigma \Rightarrow \underline{\Gamma}\big) \tag{3.10}$$

where $\mathcal{G}_{c'a}$ is the set of all the collections of compatible contours $\underline{\Gamma} = \{\Gamma_1, \dots, \Gamma_\kappa\}$ such that $\Gamma_i \notin \Omega^b$, $\sum_i \mathcal{F}_{\Gamma_i}(m_i^\star) \ge c' a L_\gamma^{d-1}$. Notice that $\kappa \le L_\gamma^{d-b}$ since the total length of the contours can not exceed $L_\gamma^d$. Then applying (3.6) for $\underline{\Gamma} \in \mathcal{G}_{c'a}$

$$\log \mu_{\gamma,L}\Big(\sigma \Rightarrow \underline{\Gamma}\Big) \le -\beta\gamma^{-d}\left[c' a L_\gamma^{d-1}/2 + \Big(c\zeta^2 2^{-kd} - o_\gamma(1) 2^{hd}\Big) \sum_{i=1}^{\kappa} \#\Gamma_i\right] \tag{3.11}$$

Thus for $\gamma$ small enough the r.h.s. of (3.10) is bounded above by

$$\exp\big(-c' a \beta \gamma^{-d} L_\gamma^{d-1}/2\big) \left[1 + \sum_{\Gamma : |\Gamma| \ge L_\gamma^b} \exp\big(-c\beta\gamma^{-d}\zeta^2 2^{-(k+h)d} |\Gamma|/2\big)\right]^{L_\gamma^{d-b}} \tag{3.12}$$

Moreover, using a well known combinatorial argument (see for instance Theorem 6.3 of [3]), if $\gamma$ is sufficiently small, then the previous term is less than

$$\exp\big(-c' a \beta \gamma^{-1} L^{d-1}/2\big) \left[1 + L_\gamma^d \exp\big(-c\beta\zeta^2 2^{-(k+h)d} \gamma^{b-d} L^b/8\big)\right]^{L_\gamma^{d-b}} \tag{3.13}$$

and the Lemma follows. □

**Lemma 3.3** *For any $\delta > 0$ and for $\gamma$ small enough,*

$$\limsup_{L\to\infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,L}\big(\|T-\eta\| > \delta L_\gamma^d\big) = -\infty \tag{3.14}$$

*Proof.* From the definition of $T$, we get that

$$\|T-\eta\| \le 2^{hd+1} \sum_{\Gamma\in\Omega^b} \mathrm{vol}(\Gamma) + 2 \sum_{\Gamma\notin\Omega^b} |\Gamma| \tag{3.15}$$

By the Peierls estimates (3.6), for any $\delta > 0$

$$\limsup_{L\to\infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,L}\Big(\sum_{\Gamma\notin\Omega^b} |\Gamma| > \delta L_\gamma^d\Big) = -\infty \tag{3.16}$$

provided $\gamma$ is small enough. So we are reduced to study the cost of the event

$$\mathcal{B}(\delta) := \Big\{ \sum_{\Gamma\in\Omega^b} \mathrm{vol}(\Gamma) \ge \delta |L_\gamma|^d \Big\} \tag{3.17}$$

Let $\mathcal{D}^{(\ell)}$ be the partition of $L_\gamma\mathcal{T}$ into cubes $A_i$ of side $\ell = 10(L_\gamma)^b$ and let $N = \ell^{-d}L_\gamma^d$ be the number of these cubes. We call $d_a B$, $a \in \mathbb{R}^d$, $B \subset \mathbb{R}^d$, the translate by $a$ of $B$.

*A geometric remark.* There are $n$ vectors $\{e_j\}$ such that the following holds. Let $\Gamma$ be a contour in $\Omega^b$ and $\Gamma \cap A_i \neq \emptyset$. Then there is $j \in \{1,\dots n\}$ so that $\Gamma < d_{\ell e_j} A_i$, by this meaning that $\Gamma$ is strictly contained in $d_{\ell e_j} A_i$ and the distance from the complement of $d_{\ell e_j} A_i$ is $> 2^{h+10}$, $2^h$ the side of the cubes in the definition of the block spins. As a consequence,

$$\sum_{A_i} \sum_{\{e_j\}} \sum_{\Gamma\in\Omega^b} \mathbf{1}_{\{\Gamma < d_{\ell e_j} A_i\}} \mathrm{vol}(\Gamma) \ge \sum_{\Gamma\in\Omega^b} \mathrm{vol}(\Gamma) \tag{3.18}$$

If we define for any vector $e \in \mathbb{R}^d$

$$\mathcal{B}_e(\delta) := \Big\{ \sum_{A_i} \sum_{\Gamma\in\Omega^b} \mathbf{1}_{\{\Gamma < d_{\ell e} A_i\}} \mathrm{vol}(\Gamma) \ge \delta |\gamma L|^d \Big\} \tag{3.19}$$

then

$$\mathcal{B}(\alpha) \subset \bigcup_{j=1}^{n} \mathcal{B}_{e_j}(\alpha/n) \tag{3.20}$$

It is therefore enough to prove that for any $\delta > 0$ and any $e \in \mathbb{R}^d$

$$\limsup_{L\to\infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,L}\big(\mathcal{B}_e(\delta)\big) = -\infty \tag{3.21}$$

For notational simplicity we take in the following $e = 0$, dropping when possible the subscript 0 ($e = 0$).

For each cube $A_i$ we define a random variable $\xi_i$ with values in $\{0,1\}$ as follows. We set $\xi_i = 1$ if

$$\sum_{\Gamma\in\Omega^b} \mathbf{1}_{\{\Gamma<A_i\}} \frac{\mathrm{vol}(\Gamma)}{|A_i|} \geq \delta', \qquad \delta' = \frac{\delta}{2} \tag{3.22}$$

Otherwise we set $\xi_i = 0$.

We want to prove that

$$\mathcal{B}_0(\delta) \subset \Big\{\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{\xi_i=1\}} \geq \delta'\Big\} \tag{3.23}$$

Calling $M$ the number of $i$'s such that $\xi_i = 1$, we suppose, by contradiction, that $M < \delta' N$. Then

$$\frac{1}{|\gamma L|^d}\sum_{i=1}^{N}\sum_{\Gamma\in\Omega^b} \mathbf{1}_{\{\Gamma<A_i\}}\mathrm{vol}(\Gamma) \leq \frac{1}{N}\Big(\delta'(N-M)+M\Big) < \delta \tag{3.24}$$

and (3.23) is proved.

Let $\partial A_i$ be the union of all the blocks in $A_i$ connected to $A_i^c$ and $\partial\mathcal{D}$ the union of all $\partial A_i$. By conditioning on $S_{\partial\mathcal{D}}$ we get

$$\mu_{\gamma,L}\big(\mathcal{B}_0(\delta)\big) = E_{\mu_{\gamma,L}}\Bigg(\sum_{\{a_i\}\in\{0,1\}^N} \mathbf{1}_{\{\sum a_i>\delta' N\}}\prod_{i=1}^{N}\mu_{\gamma,L}\Big(\xi_i = a_i \Big| S_{\partial\mathcal{D}}\Big)\Bigg) \tag{3.25}$$

Thus

$$\mu_{\gamma,L}\big(\mathcal{B}_0(\delta)\big) \leq 2^N \sup_{S_{\partial\mathcal{D}}}\ \sup_{\{\sum a_i>\delta' N\}} \prod_{\{a_i=1\}} \mu_{\gamma,L}\Big(\xi_i = 1 \Big| S_{\partial\mathcal{D}}\Big) \tag{3.26}$$

Let $B_i$ be the intersection of $A_i$ and the union of all the contours that intersect $\partial A_i$. The set of spin configurations that give rise to $B_i$ is not in the $\sigma$-algebra generated by the spins in $B_i$ itself. We then define $\bar{B}_i$ which is obtained as follows. We first add to $B_i$ all the block cubes that are $\star$ connected to $B_i$ and then repeat the operation starting from this new set, call $B_i^\star$ this second set. We next consider all the block cubes that are $\star$ connected to $\partial\mathcal{D}$, the union of this set and $B_i^\star$ is the set $\bar{B}_i$.

The set of spin configurations that give rise to $B_i$ is in the $\sigma$-algebra generated by the spins in $\bar{B}_i$. Moreover if $\Gamma < A_i$ then $\Gamma \cap B_i = \emptyset$.

After conditioning in (3.26) on $S_{\bar{B}_i}$ we use the Chebishev inequality and get

$$\mu_{\gamma,L}\Big(\xi_i = 1 \Big| S_{\bar{B}_i}\Big) \leq \frac{1}{\delta'}\sum_{C\subset A_i\setminus B_i}\frac{|C|}{|A_i|}\sum_{\Gamma}\mathbf{1}_{\{C\subset \mathrm{Int}(\Gamma)\}}\mu_{\gamma,L}\Big(\Gamma\Big|S_{\bar{B}_i}\Big) \tag{3.27}$$

where by an abuse of notation the sum is over all $\star$-connected sets $\Gamma$ with $|\Gamma| \leq L_\gamma^b$ and such that $\Gamma \subset A_i \setminus B_i$; $C$ is a block cube and $\mu_{\gamma,L}\left(\Gamma \middle| S_{\bar{B}_i}\right)$ is the probability that $\Gamma$ is the spatial support of a contour.

Then

$$\mu_{\gamma,L}\left(\xi_i = 1 \middle| S_{\bar{B}_i}\right) \leq \frac{1}{\delta'} \frac{|C|}{|A_i|} \sum_{l=4}^{L_\gamma^b} \sum_{\Gamma \subset A_i \setminus B_i, |\Gamma| = l} l^d \mu_{\gamma,L}\left(\Gamma \middle| S_{\bar{B}_i}\right) \tag{3.28}$$

By the Peierls estimate

$$\mu_{\gamma,L}\left(\xi_i = 1 \middle| S_{\bar{B}_i}\right) \leq \frac{1}{\delta'} c e^{-c'\gamma^{-d}} \tag{3.29}$$

Then, from (3.26)

$$\mu_{\gamma,L}\left(\mathcal{B}_0(\delta)\right) \leq \left(\frac{2c}{\delta'}\right)^N e^{-c'\gamma^{-d}\delta' N} \tag{3.30}$$

recalling $N = \ell^{-d} L_\gamma^d$, $\ell = 10 L_\gamma^b$ and $0 < b < 1/d$, we get (3.21) $\square$

**Lemma 3.4** *For any $a > 0$ and $\delta > 0$,*

$$\limsup_{\varepsilon \to 0} \sup_{u \in K_a} \sup_{\|u - v\| \leq \delta} \|u - v^{(\varepsilon)}\| \leq \delta \tag{3.31}$$

*Proof.* Let $u$ be a function in $K_a$, then for any $\alpha > 0$, there exists $w_\alpha \in BV(\mathcal{T}; \{\pm 1\})$ such that the boundary of the set $\{w_\alpha = +1\}$ is a $C^\infty$ surface, $\|u - w_\alpha\| \leq \alpha$ and $|P(u) - P(w_\alpha)| \leq \alpha$, [11]. We define the $\mathcal{D}^{(\varepsilon)}$-measurable function $\tilde{w}_\alpha^{(\varepsilon)}$ as $\pm 1$ according to the sign of the coarse grained $w_\alpha^{(\varepsilon)}$. Remark that since $w_\alpha$ has a regular boundary, the volume of the cubes $C^{(\varepsilon)} \in \mathcal{D}^{(\varepsilon)}$ where $\tilde{w}_\alpha^{(\varepsilon)} \neq w_\alpha$ is going to 0 with $\varepsilon$ and as a consequence we have

$$\limsup_{\varepsilon \to 0} \|\tilde{w}_\alpha^{(\varepsilon)} - u\| \leq \alpha \tag{3.32}$$

Let $v \in L^1(\mathcal{T})$ such that $\|u - v\| \leq \delta$. As $\tilde{w}_\alpha^{(\varepsilon)}$ is $\mathcal{D}^{(\varepsilon)}$-measurable

$$\|v^{(\varepsilon)} - \tilde{w}_\alpha^{(\varepsilon)}\| \leq \|v - \tilde{w}_\alpha^{(\varepsilon)}\| \tag{3.33}$$

We deduce from this inequality that

$$\|u - v^{(\varepsilon)}\| \leq \|u - v\| + 2\|\tilde{w}_\alpha^{(\varepsilon)} - u\| \tag{3.34}$$

and from (3.32) that for any $\alpha > 0$

$$\limsup_{\varepsilon \to 0} \sup_{\|u - v\| \leq \delta} \|u - v^{(\varepsilon)}\| \leq \delta + 2\alpha \tag{3.35}$$

The compactness of $K_a$ implies that the supremum over $u \in K_a$ in (3.31) can be written as a maximum over a finite number of elements of $K_a$. Thus, the Lemma follows from (3.35). $\square$

*Proof of Proposition 3.1.* We first relate the mesoscopic coarse grained configuration $S^{(2^h)}$ to the variable $T$: we observe that

$$\|S^{(2^h)} - m_\beta T\| \le \zeta L_\gamma^d + \int dr\, \mathbf{1}_{\{|S^{(2^h)} - m_\beta T| \ge \zeta\}} \le \zeta L_\gamma^d + \|\eta - T\| \tag{3.36}$$

Fix $a > 0$ and $\delta > 0$, then, recalling that $t(r) = T(L_\gamma r)$,

$$\mu_{\gamma,L}\big(m_\beta^{-1} s^{(\varepsilon)} \notin K_a^\delta\big) \le \mu_{\gamma,L}\big(P(t) > a\big) + \mu_{\gamma,L}\big(\|m_\beta^{-1} s^{(\varepsilon)} - t\| > \delta, P(t) \le a\big) \tag{3.37}$$

¿From Lemma 3.4, there exists $\varepsilon(\delta)$ such that for any $0 < \varepsilon < \varepsilon(\delta)$, the last term of the r.h.s. of the previous inequality is bounded above by

$$\mu_{\gamma,L}\big(\|m_\beta^{-1} S^{(2^h)} - T\| > \delta L_\gamma^d/2\big) \tag{3.38}$$

and using (3.36) with $\zeta < m_\beta \delta/4$, we obtain

$$\mu_{\gamma,L}\big(\|m_\beta^{-1} s^{(\varepsilon)} - t\| > \delta, P(t) \le a\big) \le \mu_{\gamma,L}\big(\|\eta - T\| > \delta m_\beta L_\gamma^d/4\big) \tag{3.39}$$

Finally, by (3.37) and Lemma 3.3,

$$\limsup_{\gamma\to 0} \limsup_{\varepsilon\to 0} \limsup_{L\to\infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,L}\big(m_\beta^{-1} s^{(\varepsilon)} \notin K_a^\delta\big)$$
$$\le \limsup_{\gamma\to 0} \limsup_{L\to\infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,L}\big(P(T) > a L_\gamma^{d-1}\big) \tag{3.40}$$

and Lemma 3.2 concludes the proof. $\square$

## 4. Upper bound

The upper bound (2.12) will follow from the exponential tightness (see Proposition 3.1) if for any closed subset $F$ of $L^1(\mathcal{T})$ and for any $a > 0$,

$$\lim_{\delta\to 0} \limsup_{\gamma\to 0} \limsup_{\varepsilon\to 0} \limsup_{L\to\infty} \frac{\gamma}{\beta L^{d-1}} \log \mu_{\gamma,L}\big(m_\beta^{-1} s^{(\varepsilon)} \in (F \cap K_a)^\delta\big) \le -\inf_{u\in F} \mathcal{I}(u) \tag{4.1}$$

¿From the compactness of the level set $K_a$, there exists a finite subset $F(a,\delta)$ of $F \cap K_a$ such that

$$(F \cap K_a)^\delta \subset \bigcup_{u\in F(a,\delta)} B(u, 2\delta) \tag{4.2}$$

where $B(u,\delta)$ is the ball with center $u$ and radius $\delta$ for the $L^1$-norm. Therefore

$$\begin{aligned}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}&\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}\left(m_\beta^{-1}s^{(\varepsilon)}\in (F\cap K_a)^\delta\right)\\ &\le \max_{u\in F(a,\delta)}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}\left(m_\beta^{-1}s^{(\varepsilon)}\in B(u,2\delta)\right)\end{aligned}\tag{4.3}$$

Let $u_{a,\delta,\gamma}\in F(a,\delta)$ be the function for which the above maximum is obtained. Then using again the compactness of $K_a$, there are sequences of positive numbers $\delta_n$ and $\gamma_k$ going to 0 such that $u_{a,\delta_n,\gamma_k}$ is converging in $L^1$ to some function $u_a\in F\cap K_a$ when $k$ and then $n$ go to infinity . So, for any $\alpha>0$,

$$\begin{aligned}\lim_{\delta\to 0}\limsup_{\gamma\to 0}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}&\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}\left(m_\beta^{-1}s^{(\varepsilon)}\in (F\cap K_a)^\delta\right)\\ &\le \limsup_{\gamma\to 0}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}\left(m_\beta^{-1}s^{(\varepsilon)}\in B(u_a,\alpha)\right)\end{aligned}\tag{4.4}$$

Now, from the proof of the upper bound of the weak large deviation principle in [BBBP]

$$\lim_{\alpha\to 0}\limsup_{\gamma\to 0}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\frac{\gamma}{\beta L^{d-1}}\log\mu_{\gamma,L}\left(m_\beta^{-1}s^{(\varepsilon)}\in B(u_a,\alpha)\right)\le -\mathcal{I}(u_a)\le -\inf_{u\in F}\mathcal{I}(u)\tag{4.5}$$

This inequality together with (4.4) implies (4.1).

## Appendix

In this appendix we will prove the inequality (3.8), the proof is similar to one in [5]. We first recall that the excess free energy of $m$ in $\Lambda$, $\Lambda$ a $\mathcal{D}^{(1)}$ measurable set in $\mathbb{R}^d$ (or in a torus) and $m\in L^\infty(\Lambda,[-1,1])$, is

$$\mathcal{F}_\Lambda(m)=\int_\Lambda dx\,\left[f\left(m(x)\right)-f(m_\beta)\right]+\frac{1}{4}\iint_{\Lambda\times\Lambda}dx\,dy\,J(|x-y|)\left[m(x)-m(y)\right]^2\tag{A.1}$$

where

$$f(m)=-\frac{m^2}{2}-\beta^{-1}i(m)\tag{A.2}$$

$$i(m)=-\frac{1-m}{2}\log\frac{1-m}{2}-\frac{1+m}{2}\log\frac{1+m}{2}\tag{A.3}$$

Observe that $f(m_\beta)$ is the minimum of $f(m)$ so that (A.1) is the sum of two non negative terms.

We fix $\zeta' > 0$ and for any function $m \in L^\infty(\Lambda, [-1,1])$ we consider

$$\Psi_m(x) = \begin{cases} 1 & \text{if } S^{(1)}(x) \geq m_\beta - \zeta' \\ -1 & \text{if } S^{(1)}(x) \leq -m_\beta + \zeta' \\ 0 & \text{otherwise} \end{cases} \tag{A.4}$$

Notice that the function $\Phi_m$ defined in (3.7) satifies $\Phi_m = 1$ if $\Psi_m \geq 0$ and $\Phi_m = -1$ if $\Psi_m = -1$. We denote by $N^0(m)$ the number of cubes $C$ in $\Lambda$ where $\Psi_m = 0$ and by $N^{\pm}(m)$ the number of pairs of cubes $C$ in $\Lambda$ which are connected and where $\Psi_m$ has opposite signs. Then (3.8) is a straight consequence of the following lemma

**Lemma A.1** *There is a constant $c > 0$ (depending on $\zeta'$) such that for any $m \in L^\infty(\Lambda, [-1,1])$*

$$N^0(m) + N^{\pm}(m) \leq c\mathcal{F}_\Lambda(m) \tag{A.5}$$

*Proof.* We start from a geometric remark. Let $e_1, \dots, e_n$ be the unit coordinates vectors of $\mathbb{R}^d$, $e_0 = 0$ and $d_e\mathcal{D}$ be the translate of the partition $\mathcal{D}$ by the vector $e$. If $C_1$ and $C_2$ are two connected cubes in $\mathcal{D}^{(1)}$, then there exists $0 \leq i \leq n$ and $C \in d_{e_i}\mathcal{D}^{(2)}$ such that $C_1 \cup C_2 \subset C$ and $C \subset \Lambda$. We denote by $\mathcal{D}_{(j)}$, $0 \leq j \leq d$, the collection of all the cubes of $d_{e_j}\mathcal{D}^{(2)}$ that are in $\Lambda$. We also denote by $\mathcal{D}_{(-1)}$ the unit cubes in $\Lambda$. Finally we let $N_j^{\pm}(m)$, $0 \leq j \leq d$, be the number of cubes in $\mathcal{D}_{(j)}$ where $\Psi_m$ takes both values 1 and $-1$. So

$$N^{\pm}(m) \leq \sum_{j=0}^{d} N_j^{\pm}(m) \tag{A.6}$$

Then dropping out the interaction between cubes,

$$\mathcal{F}_\Lambda(m) \geq \frac{1}{d+2} \sum_{j=-1}^{d} \sum_{C \in \mathcal{D}_{(j)}} \mathcal{F}_C(m_C) \tag{A.7}$$

where $m_C$ is the restriction of $m$ to $C$. We define

$$\chi_C(x) = \int_C dy\, J(x-y) \tag{4.8}$$

and

$$\overline{\mathcal{F}}_C(m) = \int\limits_C dx\, \chi_C(x) \left[f\left(m(x)\right) - f(m_\beta)\right] + \frac{1}{4} \iint\limits_{C\times C} dx\, dy\, J(|x-y|) \left[m(x) - m(y)\right]^2 \tag{A.9}$$

Since $\chi_C \le 1$, $\overline{\mathcal{F}}_C \le \mathcal{F}_C$. Moreover $\overline{\mathcal{F}}_C$ is a lower semicontinuous functional for the weak topology because

$$\overline{\mathcal{F}}_C(m) = -\beta^{-1} \int_C dx\, \chi_C(x) i\big(m(x)\big) - \frac{1}{2} \int\!\!\int_{C\times C} dx\, dy\, J(|x-y|) m(x) m(y) - |C| f(m_\beta) \quad (A.10)$$

By convexity the first term is lower semicontinuous while the second one is continuous. Therefore there is $c' > 0$ depending only on $\zeta'$ such that $\overline{\mathcal{F}}_C(m) \ge c'$ for any cube $C$ in $\mathcal{D}_{(-1)}$ where $\Psi_m = 0$ and any cube $C$ in $\mathcal{D}_{(j)}$ where $\Psi_m$ takes both values 1 and $-1$. $\square$

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