# Ergodic properties of Anosov maps with rectangular holes

N. Chernov[01]
Department of Mathematics
University of Alabama in Birmingham
Birmingham, AL 35294, USA
E-mail: chernov@vorteb.math.uab.edu; Fax: 1-(205)-934-9025

R. Markarian[02]
Instituto de Matemática y Estadística "Prof. Ing. Rafael Laguardia"
Facultad de Ingeniería. Universidad de la República
C.C. 30, Montevideo, Uruguay
E-mail: roma@fing.edu.uy; Fax: (598-2)-715-446

.

*To the memory of Ricardo Mañé*

**Abstract**

We study Anosov diffeomorphisms on manifolds in which some 'holes' are cut. The points that are mapped into those holes disappear and never return. The holes studied here are rectangles of a Markov partition. Such maps generalize Smale's horseshoes and certain open billiards. The set of nonwandering points of a map of this kind is a Cantor-like set called *repeller*. We construct invariant and conditionally invariant measures on the sets of nonwandering points. Then we establish ergodic, statistical, and fractal properties of those measures.

*AMS classification numbers: 58F12, 58F15, 58F11*

*Keywords*: repellers, scattering theory, chaotic dynamics, conditionally invariant measures, Anosov diffeomorphisms.

---

[01] Partially supported by NSF grant DMS-9401417.
[02] Partially supported by CONICYT (Uruguay).

# 1 Introduction and main results

Let $T : M' \to M'$ be a topologically transitive Anosov diffeomorphism of class $C^{1+\alpha}$ on a compact Riemannian manifold $M'$. Recall that a diffeomorphism $T : M' \to M'$ is said to be Anosov if at every point $x \in M'$ there is a $DT$-invariant splitting

$$\mathcal{T}_x M' = E_x^u \oplus E_x^s \tag{1.1}$$

such that

$$\begin{aligned} ||DT^{-n}v|| &\leq C_T\lambda_T^n||v|| \quad \text{for all } v \in E_x^u \text{ and } n > 0, \\ ||DT^{n}v|| &\leq C_T\lambda_T^n||v|| \quad \text{for all } v \in E_x^s \text{ and } n > 0, \end{aligned} \tag{1.2}$$

for some constants $C_T > 0$ and $\lambda_T \in (0,1)$ independent of $v$ and $x$. The splitting (1.1) is continuous in $x$. Topological transitivity of $T$ means that it has a dense orbit in $M'$.

Sinai [23] and Bowen [2] constructed Markov partitions for transitive Anosov diffeomorphisms[1]. Let $\mathcal{R}'$ be a Markov partition of $M'$ into rectangles $R_1, \ldots, R_{I'}$. We assume that these rectangles are small enough, so that the symbolic dynamics can be defined [23, 2].

Let $I < I'$. Put $H = \cup_{i=I+1}^{I'}(\operatorname{int} R_i)$ and $M = M' \setminus H$. Then $M$ is a manifold with boundary. We will study the dynamics of $T$ on $M$, thinking of $H$ as a 'hole' into which some points of $M$ will be mapped by $T$, and then they disappear (escape). Equivalently, one can think that $H$ 'absorbs' points mapped into it by $T$.

A pictorial model of this type of dynamics was proposed by Pianigiani and Yorke [22]. Imagine a Sinai billiard table (with dispersing boundary) in which the dynamics of the ball is strongly chaotic. Let one or more holes be cut in the table, so that the ball can fall through. One can also think of those holes as 'pockets' at the corners of the table. Let the initial position of the ball be chosen at random with some smooth probability distribution (e.g., equilibrium distribution). Denote by $p(t)$ the probability that the ball stays on the table for at least time $t$ and, if it does, by $\rho(t)$ its (normalized) distribution on the table at time $t$. Natural questions are: at what rate does $p(t)$ converge to zero as $t \to \infty$? what is the limit probability distribution $\lim_{t\to\infty} \rho(t)$, and does it depend on the initial distribution $\rho(0)$? These questions still remain open.

We assume that the symbolic dynamics generated by the partition $\mathcal{R} = \{R_1, \ldots, R_I\}$ of $M$ is rich enough, i.e., it is a topologically mixing subshift of finite type. General case is discussed in Section 8.

*Notations.* For any $n \geq 0$ we put

$$M_n = \cap_{i=0}^n T^i M \quad \text{and} \quad M_{-n} = \cap_{i=0}^n T^{-i} M,$$

[1] Bowen's construction actually covers larger systems – Axiom A diffeomorphisms – which we do not consider here.

and also

$$M_+ = \cap_{n\geq 1} M_n, \quad M_- = \cap_{n\geq 1} M_{-n}, \quad \Omega = M_+ \cap M_-$$

All these sets are closed, $T^{-1}M_+ \subset M_+$, $TM_- \subset M_-$ and $T\Omega = T^{-1}\Omega = \Omega$.

Denote by

$$\mathcal{U}' = \vee_{n=0}^{\infty} T^n \mathcal{R}' \quad \text{and} \quad \mathcal{S}' = \vee_{n=0}^{\infty} T^{-n} \mathcal{R}'$$

the partitions of $M'$ into unstable and stable manifolds (fibers), respectively. The restrictions of $\mathcal{U}'$ to $M$, $M_n$ and $M_+$ are denoted by $\mathcal{U}$, $\mathcal{U}_n$ and $\mathcal{U}_+$, respectively. Similarly, we have partitions $\mathcal{S}$, $\mathcal{S}_{-n}$, $\mathcal{S}_-$ of the sets $M$, $M_{-n}$, $M_-$ into stable fibers. Atoms $U \in \mathcal{U}$ and $S \in \mathcal{S}$ are closed domains on unstable and stable manifolds, respectively, whose boundary has Riemannian volume zero. Riemannian volume on fibers is induced by the Riemannian metric in $M$.

For any $x \in M'$ we denote by $J^u(x)$ and $J^s(x)$ the Jacobians of the map $DT$ restricted to $E_x^u$ and $E_x^s$, respectively. We also put

$$J_n^{u,s}(x) = J^{u,s}(x) J^{u,s}(Tx) \cdots J^{u,s}(T^{n-1}x)$$

the Jacobians of $DT^n$ on unstable and stable fibers.

Our first result deals with measures on unstable fibers $U \in \mathcal{U}_+$.

*Definition.* A family of probability measures, $\nu_U^u$, on unstable fibers $U \in \mathcal{U}$, is said to be *conditionally invariant* under $T$, if
(i) on every fiber $U \in \mathcal{U}$ the measure $\nu_U^u$ is absolutely continuous with respect to the Riemannian volume on $U$, and its density, $\rho_U^u(x)$, $x \in U$, is Hölder continuous (see a convention below);
(ii) for any $x \in U_1 \in \mathcal{U}$ and $Tx \in U_2 \in \mathcal{U}$ we have

$$\rho_{U_1}^u(x) = \nu_{U_1}^u(T^{-1}U_2) \cdot J^u(x) \cdot \rho_{U_2}^u(Tx) \tag{1.3}$$

*Convention.* All the densities of measures on unstable and stable fibers are assumed to be Hölder continuous with the same Hölder exponent $\alpha$, as the derivative of the map $T$.

**Theorem 1.1** *There is a unique conditionally invariant family of probability measures, $\nu_U^u$, on fibers $U \in \mathcal{U}_+$. Any other family of probability measures on $U \in \mathcal{U}_+$ with Hölder continuous densities will converge, under naturally defined action of $T$ (see Sect. 3), to this unique family.*

*Remark.* The family $\nu_U^u$, $U \in \mathcal{U}_+$, is a part of a 'bigger' conditionally invariant family of probability measures $\nu_U^u$, $U \in \mathcal{U}'$, 'inherited' from the Anosov diffeomorphism

$T : M' \to M'$ with the Markov partition $\mathcal{R}'$. The densities $\rho_U^u(x)$ of the measures $\nu_U^u$, $U \in \mathcal{U}'$ satisfy the equation [23]

$$\rho_U^u(x) / \rho_U^u(y) = \lim_{n\to\infty} \frac{J_n^u(T^{-n}y)}{J_n^u(T^{-n}x)} \tag{1.4}$$

for all $x, y \in U \in \mathcal{U}'$. Note that this equation defines the densities $\rho_U^u$ and measures $\nu_U^u$ completely, because of normalization.

*Remark.* If the Anosov diffeomorphism $T : M' \to M'$ is of class $C^2$, then the densities $\rho_U^u$ are at least Lipschitz continuous on every unstable fiber $U$, see [23].

*Remark.* The invariance condition (1.3) implies the following. Let $n \geq 1$, $U \in \mathcal{U}$ and $T^n(U \cap M_{-n}) = U_1 \cup \cdots \cup U_L$ for some fibers $U_1, \ldots, U_L \in \mathcal{U}$. Then

$$\nu_U^u(T^{-n}(A \cap M_n) \cap U) = \sum_{i=1}^{L} \nu_U^u(T^{-n}U_i) \cdot \nu_{U_i}^u(A \cap U_i) \tag{1.5}$$

for any Borel set $A \subset M$. This is the analog of the Chapman-Kolmogorov equation in the theory of Markov processes, see [23].

The next three theorems are related to the evolution of measures on $M$ under the action of $T$. Denote by $\mathcal{M}$ the class of all Borel measures on $M$. For any $\mu \in \mathcal{M}$ we put $||\mu|| = \mu(M)$. We denote by $T_* : \mathcal{M} \to \mathcal{M}$ the adjoint operator defined by

$$(T_*\mu)(A) = \mu(T^{-1}(A \cap M_1))$$

for any Borel set $A \subset M$. We denote by $T_+$ the (nonlinear) transformation of $\mathcal{M}$ defined by the normalization of the measure $T_*\mu$:

$$T_+\mu = \frac{T_*\mu}{||T_*\mu||} = \frac{T_*\mu}{\mu(M_{-1})} \tag{1.6}$$

We denote by $\mathcal{M}_n$, $n \geq 1$, the class of Borel measures supported on $M_n$. Obviously, $T_*^n\mathcal{M} = \mathcal{M}_n$. We denote by $\mathcal{M}_+^u \subset \mathcal{M}$ the class of measures supported on $M_+$ whose conditional measures on fibers $U \in \mathcal{U}_+$ coincide with the above conditionally invariant measures $\nu_U^u$. Any measure $\mu \in \mathcal{M}_+^u$ is then completely defined by its factor measure[2], $\hat{\mu}$, on the set $\mathcal{U}_+$ (this set can be naturally equipped with a metric, see Sect. 2).

*Definition.* A measure $\mu \in \mathcal{M}_+^u$ is said to be conditionally invariant under $T$ if $T_+\mu = \mu$, i.e. there is a $\lambda > 0$ such that $\mu(T^{-1}A \cap M_+) = \lambda\mu(A \cap M_+)$ for any Borel set $A \subset M$.

[2] For any measure $\mu \in \mathcal{M}$ its factor measure $\hat{\mu}$ on $\mathcal{U}$ is defined by $\hat{\mu}(W) = \mu(\cup_{U\in W} U)$ for any Borel subset $W \subset \mathcal{U}$.

**Theorem 1.2** *The map $T$ has a unique conditionally invariant probability measure $\mu_+ \in \mathcal{M}_+^u$. For any other $\mu \in \mathcal{M}_+^u$ the sequence $T_+^n \mu$ weakly converges, as $n \to \infty$, to $\mu_+$.*

We also call this unique measure $\mu_+$ the *eigenmeasure* of the map $T$, and the corresponding factor $\lambda_+ = \lambda \in (0, 1)$ the *eigenvalue* of $T$.

**Theorem 1.3** *For any smooth measure $\mu$ on $M$ (see a convention below) the sequence $T_+^n \mu$ weakly converges, as $n \to \infty$, to the eigenmeasure $\mu_+$. Furthermore, the sequence $\lambda_+^{-n} \cdot T_*^n \mu$ weakly converges, as $n \to \infty$, to the measure $c[\mu] \cdot \mu_+$, where $c[\mu] > 0$ is a linear functional on smooth measures on $M$.*

*Remark.* The conditionally invariant measure $\mu_+$ constructed in this way is very natural according to the above Pianigiani-Yorke physical motivation [22]. This measure coincides with Sinai-Bowen-Ruelle measure in the case $H = \emptyset$.

*Convention.* We call a measure on $M$ smooth if it is absolutely continuous with respect to the Riemannian volume on $M$, and its conditional measures on unstable fibers have Hölder continuous densities (cf. also the previous convention!).

This theorem shows that the eigenmeasure $\mu_+$ can be naturally obtained by iterating smooth measures under $T$ on $M$.

One can think of an experiment in which we place $N = N(0)$ points (particles) in $M$ at random according to a smooth probability distribution $\mu$. Then those points are mapped by successive iterations of $T$. The number of points that stay in $M$ (do not escape) after $n$ iterations, $N(n)$, is approximately

$$N(n) \sim N(0) \cdot c[\mu] \cdot e^{-n \ln \lambda_+^{-1}} \tag{1.7}$$

We call $\gamma_+ = \ln \lambda_+^{-1}$ the *escape rate*, cf. [10, 12, 11].

Next, we show that the eigenmeasure $\mu_+$ can be also obtained by iterating singular measures supported on individual unstable fibers.

For any unstable fiber $U \in \mathcal{U}$ let $\mu_U^u \in \mathcal{M}$ be a (canonical) singular probability measure supported on $U$, which coincides on $U$ with the measure $\nu_+^u$, described in the remark after Theorem 1.1.

**Theorem 1.4** *For any $U \in \mathcal{U}$ and any singular measure $\mu_U \in \mathcal{M}$ supported on $U$ with a Hölder continuous density with respect to the Riemannian volume on $U$, the sequence $T_+^n \mu_U$ weakly converges, as $n \to \infty$, to $\mu_+$. Furthermore, the sequence of measures $\lambda_+^{-n} \cdot T_*^n \mu_U^u$ weakly converges, as $n \to \infty$, to a measure supported on $M_+$ and proportional to $\mu_+$.*

**Proposition 1.5** *The function $e(U)$ on the set of unstable fibers $U \in \mathcal{U}$ defined by*

$$\lim_{n\to\infty} \lambda_+^{-n} \cdot T_*^n \mu_U^u = e(U) \cdot \mu_+ \tag{1.8}$$

*is bounded away from 0 and $\infty$ and its restriction on the set of fibers $U \in \mathcal{U}_+$ satisfies the equation*

$$\int_{\mathcal{U}_+} e(U)\, d\hat{\mu}_+(U) = 1 \tag{1.9}$$

*where $\hat{\mu}_+$ is the factor measure of the eigenmeasure $\mu_+$.*

Next, since the set $M_+$ is invariant under $T^{-1}$, it makes sense to define the inverse images of $\mu_+$ under $T_*$, i.e. $T_*^{-n}\mu_+$ for $n \geq 1$, by

$$(T_*^{-n}\mu_+)(A) = \mu_+(T^n[A \cap M_{-n}]) \tag{1.10}$$

for any Borel set $A \subset M$. In virtue of Theorem 1.2 the measure $T_*^{-n}\mu_+$, $n \geq 1$, simply coincides with the conditional measure $\mu_+(\cdot/M_{-n})$ defined by

$$\mu_+(A/M_{-n}) = \mu_+(A \cap M_{-n})/\mu_+(M_{-n}) = \lambda_+^{-n} \cdot \mu_+(A \cap M_{-n}) \tag{1.11}$$

**Theorem 1.6** *The sequence of measures $T_*^{-n}\mu_+ = \mu_+(\cdot/M_{-n})$ weakly converges, as $n \to \infty$, to a probability measure, $\eta_+ \in \mathcal{M}$, supported on the set $\Omega = M_+ \cap M_-$. The measure $\eta_+$ is $T$-invariant, i.e.*

$$\eta_+(T^{-1}A) = \eta_+(TA) = \eta_+(A) \tag{1.12}$$

*for every Borel set $A \subset M$.*

**Proposition 1.7** *The factor measure $\hat{\eta}_+$ of the measure $\eta_+$ on the set of unstable fibers $U \in \mathcal{U}_+$ is absolutely continuous with respect to the factor measure $\hat{\mu}_+$ of the eigenmeasure $\mu_+$, and its Radon-Nikodym derivative is*

$$\frac{d\hat{\eta}_+}{d\hat{\mu}_+}(U) = e(U) \tag{1.13}$$

*where $e(U)$ is the function introduced in Proposition 1.5.*

We call the closed set $\Omega = M_+ \cap M_-$ the *repeller* of the map $T$. It is normally a Cantor-like set. The $T$-invariant measure $\eta_+$ on $\Omega$ can be obtained naturally by iterating smooth measures on $M$ as follows. For any probability measure $\mu \in \mathcal{M}$ and $n, m \geq 1$ we denote by $\mu_{n,m}$ the measure $T_+^n\mu$ conditioned on $M_{-m}$, i.e.

$$\mu_{n,m}(A) = T_+^n\mu(A \cap M_{-m}) \cdot [T_+^n\mu(M_{-m})]^{-1} \tag{1.14}$$

for any Borel $A \subset M$.

**Theorem 1.8** *For any smooth probability measure $\mu$ on $M$ the sequence of measures $\mu_{n,m}$ weakly converges, as $m, n \to \infty$, to the invariant measure $\eta_+$ on the repeller $\Omega$. Moreover, the sequence of measures $\mu^*_{n,m}$ defined by*

$$\mu^*_{n,m}(A) = \lambda_+^{-n-m} \cdot T_*^n\mu(A \cap M_{-m}) \tag{1.15}$$

*weakly converges, as $m, n \to \infty$, to the measure $c[\mu] \cdot \eta_+$, where $c[\mu]$ is the positive linear functional on measures, involved in Theorem 1.3.*

Next, we establish the ergodic properties of the invariant measure $\eta_+$ on the repeller $\Omega$.

**Theorem 1.9** *The measure $\eta_+$ is an equilibrium measure for the Hölder continuous potential*

$$g_+(x) = -\log J^u(x) \tag{1.16}$$

*on $\Omega$ and the topological pressure $P(\eta_+) = -\log\lambda_+^{-1} = -\gamma_+$. Thus, $\eta_+$ is a Gibbs measure.*

**Corollary 1.10** *The measure $\eta_+$ is ergodic, mixing, K-mixing and Bernoulli. Its correlations decay exponentially fast and it satisfies the central limit theorems and its invariance principle.*

*Remark.* There are certainly other Gibbs invariant measures on $\Omega$, see [7]. Some particularly interesting ones are the measure of maximal entropy and the Hausdorff measure [14]. Our measure $\eta_+$ is the only one generated by originally smooth measures $\mu$ on $M$, in the sense of Theorems 1.3 and 1.8 and the original Pianigiany-Yorke philosophy [22]. Let us note that Theorems 1.3 and 1.8 cannot be obtained by the study of the symbolic dynamics on the repeller $\Omega$ alone.

**Theorem 1.11** *The sum of positive Lyapunov exponents of the map $T$ is*

$$\chi^{+}_{\eta_+} = \int_\Omega \log J^u(x)\, d\eta_+(x) > 0 \qquad \text{a.e.} \tag{1.17}$$

*and the sum of negative Lyapunov exponents of $T$ is*

$$\chi^{-}_{\eta_+} = \int_\Omega \log J^s(x)\, d\eta_+(x) < 0 \qquad \text{a.e.} \tag{1.18}$$

*The variational principle*

$$-\gamma_+ = h_{\eta_+}(T) - \int_\Omega \log J^u(x)\, d\eta_+(x) = \sup_\eta \{h_\eta(T) - \int_\Omega \log J^u(x)\, d\eta(x)\} \tag{1.19}$$

*holds, where $h_\eta(T)$ denotes the Kolmogorov-Sinai entropy of the measure $\eta$, and the supremum is taken over all $T$-invariant probability measures on the repeller $\Omega$. The left equation in (1.19) is equivalent to*

$$\chi^{+}_{\eta_+} = h_{\eta_+}(T) + \gamma_+ \tag{1.20}$$

The equation (1.20) generalizes Pesin's formula for smooth hyperbolic maps, for which $h = \chi^+$ and $\gamma_+ = 0$. This equation can be understood as follows. The exponential rate of separation of nearby trajectories, characterized by $\chi^+$, contributes to both the chaoticity of the dynamics on the repeller, measured by $h(T)$, and the scattering away from the repeller measured by the escape rate $\gamma_+$.

In a particular case, where $\dim M' = 2$, let $\delta^u_+$ and $\delta^s_-$ be the Hausdorff dimensions of the invariant measure $\eta_+$ on unstable fibers $U \subset M_+$ and on stable fibers $U \subset M_-$, respectively.

**Theorem 1.12** *Let* $\dim M = 2$. *According to Manning's formula [18], we have*

$$h_{\eta_+}(T) = \delta^u_+ \chi^{+}_{\eta_+} = -\delta^s_+ \chi^{-}_{\eta_+} \tag{1.21}$$

*This agrees with Young's formula [25] for the Hausdorff dimension of the measure $\eta_+$:*

$$HD(\eta_+) = h_{\eta_+}(T) \left( \frac{1}{\chi^{+}_{\eta_+}} - \frac{1}{\chi^{-}_{\eta_+}} \right) = \delta^u_+ + \delta^s_+ \tag{1.22}$$

By reversing the time, we can define the eigenmeasure $\mu_-$ on $M_-$ for the map $T^{-1}$, whose eigenvalue is $\lambda_- \in (0,1)$. We then can define the corresponding invariant measure $\eta_-$ on the repeller $\Omega$. These also have all the properties described in the above theorems. The measure $\eta_-$ and the values of $\lambda_-$ and $\chi^{\pm}_{\eta_-}$ are, generally, different from the previously described measure $\eta_+$ and the quantities $\lambda_+$ and $\chi^{\pm}_{\eta_+}$, see some examples in [4]. However, there are remarkable exceptions.

*Definition.* We say that the repeller $\Omega$ is time-symmetric if $\eta_+ = \eta_-$, $\lambda_+ = \lambda_-$, $\chi^+_{\eta_+} = \chi^+_{\eta_-} = |\chi^-_{\eta_+}| = |\chi^-_{\eta_-}|$.

**Theorem 1.13** *The measures $\eta_+$ and $\eta_-$ on the repeller $\Omega$ coincide if and only if there is a constant $Z > 0$ such that for every periodic point $x \in \Omega$, $T^k x = x$, we have*

$$\det DT^k(x) = J^u_k(x) \cdot J^s_k(x) = Z^k$$

*Moreover, the repeller $\Omega$ is time-symmetric if and only if $Z = 1$.*

**Corollary 1.14** *If the original Anosov diffeomorphism $T : M' \to M'$ preserves an absolutely continuous invariant measure on $M'$, then the repeller $\Omega$ is time-symmetric.*

The history of the subject goes back to 1979, when Pianigiani and Yorke [22] constructed conditionally invariant measures for expanding (noninvertible) maps. Their results are analogous to our Theorems1.2 and 1.3. In 1981-86 Čencova [3, 4] undertook a detailed study of both invariant and conditionally invariant measures for smooth Smale's horseshoes (her results are a particular case of our Theorems 1.1-1.8). In 1994, Collet, Martinez and Schmitt [6] constructed invariant measures on the sets of nonwandering points for Pianigiani-Yorke transformations (their results are similar to our Theorems 1.6-1.9). In a later manuscript [7] the same authors constructed conditionally invariant measures for some symbolic subshifts of finite type. Smooth hyperbolic systems other than horseshoes were first considered in this context by Lopes and Markarian recently [16]. They studied an open billiard system – a particle bouncing off three circular scatterers placed sufficiently far apart. Their results are a particular case of our Theorems 1.2, 1.3, 1.6, and 1.9-1.12. Theorem 1.13 applies to open billiards, answering a question posed in [16]. Let us also point out physical papers by Gaspard et. al. [9, 10, 11, 12, 15] in which the dynamics on repellers was discussed and some equations, like our (1.7) and (1.20), were conjectured and their connections with other equations in statistical physics established.

¿From measure-theoretic point of view, our systems resemble probabilistic Markov chains with absorbing states. For such chains, conditionally invariant distributions (called quasi-stationary distributions) have been studied in [8, 19].

The purpose of the present paper is threefold. First, we cover much larger classes of smooth hyperbolic systems with 'holes' than the previous papers did. Second, we collect all the existing results in this direction scattered in other papers, add some new ones (e.g., 1.13 and 1.14), and present the complete (up-to-date) program for studying smooth hyperbolic repellers. Third, we simplify and improve the matrix techniques for the construction of conditionally invariant measures used by Čencova [4]. The matrix method she used goes back to Sinai [24], but its realizations are sometimes lengthy and heavy, as it unfortunately happened to [4]. In our framework, this method works quite effectively and easily. Moreover, at present it is nearly the only workable method in the context of systems with countable Markov partitions, like billiards with 'holes', open Lorentz gases [11, 12] and other models of physical interest. We sharpen the matrix method preparing it for an attack on billiards, but such an attack is beyond the scopes of this paper.

The paper is organized as follows. Section 2 provides necessary results on Markov partitions and symbolic dynamics for Anosov diffeomorphisms. Section 3 contains a proof of Theorem 1.1 and other properties of conditional measures on unstable fibers. In Section 4 we describe, in general terms, the matrix techniques for constructing invariant measures. Then we construct the conditionally invariant measure $\mu_+$ proving Theorem 1.2. In Section 5 we prove the limit theorems 1.3 and 1.4 along with Proposition 1.5. In Section 6 we construct the invariant measure $\eta_+$ and prove statements 1.6-1.8. In Section 7 we prove the ergodic and fractal properties of the measure $\eta_+$ described by the statements 1.9-1.14. In Section 8 we discuss possible generalizations of our main results and related open problems. Appendix provides necessary techniques from the theory of positive matrices.

**Acknowledgements**. R.M. is indebted to S. Martínez for introducing him to the subject and stimulating discussions. This work was initiated during the authors' visits at Princeton university, for which we are grateful to Ya. Sinai and J. Mather. Special thanks go to Ya. Sinai who mentioned to us Čencova's papers. This work was essentially completed when N.Ch. visited IMERL, Facultad de Ingeniería, Uruguay, for which he is the most indebted. N.Ch. acknowledges the support of NSF grant DMS-9401417.

## 2 Background on Anosov diffeomorphisms

This section provides necessary tools from the theory of Anosov diffeomorphisms. It is known that Anosov diffeomorphisms enjoy strong ergodic properties if they are of class $C^{1+\alpha}$, not just $C^1$, i.e.

$$||DT(x) - DT(y)|| \le C_\alpha \cdot [d(x, y)]^\alpha$$

for some $C_\alpha > 0$, where $d(x, y)$ is the distance in the Riemannian metric. The constant $\alpha \in (0, 1]$ will be fixed throughout the paper.

The local unstable manifolds $W^u_\varepsilon(x)$, $x \in M'$, are defined by

$$W^u_\varepsilon(x) = \{y \in M' : d(T^n x, T^n y) \le \varepsilon \quad \forall n \le 0\}$$

for small $\varepsilon > 0$. Similarly, local stable manifolds $W^s_\varepsilon(x)$ are defined taking positive $n$.

It is known that these manifolds are 'as smooth as the map' $T$, see [1]. Precisely, they are of class $C^{1+\alpha}$, i.e. the tangent space $E^u_x$ is Hölder continuous along each $W^u$, with the Hölder exponent $\alpha$, and the same is true for $E^s_x$ along stable manifolds. The tangent bundles $E^u_x$ and $E^s_x$ over the whole of $M'$ are also Hölder continuous [1, 17], but the exponent may be different from $\alpha$.

Therefore, the Jacobians $J^u(x)$ and $J^s(x)$ are Hölder continuous function on $M'$. Moreover, the restrictions of $\log J^u(x)$ on unstable manifolds are Hölder continuous with the exponent $\alpha$:

$$|\log J^u(x) - \log J^u(y)| \le C_J \cdot [d_u(x,y)]^\alpha \tag{2.1}$$

with some $C_J > 0$, for all $x, y \in U$, $U \in \mathcal{U}'$ (the same is true for $J^s$, of course). Here and elsewhere $d_u$ and $d_s$ are intrinsic metrics on unstable and stable manifolds, respectively, induced by the Riemannian metric on $M'$.

For any $x, y \in M'$ we put

$$[x, y] = W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$$

There is a $\delta > 0$ such that if $d(x,y) < \delta$, then $[x,y]$ consists of a single point. A subset $R \subset M'$ is called a rectangle if $\operatorname{diam} R < \delta$ and $[x,y] \in R$ whenever $x, y \in R$. A rectangle $R$ is called *proper* if $R = \overline{\mathrm{int} R}$ and for any point $x \in R$ the sets $W^u_\varepsilon(x) \cap \partial R$ and $W^u_\varepsilon(x) \cap \partial R$ have zero Riemannian volumes in the manifolds $W^u_\varepsilon(x)$ and $W^s_\varepsilon(x)$, respectively. For $x \in R$ we put

$$W^{u,s}(x, R) = W^{u,s}_\varepsilon(x) \cap R$$

Recall [2] that $R' \subset R$ is called a $u$-subrectangle in a rectangle $R$ if $W^u(R,x) \subset R'$ for all $x \in R'$. Similarly, $R' \subset R$ is an $s$-subrectangle in $R$ if $W^s(R,x) \subset R'$ for all $x \in R'$.

A Markov partition of $M'$ is a finite covering $\mathcal{R}' = \{R_1, R_2, \ldots, R_{I'}\}$ of $M'$ by proper rectangles such that

(i) $\mathrm{int} R_i \cap \mathrm{int} R_j = \emptyset$ for $i \ne j$;

(ii) if $x \in \mathrm{int} R_i$ and $Tx \in \mathrm{int} R_j$, then $TW^u(x, R_i) \supset W^u(Tx, R_j)$ and $TW^s(x, R_i) \subset W^s(Tx, R_j)$

Equivalently, for any $R_i, R_j$ and $n \ge 1$ such that $\mathrm{int}(T^n R_i \cap R_j) \ne \emptyset$ the set $T^n R_i \cap R_j$ is a $u$-subrectangle in $R_j$ and $R_i \cap T^{-n} R_j$ is an $s$-subrectangle in $R_i$.

Every topologically transitive Anosov diffeomorphism $T : M' \to M'$ has Markov partitions of arbitrary small diameter.

We work with a fixed Markov partition $\mathcal{R}'$ of a sufficiently small diameter.

For every $z \in R_i$ we define the projection $h^s_z : R_i \to W^u(z, R_i)$ by $h^s_z(x) = [x, z]$. For every $x \in R_i$ this is a one-to-one map from $W^u(x, R_i)$ to $W^u(z, R_i)$, which is called canonical isomorphism or holonomy map. This map is absolutely continuous in the sense that its Jacobian with respect to Riemannian volume on unstable fibers is bounded and positive. Moreover, the Jacobian $Dh_z^s(x)$ of the map $h_z^s : W^u(x, R_i) \to W^u(z, R_i)$ satisfies the Anosov-Sinai formula [1]

$$Dh_z^s(x) = \lim_{n\to\infty} J_n^u(x)/J_n^u(h_z^s(x))$$

The Jacobian of the holonomy map is Hölder continuous in the following sense: for any $x, y \in W^u(x, R_i)$ we have

$$|Dh_z^s(x) - Dh_z^s(y)| \le C' \cdot [d_u(x, y)]^{\alpha'} \tag{2.2}$$

and

$$|Dh_z^s(x)| \le \exp\left(C' \cdot [d_s(x, h_z^s(x))]^{\alpha'}\right) \tag{2.3}$$

for some constants $C' > 0$, $\alpha' > 0$. (For proofs of these results, see for example, the book by Mañé [17], Chapter 3, Lemmas 2.7 and 3.2).

We now recall the basic definitions of symbolic dynamics. A transition matrix $A' = (A'_{ij})$ of size $I' \times I'$ is defined by

$$A'_{ij} = \begin{cases} 1 & \text{if } \operatorname{int} R_i \cap T^{-1}(\operatorname{int} R_j) \ne \emptyset \\ 0 & \text{otherwise} \end{cases}$$

In the space $\Sigma' = \{1, 2, \ldots, I'\}^{\mathbb{Z}}$ of doubly infinite sequences $\underline{\omega} = \{\omega_i\}_{-\infty}^{\infty}$ with the product topology we consider a closed subset

$$\Sigma'_{A'} = \{\underline{\omega} \in \Sigma' : A'_{\omega_i \omega_{i+1}} = 1 \text{ for all } -\infty < i < \infty\}$$

The left shift homeomorphism $\sigma : \Sigma'_{A'} \to \Sigma'_{A'}$ is defined by $(\sigma(\underline{\omega}))_i = \omega_{i+1}$. This symbolic system is called a subshift of finite type, or a topological Markov chain.

There is a natural projection $\Pi : \Sigma'_{A'} \to M'$, continuous, surjective and commuting with the dynamics: $\Pi \circ \sigma = T \circ \Pi$. This projection is one-to-one on the set $M' \setminus \cup_{j\in\mathbb{Z}} T^j(\partial R')$.

Now, the covering $\mathcal{R} = \{R_1, \ldots, R_I\}$ of $M = M' \setminus H$ defines a $I \times I$ submatrix $A = (A_{ij})$ of $A'$. We call $A$ the transition matrix for the restriction of $T$ on $M$. It defines a new subshift of finite type by

$$\Sigma_A = \{\underline{\alpha} \in \Sigma = \{1, \ldots, I\}^{\mathbb{Z}},\ A_{\alpha_i, \alpha_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\}.$$

**Mixing assumption.** The matrix $A$ is irreducible and aperiodic. This means that $\sigma : \Sigma_A \to \Sigma_A$ is topologically mixing. Equivalently, there is a $k_0 \ge 1$ such that $A^{k_0}$ has all positive entries. We call $k_0$ the *mixing power* of $A$.

Next, for every $n \ge 0$ we denote by $\mathcal{R}_n$ the restriction of the partition $\mathcal{R}' \vee T\mathcal{R}' \vee \ldots \vee T^n\mathcal{R}'$ of $M'$ to the set $M_n$. It is a partition of $M_n$ into $u$-subrectangles of the Markov rectangles $R_i$. Likewise, $\mathcal{R}_{-n}$ is the restriction of $\mathcal{R}' \vee T^{-1}\mathcal{R}' \vee \ldots \vee T^{-n}\mathcal{R}'$ to $M_{-n}$, which is a partition of $M_{-n}$ into $s$-subrectangles of $R_i$. Also, for any $n \geq 1$ let $\mathcal{R}_n^+$ be the restriction of the partition $\mathcal{R}_n$ of $M_n$ to the set $M_+ \subset M_n$. Note that each atom of $\mathcal{R}_n^+$ consists of some fibers $U \in \mathcal{U}_+$.

We equip the sets $\mathcal{U}$ and $\mathcal{S}$ defined in Introduction with the following metrics. For any $U, U' \in \mathcal{U}$ we put

$$d_{\mathcal{U}}(U, U') = \sup\{d_s(x, [x, y]) : x \in U, y \in U'\}$$

if $U, U'$ belong in one Markov rectangle $R_i$, otherwise we set $d_{\mathcal{U}}(U, U') = \operatorname{diam} M$. Similarly, we define a metric $d_{\mathcal{S}}$ on $\mathcal{S}$.

For any atom $B \in \mathcal{R}_m$, $m \geq 0$, we put

$$\mathcal{U}_B = \{U \in \mathcal{U} : U \subset B\}$$

For any $\mu \in \mathcal{M}$ and $U \in \mathcal{U}$ we will denote by $\mu_U$ the conditional probability measure of $\mu$ on $U$. Note that if two measures, $\mu$ and $\mu'$ are proportional, then $\mu_U = \mu'_U$ for all $U \in \mathcal{U}$. For any $U \in \mathcal{U}'$ we denote by $m_U$ the Riemannian volume on $U$. The conditional measures satisfy the following properties.

Let $n \geq 1$, $U \in \mathcal{U}$ and $T^n(U \cap M_{-n}) = U_1 \cup \cdots \cup U_l$ for some fibers $U_i \in \mathcal{U}_n$. Then

$$\mu_U[T^{-n}(A \cap M_n) \cap U] = \sum_{i=1}^{l} \mu_U(T^{-n}U_i) \cdot (T_*^n \mu)_{U_i}(A \cap U_i) \tag{2.4}$$

for any Borel subset $A \subset M$. In addition, if the measure $\mu_U$ is absolutely continuous with respect to the Riemannian volume $m_U$ on $U$ with density $f_\mu(x) = d\mu_U/dm_U(x)$ and $T^n x \in U_i$, then the measure $(T_*^n \mu)_{U_i}$ has a density on $U_i$, which is

$$f_{T_*^n \mu}(T^n x) = [\mu_U(T^{-n}U_i)]^{-1} f_\mu(x)/J_n^u(x) \tag{2.5}$$

We denote by $\mathcal{H}(G)$, $G > 0$, the class of measures $\mu \in \mathcal{M}$ such that their conditional measures $\mu_U$ on unstable fibers $U \in \mathcal{U}$ are absolutely continuous with respect to the Riemannian volume $m_U$ with densities $f_\mu(x)$ whose logarithms are Hölder continuous with the exponent $\alpha$ and constant $G > 0$:

$$|\log f_\mu(x) - \log f_\mu(y)| \leq G \cdot [d_u(x, y)]^\alpha \tag{2.6}$$

for all $x, y \in U$ and $U \in \mathcal{U}$.

# 3 Conditionally invariant measures on unstable fibers

In this Section we prove Theorem 1.1 and some lemmas on the evolution of measures under $T_*$, that will be used in the forthcoming sections.

*Proof of Theorem 1.1.* Our Theorem 1.1 is in fact an adapted version of a result by Sinai for ordinary Anosov systems (without holes). In our notations, his result reads

**Fact** [23, Lemma 2.3]. Let $T$ be a $C^2$ transitive Anosov diffeomorphism. Then there exists a unique family of conditionally invariant probability measures $\nu_U^u$ on unstable fibers $U \in \mathcal{U}'$ satisfying (1.3) with Lipschitz continuous densities $\rho_U^u(x) = d\nu_U^u/dm_U(x)$.

*Remarks.* Actually, Sinai constructed measures on stable fibers, but this does not matter because one can take $T^{-1}$ instead of $T$. Our map $T$ need not be $C^2$, it may be less regular than Sinai's. This is why our densities are only Hölder continuous.

We now start the proof. Let $\mu \in \mathcal{H}(G)$. Our proof works for measures defined on $M'$ with (2.6) valid on all $U \in \mathcal{U}'$. Take a fiber $U \in \mathcal{U}'$. The measure $\mu_n = T_*^n\mu$ on $M'$ conditioned on $U$ has a density $f_n(x) = d\mu_{n,U}/dm_U(x)$. Due to (2.5), we have

$$\frac{f_n(x)}{f_n(y)} = \frac{J^u(T^{-1}y)\cdots J^u(T^{-n}y)}{J^u(T^{-1}x)\cdots J^u(T^{-n}x)} \cdot \frac{f_\mu(T^{-n}x)}{f_\mu(T^{-n}y)} \tag{3.1}$$

for every $x, y \in U$.

Note that

$$d_u(T^{-n}x, T^{-n}y) \leq C_T\lambda_T^n \cdot d_u(x,y)$$

Since both $J^u$ and $f_\mu$ are Hölder continuous on unstable fibers, see (2.1) and (2.6), we have

$$|\log J^u(T^{-n}y) - \log J^u(T^{-n}x)| \leq C_J \cdot C_T^\alpha \lambda_T^{\alpha n}[d_u(x,y)]^\alpha \tag{3.2}$$

and

$$|\log f_\mu(T^{-n}x) - \log f_\mu(T^{-n}y)| \leq G \cdot C_T^\alpha \lambda_T^{\alpha n}[d_u(x,y)]^\alpha \tag{3.3}$$

Hence, the ratio in (3.1) converges, as $n \to \infty$, to

$$r(x,y) = \lim_{n\to\infty} f_n(x)/f_n(y) = \lim_{n\to\infty} J_n^u(T^{-n}y)/J_n^u(T^{-n}x)$$

Moreover, this convergence is uniformly exponential in $n$:

$$|\log[f_n(x)/f_n(y)] - \log r(x,y)| \leq (c_1 + c_2G)\lambda_T^{\alpha n}$$

with some $c_1, c_2$ independent of $x, y, U, \mu$ and $G$.

We define a function $\rho_U^u(x)$ on $U$ by

$$\rho_U^u(x) = \left(\int_U r(x,x_0)\, dm_U(x)\right)^{-1} r(x,x_0)$$

for any $x_0 \in U$. The function $\rho_U^u(x)$ so defined does not depend on $x_0$ and is a density of a probability measure, $\nu_U^u$, on $U$. It is a direct calculation that

$$\rho_U^u(x) = \lim_{n\to\infty} f_n(x) \tag{3.4}$$

and

$$|\log f_n(x) - \log \rho^u_U(x)| \leq (c_3 + c_4 G)\lambda_T^{\alpha n} \tag{3.5}$$

with some $c_3, c_4$ independent of $x, y, U, \mu$ and $G$.

Obviously, the function $\rho^u_U(x)$ is bounded away from zero and infinity, and it is Hölder continuous with the exponent $\alpha$:

$$|\log \rho^u_U(x) - \log \rho^u_U(y)| \leq G_* \cdot [d_u(x,y)]^\alpha \tag{3.6}$$

with

$$G_* = C_J \cdot C_T^\alpha \cdot \sum_{n=0}^{\infty} \lambda_T^{\alpha n}$$

which is independent of $U$.

The conditional invariance (1.3) now follows from the fact that

$$f_n(x) = \mu_{n,U}(T^{-1}U') \cdot J^u(x) \cdot f_{n+1}(Tx)$$

for all $x \in U$, $Tx \in U'$, which is just a particular case of (2.5). Taking the limit as $n \to \infty$ yields (1.3).

The uniqueness of the conditionally invariant family of measures follows from the convergence to it of any other family of measures with Hölder continuous densities on unstable fibers, under the iterates of $T_*$, due to (3.4).

The restriction of the conditionally invariant family of measures $\nu^u_U$ to $\mathcal{U}_+$ will then satisfy Theorem 1.1. Theorem 1.1 is now proved.

We now establish a few useful lemmas.

**Lemma 3.1** *There is a constant $G_0 > 0$, and for any $G > 0$ there is an integer $n_G \geq 1$ such that if $\mu \in \mathcal{H}(G)$, then $T_*^n \mu \in \mathcal{H}(G_0)$ for all $n \geq n_G$.*

*Proof.* It follows from (3.1)-(3.3) that if $\mu \in \mathcal{H}(G)$, then $T_*^n\mu \in \mathcal{H}(G_n)$ with

$$G_n \leq G \cdot C_T^\alpha \lambda_T^{\alpha n} + G_*$$

Lemma 3.1 is then established for any $G_0 > G_*$.

Lemma 3.1 means the following. If the densities of the conditional measures $\mu_U$ on $U \in \mathcal{U}$ oscillate wildly ($G$ is big), then the map $T$ stretching unstable fibers will quickly 'smooth out' those densities. In fact, the Hölder constant $G_n$ decreases basically like a geometric progression as $n$ grows. There is a natural bound, $G_*$, however, under which the values of $G_n$ will not drop.

**Lemma 3.2** *The function $\rho^u_U(x)$ and its logarithm are Hölder continuous (with some exponent $\alpha' > 0$) on every Markov rectangle $R_i \in \mathcal{R}'$.*

*Proof.* The Hölder continuity of $\rho^u_U(x)$ along every unstable fiber $U \in \mathcal{U}'$ (with the exponent $\alpha$) was established by (3.6). Its Hölder continuity along stable fibers (with some positive exponent) follows from the Hölder continuity of $J^u(x)$ along stable manifolds and the Hölder continuity of the holonomy map (2.3).

Let $\mu \in \mathcal{H}(G_0)$. For any $n \geq 0$ and $U \in \mathcal{U}_n$ denote by $\mu_{n,U}$ the measure $\mu_n = T^n_* \mu$ conditioned on $U$.

**Lemma 3.3** *For any $U \in \mathcal{U}_n$ the above measure $\mu_{n,U}$ is equivalent to $\nu^u_U$ and*

$$e^{-c\lambda^n} \leq \frac{d\mu_{n,U}}{d\nu^u_U} \leq e^{c\lambda^n}$$

*where $c > 0$ and $\lambda \in (0,1)$ are independent of $U, n, \mu$.*

*Proof.* This follows from (3.5) with $\lambda = \lambda^\alpha_T$ and $c = c_3 + c_4 G_0$.

In the notations of the previous lemma, let $m \geq 0$ and $B \in \mathcal{R}_m$ be an atom of the partition $\mathcal{R}_m$ of the set $M_m$, and $U, U' \subset B$ two unstable fibers. Let $A \subset U$ and $A' \subset U'$ be two canonically isomorphic Borel subsets, i.e. $A' = h^s_z(A)$ for any $z \in U'$.

**Lemma 3.4** *For any $n \geq m$ we have*

$$e^{-c\lambda^m} \leq \frac{\nu^u_U(A)}{\nu^u_{U'}(A')} \leq e^{c\lambda^m} \tag{3.7}$$

*and*

$$e^{-c\lambda^m} \leq \frac{\mu_{n,U}(A)}{\mu_{n,U'}(A')} \leq e^{c\lambda^m} \tag{3.8}$$

*with some $c > 0$ and $\lambda \in (0,1)$ independent of $U, n, \mu$.*

*Proof.* First, note that

$$d_{\mathcal{U}}(U, U') \leq D_s C_T \lambda^m_T$$

where $D_s$ is the maximum diameter of stable fibers $S \in \mathcal{S}'$. The bound (3.7) now follows from Lemma 3.2 and the Hölder continuity of the Jacobian of the holonomy map (2.3). The bound (3.8) follows from (3.7) and the previous lemma.

*Convention.* Without loss of generality, we can assume that the values of $c$ and $\lambda$ are the same in both lemmas.

The next three statements involve the mixing power $k_0$ of the transition matrix A.

**Lemma 3.5** *There is a constant $\beta > 0$ such that for any $\mu \in \mathcal{H}(G_0)$ and $R_j \in \mathcal{R}$ we have*

$$\inf_{U \in \mathcal{U}} \mu_U(T^{-k_0}(R_j \cap M_{k_0}) \cap U) \geq \beta \tag{3.9}$$

*Proof.* In virtue of the mixing assumption, for any $U \in \mathcal{U}$ and any $R_j \in \mathcal{R}$ we have $\nu_U^u(T^{-k_0}(R_j \cap M_{k_0}) \cap U) > 0$. For every $i = 1, \ldots, I$ we pick an arbitrary 'representative' fiber $\tilde{U}_i \subset R_i$ and from Lemmas 3.3 and 3.4 it follows that for any other $U \subset R_i$ we have

$$\mu_U(T^{-k_0}(R_j \cap M_{k_0}) \cap U) \geq e^{-2c} \nu_{\tilde{U}_i}^u(T^{-k_0}(R_j \cap M_{k_0}) \cap \tilde{U}_i)$$

The bound (3.9) follows with

$$\beta = e^{-2c} \cdot \min_{j,i} \nu_{\tilde{U}_i}^u(T^{-k_0}(R_j \cap M_{k_0}) \cap \tilde{U}_i) > 0$$

**Lemma 3.6** *There is a $\beta > 0$ such that for any $\mu \in \mathcal{H}(G_0)$ and $R_j \in \mathcal{R}$ and all $k \geq k_0$ we have*

$$\inf_{U \in \mathcal{U}} \mu_U(T^{-k}(R_j \cap M_k) \cap U) \geq \beta \cdot \sup_{U \in \mathcal{U}} \mu_U(T^{-k}(R_j \cap M_k) \cap U) \tag{3.10}$$

*Proof.* Put $m = k - k_0$. For $U \in \mathcal{U}$, let $T^{k_0}(U \cap M_{-k_0}) = U_1 \cup \cdots \cup U_L$ for some fibers $U_l \in \mathcal{U}_{k_0}$. From (2.4) we obtain

$$\begin{aligned}
\mu_U(T^{-k}(R_j \cap M_k) \cap U) &= \sum_{l=1}^{L} \mu_U(T^{-k_0}U_l) \cdot (T_*^{k_0}\mu)_{U_l}(T^{-m}(R_j \cap M_m) \cap U_l) \\
&= \sum_{i=1}^{I} \sum_{l:U_l \subset R_i} \mu_U(T^{-k_0}U_l) \cdot (T_*^{k_0}\mu)_{U_l}(T^{-m}(R_j \cap M_m) \cap U_l)
\end{aligned}$$

Using once again 'representatives' $\tilde{U}_i \subset R_i$ and Lemmas 3.3 and 3.4, we get an upper bound,

$$\begin{aligned}
\mu_U(T^{-k}(R_j \cap M_k) \cap U) &\leq e^{2c} \cdot \sum_{i=1}^{I} \mu_U(T^{-k_0}(R_i \cap M_{k_0}) \cap U) \cdot \nu_{\tilde{U}_i}^u(T^{-m}(R_j \cap M_m) \cap \tilde{U}_i) \\
&\leq e^{2c} \cdot \sum_{i=1}^{I} \nu_{\tilde{U}_i}^u(T^{-m}(R_j \cap M_m) \cap \tilde{U}_i)
\end{aligned}$$

By invoking (3.9), we get a lower bound,

$$\begin{aligned}
\mu_U(T^{-k}(R_j \cap M_k) \cap U) &\geq e^{-2c} \cdot \sum_{i=1}^{I} \mu_U(T^{-k_0}(R_i \cap M_{k_0}) \cap U) \cdot \nu_{\tilde{U}_i}^u(T^{-m}(R_j \cap M_m) \cap \tilde{U}_i) \\
&\geq e^{-2c} \cdot \beta \cdot \sum_{i=1}^{I} \nu_{\tilde{U}_i}^u(T^{-m}(R_j \cap M_m) \cap \tilde{U}_i)
\end{aligned}$$

Then we decrease the value of $\beta$ by a factor of $e^{-4c}$ and complete the proof.

**Corollary 3.7** *There is a $\beta > 0$ such that for any $\mu \in \mathcal{H}(G_0)$ and any s-subrectangle $D \in R_i$ (in particular for any atom $D \in \mathcal{R}_{-m}$, $m \geq 0$) and all $k \geq k_0$ we have*

$$\inf_{U \in \mathcal{U}} \mu_U(T^{-k}(D \cap M_k) \cap U) \geq \beta \cdot \sup_{U \in \mathcal{U}} \mu_U(T^{-k}(D \cap M_k) \cap U) \ , \tag{3.11}$$

Without loss of generality, the values of $\beta \in (0, 1)$ are assumed to be the same in these statements.

# 4 Conditionally invariant measure $\mu_+$ on $M_+$

In this section we prove Theorem 1.2. First, we describe the concepts on which our proofs in this and the following sections are based.

We invoke the Perron-Frobenius theorem for positive matrices and related techniques developed by Sinai and Čencova. One can think of the matrices we will work with as finite-dimensional approximations to the usual Perron-Frobenius operator on (infinite-dimensional) space of measures. To clarify this connection, let us sketch how these matrix techniques work for an arbitrary measurable transformation $T : M \to M$.

The adjoint operator, $T_*$, on the space of measures on $M$ acts by $T_*\mu(A) = \mu(T^{-1}A)$ for any measurable subset $A \subset M$. Constructions of the invariant measures and studies of their statistical properties usually rely on the convergence of the sequence of measures $\mu_n = T_*^n \mu$, as $n \to \infty$, to a $T$-invariant measure $\mu_0$ on $M$. To study this convergence, one can take an increasing sequence of finite partitions $\xi_1 < \xi_2 < \cdots$ of $M$, where $\xi_m = \{A_1^{(m)}, \ldots, A_{k_m}^{(m)}\}$, that converges to a partition into single points. Then one can represent any measure $\mu$ on $M$ by a sequence of (row) vectors $\mathbf{p}_m(\mu)$ with components $(\mathbf{p}_m(\mu))_i = \mu(A_i^{(m)})$, $1 \leq i \leq k_m$. A probability measure $\mu$ is represented by unit vectors, $|\mathbf{p}_m(\mu)| = 1$, the norm $|\cdot|$ for (row) vectors being defined below. Then, under certain regularity conditions that we leave out here, the weak convergence of a sequence of measures $\mu_n$, as $n \to \infty$, to a measure $\mu_0$ is equivalent to the componentwise convergence of the sequence of vectors $\mathbf{p}_m(\mu_n)$, as $n \to \infty$, to the vector $\mathbf{p}_m(\mu_0)$ for every $m \geq 1$.

For a fixed $m \geq 1$ and a measure $\mu$, the vectors $\mathbf{p}_m(\mu)$ and $\mathbf{p}_m(T_*\mu)$ are related by

$$\mathbf{p}_m(T_*\mu) = \mathbf{p}_m(\mu)\Pi_m(\mu) \tag{4.1}$$

where $\Pi_m(\mu)$ is a $k_m \times k_m$ matrix with components $(\Pi_m(\mu))_{ij} = \mu(T^{-1}A_j^{(m)} \cap A_i^{(m)})/\mu(A_i^{(m)})$ (we assume $\mu(A_i^{(m)}) \neq 0$ for all $m, i$). Therefore, we have

$$\mathbf{p}_m(T_*^n\mu) = \mathbf{p}_m(\mu)\Pi_m(\mu) \cdot \Pi_m(T_*\mu) \cdots \Pi_m(T_*^{n-1}\mu) \tag{4.2}$$

If the partitions $\xi_m$ have nice geometric properties (e.g., they are Markov partitions or alike), then the matrices in (4.2) are very close to each other, and so one can replace their product by $\tilde{\Pi}_m^n$ with some matrix $\tilde{\Pi}_m$ close to all of the matrices in (4.2). All these matrices have nonnegative entries, and usually some power, $\tilde{\Pi}_m^{n_m}$, $n_m \geq 1$, has all positive entries. In that case Perron-Frobenius theorem for positive matrices, see Appendix, applies. It provides a (unique) positive unit eigenvector, $\tilde{\mathbf{p}}_m$, for the matrix $\tilde{\Pi}_m$, corresponding to its largest eigenvalue $\tilde{\lambda}_m > 0$ (of multiplicity one). We call $\tilde{\mathbf{p}}_m$ the Perron eigenvector and $\tilde{\lambda}_m$ the Perron eigenvalue. Moreover, for any other positive unit vector $\mathbf{q}_m$ the sequence of vectors $\mathbf{q}\tilde{\Pi}_m^n$ converges, as $n \to \infty$, to $\tilde{\mathbf{p}}_m$ (exponentially fast in $L = n/n_m$). These facts can be used to prove that for some suitable probability measures $\mu$ the vectors $\mathbf{p}_m(T_*^n\mu)$ will be close to the Perron eigenvector $\mathbf{p}_m$ for large enough $n$.

Now, the limit of the Perron eigenvectors $\tilde{\mathbf{p}}_m$, as $m \to \infty$, defines a measure $\mu_0$ on $M$, which will be the weak limit of $T_*^n\mu$, as $n \to \infty$. The details of this scheme depend on the specific dynamical system and specific sequence of partitions $\xi_m$. Various versions of this matrix method work well for systems with sufficiently strong hyperbolic or expanding properties.

We prefer this matrix machinery to the Perron-Frobenius functional operator techniques for two reasons. First, it allows us to compute some characteristics of limit invariant measures which are not readily available otherwise, like the ones in our Propositions 1.5 and 1.7. Second, this machinery looks flexible enough to work well for nonuniformly hyperbolic systems, in particular billiards, where other techniques fail.

We now make a few conventions. As it is already clear, we will study vectors $\mathbf{p}$ whose components correspond to atoms $A \in \xi$ of some finite partitions $\xi$ of $M$. We will not enumerate or even order those atoms, so our 'vectors' will be just collections of numbers, denoted by $\mathbf{p}_A$, $A \in \xi$. Likewise, we will work with 'matrices' $\Pi$ whose entries correspond to (ordered) pairs $A, B$ of atoms of the partition $\xi$, and we denote them by $\Pi_{A,B}$. Despite the lack of order, we think of our vectors as row vectors, and the product $\mathbf{q} = \mathbf{p}\Pi$ is naturally defined to be another (row) vector with components

$$\mathbf{q}_B = \sum_{A \in \xi} \mathbf{p}_A \Pi_{A,B}$$

Next, for any (row) vector $\mathbf{p}$ we define its norm by

$$|\mathbf{p}| = \sum_{A \in \xi} |\mathbf{p}_A|$$

and we call a positive vector $\mathbf{p}$ a unit vector if $|\mathbf{p}| = 1$. For a positive matrix $\Pi$ the ratio of rows, $P$, is defined by

$$P = \max_{A', A'', B \in \xi} \Pi_{A',B} / \Pi_{A'',B}$$

Any two positive matrices, $\Pi$ and $\Pi'$, are said to be close with the constant of proximity $R \geq 1$ if for all $A, B \in \xi$ we have

$$R^{-1} \leq \Pi_{A,B} / \Pi'_{A,B} \leq R$$

We now begin the proof of Theorem 1.2. Recall that any measure $\mu \in \mathcal{M}_+^u$ is supported on $M_+$, has conditional measures $\nu_U^u$ on fibers $U \in \mathcal{U}_+$ and is then completely defined by its factor measure $\hat{\mu}$ on $\mathcal{U}_+$. Due to Theorem 1.1 the operator $T_*$ and the transformation $T_+$ leave $\mathcal{M}_+^u$ invariant. The conditionally invariant measures $\mu \in \mathcal{M}_+^u$ are fixed points of the transformation $T_+$.

Consider the increasing sequence of partitions $\mathcal{R}_1^+ < \mathcal{R}_2^+ < \cdots$ of $M_+$ defined in Sect. 2

Any measure $\mu \in \mathcal{M}_+^u$ can be represented by a sequence of (row) vectors

$$\mathbf{p}_m(\mu) = \{\mu(B) : B \in \mathcal{R}_m^+\}$$

The weak convergence of a sequence of measures, $\mu_n \to \mu$, in $\mathcal{M}_+^u$, is equivalent to the componentwise convergence $\mathbf{p}_m(\mu_n) \to \mathbf{p}_m(\mu)$, as $n \to \infty$, for every $m \geq 1$.

According to (4.1), for any $\mu \in \mathcal{M}_+^u$ and $k \geq 1$ we have

$$\mathbf{p}_m(T_*^k \mu) = \mathbf{p}_m(\mu)\Pi_m^{(k)}(\mu) \tag{4.3}$$

where $\Pi_m^{(k)}(\mu)$ is a matrix with components

$$\{\mu(T^{-k}[B'' \cap M_k] \cap B')/\mu(B') : B', B'' \in \mathcal{R}_m^+\} \tag{4.4}$$

(here $B'$ is the 'row number' and $B''$ is the 'column number'). Note that if $\mu'$ is proportional to $\mu$, $\mu' = a \cdot \mu$ with some constant $a > 0$, then $\Pi_m^{(k)}(\mu') = \Pi_m^{(k)}(\mu)$ for all $m, k \geq 1$.

*Remark.* Some entries of $\Pi_m^{(k)}(\mu)$ may not be defined by (4.4) if $\mu(B') = 0$. In that case we can define them arbitrarily without doing any harm to the equation (4.3). We simply pick a $U \subset B'$ and set the component (4.4) to $\nu_U^u(T^{-k}[B'' \cap M_k] \cap U)$.

Next, the equation (4.3) directly implies that

$$\mathbf{p}_m(T_+^k \mu) = \frac{\mathbf{p}_m(\mu)\Pi_m^{(k)}(\mu)}{|\mathbf{p}_m(\mu)\Pi_m^{(k)}(\mu)|} \tag{4.5}$$

**Lemma 4.1** *For any $m \geq 1$ and $k \geq k_0$ the matrices $\Pi_m^{(m+k)}(\mu)$, $\mu \in \mathcal{M}_+^u$, satisfy two conditions:*

(i) *the ratio of its rows is bounded by $P = \beta^{-1}$:*

$$\beta \leq \frac{\mu(T^{-m-k}[B'' \cap M_{m+k}] \cap B_1')/\mu(B_1')}{\mu(T^{-m-k}[B'' \cap M_{m+k}] \cap B_2')/\mu(B_2')} \leq \beta^{-1} \tag{4.6}$$

*for all $B_1', B_2', B'' \in \mathcal{R}_m^+$;*

(ii) *the matrices $\Pi_m^{(m+k)}(\mu_1)$ and $\Pi_m^{(m+k)}(\mu_2)$, for any $\mu_1, \mu_2 \in \mathcal{M}_+^u$, are close to each* *other with the constant of proximity $R = \exp(c\lambda^m)$, i.e.*

$$e^{-c\lambda^m} \le \frac{\mu_1(T^{-m-k}[B'' \cap M_{m+k}] \cap B')/\mu_1(B')}{\mu_2(T^{-m-k}[B'' \cap M_{m+k}] \cap B')/\mu_2(B')} \le e^{c\lambda^m} \tag{4.7}$$

*for all $B', B'' \in \mathcal{R}_m^+$.*

*Proof.* Put $\mathcal{U}_{+,B} = \{U \in \mathcal{U}_+ : U \subset B\}$ for $B \in \mathcal{R}_m^+$. Then the components of the matrix $\Pi_m^{(m+k)}(\mu)$ can be expressed by

$$\frac{\mu(T^{-m-k}[B'' \cap M_{m+k}] \cap B')}{\mu(B')} = \frac{1}{\hat{\mu}(\mathcal{U}_{B',+})} \int_{\mathcal{U}_{B',+}} \nu_U^u(T^{-m-k}[B'' \cap M_{m+k}] \cap U)\, d\hat{\mu}(U) \tag{4.8}$$

For any $B'' \in \mathcal{R}_m^+$ there is an atom $D \in \mathcal{R}_{-m}$ such that $T^{-m}B'' = M_+ \cap D$. So, for any $k \ge 0$ we have $T^{-m-k}[B'' \cap M_{m+k}] = T^{-k}[D \cap M_k] \cap M_+$. Now, the estimate (4.6) follows from (4.8) and Corollary 3.7.

To prove (4.7), notice that the set $T^{-k}[D \cap M_k]$ is a finite union of $s$-subrectangles (some atoms of $\mathcal{R}_{-m-k}$). Thus, for any two unstable fibers $U, U' \subset B'$ the sets $T^{-k}[D \cap M_k] \cap U$ and $T^{-k}[D \cap M_k] \cap U'$ are canonically isomorphic, and Lemma 3.4 implies

$$e^{-c\lambda^m} \le \frac{\nu_U^u(T^{-k}[D \cap M_k] \cap U)}{\nu_{U'}^u(T^{-k}[D \cap M_k] \cap U')} \le e^{c\lambda^m} \tag{4.9}$$

This and (4.8) prove (4.7). Lemma 4.1 is proved.

We continue the proof of Theorem 1.2. For any $m \ge 1$ and $B \in \mathcal{R}_m^+$ we pick an arbitrary 'representative' unstable fiber $\tilde{U}_B \subset B$. For any $m \ge 1$, $k \ge 1$, denote by $\tilde{\Pi}_m^{(k)}$ the matrix with components

$$\{\nu_{\tilde{U}_{B'}}^u(T^{-k}[B'' \cap M_k] \cap \tilde{U}_{B'}) : B', B'' \in \mathcal{R}_m^+\} \tag{4.10}$$

Note that $\tilde{\Pi}_m^{(k)} = \Pi_m^{(k)}(\tilde{\mu})$ for any measure $\tilde{\mu} \in \mathcal{M}_+^u$ supported on the union of representative fibers $\tilde{U}_B$, $B \in \mathcal{R}_m^+$, and such that $\tilde{\mu}(\tilde{U}_B) > 0$ for all $B \in \mathcal{R}_m^+$. Thus, the matrix $\tilde{\Pi}_m^{(m+k)}$, $k \ge k_0$, satisfies the bound (4.6) on the ratio of rows and is close to any $\Pi_m^{(m+k)}(\mu)$, $\mu \in \mathcal{M}_+^u$, with the constant of proximity $P_m$, see (4.7).

According to the Perron-Frobenius theorem, provided in Appendix, the matrix $\tilde{\Pi}_m^{(m+k_0)}$ has a positive unit (row) eigenvector, $\tilde{\mathbf{p}}_m$, corresponding to its largest eigenvalue.

We put

$$\gamma = \max\{\lambda, 1 - \beta/2\}$$

and fix an $m_0$ such that

$$(1-\beta)e^{2c\lambda^{m_0}} < \gamma$$

**Proposition 4.2** *There is a constant $C_1 > 0$ such that for all $m \geq m_0$, $m_1 = m + k_0$, and $n \geq m$ we have*

$$|\mathbf{p}_m(T_+^n\mu) - \tilde{\mathbf{p}}_m| \leq C_1(\gamma^{[n/m_1]} + \lambda^m)$$

*Proof.* Put $L = [n/m_1]$ and $l = n - m_1 L$, so that $n = m_1 L + l$, $0 \leq l < m_1$. Then

$$\mathbf{p}_m(T_*^n\mu) = \mathbf{p}_m(T_*^l\mu)\Pi_m^{(m_1)}(T_*^l\mu)\Pi_m^{(m_1)}(T_*^{m_1+l}\mu)\cdots\Pi_m^{(m_1)}(T_*^{(L-1)m_1+l}\mu)$$

and

$$\mathbf{p}_m(T_+^n\mu) = \frac{\mathbf{p}_m(T_*^n\mu)}{|\mathbf{p}_m(T_*^n\mu)|}$$

Theorem A.6 now implies Proposition 4.2.

Next, for any $m > l \geq 1$ and any vector $\mathbf{p}_m$ whose components correspond to atoms $B \in \mathcal{R}_m^+$, we denote by $\mathbf{p}_{m\downarrow l}$ the vector with components

$$(\mathbf{p}_{m\downarrow l})_{B'} = \sum_{B \subset B'} (\mathbf{p}_m)_B$$

corresponding to atoms $B' \in \mathcal{R}_l^+$.

**Proposition 4.3** *For any $l \geq 1$ there exists a limit*

$$\mathbf{r}_l = \lim_{m\to\infty} \tilde{\mathbf{p}}_{m\downarrow l}$$

*The sequence of vectors $\mathbf{r}_l$ satisfies the equations*

$$|\mathbf{r}_l| = 1 \quad \text{and} \quad \mathbf{r}_{l\downarrow k} = \mathbf{r}_k \tag{4.11}$$

*for all $l > k \geq 1$. Moreover, for all $m \geq l$ we have*

$$|\tilde{\mathbf{p}}_{m\downarrow l} - \mathbf{r}_l| \leq 4C_1\gamma^m \tag{4.12}$$

*Proof.* Let $\mu \in \mathcal{M}_+^u$, $l \geq 1$ and $n > m (\geq l)$ be large enough. For any $s \geq n(n + k_0)$ Proposition 4.2 yields

$$|\mathbf{p}_m(T_+^s\mu) - \tilde{\mathbf{p}}_m| \leq 2C_1\gamma^m$$

and

$$|\mathbf{p}_n(T_+^s\mu) - \tilde{\mathbf{p}}_n| \leq 2C_1\gamma^n$$

By using an obvious fact that $\mathbf{p}_{n\downarrow m}(T_+^s\mu) = \mathbf{p}_m(T_+^s\mu)$, we get

$$|\tilde{\mathbf{p}}_{m\downarrow l} - \tilde{\mathbf{p}}_{n\downarrow l}| \leq |\tilde{\mathbf{p}}_m - \tilde{\mathbf{p}}_{n\downarrow m}| \leq 4C_1\gamma^m \tag{4.13}$$

Thus, for any $l \geq 1$ the sequence of vectors $\tilde{\mathbf{p}}_{n\downarrow l}$, $n \geq 1$, is a Cauchy sequence, so it converges to a vector that we denote by $\mathbf{r}_l$. Now (4.12) follows from (4.13). It, in turn, readily implies (4.11). Proposition 4.3 is proved.

Due to (4.11), the sequence of vectors $\mathbf{r}_l$, $l \geq 1$, specifies a probability measure $\mu_+ \in \mathcal{M}_+^u$ such that $\mathbf{p}_l(\mu_+) = \mathbf{r}_l$ for all $l \geq 1$.

**Corollary 4.4** *For any measure $\mu \in \mathcal{M}_+^u$ the sequence $\{T_+^n\mu\}$ weakly converges, as $n \to \infty$, to $\mu_+$. Moreover, for all $l \geq 1$ and $n > \max\{m_0^2, l^2\}$ we have*

$$|\mathbf{p}_l(T_+^n\mu) - \mathbf{p}_l(\mu_+)| \leq C_2\gamma^{\sqrt{n}},$$

*with some constant $C_2 > 0$.*

Clearly, $T_+\mu_+ = \mu_+$ and

$$T_*\mu_+ = \lambda_+\mu_+ \quad \text{with} \quad \lambda_+ = \mu_+(M_{-1})$$

Theorem 1.2 is now proved.

# 5 Limit theorems for the measure $\mu_+$

Here we prove Theorems 1.3 and 1.4. The proofs require the extension of the previous analysis from the class of measures $\mathcal{M}_+^u$ to the larger classes $\mathcal{M}_n$.

For any measure $\mu \in \mathcal{M}_n$ we denote by $\mu_U$ its conditional measures on unstable fibers $U \subset M_n$, and by $\hat{\mu}$ its factor measure on $\mathcal{U}_n$. For any measure $\mu \in \mathcal{M}_n$ we can consider a finite sequence of vectors,

$$\mathbf{p}_m(\mu) = \{\mu(B) : B \in \mathcal{R}_m\}$$

for $1 \leq m \leq n$. Note that if we have a sequence of measures $\mu_n \in \mathcal{M}_n$, for which the sequence of factor measures $\hat{\mu}_n$ weakly converges, then its limit is a factor measure $\hat{\mu}$ of some $\mu \in \mathcal{M}_+^u$. This is equivalent to a componentwise convergence $\mathbf{p}_m(\mu_n) \to \mathbf{p}_m(\mu)$, as $n \to \infty$, for every $m \geq 1$.

According to (4.1), for any $n \geq m \geq 1$, $k \geq 1$ and $\mu \in \mathcal{M}_n$ we have $T_*^k \mu \in \mathcal{M}_{n+k}$ and

$$\mathbf{p}_m(T_*^k \mu) = \mathbf{p}_m(\mu) \cdot \Pi_m^{(k)}(\mu)$$

where $\Pi_m^{(k)}(\mu)$ is the matrix with components

$$\{\mu(T^{-k}[B'' \cap M_k] \cap B')/\mu(B') : \; B', B'' \in \mathcal{R}_m\}$$

(here, as in (4.4), $B'$ is the 'row number' and $B''$ is the 'column number'). The equation (4.5) holds without changes. The remark before Lemma 4.1 also applies, but now $B', B''$ are atoms of $\mathcal{R}_m$ instead of $\mathcal{R}_m^+$. The following lemma is an analog of Lemma 4.1:

**Lemma 5.1** *Let $\mu \in \mathcal{H}(G)$ with some $G > 0$. Then for any $m \geq 1$, $k \geq k_0$ and $n \geq m + n_G$ the matrix $\Pi_m^{(m+k)}(\mu_n)$ for the measure $\mu_n = T_*^n \mu \in \mathcal{M}_n$ satisfies two conditions:*

(i) *the ratio of its rows is bounded by $\beta^{-1}$:*

$$\beta \leq \frac{\mu_n(T^{-m-k}[B'' \cap M_{m+k}] \cap B_1')/\mu_n(B_1')}{\mu_n(T^{-m-k}[B'' \cap M_{m+k}] \cap B_2')/\mu_n(B_2')} \leq \beta^{-1} \tag{5.1}$$

*for all $B_1', B_2', B'' \in \mathcal{R}_m$;*

(ii) *for all $B', B'' \in \mathcal{R}_m$ we have*

$$e^{-2c\lambda^m} \leq \frac{\mu_n(T^{-m-k}[B'' \cap M_{m+k}] \cap B')/\mu_n(B')}{\mu_+(T^{-m-k}[B'' \cap M_{m+k}] \cap B')/\mu_+(B')} \leq e^{2c\lambda^m} \tag{5.2}$$

*Proof.* Note that $\mu_n \in \mathcal{H}(G_0)$ due to Lemma 3.1. for $B \in \mathcal{R}_m$. Then the components of the matrix $\Pi_m^{(m+k)}(\mu_n)$ can be expressed by

$$\frac{\mu_n(T^{-m-k}[B'' \cap M_{m+k}] \cap B')}{\mu_n(B')} = \frac{1}{\hat{\mu}_n(\mathcal{U}_{B'})} \int_{\mathcal{U}_{B'}} \mu_{n,U}(T^{-m-k}[B'' \cap M_{m+k}] \cap U) \, d\hat{\mu}_n(U) \tag{5.3}$$

For any $B'' \in \mathcal{R}_m$ the set $D = T^{-m}B''$ is an atom of $\mathcal{R}_{-m}$. So, for any $k \geq 0$ we have $T^{-m-k}[B'' \cap M_{m+k}] = T^{-k}[D \cap M_k]$. Now, the estimate (5.1) follows from (5.3) and Corollary 3.7.

The first part of the proof of (5.2) repeats word by word that of (4.7), but then (4.9) must be combined with Lemma 3.3. This gives

$$e^{-2c\lambda^m} \le \frac{\mu_{n,U}(T^{-k}[D\cap M_k]\cap U)}{\nu^u_{U'}(T^{-k}[D\cap M_k]\cap U')} \le e^{2c\lambda^m}$$

for all $U, U' \subset B'$. This and (4.8) with (5.3) prove (5.2). Lemma 5.1 is proved.

The following proposition is an analog of Proposition 4.2:

**Proposition 5.2** *There is a constant $C_3 > 0$ such that for all $m \ge m_0$, $m_1 = m + k_0$, $n \ge m$ and any $G > 0$, $\mu \in \mathcal{H}(G)$ we have*

$$|\mathbf{p}_m(T_+^{n+n_G}\mu) - \tilde{\mathbf{p}}_m| \le C_3(\gamma^{[n/m_1]} + \lambda^m)$$

It is enough to prove this for $\mu \in \mathcal{H}(G_0)$ and $n_G = 0$. The proof then repeats that of Proposition 4.2 word by word.

Combining Propositions 5.2 and 4.3 gives

**Corollary 5.3** *Let $G > 0$ and $\mu \in \mathcal{H}(G)$. For every $l \ge 1$ the sequence of vectors $\mathbf{p}_l(T_+^n\mu)$ converges to $\mathbf{r}_l = \mathbf{p}_l(\mu_+)$. Moreover, for all $n \ge \max\{m_0^2, l^2\}$ we have*

$$|\mathbf{p}_l(T_+^{n+n_G}\mu) - \mathbf{p}_l(\mu_+)| \le C_4\gamma^{\sqrt{n}},$$

*with some constant $C_4 > 0$.*

**Corollary 5.4** *Let $G > 0$ and $\mu \in \mathcal{H}(G)$. The sequence of factor measures $\hat{\mu}_n$, where $\mu_n = T_+^n\mu$, weakly converges, as $n \to \infty$, to the factor measure $\hat{\mu}_+$ on $\mathcal{U}_+$.*

We now begin the proofs of Theorems 1.3 and 1.4.

**Proposition 5.5** *Let $G > 0$ and $\mu \in \mathcal{H}(G)$. The sequence of measures $\mu_n = T_+^n\mu$, $n \ge 1$, weakly converges to the measure $\mu_+$.*

*Proof.* Since $T_+^{n_G}\mu \in \mathcal{H}(G_0)$, we may assume that $\mu \in \mathcal{H}(G_0)$. It is enough to show that for every $l \geq 0$, $k \geq 0$, every atom $B \in \mathcal{R}_l$ and every atom $D \in \mathcal{R}_{-k}$ we have a convergence

$$\mu_n(B \cap D) \to \mu_+(B \cap D) \quad \text{as} \quad n \to \infty \tag{5.4}$$

In the following, $B$ and $D$ may be also unions of some atoms of $\mathcal{R}_l$ and $\mathcal{R}_{-k}$, respectively, in one Markov rectangle $R_i \in \mathcal{R}$. Let $n \geq \max\{m_0^2, l^2\}$. Put $m = [\sqrt{n}]$. Then $B$ is the union of some atoms of $\mathcal{R}_m$, let us denote them by $B_1, \ldots, B_L$. In every $B_i$ we pick a 'representative' fiber $\tilde{U}_i \subset B_i$. Note that

$$\mu_n(B \cap D) = \int_{\mathcal{U}_B} \mu_{n,U}(D \cap U)\, d\hat{\mu}_n(U)$$

where $\mu_{n,U}$ is $\mu_n$ conditioned on the fiber $U$ and $\hat{\mu}_n$ is its factor measure on $\mathcal{U}_n$. Due to Lemmas 3.3 and 3.4 we have

$$\begin{aligned} \mu_n(B \cap D) &\leq e^{c\lambda^m} \sum_{i=1}^{L} \mu_{n,\tilde{U}_i}(D) \cdot \mu_n(B_i) \\ &\leq e^{2c\lambda^m} \sum_{i=1}^{L} \mu^u_{\tilde{U}_i}(D) \cdot \mu_n(B_i) \end{aligned}$$

The corresponding estimate from below with negative exponents also holds. In the same way Lemma 3.4 yields

$$\mu_+(B \cap D) \leq e^{c\lambda^m} \sum_{i=1}^{L} \mu^u_{\tilde{U}_i}(D) \cdot \mu_+(B_i)$$

and the corresponding lower bound with the negative exponent.

Now, Corollary 5.3, in which we can set $l = m$, implies

$$\begin{aligned} \mu_n(B \cap D) &\leq e^{3c\lambda^m} \mu_+(B \cap D) + e^{3c} \cdot \sup_{U \in \mathcal{U}_B} \mu^u_U(D) \cdot |\mathbf{p}_m(\mu_n) - \mathbf{p}_m(\mu_+)| \\ &\leq e^{3c\lambda^m} \mu_+(B \cap D) + e^{3c} \cdot \sup_{U \in \mathcal{U}_B} \mu^u_U(D) \cdot C_4 \gamma^m \end{aligned} \tag{5.5}$$

and, respectively,

$$\mu_n(B \cap D) \geq e^{-3c\lambda^m} \mu_+(B \cap D) - e^{3c} \cdot \sup_{U \in \mathcal{U}_B} \mu^u_U(D) \cdot C_4 \gamma^m \tag{5.6}$$

These two bounds readily imply (5.4). Proposition 5.5 is proved.

The first statement of Theorem 1.3 now follows immediately. To prove the second, it is enough to establish the following:

**Proposition 5.6** *For any $G > 0$ and $\mu \in \mathcal{H}(G)$ the limit*

$$c[\mu] = \lim_{n \to \infty} \lambda_+^{-n} ||T_*^n \mu|| \tag{5.7}$$

*exists and $c[\mu] > 0$.*

*Proof.* Clearly,

$$||T_*^n\mu|| = \prod_{i=0}^{n-1} ||T_*(T_+^i\mu)|| = \prod_{i=0}^{n-1} (T_+^i\mu)(M_{-1}) \tag{5.8}$$

Let $\mu_n = T_+^n\mu$ for $n \geq 0$. It is enough to show that the series

$$\sum_{n=0}^{\infty} \log \frac{\mu_n(M_{-1})}{\lambda_+} = \sum_{n=0}^{\infty} \log \frac{\mu_n(M_{-1})}{\mu_+(M_{-1})}$$

converges. Note that $\mu_{n_G} \in \mathcal{H}(G_0)$, so we may again assume that $\mu \in \mathcal{H}(G_0)$.

Now, let $R_i \in \mathcal{R}$ and $D = R_i \cap M_{-1}$. Let $P_0^{-1} = \min_j \mu_+(R_j \cap M_{-1})$. Then the bounds (5.5) and (5.6) combined with (3.7) imply that

$$e^{-3c\lambda^m} - P_0 \cdot C_4\gamma^m \leq \mu_n(D)/\mu_+(D) \leq e^{3c\lambda^m} + P_0 \cdot C_4\gamma^m$$

for all $n > m_0^2$ with $m = [\sqrt{n}]$. Therefore,

$$\left|\log \frac{\mu_n(M_{-1})}{\mu_+(M_{-1})}\right| \leq C_5\gamma^{\sqrt{n}}$$

with some constant $C_5 > 0$. Proposition 5.6 is proved.

Theorem 1.3 is then proved also.

*Remark.* For every $G > 0$ the convergence in (5.7) is uniform in $\mu \in \mathcal{H}(G)$. In particular, if $\mu \in \mathcal{H}(G_0)$, then for all $m \geq m_0^2$

$$|\log c[\mu] - \log(\lambda_+^{-m}||T_*^m\mu||)| \leq C_5 \cdot \sum_{n=m}^{\infty} \gamma^{\sqrt{n}} \tag{5.9}$$

We conclude this section with proofs of Theorem 1.4 and Proposition 1.5. The first part of Theorem 1.4 is a particular case of Proposition 5.5. Next, since $\mu_U^u \in \mathcal{H}(G_0)$, Proposition 5.6 applies and ensures the second part of Theorem 1.4 with

$$e(U) = c[\mu_U^u] = \lim_{n\to\infty} \lambda_+^{-n}\mu_U^u(M_{-n}) \tag{5.10}$$

In virtue of Corollary 3.7, the function $e(U)$ is positive and bounded:

$$\sup_{U\in\mathcal{U}} e(U) \leq \beta^{-1} \inf_{U\in\mathcal{U}} e(U) \tag{5.11}$$

This bound and (5.9) imply the following:

**Corollary 5.7** *There is $C_6 > 1$ such that for any $m \geq 0$ and any $U \in \mathcal{U}$ we have*

$$C_6^{-1} \leq \lambda_+^{-m}\mu_U^u(M_{-m}) \leq C_6$$

Due to (5.9), the normalization (1.9) will follow if we show that for all $n \geq 0$

$$\lambda_+^{-n} \int_{\mathcal{U}_+} ||T_*^n \mu_U^u||\, d\hat{\mu}_+(U) = 1$$

This equation is verified as follows:

$$\int_{\mathcal{U}_+} ||T_*^n \mu_U^u||\, d\hat{\mu}_+(U) = \int_{\mathcal{U}_+} \mu_U^u(M_{-n})\, d\hat{\mu}_+(U) = \mu_+(M_{-n}) = \lambda_+^n$$

Proposition 1.5 is proved.

*Remark.* There is an alternative proof of Theorem 1.4, along the lines of [4], based on the following observation. Recall that the matrix $\Pi_m^{m+k_0}(\mu_+)$, cf. Sect. 4, has the largest eigenvalue $\lambda_+^{m+k_0}$ and the Perron row eigenvector $\mathbf{p}_m(\mu_+)$. According to the Perron-Frobenius theorem, see Appendix, it also has a positive column eigenvector, $\mathbf{p}_m^*(\mu_+)$, such that

$$\Pi_m^{m+k_0}(\mu_+)\mathbf{p}_m^*(\mu_+) = \lambda_+^{m+k_0}\mathbf{p}_m^*(\mu_+)$$

The sequence of vectors $\mathbf{p}_m^*(\mu_+)$ 'converges', as $m \to \infty$, to the function $e(U)$ on $\mathcal{U}_+$ in the following sense: for any $U \in \mathcal{U}_+$ the numerical sequence

$$\{(\mathbf{p}_m^*(\mu_+))_B, \quad \text{where} \quad B \in \mathcal{R}_m^+ \quad \text{is such that} \quad B \supset U\}$$

converges, as $m \to \infty$, to $e(U)$ exponentially fast in $m$. We do not elaborate this proof here, it is given in full detail in [4] for the case where $T$ is a smooth horseshoe.

# 6 Invariant measure $\eta_+$ on the repeller $\Omega$

Here we prove Theorems 1.6 and 1.8.

For any $n \geq 1$ the measure $\mu_+^{(n)} = T_*^{-n}\mu_+$ defined by (1.10) is supported on $M_+ \cap M_{-n}$. Its conditional measures on $U \cap M_{-n}$, $U \subset M_+$, i.e. $\nu_U^u(\cdot/M_{-n})$, they are absolutely continuous with respect to the Riemannian volume on $U$ with densities

$$\rho_{n,U}^u(x) = [\nu_U^u(U \cap M_{-n})]^{-1}\rho_U^u(x), \quad x \in U \cap M_{-n}$$

Its factor measure, $\hat{\mu}_+^{(n)}$, on $\mathcal{U}_+$ is absolutely continuous with respect to $\hat{\mu}_+$, and its Radon-Nikodym derivative is

$$\frac{d\hat{\mu}_+^{(n)}}{d\hat{\mu}_+}(U) = \lambda_+^{-n} \cdot \nu_U^u(U \cap M_{-n})$$

for $U \subset \mathcal{U}_+$, in virtue of (1.11). Due to (5.10), we have

$$\lim_{n\to\infty} \frac{d\hat{\mu}_+^{(n)}}{d\hat{\mu}_+}(U) = e(U)$$

for any $U \in \mathcal{U}_+$.

**Corollary 6.1** *The sequence of measures $\hat{\mu}_+^{(n)}$ weakly converges to the measure $\hat{\mu}_0$ on $\mathcal{U}_+$ defined by*

$$d\hat{\mu}_0(U) = e(U)d\hat{\mu}_+(U)$$

We now complete the proof of Theorem 1.6. Let $k \geq 1$, $l \geq 1$, and consider two arbitrary atoms $B \in \mathcal{R}_l$ and $D \in \mathcal{R}_{-k}$. For all $n \geq k$ we have

$$\begin{aligned} \mu_+^{(n)}(B \cap D) &= \mu_+(T^n[(B \cap D) \cap M_{-n}]) \\ &= \mu_+(T^{n-k}[T^k(B \cap D) \cap M_{-n+k}]) = \mu_+^{(n-k)}(T^k(B \cap D)) \end{aligned}$$

The set $T^k(B \cap D)$ is an atom of $\mathcal{R}_{k+l}$. Due to Corollary 6.1 we have

$$\lim_{n\to\infty} \mu_+^{(n)}(B \cap D) = \lim_{n\to\infty} \mu_+^{(n-k)}(T^k(B \cap D)) = \hat{\mu}_0\{U \in \mathcal{U}_+ : U \subset T^k(B \cap D)\}$$

Hence, the sequence of measures $\mu_+^{(n)} = T_*^{-n}\mu_+$ weakly converges, as $n \to \infty$, to a measure $\eta_+$, which is supported on the closed set $M_+ \cap (\cap_{n\geq 1} M_{-n}) = \Omega$. The invariance of $\eta_+$ under $T$ follows from two equations:

$$\mu_+^{(n)}(T(B \cap D)) = \mu_+^{(n-k+1)}(T^k(B \cap D))$$

and

$$\mu_+^{(n)}(T^{-1}(B \cap D)) = \mu_+^{(n-k-1)}(T^k(B \cap D))$$

($B$ and $D$ are the same as above). By taking the limit as $n \to \infty$, we obtain (1.12). Theorem 1.6 is proved.

Proposition 1.7 follows from Corollary 6.1.

We now prove Theorem 1.8.

**Proposition 6.2** *For any $G > 0$ and any measure $\mu \in \mathcal{H}(G)$ the sequence of measures $\mu_{n,m}$ defined by (1.14) weakly converges, as $m, n \to \infty$, to $\eta_+$.*

*Proof.* Since $T_+^{n_G}\mu \in \mathcal{H}(G_0)$, we may assume that $\mu \in \mathcal{H}(G_0)$. It is enough to show that for every $k, l \geq 0$, every atom $B \in \mathcal{R}_l$ and every atom $D \in \mathcal{R}_{-k}$ we have

$$\lim_{m,n\to\infty} \mu_{n,m}(B \cap D) = \eta_+(B \cap D)$$

Let $m > k$ and $n > l$. Note that $\eta_+(B \cap D) = \eta_+(T^k(B \cap D))$ and $\mu_{n,m}(B \cap D) = \mu_{n+k,m-k}(T^k(B \cap D))$, and $T^k(B \cap D)$ is an atom of $\mathcal{R}_{k+l}$. Thus, it is enough to show that

(6.1) $$\lim_{m,n\to\infty} \mu_{n,m}(B) = \eta_+(B)$$

Recall that $\mu_+^{(n)}(B) \to \eta_+(B)$ as $n \to \infty$ by Theorem 1.6. Thus, (6.1) is equivalent to the following:

$$\lim_{m,n\to\infty} \frac{\mu_{n,m}(B)}{\mu_+^{(m)}(B)} = 1 \tag{6.2}$$

Let $\mu_n = T_+^n \mu$. The ratio of the above measures can be rewritten as

$$\frac{\mu_{n,m}(B)}{\mu_+^{(m)}(B)} = \frac{\mu_n(B \cap M_{-m})}{\mu_+(B \cap M_{-m})} \cdot \frac{\mu_+(M_{-m})}{\mu_n(M_{-m})}$$

First, we will show that

$$\lim_{m,n\to\infty} \frac{\mu_n(B \cap M_{-m})}{\mu_+(B \cap M_{-m})} = 1 \tag{6.3}$$

A direct application of bounds (5.5), (5.6) and Corollary 5.7 gives

$$\mu_n(B \cap M_{-m}) \le e^{3c\lambda^{\sqrt{n}}} \mu_+(B \cap M_{-m}) + e^{4c} \cdot C_6 \lambda_+^m \cdot C_4 \gamma^{\sqrt{n}}$$

and

$$\mu_n(B \cap M_{-m}) \ge e^{-3c\lambda^{\sqrt{n}}} \mu_+(B \cap M_{-m}) - e^{4c} \cdot C_6 \lambda_+^m \cdot C_4 \gamma^{\sqrt{n}}$$

for all $n \ge \max\{m_0^2, l^2\}$ and $m \ge m_0^2$. Due to Corollary 5.7 we have

$$C_6^{-1} \mu_+(B) \le \frac{\mu_+(B \cap M_{-m})}{\lambda_+^m} \le C_6 \mu_+(B)$$

Combining all the previous bounds yields (6.3).

The equation (6.3) holds, in particular, for $l = 0$ and $B = R_i$, $1 \le i \le I$. Since $M_{-m} = \cup_{i=1}^I (R_i \cap M_{-m})$, we immediately obtain

$$\lim_{m,n\to\infty} \frac{\mu_+(M_{-m})}{\mu_n(M_{-m})} = 1$$

thus completing the proof of (6.2) and Proposition 6.2.

The first part of Theorem 1.8 is then established.

**Proposition 6.3** *For any $G > 0$ and $\mu \in \mathcal{H}(G)$ the limit (see (1.15))*

$$c[\mu] = \lim_{m,n\to\infty} ||\mu_{n,m}^*||$$

*exists, and $c[\mu]$ is the same as in Proposition 5.6.*

*Proof.* We have

$$\begin{aligned} \lim_{m,n\to\infty} ||\mu_{n,m}^*|| &= \lim_{m,n\to\infty} \lambda_+^{-n-m} (T_*^n \mu)(M_{-m}) \\ &= \lim_{m,n\to\infty} \lambda_+^{-n-m} \mu(M_{-n-m}) \end{aligned}$$

which is equal to $c[\mu]$ due to Proposition 5.6. Proposition 6.3 is proved.

The proof of Theorem 1.8 is completed.

# 7 Ergodic properties of the measure $\eta_+$

Here we prove the ergodic and fractal properties of the invariant measure $\eta_+$ on the repeller, given by statements 1.9-1.14.

Let $k, l \geq 0$, and take arbitrary atoms $B \in \mathcal{R}_l$ and $D \in \mathcal{R}_{-k}$. Assume that $\mathrm{int}(B \cap D) \neq \emptyset$ and pick a point $x \in B \cap D$.

**Lemma 7.1** *There is a constant $C_7 > 1$ independent of $x, B, D, k, l$ such that*

$$C_7^{-1} \leq \lambda_+^{k+l} J_{k+l}^u(T^{-l}x) \cdot \eta_+(B \cap D) \leq C_7$$

*Proof.* The set $E = T^k(B \cap D)$ is an atom of $\mathcal{R}_{k+l}$, and due to (1.13)

$$\eta_+(B \cap D) = \eta_+(T^k(B \cap D)) = \int_{\mathcal{U}_E} e(U)\, d\hat{\mu}_+(U)$$

In virtue of (1.9) and (5.11) we have

$$\beta \leq \inf_{U \in \mathcal{U}} e(U) \leq \sup_{U \in \mathcal{U}} e(U) \leq \beta^{-1}$$

so that

$$\beta \leq \frac{\eta_+(B \cap D)}{\mu_+(E)} \leq \beta^{-1}$$

Next, the conditional invariance of $\mu_+$ implies that

$$\mu_+(E) = \lambda_+^{-k-l} \mu_+(F) = \lambda_+^{-k-l} \int_{\mathcal{U}(F)} \nu_U^u(F \cap U)\, \hat{\mu}_+(U)$$

where $F = T^{-l}(B \cap D)$ is an atom of $\mathcal{R}_{-k-l}$ and $\mathcal{U}(F) = \{U \in \mathcal{U} : U \cap F \neq \emptyset\}$.

To estimate this last integral, recall that the measures $\nu_U^u$ on unstable fibers $U \in \mathcal{U}$ have densities uniformly bounded away from zero and infinity, and note that for all $U \subset \mathcal{U}(F)$

$$0 < \mathrm{const} \leq m_U(F \cap U) \cdot J_{k+l}^u(T^{-l}x) \leq \mathrm{const} < \infty$$

which follows from the absolute continuity of stable and unstable foliations, Sect. 2. This completes the proof of Lemma 7.1.

This lemma immediately implies that $\eta_+$ is a Gibbs measure with the potential $g_+(x) = -\log J^u(x)$ and the topological pressure $P(\eta_+) = \log \lambda_+ = -\gamma_+$, see [2].

Theorems 1.9 and 1.11 are now proved. Corollary 1.10 mostly follows from [2], for more advanced limit theorems than the central limit theorem see, e.g., [13].

Theorem 1.12 is self-evident.

We now turn to Theorem 1.13.

The measure $\eta_-$ is also a Gibbs measure, with potential

$$g_-(x) = \log J^s(T^{-1}x)$$

and topological pressure $P(\eta_-) = -\log \lambda_-^{-1} = -\gamma_-$. The next lemma is a direct consequence of [2, Proposition 4.5].

**Lemma 7.2** *The following three conditions are equivalent:*

(i) $\eta_+ = \eta_-$;

(ii) *there is a constant* $Z > 0$ *such that for any periodic point* $x \in \Omega$, $T^k x = x$, *we have* $J_k^u(x) \cdot J_k^s(x) = Z^k$;

(iii) *the functions* $g_+(x)$ *and* $g_-(x)$ *are cohomologous, i.e. there is a constant* $R$ *and a Hölder continuous function* $u(x)$ *such that* $g_+(x) - g_-(x) = R + u(Tx) - u(x)$.

*If those conditions are satisfied, then*

$$-\ln Z = R = P(\eta_+) - P(\eta_-) = \gamma_- - \gamma_+$$

Theorem 1.13 now follows immediately. This theorem, combined with Proposition 4.14 from [2], gives Corollary 1.14.

Possible applications of Theorem 1.13 and Corollary 1.14 cover hyperbolic repellers constructed on the base of Hamiltonian systems (those preserve Liouville measures that are absolutely continuous). In particular, these include billiard systems, like the open billiard with three circular scatterers studied in [16], where repellers are thus always time-symmetric.

Another interesting class of repellers are *linear* repellers. Let the rectangles $R_1, \ldots, R_I$ be subset in $\mathbb{R}^d$ and all $E_x^u$ and all $E_x^s$ be parallel. Let the map $T$ be linear on each $R_i$, with the constant derivative $DT$ =const on $M$, so that the functions $J^u(x) = J^u$ and $J^s(x) = J^s$ are constant on $M$. In this case the measures $\eta_+$ and $\eta_-$ always coincide, and both coincide with the measure of maximal entropy on $\Omega$, see [2] for definitions and details. In this case the repeller $\Omega$ is, however, time symmetric if and only if $\det DT = J^u \cdot J^s = 1$, i.e. if $T$ preserves the Lebesgue measure in $\mathbb{R}^d$.

# 8 Generalizations and open problems

In our arguments, we never essentially relied on the fact that $T$ was a diffeomorphism of a connected manifold, in fact the action of $T$ on $H = M' \setminus M$ never came into play. All our results hold true under the following, more general assumptions:

Let $M$ be a finite union of disjoint closed domains $R_1, \ldots, R_I$ in a smooth Riemannian manifold $\mathcal{M}$. Let $T : M \to \mathcal{M}$ be a diffeomorphism of $M$ onto its image, which is $C^{1+\alpha}$ up to the boundary $\partial M$. We assume the Anosov splitting (1.1) at every $x \in M$, and require (1.2) if the corresponding iterations of $T$ are defined. Let the bundles $E_x^{u,s}$ be Hölder continuous and integrable over every $R_i$, so that $R_i$ is foliated by Hölder continuous families of $C^{1+\alpha}$ submanifolds $W_x^{u,s}$ such that $\mathcal{T}W_x^{u,s} = E_x^{u,s}$ at every $x \in R_i$. Assume that every $R_i$ is a rectangle and $\{R_1 \ldots, R_I\}$ is a Markov partition of $M$ in the sense of Section 2.

Under these assumptions our results remain true. The above setting is very convenient for horseshoe-like maps, studied in [4, 21].

We now discuss what happens if we relax the mixing assumption in Section 2. First, we can classify the rectangles like one does states of Markov chains. We call a rectangle $R_i$ recurrent if its points come back to itself under $T$, i.e. $\text{int}R_i \cap T^n(R_i \cap M_{-n}) \neq \emptyset$ for some $n \geq 1$. In the trivial case, where all the rectangles are nonrecurrent (transient), the sets $M_+, M_-$ and $\Omega$ are empty, and the phase space $M$ 'escapes' entirely.

The recurrent rectangles can be grouped, in each group points from any rectangle can be mapped into any other rectangle, so that the symbolic dynamics within every group is transitive.

Let us assume first that there is only one transitive group of rectangles $R_1, \cdots, R_{I_0}$, and put $M_0 = R_1 \cup \ldots \cup R_{I_0}$. This group is periodic if there is a $k \geq 1$ such that the periods of all the periodic points in $M_0$ are multiples of $k$. In that case this group can be divided into $k$ subgroups cyclicly permuted by $T$, and the restriction of $T^k$ to any subgroup is topologically mixing. The study of the map $T$ admits a standard reduction to that of $T^k$, well known in the theory of Axiom A diffeomorphisms [2], so that we can restrict ourselves to the case $k = 1$. Then the repeller $\Omega$ belongs in $M_0$. The nonrecurrent rectangles $R_i$, $i > I_0$, can be of three types: isolated (such that $\text{int}T^nR_i \cap M_0 = \emptyset$ for all $n \in \mathbb{Z}$), incoming (such that $\text{int}T^nR_i \cap M_0 \neq \emptyset$ for some $n > 0$) and outgoing (such that $\text{int}T^nR_i \cap M_0 \neq \emptyset$ for some $n < 0$). The set $M_+$ intersects only recurrent and outgoing rectangles, $M_-$ only recurrent and incoming ones. The measures $\mu_\pm$ conditioned on $M_0$ coincide with the corresponding measures for the restriction of $T$ to $M_0$. The measures $\eta_\pm$ and the escape rates $\gamma_\pm$ will be the same for $T|_M$ and $T|_{M_0}$. So, nonrecurrent rectangles do not really affect the properties of the repeller $\Omega$ studied in this paper, they only may enlarge the sets $M_\pm$ and 'stretch' the measures $\mu_\pm$ accordingly.

A more involved situation occurs when there are two or more groups of recurrent rectangles. For simplicity, consider two groups, $R'_1, \ldots, R'_{I_0}$ and $R''_1, \ldots, R''_{J_0}$, and put $M'_0 = \cup R'_i$ and $M''_0 = \cup R''_j$. If there is no connection between these groups, i.e. $\text{int}(T^nM'_0 \cap T^mM''_0) = \emptyset$ for all $m, n \in \mathbb{Z}$, then we have two trivially independent repellers in $M'_0$ and $M''_0$, respectively. On the contrary, if there is a route from $M'_0$ to $M''_0$, i.e. $\text{int}(T^nM'_0 \cap M''_0) \neq \emptyset$ for some $n \geq 1$, then the picture gets intricate. The rate of escape from $M'_0$ is still the same as for the map $T|_{M'_0}$, as if $M''_0$ did not exist. The escape from $M''_0$, however, is combined with the influx of points from $M'_0$. The resulting escape rate from $M$ will be than influenced by three factors: the escape rates from $M'_0$, $M''_0$ and by the fraction of $M_0'$ transmitted to $M_0''$ after escaping from $M_0'$. We did not investigate here these interesting phenomena.

Another natural extension would be to study Axiom A diffeomorphisms rather than Anosov ones. Let $T : M' \to M'$ be an Axiom A diffeomorphism with the basic set $\Omega$. Let $M \subset M'$ be a proper closed subdomain such that $\Omega = \cap_{-\infty}^{\infty} T^n M$. Then it might be possible to construct conditionally invariant measures on $M_+ = \cap_0^{\infty} T^n M$ and invariant measures on $\Omega$ in the same way as we did for Anosov diffeomorphisms. We leave this for future researches.

Lastly, there are nonuniformly hyperbolic diffeomorphisms and hyperbolic maps with singularities, like billiards, which have countable Markov partitions and the derivatives growing to infinity at singularities. Extension of our results to those models is the most challenging problem at present.

# Appendix

This appendix contains the Perron-Frobenius theorem on positive matrices and related results. Most of these results are taken from [4].

Let $V_m$ be the space of row $m$-vectors, and $V_m^*$ the space of column $m$-vectors. We equip them with norms

$$|\mathbf{a}| = \sum_{i=1}^{m} |a_i|, \qquad |\mathbf{b}^*| = \max_{1 \le i \le m} |b_i| \tag{A.1}$$

and scalar product

$$(\mathbf{a},\ \mathbf{b}^*) = a_1 b_1 + \cdots + a_m b_m$$

for all $\mathbf{a} \in V_m$ and $\mathbf{b}^* \in V_m^*$. We call vectors $\mathbf{a} \in V_m$ and $\mathbf{b}^* \in V_m^*$ positive if their components are all positive.

Note that $|(\mathbf{a}, \mathbf{b}^*)| \le |\mathbf{a}||\mathbf{b}^*|$.

Let $\mathcal{P}_m$ be the set of $m \times m$ matrices with positive entries:

$$\mathbf{A} = (A_{ij}) \in \mathcal{P}_m \quad \text{if} \quad A_{ij} > 0 \quad \forall\ 1 \le i, j \le m$$

Stochastic ($\sum_j A_{ij} = 1$) and substochastic ($\sum_j A_{ij} \le 1$) matrices are the best studied classes of matrices in $\mathcal{P}_m$.

**Theorem A.1 (Perron-Frobenius theorem)** *Every positive matrix $\mathbf{A} \in \mathcal{P}_m$ has a positive row eigenvector $\mathbf{p}$ and a positive column eigenvector $\mathbf{p}^*$:*

$$\mathbf{pA} = \lambda \mathbf{p} \quad \text{and} \quad \mathbf{Ap}^* = \lambda \mathbf{p}^*$$

*where $\lambda > 0$ is the largest (in absolute value) eigenvalue of the matrix $\mathbf{A}$. These vectors are unique up to a scalar multiple, i.e. the multiplicity of $\lambda$ is one.*

We put $\mathbf{p}$ and $\mathbf{p}^*$ for the Perron eigenvectors of $\mathbf{A}$ normalized so that

$$\sum_{i=1}^{m} p_i = |\mathbf{p}| = 1 \quad \text{and} \quad \sum_{i=1}^{m} p_i p_i^* = (\mathbf{p}^*, \mathbf{p}) = 1 \tag{A.2}$$

For a fixed matrix $\mathbf{A} \in \mathcal{P}_m$, we introduce other norms in $V_m$ and $V_m^*$ by

$$||\mathbf{a}||_r = \sum_{i=1}^{m} |a_i| p_i^*, \qquad ||\mathbf{b}^*||_c = \max_{1\le j\le m} (|b_j^*|/p_j^*) \tag{A.3}$$

If the components of $\mathbf{a}$ are non-negative, then

$$||\mathbf{a}||_r = (\mathbf{a}, \mathbf{p}^*) \quad \text{and} \quad ||\mathbf{a}\mathbf{A}||_r = \lambda ||\mathbf{a}||_r$$

Note that $|(\mathbf{a}, \mathbf{b}^*)| \le ||\mathbf{a}||_r ||\mathbf{b}^*||_c$.

We will say that $P \ge 1$ is an *estimate of the ratio of rows* of $\mathbf{A} \in \mathcal{P}_m$ if

$$P^{-1} \le A_{ij}/A_{kj} \le P \quad \forall\, 1 \le i, j, k \le m$$

If $\mathbf{A}$ satisfies this estimate, we write $\mathbf{A} \in \mathcal{P}_m(P)$.

If $\mathbf{A} \in \mathcal{P}_m(P)$, then the components of its Perron eigenvectors satisfy

$$P^{-1} \le p_i^*/p_j^* \le P, \quad \lambda P^{-1} \le A_{ij}/p_j \le \lambda P, \quad P^{-1} \le p_i^* \le P$$

for all $1 \le i, j \le m$. The norms defined by (A.1) and (A.3) are then equivalent:

$$P^{-1}|\mathbf{a}| \le ||\mathbf{a}||_r \le P|\mathbf{a}| \quad \text{and} \quad P^{-1}|\mathbf{b}^*| \le ||\mathbf{b}^*||_c \le P|\mathbf{b}|$$

for all $\mathbf{a} \in V_m$ and $\mathbf{b}^* \in V_m^*$.

The following estimate on the so called coefficients of ergodicity is also satisfied if $\mathbf{A} \in \mathcal{P}_m(P)$:

$$\sum_{j=1}^{m} p_j^* \cdot \inf_{1\le i\le m} (A_{ij}/p_i^*) \ge \lambda P^{-1}$$

Denote by $L_m$ and $L_m^*$ the orthogonal complements to the Perron eigenvectors:

$$L_m = \{\mathbf{a} \in V_m : (\mathbf{a}, \mathbf{p}^*) = 0\} \quad \text{and} \quad L_m^* = \{\mathbf{b}^* \in V_m^* : (\mathbf{p}, \mathbf{b}^*) = 0\}$$

Then we have the decompositions

$$\mathbf{a} = (\mathbf{a}, \mathbf{p}^*)\mathbf{p} + \mathbf{a}_0 \quad \text{with} \quad \mathbf{a}_0 \in L_m$$

and

$$\mathbf{b}^* = (\mathbf{p}, \mathbf{b}^*)\mathbf{p}^* + \mathbf{b}_0^* \quad \text{with} \quad \mathbf{b}_0^* \in L_m^*$$

**Lemma A.2** *If $\mathbf{A} \in \mathcal{P}_m(P)$, then for any $\mathbf{a} \in L_m$ we have*

$$||\mathbf{aA}||_r \leq \lambda(1 - P^{-1})||\mathbf{a}||_r$$

*and for any $\mathbf{b}^* \in L_m^*$ we have*

$$||\mathbf{Ab}^*||_c \leq \lambda(1 - P^{-2})||\mathbf{b}^*||_c$$

If $\lambda < 1$ (this is the case if $\mathbf{A}$ is a proper substochastic matrix), then this lemma says that the contraction in the orthogonal subspaces $L_m$ and $L_m^*$ is stronger than that in the eigenspaces spanned by the Perron eigenvectors.

**Corollary A.3** *If $\mathbf{A} \in \mathcal{P}_m(P)$ and $\theta = 1 - P^{-1}$, then*

$$\lim_{n\to\infty} \lambda^{-n}\mathbf{A}^n = \mathbf{p}^* \otimes \mathbf{p}$$

*where $(\mathbf{p}^* \otimes \mathbf{p})_{ij} = p_i^* p_j$ is the tensor product of $\mathbf{p}^*$ and $\mathbf{p}$. Moreover,*

$$||(\lambda^{-n}\mathbf{A}^n - \mathbf{p}^* \otimes \mathbf{p})_k||_r \leq 2\theta^n p_k^*$$

*where $B_k$, for a matrix $\mathbf{B}$, means the $k$-th row.*

*Remark.* If $\mathbf{A}$ is a stochastic matrix ($\sum_j A_{ij} = 1$), then $\lambda = 1$ and $p_j^* = 1$ for all $1 \leq j \leq n$, and we recover a well known ergodic theorem for finite Markov chains.

We now compare the action on positive row vectors by two positive matrices which are close to each other. We say that $\mathbf{B} \in \mathcal{P}_m$ is close to $\mathbf{A} \in \mathcal{P}_m$, with the *constant of proximity* $R \geq 1$ if

(A.4) $$R^{-1} \leq B_{ij}/A_{ij} \leq R \quad \forall\, 1 \leq i,j \leq m$$

In the following statements, $\mathbf{A} \in \mathcal{P}_m(P)$ is a fixed matrix, $\mathbf{p}$ is its Perron row eigenvector normalized by (A.2) and $\lambda$ is the corresponding eigenvalue. We also set $\theta = 1 - P^{-1}$.

**Lemma A.4** *Let $\mathbf{q}$ be an arbitrary positive row vector such that $||\mathbf{q}||_r = 1$. Let $\mathbf{B} \in \mathcal{P}_m$ be another matrix close to $\mathbf{A}$ with the constant of proximity $R \geq 1$. Then*

$$R^{-1}||\mathbf{pA}||_r \leq ||\mathbf{qB}||_r \leq R||\mathbf{pA}||_r$$

*and*

$$||\mathbf{qB} - \mathbf{pA}||_r \leq \lambda\theta||\mathbf{q} - \mathbf{p}||_r + \lambda(R - 1)$$

**Lemma A.5** *Let* $\mathbf{B} \in \mathcal{P}_m$ *be as in Lemma A.4. For any positive row vector* $\mathbf{q} \in V_m$ *we have*

$$\left|\left| \frac{\mathbf{qB}}{||\mathbf{qB}||_r} - \frac{\mathbf{pA}}{||\mathbf{pA}||_r} \right|\right|_r \leq \theta R ||\mathbf{q} - \mathbf{p}||_r + 2R(R-1)$$

**Theorem A.6** *Let* $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n \in \mathcal{P}_m$ *be matrices, all close to* $\mathbf{A}$ *with the same constant of proximity* $R \geq 1$*. For any positive row vector* $\mathbf{q} \in V_m$ *we put* $\mathbf{q}_n = \mathbf{qB}_1 \cdots \mathbf{B}_n$*. In addition, assume that* $\theta R < 1$*. Then we have*

$$\left|\left| \frac{\mathbf{q}_n}{||\mathbf{q}_n||_r} - \mathbf{p} \right|\right|_r \leq 2\theta^n R^n + 2R(R-1)(1-\theta R)^{-1}$$

*and*

$$\left| \frac{\mathbf{q}_n}{|\mathbf{q}_n|} - \mathbf{p} \right| \leq 4P\theta^n R^n + 4PR(R-1)(1-\theta R)^{-1}$$

# References


[1] D.V. Anosov and Ya.G. Sinai, *Some smooth ergodic systems*, Russ. Math. Surveys **22** (1967), 103–167.

[2] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lect. Notes Math. **470**, Springer-Verlag, Berlin, 1975.

[3] N.N. Čencova, *A natural invariant measure on Smale's horseshoe*, Soviet Math. Dokl. **23** (1981), 87–91.

[4] N.N. Čencova, *Statistical properties of smooth Smale horseshoes*, in: *Mathematical Problems of Statistical Mechanics and Dynamics*, R.L. Dobrushin, Editor, pp. 199–256, Reidel, Dordrecht, 1986.

[5] N.I. Chernov, G.L. Eyink, J.L. Lebowitz and Ya.G. Sinai, *Steady-state electrical conduction in the periodic Lorentz gas*, Comm. Math. Phys. **154** (1993), 569–601.

[6] P. Collet, S. Martinez and B. Schmitt, *The Yorke-Pianigiani measure and the asymptotic law on the limit Cantor set of expanding systems*, Nonlinearity **7** (1994), 1437–1443.

[7] P. Collet, S. Martinez and B. Schmitt, *The Pianigiani-Yorke measure for topological Markov chains*, manuscript, 1994.

[8] P. Ferrari, H. Kesten, S. Martinez and P. Picco, *Existence of quasi stationary distribution. A renewal dynamical approach*, to appear in Annals Probab.

[9] P. Gaspard and F. Baras, *Chaotic scattering and diffusion in the Lorentz gas*, Phys. Rev. E **51** (1995), 5332–5352.

[10] P. Gaspard and J.R. Dorfman, *Chaotic scattering theory, thermodynamic formalism, and transport coefficients*, preprint, 1995.

[11] P. Gaspard and G. Nicolis, *Transport properties, Lyapunov exponents, and entropy per unit time*, Phys. Rev. Lett. **65** (1990), 1693–1696.

[12] P. Gaspard and S. Rice, *Scattering from a classically chaotic repellor*, J. Chem. Phys. **90** (1989), 2225–2241.

[13] Y. Guivarc'h and J. Hardy, *Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov*, Ann. Inst. H. Poincaré, **24** (1988), 73–98.

[14] M. Keane and M. Mori, Dynamical systems on the Cantor sets associated with piecewise linear transformations, manuscript.

[15] O. Legrand and D. Sornette, *Coarse-grained properties of the chaotic trajectories in the stadium*, Phys. D **44** (1990), 229–247.

[16] A. Lopes and R. Markarian, *Open billiards: Cantor sets, invariant and conditionally invariant probabilities*, to appear in SIAM J. of Applied Math.

[17] R. Mañé, *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, Berlin, 1987.

[18] A. Manning, *A relation between Lyapunov exponents, Hausdorff dimension and entropy*, Ergod. Th. & Dynam. Sys. **1** (1981), 451–459.

[19] S. Martinez and M.E. Vares, *Markov chain associated to the minimal Q.S.D. of birth-rate chains*, to appear in J. Applied Probab.

[20] L. Mendoza, *The entropy of $C^2$ surface diffeomorphisms in terms of Hausdorff dimension and a Lyapunov exponent*, Ergod. Th. & Dynam. Sys. **5** (1985), 273–283.

[21] Z. Nitecki, *Differentiable Dynamics*, MIT Press, Cambridge, Mass., 1971.

[22] G. Pianigiani and J. Yorke, *Expanding maps on sets which are almost invariant: decay and chaos*, Trans. Amer. Math. Soc. **252** (1979), 351–366.

[23] Ya.G. Sinai, *Markov partitions and C-diffeomorphisms*, Funct. Anal. Its Appl. **2** (1968), 61–82.

[24] Ya.G. Sinai, *Gibbs measures in ergodic theory*, Russ. Math. Surveys **27** (1972), 21–69.

[25] L.-S. Young, *Dimension, entropy and Lyapunov exponents*, Ergod. Th. & Dynam. Sys. **2** (1982), 109–124.