Positive Commutators and Spectrum of Nonrelativistic QED

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Abstract

In this paper we consider the Hamiltonian of the standard model of non-relativistic QED. In this model non-relativistic quantum particles interact with quantized electro-magnetic field and their interaction is subjected to an ultra-violet cut-off. We prove absence of excited states and absolute continuity of the spectrum for sufficiently small charges under conditions on the coupling functions which milder than those of [2]. We use the method of positive commutators with the "conjugate" operator deformed appropriately in order to accomodate the interaction term.

1 Introduction

In this paper we develop the method of positive commutators for nonrelativistic quantum electrodynamics (QED). The latter is described by the Schrödinger operator (in the Coulomb gauge and in the units in which $\hbar = 1$ and $c = 1$)

$$H(e) \;=\; \sum_{j=1}^{N} \frac{1}{2m_j}\Big(p_j - e_j A(x_j)\Big)^2 + V(x)\otimes \mathbf{1}_f + \mathbf{1}_{\mathrm{part}} \otimes H_f\,, \tag{1.1}$$

corresponding to a system of N quantum particles with masses $m_1, \ldots, m_N$, charges $e_1, \ldots, e_N$, coordinates $x_1, \ldots, x_N$ and (quantum) momenta $p_1, \ldots, p_N$, interacting with quantized electromagnetic field. The latter is described by the (transverse) vector potential $A(y)$, the scalar (longitudinal) potential $V(x)$, $x = (x_1, \ldots, x_N)$, and the field Hamiltonian H_f. Here, $e = (e_1, \ldots, e_N)$ and $H(e)$ and $A(y)$ act on $\mathcal{H}_{\mathrm{part}} \otimes \mathcal{H}_f$, where $\mathcal{H}_{\mathrm{part}}$ and $\mathcal{H}_f$ are the particle and radiation Hilbert spaces, respectively. Usually, $\mathcal{H}_{\mathrm{part}}$ is $L^2(X)$ or a subspace thereof, where X is the particle configuration space, and $\mathcal{H}_f$ is a Bosonic Fock space. Note that in our units the charge of the electron, $-e$, is $-2\sqrt{\pi\alpha}$, where $\alpha = \frac{1}{137}$, the fine structure constant.

For $|e_j|$'s sufficiently small we construct a deformation of the second quantized generator of dilations for photons so that the resulting operator has a positive commutator with $H(e)$ locally in small neighbourhoods of the excited eigenvalues of the particle Hamiltonian

$$H_{\mathrm{part}} \;=\; \sum_{j=1}^{N} \frac{1}{2m_j} p_j^2 + V(x)\,, \tag{1.2}$$

acting on $\mathcal{H}_{\mathrm{part}}$. This implies that in those neighbourhoods (i) $H(e)$ has no eigenvalues, (ii) $H(e)$ has purely absolutely continuous spectrum and (iii) $H(e)$ satisfies the limiting absorption principle. Conclusion (i) is derived via a virial theorem and (ii) and (iii), with a help of the Kato-Mourre theory extended further in this paper. Note that the limiting absorption principle constitutes the first step in the analysis of the time behaviour of a system in question.

A result similar to our spectral result was obtained first under more stringent conditions in [1, 2, 3] (see the paragraph after Theorem 3.1). For massive quantum fields such results were given in [16, 6, 8, 7, 10, 11]. Also, [8] introduced commutator estimates which inspired our method.

The commutator methods were initiated in [18, 12], developed in [13] and turned into a deep theory in [14]. They were shown in [14, 15, 17, 20] to be a powerful tool in understanding spectral properties of quantum Hamiltonians and in controlling the time evolution generated by these Hamiltonians. We see the present paper as a step toward these goals, more precisely, toward a dynamical theory of photons interacting with matter.

2 Nonrelativistic QED Hamiltonian

Consider a particle system interacting with the electromagnetic field. Both the particles and transverse part of the field are quantized. The corresponding Hamiltonian is given by (1.1). We assume that $V(x)$ is the classical potential of interaction between the particles (in particular, it absorbs the longitudinal electric potential). The radiation Hamiltonian, H_f, can be expressed in terms of the (photon) creation and annihilation operators, $a^*(k)$ and $a(k)$ as

$$H_f \;=\; \int \omega(k)\, a^*(k) \cdot a(k)\, dk\;, \tag{2.1}$$

where $\omega = \omega(k) = |k|$ is the photon energy. Here $a^*(k)$ and $a(k)$ are operator-valued transverse vectors, i.e., they satisfy $k \cdot a^*(k) = 0$ and $k \cdot a(k) = 0$. For each $x \in \mathbb{R}^3$, the vector potential $A(x)$ acts on $\mathcal{H}_f$ as an operator-valued distribution and is given by the following expression.

$$A(x) \;=\; \int \Big(e^{-ikx} \otimes a^*(k) \;+\; e^{ikx} \otimes a(k) \Big) \frac{\kappa(k)}{\sqrt{\omega(k)}} dk\;, \tag{2.2}$$

where κ is an ultraviolet cut-off, i.e., a real function on $\mathbb{R}^3$ vanishing sufficiently fast as $|k| \to \infty$. In order to keep the infrared features of the original system, one can take $\kappa \equiv 1$ in, say, the unit ball, $\{|k| \leq 1\}$. However, the particular form of κ will be of no importance in our analysis. Only certain estimates on κ will be needed. Thus we can, as well, consider the vector potentials of a more general form

$$A(x) \;=\; \int \Big(G_x(k) \otimes a^*(k) \;+\; \overline{G}_x(k) \otimes a(k) \Big) dk\;, \tag{2.3}$$

where the function $G_x(k)$ will be assumed to satisfy several conditions (depending on the problem we tackle), the most important of which is

$$\sup_x \left\{ \int \left(1 + \frac{1}{\omega(k)}\right) |G_x(k)|^2 \, dk \right\} \ < \ \infty \, , \tag{2.4}$$

as this condition ensures semiboundedness of $H(e)$ and essential self-adjointness on the domain of $H(e = 0)$.

We remark that we have ignored the Zeeman term,

$$-\sum \frac{e_i}{2m_i} \, \sigma_i \cdot B(x_i) \, , \tag{2.5}$$

which is due to the interaction of the particle spins, σ_i, with the magnetic field $B(x) = \operatorname{curl} A(x)$.

In order to simplify notation and exposition, we demonstrate our approach on the model of a particle system interacting with a massless scalar field instead of the vector one. The Hamiltonian for such a model is given by

$$H \ = \ H_{\text{part}} \otimes \mathbf{1}_f + \mathbf{1}_{\text{part}} \otimes H_f + I \, , \tag{2.6}$$

acting on $\mathcal{H}_{\text{part}} \otimes \mathcal{F}$, where H_{part} is a particle (atomic) Hamiltonian,

$$H_{\text{part}} \ = \ \sum \frac{1}{2m_i} p_i^2 + V(x) \tag{2.7}$$

(with the same notation as above), acting on a dense domain in $\mathcal{H}_{\text{part}}$. H_f is a scalar field Hamiltonian,

$$H_f \ = \ \int \omega(k) \, a^*(k) \, a(k) \, dk \, , \tag{2.8}$$

with $\omega = \omega(k) = |k|$, as above, acting on $\mathcal{F}$, and I is the interaction given by

$$I \ := \ a^*(G_x) + a(G_x) \ = \ \int \Big(G_x(k) \otimes a^*(k) \ + \ \overline{G}_x(k) \otimes a(k) \Big) \, dk \, , \tag{2.9}$$

with $G_x(k)$ satisfying (2.4) (for referencing convenience we use here the same notation for coupling functions as in the vector case). Here $a^*(k)$ and $a(k)$ are creation and annihilation operators of a quantum scalar field acting on $\mathcal{F}$. $\mathcal{H}_{\text{part}}$ is either $L^2(X)$ of the particle configuration space X, which typically is $\mathbb{R}^{3N}$, or a fermionic subspace of $L^2(X)$. Finally, $\mathcal{F}$ is the Fock space for a scalar field, i.e., the Fock space built upon the one-photon space $L^2(\mathbb{R}^3)$. (We will still refer to the scalar field in question as *photon field.*) Note that for the scalar field the coupling cannot be minimal, i.e., through replacing the momentum operator by the covariant one.

This model contains all the difficulties of the vector one, but the infrared problem comes out in its pure form unencumbered by vector notation and inessential particulars of the model in question. It is straightforward to include in (2.9) also terms quadratic in a and a^*. We do not pursue this in order not to muddle the simple ideas involved. Finally we mention that the scalar model is a generalization of the so-called electric dipole approximation in QED.

Throughout the paper we assume that: $V(x)$ is a real function of the form $\sum V_i \circ \pi_i$, where π_i are linear maps from X ($=$ the particle configuration space) to $\mathbb{R}^{n_i}$ and $V \in L^{p_i}(\mathbb{R}^{n_i}) + L^\infty(\mathbb{R}^{n_i})$ with $p_i = 2$ if $n_i \le 3$, $p_i > 2$ if $n_i = 4$ and $p_i = \frac{n_i}{2}$ if $n_i \ge 5$. This is not the most general assumption compatible with our analysis, but it is simple and explicit. Under this assumption H_{part} is self-adjoint on the domain of $\sum_{j=1}^{N} \frac{1}{2m_j} p_j^2$. Its spectrum, $\sigma(H_{\mathrm{part}})$, is of the form $\{E_j\} \cup [\Sigma, \infty)$, where $E_0 \le E_1 \le E_2 \le \cdots \le \Sigma$. Moreover, $\{E_j\} \subseteq \sigma_{pp}(H_{\mathrm{part}})$ and $\sigma_{cont}(H_{\mathrm{part}}) = [\Sigma, \infty)$.

In what follows $E_\Delta(H)$ stands for a smoothed out cut-off function of a self-adjoint operator H, supported in Δ. Below we make use of the exponential decay of eigenfunctions of H_{part} in the sense that we use the existence of a constant, $C_{\Delta,M}$, such that

$$\left\| \langle x \rangle^M E_\Delta(H_{\mathrm{part}}) \right\| \le C_{\Delta,M} , \tag{2.10}$$

for all $M \ge 0$ and any compact set $\Delta \subseteq (-\infty, \Sigma)$. Here, $\langle x \rangle := \sqrt{1 + x^2}$.

3 Results

First, we consider the Hamiltonian H, as defined in (2.6)–(2.9). We show in the next section that H is self-adjoint, provided (2.4) holds. For our main result we impose a somewhat stronger condition on $G_x = G_x(k)$.

$$g := \sup_x \left\{ \int \left(1 + \frac{1}{\omega(k)}\right) |G_x(k)|^2 \, dk \; + \; \langle x \rangle^{-M} \int \frac{\left|(k \cdot \nabla_k)^2 G_x(k)\right|^2}{\omega(k)} \, dk \right\}^{1/2} < \infty . \tag{3.1}$$

Note that the last term in (3.1) can be replaced by $\|\omega^{-1/2}(k \cdot \nabla_k G_x - x \cdot \nabla_x G_x)\|_2$. If we use this alternative, then we have to make additional assumptions on the classical potential $V(x)$ typical for the Mourre theory (see e.g. [4, 9]).

Now let E^i and $\psi^{is}_{\rm part}$, $s = 1, \ldots, m_i$, be the eigenvalues and corresponding eigenfunctions of $H_{\rm part}$, where $i = 0, 1, \ldots$, and $E^0 < E^1 < \ldots$. For $i, j \geq 0$, we assume that $\int_{|k|=\omega} (A_{ij})^* A_{ij} dS_\omega$ is continuous in ω and vanishes at $\omega = 0$. Here A_{ij} are the $m_i \times m_j$ matrices with the entries $\langle \psi^{i\ell}_{\rm part}, G_x \psi^{jr}_{\rm part} \rangle$, $\ell = 1, \ldots, m_i$ and $r = 1, \ldots, m_j$, and dS_ω is the area element on the sphere $\{k \in \mathbb{R}^3 \mid |k| = \omega\}$. For $j \geq 1$ (i.e., for an excited states $\psi^{js}_{\rm part}$) we define the self-adjoint matrix Γ_j by

$$\Gamma_j \; = \sum_{i: E^i \leq E^j} \int (A_{ij})^* A_{ij} \, \delta(\omega - E^j + E^i) \, d^3k . \tag{3.2}$$

The eigenvalues of this matrix are the resonance widths at the second order associated with the eigenvalue E^j, as in Fermi's Golden Rule. We assume that we are given a family of coupling functions G_x (labeled by G_x) satisfying

$$\delta_j \; = \; \liminf_{g \to 0} g^{-2} \Gamma_j \; > \; 0 . \tag{3.3}$$

Now we consider the Hamiltonian $H(e)$, defined in (1.1). Using Kato's inequality one can easily show that $H(e)$ is self-adjoint under condition (2.4) (see [2]). We impose on G_x the same additional conditions as in the case of H, i.e. (3.1) and the condition formulated in the paragraph preceding Eqn. (3.2). Moreover, we define the matrix

$$\Gamma_j(e) \; = \sum_{i:\, E^i < E^j} \int (B_{ij})^* B_{ij} \, \delta(\omega - E^j + E^i) \, d^3k , \tag{3.4}$$

where B_{ij} are matrices with the entries $\langle \psi^{i\ell}_{\rm part}, \sum_{a=1}^{N} \frac{e_a}{m_a} p^\perp_a G_{x_a} \psi^{jr}_{\rm part} \rangle$ and $p^\perp = p - (p \cdot \hat{k}) \hat{k}$, with $\hat{k} = \frac{k}{|k|}$, the projection of p onto the plane, $k^\perp$, perpendicular to k. For the operator $H(e)$ we use the assumption

$$\delta_j^{\mathrm{EM}} \;=\; \liminf_{|e|\to 0} |e|^{-2}\Gamma_j(e) \;>\; 0 \tag{3.5}$$

where $|e| = \max_i |e_i|$. The main result of this paper is

Theorem 3.1 *Assume (3.1) and, for a given $j \geq 1$, (3.5) (resp. (3.3)). Then for $|e|$ (resp. g) sufficiently small, the spectrum of $H(e)$ (resp. H) in any interval, containing E^j but containing no other parts of the spectrum of H_{part} and whose distance to $\operatorname{spec} H_{\mathrm{part}} \cap (-\infty, E^j)$ is $\gg |e|$ (resp. $\gg g$), is purely absolutely continuous. Moreover, $H(e)$ (resp. H) has the local decay property (formulated below) in such an interval.*

The first statement of the theorem was proved in [1, 2, 3] under additional assumptions of analyticity for G_x and $\int \sup_x \frac{|G_x|^2}{\omega^{1+\beta}} < \infty$ for some $\beta > 0$, which is a stronger condition in the infrared region, $k \to 0$.

Now we formulate the local decay property mentioned in Theorem 3.1. To this end, we introduce the anti-selfadjoint operator

$$A \;=\; \mathbf{1}_{\mathrm{part}} \otimes \frac{1}{2} \int a^*(k)\Big(k \cdot \nabla_k + \nabla_k \cdot k\Big) a(k)\, d^3k\,. \tag{3.6}$$

This operator is a second quantization of the generator of dilations in the one-photon momentum space times $\sqrt{-1}$, i.e., $\frac{1}{2}(k \cdot \nabla_k + \nabla_k \cdot k)$. In what follows, whenever no danger of confusion arises, we omit the trivial factors $\mathbf{1}_{\mathrm{part}}\otimes$ and $\otimes \mathbf{1}_f$. We say that H has the *local decay property* in a spectral interval Δ (with respect to an operator A), if the following estimate holds

$$\int_{-\infty}^{\infty} \left\| \langle A \rangle^{-\alpha} e^{-iHt} \psi \right\|^2 dt \;\leq\; C_\alpha \|\psi\|^2\,, \tag{3.7}$$

for any $\alpha > 1/2$ and any $\psi \in \operatorname{Ran} E_\Delta(H)$. (In fact, a slightly stronger property, the limiting absorption principle with the Hölder constant $\theta < \alpha - \frac{1}{2}$, holds in our case.)

Theorem 3.1 follows from a positive commutator estimate derived below (Theorem 5.1) and from the Kato-Mourre theory mentioned in the introduction and expounded upon in Section 5. We prove only the part of Theorem 3.1 concerning the operator H. The corresponding part for the operator $H(e)$, given in (1.1), is proven in exactly the same way but with a couple more simple estimates related to the quadratic part $\sum \frac{e_j^2}{2m_j} A(x_j)^2$ of the perturbation $H(e) - H(0)$.

Observe that absolute continuity of the spectrum and the local decay property outside of $O(g^2)$ (resp. $O(|e|^2)$) neighbourhood of the eigenvalues and thresholds of $H_{\rm part}$ was proven in [1, 2].

4 Relative Bounds on the Interaction

In this section we collect some elementary bounds needed for the proof of Theorem 3.1. In what follows by $H_f^{-1/2}$, we always understand $H_f^{-1/2}\overline{P}_\Omega$, where $\overline{P}_\Omega$ is the projection onto the orthogonal complement of the vacuum state Ω in the Fock space.

Lemma 4.1 *(Relative bounds)*

$$\|a(f)\psi\|_{\rm Fock} \;\leq\; \left(\int \frac{|f|^2}{\omega}\right)^{1/2} \left\|(H_f)^{1/2}\psi\right\|_{\rm Fock} \tag{4.1}$$

and

$$\|a^*(f)\psi\|^2_{\rm Fock} \;\leq\; \left(\int \frac{|f|^2}{\omega}\right) \left\|(H_f)^{1/2}\psi\right\|^2_{\rm Fock} + \left(\int |f|^2\right) \|\psi\|^2_{\rm Fock}\,. \tag{4.2}$$

Proof: We drop the subindex "Fock" in the proof. By Schwarz' inequality we have

$$\|a(f)\psi\| \;\leq\; \int |f(k)|\,\|a(k)\psi\| \;\leq\; \left(\int \frac{|f|^2}{\omega}\right)^{1/2} \left(\int \omega(k)\,\|a(k)\psi\|^2\right)^{1/2}. \tag{4.3}$$

This, due to

$$\int \omega(k)\,\|a(k)\psi\|^2 \;=\; \langle \psi,\, H_f\psi\rangle\;, \tag{4.4}$$

implies (4.1). Inequality (4.2) follows from

$$a(f)a^*(f) \;=\; a^*(f)a(f) \;+\; \langle f,\, f\rangle \mathbf{1}\,, \tag{4.5}$$

$\langle \psi,\, a^*(f)a(f)\psi\rangle = \|a(f)\psi\|^2$ and (4.1). ∎

We rewrite bound (4.1) as

$$\left\| a(f)\, H_f^{-1/2} \right\|_{\text{Fock}} \;\leq\; \left(\int \frac{|f|^2}{\omega} \right)^{1/2} , \tag{4.6}$$

$$\left\| H_f^{-1/2}\, a^*(f) \right\|_{\text{Fock}} \;\leq\; \left(\int \frac{|f|^2}{\omega} \right)^{1/2} . \tag{4.7}$$

These two bounds are equivalent since the expressions under the norm signs are adjoint to each other. Moreover, (4.1) implies

$$\pm \langle a^*(f) + a(f) \rangle_\psi \;\leq\; 2 \left(\int \frac{|f|^2}{\omega} \right)^{1/2} \left\| H_f^{1/2} \psi \right\| \cdot \| \psi \| \, , \tag{4.8}$$

which yields, for any $\alpha > 0$,

$$\pm \Big(a^*(f) + a(f) \Big) \;\leq\; \alpha H_f \;+\; \frac{1}{\alpha} \int \frac{|f|^2}{\omega} \, . \tag{4.9}$$

Furthermore, inequalities (4.1) and (4.2) imply

$$\left\| \, \Big(a^*(f) + a(f) \Big) \psi \, \right\| \leq \; \left(\int |f|^2 \right)^{1/2} \| \psi \| \;+\; 2 \left(\int \frac{|f|^2}{\omega} \right)^{1/2} \left\| H_f^{1/2} \psi \right\| . \tag{4.10}$$

Eqn. (4.9) implies that I is H_f-form bounded with relative bound zero, provided (2.4) holds, while Eqn. (4.10) implies that I is $H_f^{1/2}$-bounded with relative bound $2 \sup_x \left(\int \frac{|G_x|^2}{\omega} \right)^{1/2}$, provided (2.4) holds. The latter of these two statements implies that, if (2.4) is satisfied, then H is self-adjoint on the domain of H_f.

5 Positive Commutators

In this section we formulate our key technical result. When we talk below about a communtator of two, in general unbounded, operators, say H and A, we understand that $D(H) \cap D(A)$ is dense (if necessary, in $D(H)$, in the graph norm) and $[H, A]$ is defined first as a form on $D(H) \cap D(A)$ and then extended to a bounded or relatively bounded operator.

We fix $j \geq 1$ once and for all. Let $P_{\text{part}} = P_{\text{part}}^j$ be the projection operator on the eigenspace of H_{part} corresponding to the eigenvalue E^j. For a fixed energy scale ρ we define the projection operator

$$P \;=\; P_{\text{part}} \otimes \chi_{H_f \leq \rho} \tag{5.1}$$

and, as usual, $\overline{P} = \mathbf{1} - P$. Now we define a family of operators

$$A_V \;=\; A \,+\, \overline{P}VP - PV^*\overline{P}\,, \tag{5.2}$$

where A_0 is the second quantized dilatation generator defined in (3.6), and

$$V \;=\; \theta \overline{R}_\varepsilon^2 I\,, \qquad \overline{R}_\varepsilon \;=\; R_\varepsilon \overline{P}\,, \tag{5.3}$$

with

$$R_\varepsilon \;=\; \Big[(H_0 - E^j)^2 \,+\, \varepsilon^2\Big]^{-1/2}. \tag{5.4}$$

Note that $\varepsilon R_\varepsilon^2 \to \delta(H_0 - E^j)$, as $\varepsilon \to 0$. Also, we remark that A_V depends on four parameters, g, ε, θ and ρ.

Let Δ be an energy interval containing E^j but no other parts of the spectrum of $H_{\rm part}$, and let

$$\theta_1 \;=\; \inf \Delta \;-\; \sup\Big\{\,\sigma(H_{\rm part}) \cap (-\infty, \inf \Delta)\,\Big\} \;>\; 0\,, \tag{5.5}$$

i.e., the distance, θ_1, of $\inf \Delta$ to the part of the spectrum of $H_{\rm part}$ on the left from it is assumed to be positive. The key technical result of this paper is

Theorem 5.1 *Assume conditions (3.1) and (3.3) hold and let the parameters involved satisfy*

$$\rho \;\ll\; \varepsilon \;\ll\; \min\{\theta, 1\} \;\le\; \theta \;\le\; 2\theta_1\,, \tag{5.6}$$

$$g \;\ll\; \varepsilon^{3/4}\rho^{1/2}\,. \tag{5.7}$$

Then

$$E_\Delta(H)\,[H, A_V]\,E_\Delta(H) \;\ge\; \frac{\theta(2-\alpha)\,\gamma_j}{\varepsilon}\,E_\Delta(H)^2\,, \tag{5.8}$$

where γ_j is the smallest eigenvalue of Γ_j and $\alpha = O\left(\rho^{1/2}\varepsilon^{-1/2} + \varepsilon\theta^{-1}\right) + o_\varepsilon(1)$ and $o_\varepsilon(1) \to 0$, as $\varepsilon \to 0$.

This theorem is proven in Section 7.

Proof of Theorem 3.1: The absence of eigenvalues in a vicinity of the E^j described in Theorem 3.1 follows directly from Theorem 5.1. Indeed, if such an eigenvalue existed, then (5.8) on the corresponding eigenfunction ψ would imply

$$\langle \psi,\, [H, A_V]\,\psi\rangle \;\ge\; \frac{\theta(2-\alpha)\,\gamma_j}{\varepsilon}\,\|\psi\|^2 \tag{5.9}$$

(here we used that $E_\Delta(H)\psi = \psi$). But this inequality would contradict the virial relation

$$\langle\psi,\ [H, A_V]\,\psi\rangle \;=\; 0\,. \tag{5.10}$$

To convince ourselves that the latter holds we observe that (i) $D(A)\cap D(H)$ is dense in $D(H)$ and $D(A)\cap D(H_0A)$ is dense in $D(H_0)$, both in the graph norms, (ii) $D(H_0) = D(H)$ and $[H_0, A_V]$ is H-bounded and (iii) $[H, A_V]\phi(H)$ extends to a bounded opeator, provided $\phi \in C_0^\infty$ and $\mathrm{supp}\phi \subset \Big(-\infty, \Sigma - \sup_x \int \frac{|G_x|^2}{\omega}\Big)$, where, recall, $\Sigma = \inf\sigma_{\mathrm{cont}}(H_{\mathrm{part}})$. The first two properties are straightforward if one writes $A_V = A + Q_1$, where

$$Q_1 := (\overline{P}VP - PV^*\overline{P})$$

and notices that $[H_f, A] = H_f$ and Q_1 is a bounded operator. To demonstrate (iii) we compute

$$[H, A] = H_f + a^*((k\cdot\nabla_k + \frac{3}{2})G_x) + a((k\cdot\nabla_k + \frac{3}{2})G_x)\,.$$

Next, we use the following estimate proven in [2]: if $\phi \in C_0^\infty$ with $\mathrm{supp}\phi \subset \Big(-\infty, \Sigma - \sup_x \int \frac{|G_x|^2}{\omega}\Big)$, then

$$\|e^{\alpha|x|}\phi(H)\| \le C_\alpha \tag{5.11}$$

for α sufficiently small $\Big(\alpha < \Sigma - \sup\mathrm{supp}\phi - \sup_x \int \frac{|G_x|^2}{\omega}\Big)$. The last two relations together with estimates (4.1) and (4.2) for $f := \langle x\rangle^{-M}(k\cdot\nabla_k + \frac{3}{2})G_x$ and Condition (3.1), which assures that $\sup_x \|f\| < \infty$, imply statement (iii). Now condition (iii) is weaker than the one (the H-boundedness of $[H, A_V]$) under which the virial relation is proven in [17, 6]. However, replacing A_V by $\phi(H)A_V\phi(H)$ shows that the proof of [17, 6] goes through under conditions (i)-(iii).

To prove the statements about absolute continuity and local decay we use abstract Kato-Mourre theory. A standard variant of this theory (see e.g. [4, 9]) requires H-boundedness of the commutators $[A_V, H]$ and $\Big[A_V, [A_V, H]\Big]$. In our case these commutators are not H-bounded for two reasons. First, $[A, H]$ and $\Big[A, [A, H]\Big]$ are H-bounded under Condition (3.1) only for $M = 0$. This follows from the straightforward computation

$$\mathrm{ad}_A^n(H) = H_f + a^*((k\cdot\nabla_k + \frac{3}{2})^n G_x) + a((k\cdot\nabla_k + \frac{3}{2})^n G_x)\,, \tag{5.12}$$

where we used the standard notation $\mathrm{ad}_A(H) = [H, A]$. The second reason is that the second part of the operator A_V (see Eqns (5.2)–(5.4)) contains the cut-off function $\chi_{H_f\le\rho}$, entering into P, and this cut-off function, not being differentiable in H_f, has a very singular commutator with the Hamiltonian H (or any other operator not commuting with H_f). To remedy the first problem we weaken the conditions in the Mourre theory (see Lemma 5.4 below) and use (5.11) (see Lemma 5.3 below). We go around the second problem by replacing A_V by its smooth version as follows. In definition (5.2)–(5.4) of the operator A_V we replace the projection P by the projection P_s, where

$$P_s \;=\; P_{\mathrm{part}} \otimes \chi_{H_f\le s\rho}\;. \tag{5.13}$$

Thus we just vary a bit the photon energy scale. Denote the resulting operator by $A_{V,s}$. Let now μ be a non-negative function supported in the interval $[1,2]$ and satisfying $\int\mu = 1$. Define

$$A_V^{(av)} := \int \mu(s)\; A_{V,s}\, ds\,. \tag{5.14}$$

The next two lemmas establish the desired properties of $A_V^{(av)}$.

Lemma 5.2 *Theorem 5.1 still holds if we replace A_V by $A_V^{(av)}$.*

Proof: Inequalities (5.6)–(5.7) are still true if we replace ρ by $s\rho$ with $1\le s\le 2$. Hence (5.8) holds after A_V is replaced by $A_{V,s}$, for $1\le s\le 2$. Since $\mu\ge 0$ and $\int\mu = 1$, this implies (5.8) with A_V replaced by $A_V^{(av)}$. ∎

Lemma 5.3 *Let ϕ be as in (5.11). Then the operators $[A_V^{(av)}, H]\phi(H)$ and $[A_V^{(av)}, [A_V^{(av)}, H]]\phi(H)$ are bounded.*

Proof: We write $A_V^{(av)} = A + Q$, where

$$Q := \int(\overline{P}_s V P_s - P_s V^*\overline{P}_s)\mu(s)ds\,. \tag{5.15}$$

Now we can present the first commutator as

$$[H, A_V^{(av)}] = [H, A] + [H, Q]\,. \tag{5.16}$$

Applying Eqn. (4.10) with $f := \langle x\rangle^{-M}(k \cdot \nabla_k + \frac{3}{2})^n G_x$ to (5.12) and observing that Condition (3.1) guarantees that $\sup_x \|f\|$ is finite, we conclude that $\mathrm{ad}_A^n(H)\langle x\rangle^{-M}$ are H-bounded for $n = 1, 2$. Hence, due to Eqn. (5.11), $\mathrm{ad}_A^n(H)\phi(H)$ are bounded for $n = 1, 2$. Furthermore, Q is a bounded operator and Eqns (4.3) and (4.4) show that so are the operators $H \cdot Q$ and $Q \cdot H$. Hence $[A_V^{(av)}, H]\phi(H)$ is bounded.

Now we write

$$\begin{aligned} \Big[[H, A_V^{(av)}], A_V^{(av)}\Big] &= \Big[[H, A], A\Big] + \Big[[H, A], Q\Big] \\ &+ \Big[[H, Q], Q\Big] + \Big[[H, Q], Q\Big]\,. \end{aligned} \tag{5.17}$$

By the conclusions at the end of the previous paragraph, the first three terms on the r.h.s. of (5.17), multiplied by $\phi(H)$, are bounded.

It remains to show that $\Big[A, [Q, H]\Big]$, the fourth term on the r.h.s. of (5.14), times $\phi(H)$, is bounded. To this end, we use the Jacobi identity and rewrite this term as

$$\Big[A\,,\,[Q\,,\,H]\Big] \;=\; \Big[[A\,,\,Q]\,,\,H\Big] \;+\; \Big[Q\,,\,[A\,,\,H]\Big]\,. \tag{5.18}$$

By Eqn. (5.12) with $n = 1$ it suffices to demonstrate that $[A\,,\,Q]$ and $H \cdot [A, Q]$ are is bounded. We write

$$[A\,,\,Q] \;=\; \theta(S \,+\, S^*)\,, \tag{5.19}$$

where

$$S \;=\; \Big[\,A\,,\,\int \mu(s)\,\overline{P}_s R_\varepsilon^2 I P_s\,\Big]\,. \tag{5.20}$$

Now we present S in the form

$$\begin{aligned} S \;&=\; \frac{d}{d\theta}\int \mu(s)\,e^{\theta A}\,\overline{P}_s\,R_\varepsilon^2\,I\,P_s\,e^{-\theta A}\,ds\Big|_{\theta=0} \\ &=\; \frac{d}{d\theta}\int \mu(s)\,\overline{P}_{se^{-\theta}}\,R_{\varepsilon,\theta}^2\,I_\theta\,P_{se^{-\theta}}\,ds\Big|_{\theta=0}\,, \end{aligned} \tag{5.21}$$

where $R_{\varepsilon,\theta} = e^{\theta A} R_\varepsilon e^{-\theta A}$, etc., and where we have used that $e^{\theta A} H_f e^{-\theta A} = e^\theta H_f$ and therefore

$$e^{\theta A}\, P_s\, e^{-\theta A} \;=\; P_{se^{-\theta}}\,. \tag{5.22}$$

Next using Leibnitz' rule, we transform this relation as

$$\begin{aligned} S \;&=\; \int \mu(s)\, \overline{P}_s\, [A, R_\varepsilon^2]\, \overline{P}_s\, I\, P_s\, ds \;+\; \int \mu(s)\, \overline{P}_s\, R_\varepsilon^2\, [A, I]\, P_s\, ds \\ &\quad + \int \mu(s)\, \frac{d}{d\theta}\Big(\, \overline{P}_{se^{-\theta}}\, R_\varepsilon^2\, I\, P_{se^{-\theta}} \Big)\, ds\Big|_{\theta=0}\,. \end{aligned} \tag{5.23}$$

Since $[A, H_0]$ and $[A, I]\langle x\rangle^{-M}$ are H_f-bounded, the first two terms on the r.h.s. are bounded. The last term on the r.h.s. can be rewritten as

$$-\int s\, \mu(s)\, \frac{d}{ds}\Big(\overline{P}_s\, R_\varepsilon^2\, I\, P_s\Big)\, ds \;=\; \int \frac{d(s\,\mu(s))}{ds}\, \overline{P}_s\, R_\varepsilon^2\, I\, P_s\, ds\,, \tag{5.24}$$

which shows that it is bounded, as well. Using the above analysis and Eqns. (4.1) and (4.2) one shows that $H \cdot [A, Q]$ is bounded as well. ∎

In the next lemma we weaken slightly standard hypotheses of the Mourre theory (see e.g. [6]) is order to accomodate our situation (see Lemma 5.3).

Lemma 5.4 *Assume a self-adjoint operator H and an anti-self-adjoint operator A, defined on the same Hilbert space, satisfy for some interval $\Delta \subset \mathbb{R}$ the following conditions (α) $[H, A]\phi(H)$ and $\phi(H)[[H, A], A]\phi(H)$ are bounded for some $\phi \in C_0^\infty$ s.t. $\phi \equiv 1$ on Δ and (β) $E_\Delta(H)[H, A]E_\Delta(H) \geq \theta E_\Delta(H)^2$ for some $\theta > 0$. Then the spectrum of H in Δ is absolutely continuous and H has the local decay property in Δ with respect the operator A_ϕ and therefore the operator A as well.*

Proof: Define $A_\phi = \phi(H) A \phi(H)$ and let f be a function on $\mathbb{R}$ satisfying $f \equiv 1$ on Δ and $\operatorname{supp} f \subset \{\lambda | \phi(\lambda) = 1\}$. Observe that (α') $[H, A_\phi]$ and $[f(H)[H, A_\phi]f(H), A_\phi]$ are bounded. The boundedness of the first term follows readily from the first part of condition (α). To prove the boundedness of the second term we write

$$[f[H, A_\phi]f, A_\phi] = fB[f, A]\phi - \phi[A, f]Bf + f[B, A]f\,,$$

where $f = f(H)$, $\phi = \phi(H)$ and $B = [H, A]$. The first two terms on the r.h.s. are bounded due to the boundedness of fB, Bf and $[f, A]$ and the third term is bounded by the second part of (α). Next, (β') condition (β) still holds with A replaced by A_ϕ. Now, keeping in mind the observations made above we can follow a simplified version the arguments of [17, 6] (see the proof Theorem 4.9 of [6] with Lemmas 4.12 and 4.13 omitted or the proof of Theorem 7.1 of [17] with Lemmas 7.4 and 7.5 dropped (those lemmas are replaced by the second part of (α') above) and Theorem 7.8 of [17] and use a general result of Kato (see Theorem XIII.25 of [19]) in order to derive the statements of Lemma 5.4. The fact that the local decay property w.r. to A_ϕ implies that w.r. to A follows from the observation that, by an operator calculus of [9] and property (α), $\langle A\rangle^{-1} A_\phi$ is a bounded operator. ∎

Now we can complete the proof of Theorem 3.1. The absolute continuity and local decay property, with A replaced by the operator $A_V^{(av)}$, follow from Lemmata 5.2-5.4. To pass to the local decay property w.r. to the operator A, it suffices to observe that due to (4.10), Q is a bounded operator and, therefore, $\langle A\rangle^{-\alpha} \cdot \langle A_V^{(av)}\rangle^{\alpha} \leq \mathrm{const}$, for $\alpha > 0$. ∎

6 Positivity of Truncated Commutator

Before tackling the proof of Theorem 5.1 head on we go a part way by proving a positivity of a simpler commutator. Namely, let

$$\overline{B}_{0,\Delta} \;=\; \overline{P}_\Delta \, B_0 \, \overline{P}_\Delta \,, \tag{6.1}$$

where

$$B_0 \;=\; [\, H \,,\, A_0\,] \tag{6.2}$$

and

$$\overline{P}_\Delta \;=\; \overline{P}\, E_\Delta(H_0)\,. \tag{6.3}$$

The main result of this section is

Lemma 6.1 *Assume* $g^2 \ll \rho$. *Then*

$$\overline{B}_{0,\Delta} \;\geq\; \frac{1}{2} H_f \, \overline{P}_\Delta \;\geq\; \frac{1}{2}\rho \, \overline{P}_\Delta \,, \tag{6.4}$$

and, if in addition $3|z| \leq \rho$ *, then*

$$\left\| \, |\overline{B}_{0,\Delta} - z|^{-1/2} \, \overline{P}_\Delta \, \psi \, \right\| \; \leq \; 2 \left\| \, H_f^{-1/2} \, \overline{P}_\Delta \, \psi \, \right\| , \tag{6.5}$$

where $|A| := \sqrt{A^* A}$ *, for a closed operator* A *.*

Proof: We begin with a computation:

$$B_0 \; = \; H_f \, + \, \widetilde{I} \, , \tag{6.6}$$

where $\widetilde{I} \; = \; a^*(\widetilde{G}_x) + a(\widetilde{G}_x)$, and

$$\widetilde{G}_x(k) \; := \; k \cdot \nabla_k G_x(k) \; + \; \frac{3}{2} G_x(k) \, . \tag{6.7}$$

Inequality (4.9) with $\alpha = 1/4$ and $f = \widetilde{G}_x$ yields

$$\pm \widetilde{I} \; \leq \; \frac{1}{4} \, H_f \, + \, 4 \int \frac{|\widetilde{G}_x|^2}{\omega} \tag{6.8}$$

which implies

$$B_0 \; \geq \; \frac{3}{4} \, H_f \; - \; 4 \int \frac{|\widetilde{G}_x|^2}{\omega} \, . \tag{6.9}$$

Since $H_f \geq 0$, Inequality (2.10) implies that

$$\left\| \, \langle x \rangle^M \, E_\Delta(H_0) \, \right\| \; \leq \; C_M \tag{6.10}$$

for any $M < \infty$, provided $\sup \Delta < \inf \operatorname{cont\,spec} H_{\rm part}$. The last two inequalities and the definition (3.1) of g imply that

$$\overline{B}_{0,\Delta} \; \geq \; \Big(\frac{3}{4} \, H_f \; - \; C g^2 \Big) \overline{P}_\Delta \, . \tag{6.11}$$

Next, Definition (5.1) yields that

$$\overline{P} \; = \; \overline{P}_{\rm part} \otimes \mathbf{1} \; + \; P_{\rm part} \otimes \chi_{H_f \geq \rho} \, . \tag{6.12}$$

Since, by the energy conservation,

$$\overline{P}_{\rm part} \; E_\Delta(H_0) \; = \; \sum_{i:\, E^i < E^j} P^i_{\rm part} \; E_\Delta(H_f + E^i) \, , \tag{6.13}$$

we have

$$H_f \, \overline{P}_{\rm part} \, E_\Delta(H_0) \; \geq \; \theta_1 \, \overline{P}_{\rm part} \, E_\Delta(H_0) \, , \tag{6.14}$$

where θ_1 is given in (5.5). This yields

$$H_f \overline{P} \;\geq\; \min(\theta_1, \rho) \overline{P} \;=\; \rho \overline{P}\,, \tag{6.15}$$

which together with (6.11) and the condition $g^2 \ll \rho$ implies (6.4).

The proof of (6.5) is based on the following identity,

$$\Big(\overline{B}_{0,\Delta} - z\Big)^{-1} \;=\; H_f^{-1/2} \Big(\mathbf{1} + K\Big)^{-1} H_f^{-1/2}\,, \tag{6.16}$$

where

$$K \;=\; H_f^{-1/2} \overline{P}_\Delta (\tilde{I} - z) \overline{P}_\Delta H_f^{-1/2}\,. \tag{6.17}$$

It suffices to prove that for $g^2 \ll \rho$,

$$\|K\| \;\leq\; \frac{1}{2}\,, \tag{6.18}$$

which would imply (6.5). To prove the latter inequality we write $K \;=\; K_0 + K_1$, where

$$K_0 \;=\; -z\, H_f^{-1} \overline{P}_\Delta^2 \tag{6.19}$$

and

$$K_1 \;=\; H_f^{-1/2} \overline{P}_\Delta \tilde{I} \overline{P}_\Delta H_f^{-1/2}\,. \tag{6.20}$$

Since $|z| \;\leq\; \rho/3$, we have due to (6.4) that $\|K_0\| \;\leq\; 1/3$. Next using (6.7), inequality (4.6) with $f = \widetilde{G}_x$, and inequality (6.10) again, we arrive at

$$K_1 \;=\; H_f^{-1/2} \overline{P}_\Delta O(g) \;+\; O(g) \overline{P}_\Delta H_f^{-1/2}\,, \tag{6.21}$$

which together with (6.15) yields that $\|K_1\| \;=\; O(\rho^{-1/2} g)$. Since $g^2 \ll \rho$ and since $\|K_0\| \leq 1/3$, this implies (6.18) which in turn yields (6.5). ∎

Now we boost this proposition to a more complicated result. Let

$$B_V \;:=\; [\,H\,,\,A_V\,] \quad \text{and} \quad \overline{B}_{V,\Delta} \;:=\; \overline{P}_\Delta\, B_V\, \overline{P}_\Delta\,. \tag{6.22}$$

We have

Lemma 6.2 *Assume* $g^2 \ll \rho$ *and* $g \ll \varepsilon^{3/4}\rho^{1/2}$. *Then*

$$\overline{B}_{V,\Delta} \;\geq\; \frac{1}{2} H_f\, \overline{P}_\Delta \;\geq\; \frac{1}{2} \rho\, \overline{P}_\Delta\,, \tag{6.23}$$

and, if in addition $3|z| \leq \rho$, *then*

$$\Big\|\, |\overline{B}_{V,\Delta} - z|^{-1/2}\, \overline{P}_\Delta\, \psi \,\Big\| \;\leq\; 2 \,\Big\|\, H_f^{-1/2}\, \overline{P}_\Delta\, \psi \,\Big\|\,. \tag{6.24}$$

Proof: By the definition of A_V, we have

$$\overline{B}_{V,\Delta} \;=\; \overline{B}_{0,\Delta} \;-\; E\,, \tag{6.25}$$

where

$$\begin{aligned} E \;&=\; -\overline{P}_\Delta \left[\, I\,,\, \overline{P} V P - P V^* \overline{P} \,\right] \overline{P}_\Delta \\ &=\; \overline{P}_\Delta\, I\, P\, V^* \overline{P}_\Delta \;+\; \overline{P}_\Delta\, V\, P\, I\, \overline{P}_\Delta \\ &=\; \theta\, \overline{P}_\Delta\, I\, P\, I\, R_\varepsilon^2\, \overline{P}_\Delta \;+\; \theta\, \overline{P}_\Delta\, R_\varepsilon^2\, I\, P\, I\, \overline{P}_\Delta\,. \end{aligned} \tag{6.26}$$

We claim that

$$\|E\| \;\le\; C\theta g^2 \left(\varepsilon^{-3/2} \;+\; \theta_1^{-2}\right). \tag{6.27}$$

Indeed, estimates (4.10) and (4.1) imply that

$$\|IP\| \;\le\; Cg \quad \text{and} \quad \|P\, a^*(G_x)\| \;\le\; Cg\rho^{1/2}\,. \tag{6.28}$$

Since $I = a^*(G_x) + a(G_x)$, it remains to estimate $a(G_x)\, R_\varepsilon^2\, \overline{P}_\Delta$. Remembering the definition of R_ε^2, $\overline{P}_\Delta$, and using (6.12), we obtain

$$\begin{aligned} \left\| a(G_x)\, R_\varepsilon^2\, \overline{P}_\Delta \right\| \;&\le\; \left\| a(G_x)\, R_\varepsilon^2\, \overline{P}_{\rm part} \right\| \;+\; \left\| a(G_x)\, R_\varepsilon^2\, P_{\rm part} \otimes \chi_{H_f \ge \rho} \right\| \\ &\le\; Cg\delta^{-2} \;+\; Cg \left\| H_f^{1/2} (H_f^2 + \varepsilon^2)^{-1} \chi_{H_f \ge \rho} \right\| \\ &\le\; Cg\theta_1^{-2} \;+\; Cg(\rho + \varepsilon)^{-3/2}\,. \end{aligned} \tag{6.29}$$

Collecting the last three estimates, we arrive at (6.27). The latter estimate together with (6.25) implies (6.23). Eqn. (6.24) is proven similarly to (6.5). ∎

7 Proof of Theorem 5.1

First we estimate the following operator from below,

$$B_{V,\Delta} \;:=\; E_\Delta(H_0)\, [\, H\,,\, A_V\,]\, E_\Delta(H_0)\,. \tag{7.1}$$

Using the definition of A_V (see Eqn. (5.2)), we write $B_{V,\Delta}$ in detail as

$$B_{V,\Delta} \;=\; \overline{B}_{V,\Delta} \;+\; P_\Delta\, C^*\, \overline{P}_\Delta \;+\; \overline{P}_\Delta\, C\, P_\Delta \;+\; P_\Delta\, F\, P_\Delta\,, \tag{7.2}$$

where, in accordance with (6.3),

$$P_\Delta \;=\; P\,E_\Delta(H_0)\,, \tag{7.3}$$

$$C \;=\; [\,H - E^j\,,\,V]\;+\;B_0\,, \tag{7.4}$$

$$F \;=\; B_0\;+\;V^*\,\overline{P}\,I\;+\;I\,\overline{P}\,V\,. \tag{7.5}$$

Here we used that, in virtue of the definition of V, we may identify $V \equiv \overline{P}VP$.

The key to the proof is the following inequality which follows from an application of the Feshbach projection method (a derivation is given in Appendix A):

$$\lambda_0 \;\geq\; \inf\;\mathrm{spec}\Big\{\mathcal{E}\restriction \mathrm{Ran}\,P_\Delta\Big\}\,, \tag{7.6}$$

where

$$\lambda_0 \;=\; \inf\;\mathrm{spec}\Big\{B_\Delta \restriction \mathrm{Ran}\,E_\Delta(H_0)\Big\} \tag{7.7}$$

and

$$\mathcal{E} \;=\; F\;-\;C^*\left(\overline{B}_{V,\Delta}-\lambda_0\right)^{-1}C\,. \tag{7.8}$$

We can assume here that $3\lambda_0 \leq \rho$, otherwise Theorem 5.1 follows readily due to the conditions, (5.6)–(5.7), on the parameters. With this assumption Lemma 6.2 is applicable and yields that $\left(\overline{B}_{V,\Delta}-\lambda_0\right)^{-1}$ is bounded on $\mathrm{Ran}\,\overline{P}_\Delta$. Hence (7.8) is well defined.

Our task is to estimate $\mathcal{E}$ on $\mathrm{Ran}\,P_\Delta$ from below. The first term on the r.h.s. of (7.8) can be easily analyzed. Due to (5.3),

$$V^*\,\overline{P}\,I\;+\;I\,\overline{P}\,V\;=\;2\theta\,I\,\overline{R}_\varepsilon^2\,I\,. \tag{7.9}$$

Next, Eqns. (6.6)–(6.10) imply that

$$P_\Delta\,B_0\,P_\Delta\;\geq\;\left(\frac{3}{4}\,H_f\;-\;Cg^2\right)P_\Delta\;\geq\;-Cg^2\,P_\Delta\,. \tag{7.10}$$

Hence on $\mathrm{Ran}\,P_\Delta$

$$F\;\geq\;2\theta\,I\,\overline{R}_\varepsilon^2\,I - Cg^2\,. \tag{7.11}$$

Next we analyze the operator $I\overline{R}_\varepsilon I$ restricted to $\mathrm{Ran}\,\chi_{H_f\leq\rho}$. To this end we need the pull-through Formulae

$$a(k)\,\overline{R}_\varepsilon \;=\; \overline{R}_{\varepsilon,\omega(k)}\,a(k)\,, \tag{7.12}$$

$$\overline{R}_\varepsilon\,a^*(k) \;=\; a^*(k)\,\overline{R}_{\varepsilon,\omega(k)}\,, \tag{7.13}$$

where

$$\overline{R}_{\varepsilon,\omega} \;=\; R_{\varepsilon,\omega}\overline{P}_{\text{part}} \quad \text{with} \quad R_{\varepsilon,\omega} \;=\; \Big[(H_0+\omega-E^j)^2+\varepsilon^2\Big]^{-1/2}. \tag{7.14}$$

These formulae follow from the following commutation relation (see e.g. [1, 2]),

$$a(k)\,H_f \;=\; \Big(H_f+\omega(k)\Big)\,a(k) \tag{7.15}$$

and its adjoint, which also implies that

$$a(k_1)\cdots a(k_m)\,\chi_{H_f\le\rho} \;=\; \prod_{j=1}^{m}\Big\{a(k_j)\,\chi_{H_f\le\rho}\Big\}\,. \tag{7.16}$$

Recalling (2.9) and pulling in $I\overline{R}_\varepsilon^2 I = \Big(a^*(G_x)+a(G_x)\Big)\overline{R}_\varepsilon^2\Big(a^*(G_x)+a(G_x)\Big)$, the a 's to the right and the a^* 's to the left, with the help of pull-through Formula (7.13) and (7.14), we obtain

$$I\,\overline{R}_\varepsilon^2\,I \;=\; M \;+\; L\,, \tag{7.17}$$

where

$$M \;=\; \int \overline{G_x}(k)\,\overline{R}^2_{\varepsilon,\omega(k)}\,G_x(k)\,dk \tag{7.18}$$

and, with $\omega_i=\omega(k_i)$,

$$\begin{aligned} L \;=\;& a^*(G_x)\,\overline{R}_\varepsilon^2\,a(G_x) \;+\; \iint \overline{G_x}(k_1)\,a^*(k_2)\,\overline{R}^2_{\varepsilon,\omega_1+\omega_2}\,a(k_1)\,G_x(k_2) \\ &+\; \int \overline{G_x}(k)\,\overline{R}^2_{\varepsilon,\omega(k)}\,a(k)\,a(G_x) \;+\; \text{adjoint}\,. \end{aligned} \tag{7.19}$$

Using that $\|\overline{R}_{\varepsilon,\omega}\|\le\varepsilon^{-1}$, we estimate the latter operator by

$$\begin{aligned} \Big\|\,\chi_{H_f\le\rho}\,L\,\chi_{H_f\le\rho}\Big\| \;\le\;& 2\varepsilon^{-2}\iint |G_x(k_1)G_x(k_2)|\;\|a(k_1)a(k_2)\chi_{H_f\le\rho}\| \\ &+2\left(\varepsilon^{-1}\int |G_x(k)|\;\|a(k)\chi_{H_f\le\rho}\|\right)^2\,. \end{aligned} \tag{7.20}$$

Applying inequality (7.16) to the r.h.s., we arrive at

$$\Big\|\,\chi_{H_f\le\rho}\,L\,\chi_{H_f\le\rho}\Big\| \;\le\; 4\varepsilon^{-2}\rho\int_{\{\omega\le\rho\}}\frac{|G_x|^2}{\omega}\,. \tag{7.21}$$

This in turn yields that, restricted to $\operatorname{Ran}\chi_{H_f\le\rho}$,

$$I\,\overline{R}_\varepsilon^2\,I \;=\; M \;+\; O\Big(\varepsilon^{-2}\rho\int_{\{\omega\le\rho\}}\frac{|G_x|^2}{\omega}\Big)\,. \tag{7.22}$$

This estimate together with (7.12) yields, for any $\psi \in \mathrm{Ran}\, P_\Delta$,

$$\langle F\rangle_\psi \;\geq\; 2\theta\langle M\rangle_\psi \;-\; C\left(g^2 \;+\; \varepsilon^{-2}\rho\sup_x \int_{\{\omega\leq\rho\}} \frac{|G_x|^2}{\omega}\right)\|\psi\|^2\,, \tag{7.23}$$

Thus it remains to estimate

$$G \;:=\; C^*\left(\overline{B}_{V,\Delta} - \lambda_0\right)^{-1} C\,, \tag{7.24}$$

on $\mathrm{Ran}\, P_\Delta$, from above. A greater part of the remainder of this section is devoted to this estimate.

As was mentioned after Eqn. (7.8), Lemma 4.1 is applicable to (7.24). It yields

$$|\langle\, G\,\rangle_\psi\,| \;\leq\; 2\,\Big\|\, H_f^{-1/2}\,\overline{P}_\Delta\, C\,\psi\,\Big\|^2 . \tag{7.25}$$

From now on we assume that $\psi \in \mathrm{Ran}\, P$, which implies that $\overline{P}_\Delta\psi = 0$. This relation, the definition of C (Eqn. (7.4)), and Eqn. (6.6) imply that

$$\Big\|\, H_f^{-1/2}\,\overline{P}_\Delta\, C\,\psi\,\Big\| \;\leq\; \Big\|\, H_f^{-1/2}\,\overline{P}_\Delta\,\widetilde{I}\,\psi\,\Big\| \;+\; \Big\|\, H_f^{-1/2}\,\overline{P}_\Delta\,[H - E^j\,,\,V]\,\psi\,\Big\|\,, \tag{7.26}$$

where $V = \overline{P}VP = \theta\,\overline{R}_\varepsilon^2 IP$. To estimate the first term on the r.h.s. of (7.26), we use that $\widetilde{I} = a^*(\widetilde{G}_x) + a(\widetilde{G}_x)$ (see Eqn. (6.7)) and Eqns. (6.3) and (6.4), to obtain

$$\begin{aligned}\Big\|\, H_f^{-1/2}\,\overline{P}_\Delta\,\widetilde{I}\,\psi\,\Big\| \;&\leq\; \Big\|\,\langle x\rangle^M\, E_\Delta(H_0)\Big\|\cdot\Big(\Big\|\langle x\rangle^{-M} H_f^{-1/2}\, a^*(\widetilde{G}_x)\Big\|\cdot\|\psi\| \\ &+\; \rho^{-1/2}\Big\|\langle x\rangle^{-M}\, a(\widetilde{G}_x)\, H_f^{-1/2}\Big\|\Big\|H_f^{1/2}\psi\Big\|\Big)\,.\end{aligned} \tag{7.27}$$

Since $\Big\|\langle x\rangle^M E_\Delta(H_0)\Big\| \leq C_M$ and $\Big\|H_f^{1/2}\psi\Big\|$ is bounded by $\rho^{1/2}\|\psi\|$, we obtain from (4.6)-(4.7) that

$$\Big\|\, H_f^{-1/2}\,\overline{P}_\Delta\,\widetilde{I}\,\Big\| \;\leq\; C\,g\,. \tag{7.28}$$

It remains to estimate the second term on the r.h.s. of (7.26). Using the properties of P , $\overline{P}_\Delta$, and $H - E^j = (H_0 - E^j) + I$, we obtain that

$$\begin{aligned}&H_f^{-1/2}\,\overline{P}_\Delta\,[H - E^j\,,\,V]\,P \\ &= \;\theta\, H_f^{-1/2}\,\overline{P}_\Delta\Big((H_0 - E^j)\overline{R}_\varepsilon^2 IP \;-\; \overline{R}_\varepsilon^2 IPH_f \;-\; \overline{R}_\varepsilon^2 IPIP \;+\; I\overline{R}_\varepsilon^2 IP\Big) \\ &= \;\theta\Big[\Big(\overline{P}_\Delta(H_0 - E^j)\overline{R}_\varepsilon^2\Big)\Big(H_f^{-1/2}\overline{P}IP\Big) \;-\; \overline{P}_\Delta\overline{R}_\varepsilon^2\Big(H_f^{-1/2}\overline{P}IP\Big)\Big(PH_f + PIP\Big) \\ &\qquad + H_f^{-1/2}\overline{P}_\Delta I\overline{R}_\varepsilon^2 IP\Big]\,.\end{aligned} \tag{7.29}$$

Now, we use that

$$\left\| H_f^{-1/2}\,\overline{P}\,I\,P \right\| \;\leq\; \left\| H_f^{-1/2}\,a^*(G_x) \right\| \;+\; \rho^{-1/2}\left\| a(G_x)\,P \right\| \;\leq\; 2\,g \tag{7.30}$$

and that

$$\left\| \overline{P}_\Delta\,(H_0 - E^j)\,\overline{R}_\varepsilon^2 \right\| \;\leq\; \left\| \overline{P}_\Delta\,\overline{P}_{\mathrm{part}}\,\overline{R}_\varepsilon \right\| \;\leq\; \theta_1^{-1}\,. \tag{7.31}$$

Thus, with $g^2 \ll \rho$ and $\theta \leq 2\theta_1$, we get from (7.29) that

$$\begin{aligned} \left\| H_f^{-1/2}\,\overline{P}_\Delta\,[H - E^j\,,\,V]\,P\psi \right\| \;&\leq\; C\,g\left(1 + \frac{\theta\rho}{\varepsilon^2}\right) \\ &\quad+\; C\,\theta \left\| H_f^{-1/2}\,\overline{P}_\Delta\,I\,\overline{R}_\varepsilon^2\,I\,P\psi \right\|\,. \end{aligned} \tag{7.32}$$

Finally, we estimate the second term on the r.h.s. of (7.32). Using that $I = a^*(G_x) + a(G_x)$, using elementary inequality (4.7) with $f = G_x$, and using that $\|\overline{R}_\varepsilon\| \leq \varepsilon^{-1}$, we obtain

$$\begin{aligned} \theta\left\| H_f^{-1/2}\,\overline{P}_\Delta\,I\,\overline{R}_\varepsilon^2\,I\,P\psi \right\| \;&\leq\; C\,\theta\,g\,\varepsilon^{-1}\left\| \overline{R}_\varepsilon\,I\,P\psi \right\| \\ &\quad+\; \theta\left\| H_f^{-1/2}\,\overline{P}_\Delta\,a(G_x)\,\overline{R}_\varepsilon^2\,I\,P\psi \right\|\,. \end{aligned} \tag{7.33}$$

We bound the first term on the r.h.s. of (7.33) by $C\theta g\varepsilon^{-1}\|M^{1/2}P\psi\|$. To estimate the second term on the r.h.s. of (7.33), we use (7.13) to pull $a(k)$ through $\overline{R}_\varepsilon^2$:

$$\begin{aligned} J \;&:=\; \left\| H_f^{-1/2}\,\overline{P}_\Delta\,a(G_x)\,\overline{R}_\varepsilon^2\,I\,P\psi \right\| \\ &\leq\; \int \left\| H_f^{-1/2}\,\overline{P}_\Delta\,\overline{G_x(k)}\,\overline{R}_{\varepsilon,\omega(k)}^2\,a(k)\,I\,P\psi \right\| dk\,. \end{aligned} \tag{7.34}$$

Using that

$$\begin{aligned} a(k)\,I \;&=\; I\,a(k) \;+\; G_x(k) \\ &=\; G_x(k) \;+\; a^*(G_x)\,a(k) \;+\; a(G_x)\,a(k)\,, \end{aligned} \tag{7.35}$$

we obtain

$$\begin{aligned} J \;\leq\; &\int \left\| H_f^{-1/2}\,\overline{P}_\Delta\,\overline{G_x(k)}\,\overline{R}_{\varepsilon,\omega(k)}^2\,G_x(k)\,P\psi \right\| dk \\ +\; &\int \left\| \overline{P}_\Delta\,\overline{G_x(k)}\,\overline{R}_{\varepsilon,\omega(k)}^2\,H_f^{-1/2}\,a^*(G_x)\,a(k)\,P\psi \right\| dk \\ +\; &\int \left\| H_f^{-1/2}\,\overline{P}_\Delta\,\overline{G_x(k)}\,\overline{R}_{\varepsilon,\omega(k)}^2\,a(G_x)\,a(k)\,P\psi \right\| dk\,. \end{aligned} \tag{7.36}$$

In the first term on the r.h.s. we pull $\chi_{H_f \le \rho}$ out of P, i.e., we write $P = \chi_{H_f \le \rho} P$, and use that it commutes with $\overline{G_x R^2_{\varepsilon,\omega}}$ to pull it next to $\overline{P}_\Delta$. Next, we use that $\overline{P}_\Delta \chi_{H_f \le \rho} = \overline{P}_{\rm part} \chi_{H_f \le \rho}$ and use (6.14) and $\left(\int |G_x|^2 \right)^{1/2} \le g$ to estimate this term by

$$\begin{aligned}
\theta_1^{-1/2} \int \left\| \overline{G_x(k)}\, \overline{R}^2_{\varepsilon,\omega(k)}\, G_x(k)\, P\psi \right\| dk \;&\le\; \theta_1^{-1/2} \int |G_x|\, \varepsilon^{-1} \left\| \overline{R}_{\varepsilon,\omega(k)}\, G_x(k)\, P\psi \right\| dk \\
&\le\; \theta_1^{-1/2}\, \varepsilon^{-1}\, g \left(\int \left\| \overline{R}_{\varepsilon,\omega(k)}\, G_x(k)\, P\psi \right\|^2 \right)^{1/2} \\
&=\; \theta_1^{-1/2}\, \varepsilon^{-1}\, g \left\| M^{1/2}\, P\psi \right\| . \qquad (7.37)
\end{aligned}$$

Using $\|\overline{R}^2_{\varepsilon,\omega}\| \le \varepsilon^{-2}$, (4.7) with $f = G_x$ and using (7.16), we estimate the second term on the r.h.s. of (7.36) by

$$\begin{aligned}
&\int |G_x(k)|\, \varepsilon^{-2} \left\| H_f^{-1/2}\, a^*(G_x) \right\| \cdot \|a(k)P\psi\|\, dk \qquad (7.38) \\
&\quad\le\; \int |G_x(k)|\, \varepsilon^{-2} \left(\int \frac{|G_x|^2}{\omega} \right)^{1/2} \|a(k)P\psi\|\, dk \\
&\quad\le\; \varepsilon^{-2}\, g \left(\rho \int_{\{\omega \le \rho\}} \frac{|G_x|^2}{\omega} \right)^{1/2} \|\psi\| .
\end{aligned}$$

Finally, using (6.15), $\|\overline{R}^2_{\varepsilon,\omega}\| \le \varepsilon^{-2}$ and (7.18), we estimate the third term on the r.h.s. of (7.36) by

$$\rho^{-1/2} \int |G_x(k)|\, \varepsilon^{-2}\, \|a(G_x)\, a(k)\psi\|\, dk \;\le\; \rho^{1/2}\, \varepsilon^{-2} \left(\int_{\{\omega \le \rho\}} \frac{|G_x|^2}{\omega} \right) \|\psi\| . \qquad (7.39)$$

Collecting all these estimates and using the rough estimate

$$\int_{\{\omega \le \rho\}} \frac{|G_x(k)|^2\, dk}{\omega(k)} \;\le\; g^2 , \qquad (7.40)$$

we obtain

$$J \;\le\; \theta_1^{-1/2}\, \varepsilon^{-1}\, g \left\| M^{1/2} P\psi \right\| \;+\; 2\varepsilon^{-2}\, \rho^{1/2}\, g^2\, \|\psi\| . \qquad (7.41)$$

Remember now that J is the multiple of θ in the second term on the r.h.s. of (7.33). Thus the last estimate, together with (7.32) and $g^2 \le \rho$, yields

$$\begin{aligned}
\left\| H_f^{-1/2}\, \overline{P}_\Delta\, [H - E^j , V]\, P\psi \right\| \;&\le\; C\, g \left(1 + \frac{\theta\rho}{\varepsilon^2} \right) \|\psi\| \\
&+\; C\, \theta\, \theta_1^{-1/2}\, \varepsilon^{-1}\, g \left\| M^{1/2} P\psi \right\| . \qquad (7.42)
\end{aligned}$$

We insert this estimate and (7.28) into (7.26) and obtain

$$\left\| H_f^{-1/2}\, \overline{P}_\Delta\, C\, P\psi \right\| \;\le\; C\, g\left(1+\frac{\theta\rho}{\varepsilon^2}\right)\|\psi\| \\ +\; C\,\theta\,\theta_1^{-1/2}\,\varepsilon^{-1}\, g\left\| M^{1/2}P\psi \right\|, \tag{7.43}$$

which, together with (7.25), gives

$$|\langle\, G\,\rangle_\psi\,| \;\le\; C\, g^2\left(1+\frac{\theta^2\rho^2}{\varepsilon^4}\right) \;+\; C\frac{\theta^2\, g^2}{\theta_1\,\varepsilon^2}\,\langle\, M\rangle_\psi\,, \tag{7.44}$$

for any $\psi = P\psi$. Now, Inequalities (7.23) and (7.44), inserted into Definitions (7.8) and (7.24), yield

$$\mathcal{E} \;\ge\; \left(2 \,-\, C\frac{\theta^2\, g^2}{\theta_1\,\varepsilon^2}\right)\theta\, M \;-\; C\, g^2\left(1+\frac{\theta^2\rho^2}{\varepsilon^4}\right), \tag{7.45}$$

on $\mathrm{Ran}\, P$.

Now we analyze the operator PMP. Introducing

$$P_{\mathrm{part}}^{(\le j)} \;:=\; \sum_{i:\,E^i\le E^j} P_{\mathrm{part}}^i \quad\text{and}\quad P_{\mathrm{part}}^{(>j)} \;:=\; \mathbf{1}_{\mathrm{part}} \,-\, P_{\mathrm{part}}^{(\le j)}\,, \tag{7.46}$$

and noting that $(H_{\mathrm{part}} - E^j)\, P_{\mathrm{part}}^{(>j)} \ge \delta P_{\mathrm{part}}^{(>j)}$, for some $\delta > 0$, we estimate

$$\left\| PMP \;-\; P\Big(\int \overline{G_x}(k)\, P_{\mathrm{part}}^{(\le j)}\, \overline{R}^2_{\varepsilon,\omega(k)}\, G_x(k)\, dk\Big) P \right\| \le C g^2\,. \tag{7.47}$$

Thus, remembering the definition of the matrices A_{ij} (see the paragraph preceeding (3.2)), we obtain

$$PMP \;=\; \sum_{i:\,E^i\le E^j} \int (A_{ij})^* A_{ij}\big[(H_f+\omega-E^{ji})^2+\varepsilon^2\big]^{-1} dkP \;+\; O(g^2) \tag{7.48}$$

where $E^{ji} := E^j - E^i$. Next, using the identity

$$\frac{1}{(H_f-\lambda)^2+\varepsilon^2} \;=\; \frac{1}{\lambda^2+\varepsilon^2}\left(1 \;+\; \frac{2H_f(H_f-\lambda)}{(H_f-\lambda)^2+\varepsilon^2} \;-\; \frac{H_f^2}{(H_f-\lambda)^2+\varepsilon^2}\right), \tag{7.49}$$

we obtain on $\mathrm{Ran}\, P$

$$M \;=\; \left(\sum_{i:E^i\le E^j} \int \frac{A_{ij}^* A_{ij}\; dk}{(\omega-E^{ji})^2+\varepsilon^2}\right)\cdot\big[1+O(\varepsilon^{-1}\rho)+O(\varepsilon^{-2}\rho^2)\big] \;+\; O(g^2), \tag{7.50}$$

Since $\omega^2 \int_{S^2} A^*_{ij} A_{ij} dS(k)$, where $dS(k)$ is the area element on S^2, is continuous in ω and vanishing at $\omega = 0$ and since $\delta_j > \text{const}$ (see Eqn. (3.3)), we have on $\operatorname{Ran} P$

$$M \;=\; \frac{1}{\varepsilon}\left(\sum_{i:\,E^i<E^j}\int |G_{ij}(k)|^2\,\delta\Big(\omega(k)-E^{ji}\Big)\,dk\right)\cdot\Big[1+o_\varepsilon(1)+O(\varepsilon^{-1}\rho)\Big]+O(g^2)\,, \tag{7.51}$$

where $o_\varepsilon(1)$ stands for a function of ε vanishing as $\varepsilon \to 0$ and where $O(\varepsilon^{-2}\rho^2)$ was dropped as it is much smaller than $O(\varepsilon^{-1}\rho)$.

Eqn. (7.51) inserted into (7.45) yields

$$\begin{aligned}\mathcal{E} \;&\geq\; \frac{\theta\,\Gamma_j}{\varepsilon}\left[2\;-\;O\Big(\frac{\theta g^2}{\theta_1\varepsilon^2}\Big)\;-\;o_\varepsilon(1)\;-\;O\Big(\frac{\rho}{\varepsilon}\Big)\right]\;-\;O(g^2)\left[1\;+\;\Big(\frac{\theta\rho}{\varepsilon^2}\Big)^2\right]\\ &\geq\; \frac{\theta\,\Gamma_j}{\varepsilon}\left[2\;-\;O\Big(\frac{\rho^{1/2}}{\varepsilon^{1/2}}+\frac{\varepsilon}{\theta}\Big)\;-\;o_\varepsilon(1)\right]\,,\end{aligned} \tag{7.52}$$

taking $\theta \leq 2\theta_1 \leq \text{const}$, $C \leq \delta_j \leq C'$ and $g \leq \varepsilon^{3/4}\rho^{1/2}$ into account, which implies

$$\frac{\theta g^2}{\theta_1\varepsilon^2} \;\leq\; \frac{2\rho^{1/2}}{\varepsilon^{1/2}}\,, \tag{7.53}$$

in particular. We may write (7.52) on $\operatorname{Ran} P_\Delta$ as

$$\mathcal{E} \;\geq\; \frac{\theta\,\Gamma_j\,(2-\alpha_1)}{\varepsilon}\,, \tag{7.54}$$

where

$$\alpha_1 \;=\; O\Big(\frac{\rho^{1/2}}{\varepsilon^{1/2}}+\frac{\varepsilon}{\theta}\Big)\;+\;o_\varepsilon(1)\;<\;2\,. \tag{7.55}$$

This together with (7.6)–(7.8) (see also the paragraph after Eqn. (7.8)) implies

$$B_{V,\Delta} \;\geq\; \frac{\theta\,\gamma_j\,(2-\alpha_1)}{\varepsilon}\;E_\Delta(H_0)^2\,, \tag{7.56}$$

where, recall, γ_j is the smallest eigenvalue of Γ_j.

Now we derive (5.7) from (7.56). To this end we use the estimate

$$\Big\|\,\Big(E_\Delta(H)-E_\Delta(H_0)\Big)(H_0+i)^{-1}\,\Big\| \;\leq\; C\,g\,, \tag{7.57}$$

which can be easily derived using Operator Calculus (see e.g. [9]). Recalling that $B_{V,\Delta} = E_\Delta(H_0)\,B_V\,E_\Delta(H_0)$ and $B_V = [H, A_V]$, we may write

$$E_\Delta(H)\,B_V\,E_\Delta(H) \;=\; B_{V,\Delta}\;+\;S\;+\;T\,, \tag{7.58}$$

where

$$S \ := \ \Big(E_\Delta(H) - E_\Delta(H_0)\Big)\, B_V\, E_\Delta(H_0) \ + \ \text{adjoint}\,, \tag{7.59}$$

and

$$T \ = \ \Big(E_\Delta(H) - E_\Delta(H_0)\Big)\, B_V\, \Big(E_\Delta(H) - E_\Delta(H_0)\Big)\,. \tag{7.60}$$

Estimate (7.57) implies

$$\begin{aligned} |\langle\psi,\, S\psi\rangle| \ &\le\ 2\Big\|\, B_V^{1/2}\Big(E_\Delta(H) - E_\Delta(H_0)\Big)\psi\,\Big\| \cdot \Big\|\, B_V^{1/2}\, E_\Delta(H_0)\psi\,\Big\| \\ &\le\ C\,g\,\|\psi\| \cdot \Big\|\, B_V^{1/2}\, E_\Delta(H_0)\psi\,\Big\| \\ &\le\ \beta\langle\psi,\, B_{V,\Delta}\psi\rangle \ + \ C\beta^{-1}g^2\|\psi\|^2\,, \end{aligned} \tag{7.61}$$

for any $\beta > 0$. Also,

$$\|T\| \ \le\ C\,g^2\,. \tag{7.62}$$

For $0 < \beta \le 1$, the last five estimates yield

$$E_\Delta(H)\, B_V\, E_\Delta(H) \ \ge\ (1-\beta)\, B_{V,\Delta} - C\beta^{-1}g^2\,. \tag{7.63}$$

This, together with (7.56) and (7.57) implies

$$E_\Delta(H)\, B_V\, E_\Delta(H) \ \ge\ \Big(2 - \alpha_1 - 2\beta\Big)\,\frac{\theta\,\gamma_j}{\varepsilon}\, E_\Delta(H)^2 \ - \ \frac{C\theta g^3}{\varepsilon} \ - \ \frac{Cg^2}{\beta}\,. \tag{7.64}$$

Taking $\Delta' \subset \Delta$ so that $E_{\Delta'}(H)E_\Delta(H) = E_{\Delta'}(H)$ and multiplying this inequality by $E_{\Delta'}(H)$ on both sides, we obtain

$$E_{\Delta'}(H)\, B_V\, E_{\Delta'}(H) \ \ge\ \left(2 - O\Big(\frac{\rho^{1/2}}{\varepsilon^{1/2}} + \frac{\varepsilon}{\theta} + g + 2\beta + \frac{\varepsilon}{\theta\,\beta}\Big)\right)\,\frac{\theta\,\gamma_j}{\varepsilon}\, E_{\Delta'}(H)^2\,. \tag{7.65}$$

Finally, choosing $\beta := \varepsilon^{1/2}\theta^{-1/2}$ and observing that $g \le \rho^{1/2}\varepsilon^{-1/2}$, we arrive at (5.8). ∎

A Feshbach Projection Method

Lemma A.1 *Let B be a self-adjoint operator on a Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and let (in the obvious notation) $B_{22} \ge \theta\,\mathrm{id}_{\mathcal{H}_2}$, $\theta > 0$. Then $\lambda_0 := \inf\operatorname{spec} B$ is either $\ge \theta$ or satisfies the relation*

$$\lambda_0 \ = \ \inf\ \text{spec}\Big\{B_{11} \ - \ B_{12}(B_{22} - \lambda_0)^{-1}B_{21}\Big\}\,. \tag{A.1}$$

Proof: Let $\lambda_0 < \theta$. The Feshbach projection method implies that $\lambda \in \sigma(B)$ iff

$$\lambda \in \sigma\Big(B_{11} - B_{12}(B_{22} - \lambda)^{-1} B_{21}\Big) \tag{A.2}$$

which implies (A.1). ∎

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