Asymptotic completeness in quantum field theory. A class of Galilei-covariant models

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Abstract

We describe N-body scattering in the case of identical particles (bosons and fermions).

Contents

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0 Introduction

Two basic formalisms are used in quantum mechanics to describe many-body systems. The first assumes that particles are distinguishable and the full Hilbert space is the tensor product of one-particle spaces. The second, which is physically more correct and is one of the basic ingredients of quantum field theory, assumes that particles are divided into species and within each species they are identical. The space of N-particle states is the symmetric (for bosons) or antisymmetric (for fermions) Nth power of the one-particle space.

One of the areas of non-relativistic quantum mechanics where there exists a very satisfactory mathematical understanding of physical phenomena is N-body scattering theory. Using the formalism of distinguishable particles it has been shown that wave operators exist and are complete for an arbitrary number of particles and a very large class of potentials [E], [SigSof], [Gr], [Yaf], [De], [DeGe]. This article describes the construction of wave operators and their asymptotic completeness in the case of identical particles.

We will be interested in the following class of Hamiltonians:

$$\begin{aligned} H &= \sum_{e\in\mathcal{E}} \int a_e^*(x_e)\tfrac{1}{2m_e}D_e^2 a_e(x_e)\mathrm{d}x_e \\ &+ \sum_{e,e'\in\mathcal{E}} \int\int v_{ee'}(x_e - x_{e'})a_e^*(x_e)a_{e'}^*(x_{e'})a_{e'}(x_{e'})a_e(x_e)\mathrm{d}x_e\mathrm{d}x_{e'}. \end{aligned} \tag{0.1}$$

Here the index $e \in \mathcal{E}$ parametrizes various species of "elementary particles", x_e denotes the position of the eth particle, D_e its momentum, m_e its mass, $a_e^*(x_e)$, $a_e(x_e)$ are its creation and annihilation operators. The operators $a_e^*(x_e)$, $a_e(x_e)$ satisfy the canonical commutation/anticommutation relations depending on whether the eth particle is bosonic or fermionic. The operator H acts on the physical Fock space which is the tensor product of (bosonic or fermionic) Fock spaces describing all the species of particles.

We will also introduce the asymptotic free Hamiltonian

$$K^{\mathrm{as}} = \sum_{n\in\mathcal{E}^{\mathrm{as}}} \int a_n^*(x_n)\left(\tfrac{1}{2m_n}D_n^2 + \mu_n\right)a_n(x_n)\mathrm{d}x_n.$$

Here the index $n = (n_e)_{e\in\mathcal{E}}$ parametrizes all possible composite particles—clusters formed of n_e particles of the sth species, $m_n := \sum_{e\in\mathcal{E}} n_e m_e$ is the mass of the cluster, x_n is its center of mass position, D_n is its momentum and μ_n describes its internal energies (which are the appropriate eigenvalues of H^{red} – the full Hamiltonian with a removed center of mass motion). The operator K^{as} acts on the asymptotic Fock space $\Gamma(\mathcal{H}^{\mathrm{as}})$ space, which is the tensor product of Fock spaces describing various possible composite particles. Note that the Hamiltonian $K^{\rm as}$ has a much simpler structure than H – it is a second quantized one-body Hamiltonian. On the other hand, in general, $\Gamma(\mathcal{H}^{\rm as})$ is a much bigger Hilbert space than $\Gamma(\mathcal{H})$ – composite particles are usually much more numerous than the elementary particles.

Moreover, we will need to construct a certain natural identification operator $J^{\rm as}$ that maps the asymptotic Fock space into the physical Fock space.

The main result of this paper in the short-range case says that the wave operator

$$\Omega^+ := {\rm s}-\lim_{t\to\infty} {\rm e}^{{\rm i}tH} J^{\rm as} {\rm e}^{-{\rm i}tK^{\rm as}} \tag{0.2}$$

exists and is a unitary operator from $\Gamma(\mathcal{H}^{\rm as})$ to $\Gamma(\mathcal{H})$. (The unitarity of Ω^+ is usually called asymptotic completeness).

The analogous results in the long-range case (which includes the Coulomb potentials) are a little more complicated. One has two basic choices of describing the asymptotics of ${\rm e}^{-{\rm i}tH}$.

The first possibility is to introduce the so-called interacting asymptotic Hamiltonian, which has the form

$$\begin{array}{rl} H^{\rm as} & = \sum\limits_{n\in\mathcal{E}^{\rm as}} a_n^*(x_n)\left(\frac{1}{2m_n}D_n^2+\mu_n\right)a_n(x_n){\rm d}x_n \\ & + \sum\limits_{n,n'\in\mathcal{E}^{\rm as}} \int\int v_{nn'}^{\rm as}(x_n-x_{n'})a_n^*(x_n)a_{n'}^*(x_{n'})a_{n'}(x_{n'})a_n(x_n){\rm d}x_n dx_{n'}. \end{array}$$

Then one can show the existence and completeness of the wave operator

$$\Omega^+ := {\rm s}-\lim_{t\to\infty} {\rm e}^{{\rm i}tH} J^{\rm as} {\rm e}^{-{\rm i}tH^{\rm as}} \mathbf{1}^{\rm as,free,+} \tag{0.3}$$

where $\mathbf{1}^{\rm as,free,+}$ is a certain naturally defined projection onto the states that are asymptotically free for the Hamiltonian $H^{\rm as}$ as $t\to\infty$.

Alternatively, one can show the existence and completeness of modified wave operators

$$\Omega^{\rm mod,+} := {\rm s}-\lim_{t\to\infty} {\rm e}^{{\rm i}tH} J^{\rm as} {\rm e}^{-{\rm i}tK^{\rm as}-{\rm i}G^{\rm as}(t)} \tag{0.4}$$

where $G^{\rm as}(t)$ is a certain time-dependent modifier (the N-body version of the Dollard modifier).

Scattering in quantum field theory (including asymptotic completeness) is described in many physics books. It is a conceptual basis for constructing measurable quantities such as scattering cross-sections. It is usually presented in a somewhat heuristic manner, which is partly justified by the scarcity of rigorous results on this subject. We think that it is good to know that Hamiltonians of the form (0.1) possess a very satisfactory, complete and nontrivial mathematical scattering theory.

Hamiltonians of the form (0.1) appear quite often in physics texts. They are used to describe systems of interacting nonrelativistic particles. We think that it would be justified to call (0.1) "the standard Hamiltonian of non-relativistic matter".

Hamiltonians of the form (0.1) have some special features that distinguish them among other quantum-field-theoretical Hamiltonians. One of these features is the covariance with respect to Galilean transformations. Another special feature is the fact that they conserve the number of particles of each species.

The proof of the existence and completeness of (0.2), (0.3) and (0.4) consists of two parts. In the first part, which is much more difficult, one proves the existence and completeness of wave operators for distinguishable particles. This was done in [E], [SigSof], [Gr], [Yaf], [De]. Our standard reference for these results is [DeGe]. This paper is devoted to the second part of the proof, in which one shows how the results on distinguishable particles imply an analogous result on identical particles. All the functional-analytic difficulties are solved in the first part of the proof. The difficulties of the second part reside in the combinatorics and in the notation.

The results of this paper have a rather intuitive physical content. It seems that in some form they belong to the folklore of quantum physics [GrHun]. They are related to the Haag-Ruelle theory, which in the case of relativistic quantum field theory gives a framework for scattering theory (see eg. [GJ] and references therein). A related result describing the location of the essential spectrum for identical particles was considered in [SigSig] and [DHSV]. Similar models where considered in [NT] and [Jä]. Nevertheless, we are not aware of any rigorous description of the results of our paper in the literature.

The formalism of quantum field theory (Fock spaces, field operators) is in principle adapted to treat problems with a variable number of particles. One might think that this formalism is not necessary to study particle-conserving Hamiltonians, such as (0.1). Nevertheless, this formalism turns out to be very convenient to formulate results on scattering for identical particles.

Usually one defines separately a bosonic Fock space and a fermionic Fock space. In order to present our results in a concise way we found it convenient to define from the very beginning a mixed Fock space, whose one-particle subspace is the direct sum of bosonic and fermionic one-particle spaces.

We found it also convenient to endow a Fock space with a product, which we denote by $\square$. (A similar product was considered eg. in [Slo]). This product is associative and is closely related to the usual creation operators.

The Fock space over a one-particle space $\mathcal{H}$, denoted $\Gamma(\mathcal{H})$, is sometimes called the second quantization of $\mathcal{H}$ (although, admittedly, this name is not very logical, see [We]). In order to construct the asymptotic Fock space we will have to perform the second quantization twice, obtaining "the Fock space over a Fock space over $\mathcal{H}$", denoted $\Gamma^{\mathrm{tq}}(\mathcal{H}) := \Gamma(\Gamma(\mathcal{H}))$ – "the third quantization of $\mathcal{H}$". (The name "third quantization" is probably even less appropriate than "second quantization", but we did not find a better one). We will also need a certain natural map J^{tq} from $\Gamma^{\mathrm{tq}}(\mathcal{H})$ to $\Gamma(\mathcal{H})$. In its definition it is convenient to use the product $\square$. The map J^{tq} is used to define the identification operator J^{as} that appears in (0.2), (0.3) and (0.4).

If one wants to describe results on N-body systems, one is tempted to introduce lengthy and cumbersome notation involving long sequences with lots of indices. We did our best to avoid it. In fact, we made an effort to describe our results in a possibly simple and short way. This desire forced us to reconsider some basic notions related to quantum field theory. In our paper we give an essentially self-contained introduction to basic definitions of quantum field theory that we are using. Although most of this material is well known (see eg. [RS, vol II], [GJ], [BSZ]), we introduce some objects that, it seems, are not commonly known and might be useful outside of our work. This is especially the case of the above-mentioned product $\square$ and the map J^{tq}.

Let us describe how our paper is organized.

In Section 1 we fix some notation concerning tensor products of Hilbert spaces and Cartesian products of sets.

In Section 2 we describe theorems on the existence and the completeness of wave operators for distinguishable particles. For a proof of these theorems we refer the reader to [DeGe] and references therein. This section does not present new results as compared with eg. [DeGe]. Nevertheless, the formulation of the results is different. In [De] and [DeGe] the language of generalized N-body Schrödinger operators is used, which stresses the geometric aspects of the problem. In Section 2 we use a more physical (although more narrow) formalism of (distinguishable) particles.

Various general constructions related to second quantization are described in Sections 3 and 4.

In Section 5 we describe and prove the main result of the paper, that is the existence and completeness of wave operators for identical particles.

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1 General definitions

1.1 Hilbert spaces

Let $\mathcal{H}$ be a Hilbert space. The scalar product of $\phi, \psi \in \mathcal{H}$ will be denoted by $(\phi|\psi)$. The set of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted $B(\mathcal{H}_1, \mathcal{H}_2)$. We set $B(\mathcal{H}) := B(\mathcal{H}, \mathcal{H})$. If $A = (A_1, \ldots, A_p)$ is a vector of commuting self-adjoint operators and K is a Borel subset of $\mathbf{R}^p$, then $\mathbf{1}_K(A)$ denotes the spectral projection of A onto K.

If $\{\mathcal{H}_i\}_{i=1}^{\infty}$ is a family of Hilbert spaces, then we define the algebraic direct sum

$$\mathop{\oplus}_{\mathrm{fin},i=1}^{\infty} \mathcal{H}_i, \tag{1.5}$$

consisting of finite linear combinations and the usual direct sum of Hilbert spaces

$$\overset{\infty}{\underset{i=1}{\oplus}} \mathcal{H}_i,$$

which is the closure of (1.5).

1.2 Homomorphisms of commutative C^*- and W^*-algebras

In quantum mechanics we often need to use certain families of commuting observables. Usually, this is described using a certain number of commuting self-adjoint operators. It is however more flexible and appropriate to use to this end the language of commutative C^*- and W^*-algebras. In this section we recall some basic facts about this subject and describe the notation that we will use.

If $\mathcal{X}$ is a locally compact space, then $C_\infty(\mathcal{X})$ denotes the commutative C^*-algebra of functions on $\mathcal{X}$ vanishing at ∞. Likewise, $B^\infty(\mathcal{X})$ denotes the W^*-algebra of bounded Baire functions on $\mathcal{X}$ (see Vol. I of [RS]).

Let $\mathcal{H}$ be a Hilbert space. Let

$$C_\infty(\mathcal{X}) \ni f \to \gamma(f) \in B(\mathcal{H}). \tag{1.6}$$

be a $*$-homomorphism. Clearly, (1.6) has a unique extension to a normal $*$-homomorphism on $B^\infty(\mathcal{X})$, which we will denote by the same symbol

$$B^\infty(\mathcal{X}) \ni f \to \gamma(f) \in B(\mathcal{H}).$$

One says that γ is nondegenerate if $\gamma(\mathbf{1})$ equals the identity in $B(\mathcal{H})$.

If $\mathcal{X} = \mathbf{R}^p$, then the homomorphism γ uniquely determines a vector of commuting self-adjoint operators $A = (A_1, \ldots, A_p)$ such that

$$\gamma(f) = f(A). \tag{1.7}$$

γ is nondegenerate iff A is densely defined. Note that we will sometimes use the notation of the rhs of (1.7), that is $f(A)$, even if $\mathcal{X}$ is not $\mathbf{R}^p$ for any $p \in \mathbf{N}$, and we cannot use the interpretation in terms of a vector of commuting self-adjoint operators.

Suppose that $\mathcal{X}_i$, $i = 1, 2$, are locally compact spaces and we are given $*$-homomorphisms with commuting images, which are denoted by two notations

$$C_\infty(\mathcal{X}_i) \ni f \mapsto \gamma_i(f) = f(A_i) \in B(\mathcal{H}), \quad i = 1, 2.$$

Then we have a natural $*$-homomorphism from $C_\infty(\mathcal{X}_1 \times \mathcal{X}_2)$, which can be denoted in two fashions:

$$C_\infty(\mathcal{X}_1 \times \mathcal{X}_2) \ni F \mapsto \gamma_1 \otimes \gamma_2(F) = F(A_1, A_2) \in B(\mathcal{H}).$$

Suppose that $\mathcal{E}$ is a set and we are given a family of Hilbert spaces $\{\mathcal{H}_e\}_{e\in\mathcal{E}}$. Let

$$\mathcal{H} = \underset{e\in\mathcal{E}}{\oplus} \mathcal{H}_e.$$

Equip $\mathcal{E}$ with the discrete topology. Then we have a natural $*$-homomorphism

$$C_\infty(\mathcal{E}) \ni f \mapsto f(\hat{e}) \in B(\mathcal{H})$$

defined as

$$f(\hat{e}) := \underset{e\in\mathcal{E}}{\oplus} f(e).$$

(For a space $\mathcal{E}$ with a discrete topology $C_\infty(\mathcal{E})$ denotes the set of complex-valued functions on $\mathcal{E}$ such that $f \in C_\infty(\mathcal{E})$ if for any $\epsilon > 0$ there exists a finite subset $\mathcal{E}_\epsilon \subset \mathcal{E}$ and $|f| < \epsilon$ outside $\mathcal{E}_\epsilon$).

1.3 Permutations

If $p \in \mathbf{N}$, then S_p denotes the group of permutations of $\{1, \dots, p\}$.

If $X_1, \dots, X_p$ are sets, then we define the map

$$\theta(\sigma) : \underset{q=1}{\overset{p}{\times}} X_q \to \underset{q=1}{\overset{p}{\times}} X_{\sigma q}$$

by setting

$$\theta(\sigma)(x_1, \dots, x_p) := (x_{\sigma^{-1}1}, \dots, x_{\sigma^{-1}p}).$$

In particular, if $X_1 = \dots = X_p$, then we obtain an action of S_p in X^p.

Let $\mathcal{H}_1, \dots, \mathcal{H}_p$ be Hilbert spaces and $\sigma \in S_p$. Then we define the map

$$\Theta(\sigma) : \underset{q=1}{\overset{p}{\otimes}} \mathcal{H}_q \to \underset{q=1}{\overset{p}{\otimes}} \mathcal{H}_{\sigma q}$$

by setting

$$\Theta(\sigma)(\phi_1 \otimes \dots \otimes \phi_p) := \phi_{\sigma^{-1}1} \otimes \dots \otimes \phi_{\sigma^{-1}p}.$$

In particular, if $\mathcal{H} = \mathcal{H}_1 = \dots = \mathcal{H}_p$, then we obtain an action of the group S_p on $\mathcal{H}^{\otimes p}$.

1.4 Clusters

Let $\mathcal{J}$ be a (finite or infinite) set. Anticipating the next section we will call its elements "particles".

Let $\mathcal{C}$ denote the family of finite subsets of $\mathcal{J}$ (the family of clusters). Every $j \in \mathcal{J}$ we identify with $\{j\} \in \mathcal{C}$. If $p \in \mathbf{N}$, then we set

$$\mathcal{C}_p := \{c \in \mathcal{C} \ : \ |c| = p\}.$$

($|c|$ denotes the number of elements of c). Clearly,

$$\mathcal{C} = \bigsqcup_{p\in\mathbf{N}} \mathcal{C}_p,$$

where $\bigsqcup$ denotes the disjoint union.

Suppose that we fix a total order in $c \in \mathcal{C}_p$ (denoted $<$). Then we have a unique order-preserving bijection

$$\pi : \{1, \dots, p\} \to c \tag{1.8}$$

called the numbering of c.

¿From now on we assume that a total order in $\mathcal{J}$, denoted $<$, has been fixed. This fixes a total order in every $c \in \mathcal{C}$, which will be denoted by the same symbol $<$. The corresponding map (1.8) will be denoted π_c.

1.5 Cartesian and tensor products associated with a cluster

Suppose that with every $j \in \mathcal{J}$ we associate a certain set X_j and a certain Hilbert space $\mathcal{H}_j$.

We define

$$\mathop{\times}_{j\in c} X_j := X_{\pi(1)} \times \dots \times X_{\pi(p)}, \qquad \mathop{\otimes}_{j\in c} \mathcal{H}_j := \mathcal{H}_{\pi(1)} \otimes \dots \otimes \mathcal{H}_{\pi(p)}. \tag{1.9}$$

We will often use the shorthand notation

$$X_c := \mathop{\times}_{j\in c} X_j, \qquad \mathcal{H}_c := \mathop{\otimes}_{j\in c} \mathcal{H}_j, .$$

1.6 Clusterings

Let $\mathcal{A}$ denote the family of finite subsets of $\mathcal{C}$ such that $a \in \mathcal{A}$ and $b, b' \in a$ implies $b = b'$ or $b \cap b' = \emptyset$. Elements of $\mathcal{A}$ are called clusterings. We identify $c \in \mathcal{C}$ with $\{c\} \in \mathcal{A}$. We define also

$$c^{\mathrm{free}} := \mathop{\cup}_{j\in c} \{\{j\}\} \in \mathcal{C},$$

(all clusters in c^{free} are one-element).

For any $c \in \mathcal{C}$ we set

$$\mathcal{A}_c := \left\{ a \in \mathcal{A} \ : \ c = \mathop{\cup}_{b\in a} b \right\}.$$

In other words, $a \in \mathcal{A}_c$ if a is a partition of c into disjoint subsets. The elements of $\mathcal{A}_c$ are called clusterings of c. Clearly,

$$\mathcal{A} = \bigsqcup_{c\in\mathcal{C}} \mathcal{A}_c.$$

In every $a \in \mathcal{A}$ we fix a total order, denoted $<$. Let $p \in \mathbf{N}$, $c \in \mathcal{C}_p$ and $a \in \mathcal{A}_c$. The orders in c and in a determine a certain new total order $<_a$ in c, which is defined as follows:
(1) if $j, j' \in b$, $b \in a$, then $j <_a j'$ iff $j < j'$;
(2) if $j \in b$, $j \in b'$, $b \neq b'$, $b, b' \in a$, then $j <_a j'$ iff $b < b'$.
The corresponding bijection (1.8) will be denoted π_a.

Define $\sigma_a \in S_p$

$$\sigma_a := \pi_a^{-1}\pi_c.$$

1.7 Cartesian and tensor products associated with a clustering

We have

$$\mathop{\times}_{b\in a} X_b = \mathop{\times}_{b\in a}\left(\mathop{\times}_{j\in b} X_j\right) = X_{\pi_a(1)} \times \cdots \times X_{\pi_a(p)}.$$

$$\mathop{\otimes}_{b\in a} \mathcal{H}_b = \mathop{\otimes}_{b\in a}\left(\mathop{\otimes}_{j\in b} \mathcal{H}_j\right) = \mathcal{H}_{\pi_a(1)} \otimes \cdots \otimes \mathcal{H}_{\pi_a(p)},$$

We will often use the shorthand notation

$$X_a := \mathop{\times}_{b\in a} X_b, \qquad \mathcal{H}_a := \mathop{\otimes}_{b\in a} \mathcal{H}_b.$$

Suppose that $c \in \mathcal{C}$ and $a \in \mathcal{A}_c$. Then we have natural identifications

$$\theta(\sigma_a) : X_a \to X_c, \qquad \Theta(\sigma_a) : \mathcal{H}_a \to \mathcal{H}_c. \tag{1.10}$$

2 Distinguishable particles

2.1 Elementary particles

Suppose that the set $\mathcal{J}$ labels all the particles. If $j \in \mathcal{J}$, then $m_j \in \mathbf{R}_+$ denotes the mass of the jth particle, $X_j \simeq \mathbf{R}^p$ denotes its configuration space, $x_j \in X_j$ its position, $D_j = \frac{1}{i}\nabla_{x_j}$ is its momentum, h_j is the finite dimensional Hilbert space of its internal degrees of freedom. In the case of Coulomb systems z_j is its charge. In X_j we have two Euclidean structures, the basic metric x_j^2, and the "kinetic metric" $m_j x_j^2$. We set

$$\mathcal{H}_j := L^2(X_j) \otimes h_j.$$

2.2 Clusters of elementary particles

Recall that in Subsection 1.4 for any $c \in \mathcal{C}$ we defined the Euclidean space X_c and the Hilbert space $\mathcal{H}_c$. X_c is equipped with the kinetic metric

$$\sum_{j\in c} m_j x_j^2. \tag{2.11}$$

Clearly, we have the natural identification

$$\mathcal{H}_c = L^2(X_c) \otimes \underset{j\in c}{\otimes} h_j. \tag{2.12}$$

$x_c := (x_j)_{j\in c}$ and $D_c := (D_j)_{j\in c}$ denote the position and the momentum operators of cluster c.

The center-of-mass subspace of X_c is defined as

$$X_c^{\mathrm{cm}} := \{x_c \in X_c \ : \ x_{j_1} = x_{j_2}, \quad j_1, j_2 \in c\}.$$

The space of internal coordinates of cluster $c \in \mathcal{C}$ is

$$X_c^{\mathrm{red}} := (X_c^{\mathrm{cm}})^{\perp} = \left\{ x_c \in X_c \ : \ \sum_{j\in c} m_j x_j = 0 \right\}$$

where $\perp$ denotes the orthogonal complement wrt the kinetic metric. Clearly,

$$X_c = X_c^{\mathrm{cm}} \oplus X_c^{\mathrm{red}}.$$

We set

$$m_c := \sum_{j\in c} m_j, \qquad z_c := \sum_{j\in c} z_j.$$

We define the following operators on $\mathcal{H}_c$ describing the center-of-mass position, the center-of-mass momentum and the kinetic energy:

$$x_c^{\mathrm{cm}} := \frac{1}{m_c} \sum_{j\in c} m_j x_j, \qquad D_c^{\mathrm{cm}} := \sum_{j\in c} D_j,$$

$$K_c := \sum_{j\in c} \frac{1}{2m_j} D_j^2.$$

Note that

$$X_c^{\mathrm{red}} = \{x_c \in X_c \ : x_c^{\mathrm{cm}} = 0\}, \qquad D_c^{\mathrm{cm}} := \frac{1}{i} \nabla_{x_c^{\mathrm{cm}}}.$$

We set

$$\mathcal{H}_c^{\mathrm{red}} := L^2(X_c^{\mathrm{red}}) \otimes \underset{j\in c}{\otimes} h_j.$$

We have

$$\mathcal{H}_c \simeq L^2(X_c^{\mathrm{cm}}) \otimes \mathcal{H}_c^{\mathrm{red}}. \tag{2.13}$$

With respect to the decomposition (2.13) we can write:

$$K_c = \frac{1}{2m_c}(D_c^{\mathrm{cm}})^2 \otimes \mathbf{1} + \mathbf{1} \otimes K_c^{\mathrm{red}}.$$

If $j, j' \in \mathcal{J}$ and $x \in X$, then $v_{jj'}(x) \in B(h_j \otimes h_{j'})$ is a self-adjoint operator called the potential of interaction of the jth and j'th particle. We assume that $v_{jj}(x) = 0$ and

$$v_{jj'}(x)(1 + D^2)^{-1} \text{ is compact on } L^2(X) \otimes h_{j'} \otimes h_{j'}. \tag{2.14}$$

On $\mathcal{H}_c$ we define the potential

$$V_c := \frac{1}{2} \sum_{j,j' \in c} v_{jj'}(x_j - x_{j'}),$$

Clearly, with respect to the decomposition (2.13) we can write:

$$V_c = \mathbf{1} \otimes V_c^{\mathrm{red}}.$$

Note that V_c is relatively bounded wrt K_c with an infinitesimal bound. Likewise, V_c^{red} is relatively bounded wrt K_c^{red} with an infinitesimal bound.

We define the full Hamiltonian and the reduced Hamiltonian as the self-adjoint Hamiltonians on $\mathcal{H}_c$ and $\mathcal{H}_c^{\mathrm{red}}$ respectively:

$$H_c := K + V, \qquad H_c^{\mathrm{red}} := K^{\mathrm{red}} + V^{\mathrm{red}}.$$

Clearly,

$$H_c = \frac{1}{2m_c}(D_c^{\mathrm{cm}})^2 \otimes \mathbf{1} + \mathbf{1} \otimes H_c^{\mathrm{red}}.$$

We also set

$$X_c^{\mathrm{free}} := \{x_c \in X_c \ : \ x_j \neq x_{j'},\ j \neq j',\ j, j' \in c\}.$$

2.3 Asymptotic particles

For $c \in \mathcal{C}$ we denote

$$\mathcal{H}_c^{\mathrm{as}} := \mathrm{Ran}\mathbf{1}^{\mathrm{pp}}(\mathbf{1} \otimes H_c^{\mathrm{red}}) \subset \mathcal{H}_c, \qquad h_c^{\mathrm{as}} := \mathrm{Ran}\mathbf{1}^{\mathrm{pp}}(H_c^{\mathrm{red}}) \subset \mathcal{H}_c^{\mathrm{red}}.$$

We will write x_c^{as}, D_c^{as} for x_c^{cm}, D_c^{cm} acting on $\mathcal{H}_c^{\mathrm{as}}$. We will also write X_c^{as} instead of X_c^{cm}. Clearly,

$$\mathcal{H}_c^{\mathrm{as}} = L^2(X_c^{\mathrm{as}}) \otimes h_c^{\mathrm{as}}.$$

We set

$$K_c^{\mathrm{as}} := H_c\Big|_{\mathcal{H}_c^{\mathrm{as}}}, \qquad \mu_c^{\mathrm{as}} := H_c^{\mathrm{red}}\Big|_{h_c^{\mathrm{as}}}.$$

Clearly,

$$K_c^{\mathrm{as}} = \frac{1}{2m_c}(D_c^{\mathrm{as}})^2 \otimes \mathbf{1} + \mathbf{1} \otimes \mu_c^{\mathrm{as}}.$$

2.4 Clusters of asymptotic particles

Let $a \in \mathcal{A}$. We define

$$X_a^{\mathrm{as}} := \underset{b\in a}{\times} X_b^{\mathrm{as}}, \qquad \mathcal{H}_a^{\mathrm{as}} := \underset{b\in a}{\otimes} \mathcal{H}_b^{\mathrm{as}},$$

and a self-adjoint operator acting on $\mathcal{H}_a^{\mathrm{as}}$:

$$K_a^{\mathrm{as}} := \sum_{b\in a} K_b^{\mathrm{as}}.$$

$x_a^{\mathrm{as}} := (x_b^{\mathrm{as}})_{b\in a}$ and $D_a^{\mathrm{as}} := (D_b^{\mathrm{as}})_{b\in a}$ denote the position and the momentum operators acting on $\mathcal{H}_a^{\mathrm{as}}$.

We set

$$X_a^{\mathrm{as,free}} := \{x_a^{\mathrm{as}} \in X_a^{\mathrm{as}} \ : \ x_b^{\mathrm{as}} \neq x_{b'}^{\mathrm{as}}, \ b \neq b', \ b, b' \in a\}.$$

2.5 Identification operator

Let $c \in \mathcal{C}$ and $a \in \mathcal{A}_c$. Recall that in Subsection 1.7 we defined the Hilbert space $\mathcal{H}_a$ and the vector space X_a. We have $\mathcal{H}_a^{\mathrm{as}} \subset \mathcal{H}_a$ and $X_a^{\mathrm{as}} \subset X_a$. Now we define the identification operators

$$j_a^{\mathrm{as}} : X_c^{\mathrm{as}} \to X_c, \qquad J_a^{\mathrm{as}} : \mathcal{H}_a^{\mathrm{as}} \to \mathcal{H}_c,$$

by setting

$$j_a^{\mathrm{as}} := \theta(\sigma_a)\Big|_{X_a^{\mathrm{as}}}, \qquad J_a^{\mathrm{as}} := \Theta(\sigma_a)\Big|_{\mathcal{H}_a^{\mathrm{as}}}.$$

2.6 Existence of the asymptotic velocity

The asymptotic velocity is a natural asymptotic observable that exists under very weak assumptions on the decay of potentials. It has a natural physical meaning, and it is useful in the proof of the asymptotic completeness of wave operators (for a much more restrictive class of potentials).

Theorem 2.1 *For every $j, j' \in \mathcal{J}$ assume*

$$(1 + D^2)^{-1}\nabla_x v_{jj'}(x)\mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R})(1 + D^2)^{-1} \in L^1(\mathrm{d}R). \tag{2.15}$$

Let $c \in \mathcal{C}$. Then for every $F \in C_\infty(X_c)$ there exists the limit

$$\mathrm{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH_c} F\left(\frac{x_c}{t}\right) \mathrm{e}^{-\mathrm{i}tH_c}. \tag{2.16}$$

There exists a unique vector of densely defined self-adjoint operators P_c^+ such that the limit (2.16) equals $F(P_c^+)$. Moreover, we have

$$[P_c^+, H_c] = 0,$$

$$\mathbf{1}_{X_c^{\mathrm{cm}}}(P_c^+) = \mathcal{H}_c^{\mathrm{as}}.$$

If $V_c = 0$, then $P_c^+ = D_c$

2.7 Short-range case

Let us formulate the main result of N-body scattering theory for distinguishable particles in the short-range case.

Theorem 2.2 *For every $j, j' \in \mathcal{J}$, assume*

$$(1+D^2)^{-1} v_{jj'}(x) \mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R})(1+D^2)^{-1/2} \in L^1(\mathrm{d}R). \tag{2.17}$$

Let $c \in \mathcal{C}$. Then for any $a \in \mathcal{A}_c$, there exists

$$\Omega_a^+ := \mathrm{s}-\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH_c} J_a^{\mathrm{as}} \mathrm{e}^{-\mathrm{i}tK_a^{\mathrm{as}}}. \tag{2.18}$$

Ω_a^+ is an isometry from $\mathcal{H}_a^{\mathrm{as}}$ to $\mathcal{H}_c$ and

$$\Omega_a^+ K_a^{\mathrm{as}} = H_c \Omega_a^+,$$

$$\Omega_a^+ j_a^{\mathrm{as}}(D_a^{\mathrm{as}}) = P_c^+ \Omega_a^+.$$

Moreover,

$$\mathcal{H}_c = \underset{a\in\mathcal{A}_c}{\oplus} \mathrm{Ran}\Omega_a^+$$

2.8 Long-range case – free region

The remaining part of this chapter is devoted to wave operators in the long range case. ¿From now on will assume that for any $j, j' \in \mathcal{J}$

$$v_{jj'}(x) = v_{jj'}^{\mathrm{s}}(x) + v_{jj'}^{\mathrm{l}}(x)$$

where $v_{jj'}^{\mathrm{l}}(x) \in \mathbf{R}$ and $v_{jj'}^{\mathrm{s}}(x) \in B(h_j \otimes h_{j'})$ (the long-range part of the potential is scalar in the internal variables).

In the case of $v_{jj'}^{\mathrm{l}}(x)$ we will consider two types of assumptions. Either we will assume that it is a Coulomb potential

$$v_{jj'}^{\mathrm{l}}(x) := \frac{z_j z_{j'}}{|x|},$$

or we will give more general decay conditions.

The following theorem gives a description of the scattering in the asymptotically free region. Note that the decay assumptions on the potentials are quite weak.

Theorem 2.3 *(1) For any $j, j' \in \mathcal{J}$ assume that*

$$\begin{aligned}
&(1+D^2)^{-1} v_{jj'}^{\mathrm{s}}(x) \mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R})(1+D^2)^{-1/2} \in L^1(\mathrm{d}R),\\
&\lim_{|x|\to\infty} v_{jj'}^{\mathrm{l}}(x) = 0, \\
&\nabla_x^n v_{jj'}^{\mathrm{l}}(x) \mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R}) \in \langle R\rangle^{1-n} L^1([1,\infty[, \mathrm{d}R), \qquad n = 1, 2.
\end{aligned} \tag{2.19}$$

Then we can find a function $[0,\infty[\times X_c \ni (t,\xi_c) \to S_c(t,\xi_c) \in \mathbf{R}$ *such that there exists*

$$\Omega_c^{\mathrm{free},+} := \mathrm{s}-\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH_c}\mathrm{e}^{-\mathrm{i}tS_c(t,D_c)}.$$

The operator $\Omega_c^{\mathrm{free},+}$ *is an isometry from* $\mathcal{H}_a^{\mathrm{as}}$ *to* $\mathcal{H}_c$ *and it satisfies*

$$\Omega_c^{\mathrm{free},+}K_c = H_c\Omega_c^{\mathrm{free},+},$$

$$\Omega_c^{\mathrm{free},+}D_c = P_c^+\Omega_c^{\mathrm{free},+},$$
$$\mathrm{Ran}\Omega_c^{\mathrm{free},+} = \mathrm{Ran}\mathbf{1}_c^{\mathrm{free},+},$$

where

$$\mathbf{1}_c^{\mathrm{free},+} := \mathbf{1}_{X_c^{\mathrm{free}}}(P_c^+).$$

(2) If we replace the third line of the assumption (2.19) by

$$\nabla_x v_{jj'}^{\mathrm{l}}(x)\mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R})\langle R\rangle^{1/2} \in L^1([1,\infty[,\mathrm{d}R),$$

then we can use the so-called Dollard modifier:

$$S(t,\xi_c) := \sum_{j\in c}(2m_j)^{-1}\xi_j^2 + \sum_{j,j'\in c,\ j\neq j'} g_{jj'}(t)$$

where

$$g_{jj'}(t) := \int_0^t v_{jj'}^{\mathrm{l}}\left(t'\frac{\xi_j}{m_j} - t'\frac{\xi_{j'}}{m_{j'}}\right)\mathrm{d}t'.$$

(3) In the Coulomb case we can set

$$g_{jj'}(t) := \frac{z_j z_{j'}\log t}{\left|\frac{D_j}{m_j} - \frac{D_{j'}}{m_{j'}}\right|}.$$

(4) In the short-range case, that is if $v_{jj'}^{\mathrm{l}}(x) = 0$, *we can set*

$$g_{jj'}(t) = 0.$$

Then we simply obtain the usual free region wave operator:

$$\Omega_c^{\mathrm{free},+} = \Omega_{c^{\mathrm{free}}}^+,$$

where $\Omega_{c^{\mathrm{free}}}^+$ *is the wave operator for the free channel defined in Theorem 2.2.*

2.9 Long-range case – asymptotic interacting Hamiltonian

The asymptotics of $\mathrm{e}^{-\mathrm{i}tH}$ on the whole physical space can be described in two basic ways. In this subsection we compare it with the evolution generated by a certain interacting asymptotic Hamiltonian H^{as} on the subspace of its asymptotically free states. In the next subsection we will compare it with an evolution generated by the free asymtotic Hamiltonian K^{as} with a time-dependent modifier.

For disjoint $b, b' \in \mathcal{C}$ we set

$$v_{bb'}^{\mathrm{l}}(x) := \sum_{j\in b,\ j'\in b'} v_{jj'}^{\mathrm{l}}(x).$$

In the Coulomb case we obtain

$$v_{bb'}^{\mathrm{l}}(x) = \frac{z_b z_{b'}}{|x|}.$$

Then we set as operators on $\mathcal{H}_a^{\mathrm{as}}$:

$$V_a^{\mathrm{as}} := \sum_{b,b'\in a\ \ b\neq b'} v_{bb'}^{\mathrm{l}}\left(x_b^{\mathrm{as}} - x_{b'}^{\mathrm{as}}\right).$$

$$H_a^{\mathrm{as}} := K_a^{\mathrm{as}} + V_a^{\mathrm{as}}.$$

Theorem 2.4 *Let $\mu = \sqrt{3} - 1$ and for any $j, j' \in \mathcal{J}$*

$$\begin{aligned} &(1+D^2)^{-1} v_{jj'}^{\mathrm{s}}(x) \mathbf{1}_{[1,\infty[}\left(\tfrac{|x|}{R}\right) \in L^1(\mathrm{d}R), \\ &\lim_{|x|\to\infty} v_{jj'}^{\mathrm{l}}(x) = 0, \\ &\nabla_x v_{jj'}^{\mathrm{l}}(x) \mathbf{1}_{[1,\infty[}\left(\tfrac{|x|}{R}\right)\langle R\rangle^{\mu} \in L^1([1,\infty[, \mathrm{d}R). \end{aligned} \tag{2.20}$$

Let $c \in \mathcal{C}$ and $a \in \mathcal{A}_c$. Let $P_a^{\mathrm{as},+}$ be the asymptotic velocity defined as in Theorem 2.2 for the Hamiltonian H_a^{as}. Set

$$\mathbf{1}_a^{\mathrm{as,free},+} := \mathbf{1}_{X_a^{\mathrm{as,free}}}(P_a^{\mathrm{as},+})$$

Then there exists

$$\Omega_a^+ := \mathrm{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH_c} J_a^{\mathrm{as}} \mathrm{e}^{-\mathrm{i}tH_a^{\mathrm{as}}} \mathbf{1}_a^{\mathrm{as,free},+}.$$

Moreover, Ω_a^+ is an isometry from $\mathrm{Ran}\mathbf{1}_a^{\mathrm{as,free},+} \subset \mathcal{H}_a^{\mathrm{as}}$ to $\mathcal{H}_c$ satisfying

$$\Omega_a^+ H_a^{\mathrm{as}} = H_c \Omega_a^+,$$

$$\Omega_a^+ j_a^{\mathrm{as}}(P_a^{\mathrm{as},+}) = P_c^+ \Omega_a^+$$

and

$$\mathcal{H}_c = \underset{a\in\mathcal{A}_c}{\oplus} \mathrm{Ran}\Omega_a^+$$

2.10 Long-range case – modified wave operators

In this subsection we present the second, more conventional approach to the long-range N-body scattering.

Let us define Dollard-type modifiers. For disjoint $b, b' \in \mathcal{C}$ we set:

$$g_{bb'}(t) := \int_0^t v^{\rm l}_{bb'} \left(t' \frac{D_b^{\rm as}}{m_b} - t' \frac{D_{b'}^{\rm as}}{m_{b'}} \right) {\rm d}t'. \tag{2.21}$$

where $v^{\rm l}_{bb'}(x)$ was defined in (2.21) In the case of Coulomb potentials we set

$$g_{bb'}(t) := \frac{z_b z_{b'} \log t}{\left| \frac{D_b^{\rm as}}{m_b} - \frac{D_{b'}^{\rm as}}{m_{b'}} \right|}.$$

The Dollard modifier equals

$$G_a^{\rm as}(t) := \sum_{b,b' \in a \ b \neq b'} g_{bb'}(t).$$

Theorem 2.5 *Suppose that the assumptions of Theorem 2.4 hold. Let $c \in \mathcal{C}$ and $a \in \mathcal{A}_c$. Then there exists*

$$\mathrm{s-}\lim_{t \to \infty} {\rm e}^{{\rm i}tH_c} J_a^{\rm as} {\rm e}^{-{\rm i}tK_a^{\rm as} - {\rm i}G_a^{\rm as}(t)} =: \Omega_a^{\rm mod,+}. \tag{2.22}$$

$\Omega_a^{\rm mod,+}$ *is an isometry from $\mathcal{H}_a^{\rm as}$ to $\mathcal{H}_c$,*

$$\Omega_a^{\rm mod,+} K_a^{\rm as} = H_c \Omega_a^{\rm mod,+},$$

$$\Omega_a^{\rm mod,+} j_a^{\rm as}(D_a^{\rm as}) = P_c^+ \Omega_a^{\rm mod,+}$$

and

$$\mathcal{H}_c = \mathop{\oplus}_{a \in \mathcal{A}_c} {\rm Ran} \Omega_a^{\rm mod,+}.$$

Note that Theorem 2.3 applied to $H_a^{\rm as}$ and Theorem 2.4 imply immediately Theorem 2.5.

Note also that the wave operator $\Omega_c^{\rm free,+}$ of Theorem 2.3 constructed with the Dollard modifier coincides with the wave operator $\Omega_{c^{\rm free}}^{\rm mod,+}$ of Theorem 2.5.

3 Second quantization in the category of sets

3.1 Second quantization of a set

In Section 4 we will consider the well known functor Γ, which to every Hilbert space associates the Fock space $\Gamma(\mathcal{H})$. Γ is sometimes called the "second quantization". An analogous (simpler but apparently less known) functor can be defined in the category of sets. We will denote it by the same letter Γ.

Let $\mathcal{E}$ be a set. Anticipating Section 5 we will call the elements of $\mathcal{E}$ "species".

We define $\Gamma(\mathcal{E})$ to be the set of sequences $n = (n_e)_{e\in\mathcal{E}}$ with values in $\mathbf{N}$ with all but a finite number of elements vanishing. If $p \in \mathbf{N}$, then we set

$$\Gamma_p(\mathcal{E}) := \left\{ n \in \Gamma(\mathcal{E}) \ : \ \sum_{e\in\mathcal{E}} n_e = p \right\}.$$

Clearly,

$$\Gamma(\mathcal{E}) = \bigsqcup_{p\in\mathbf{N}} \Gamma_p(\mathcal{E}). \tag{3.23}$$

Clearly, for any $p \in \mathbf{N}$ the group S_p acts on the Cartesian power $\mathcal{E}^p$. We have the natural isomorphism

$$\mathcal{E}^p/S_p \simeq \Gamma_p(\mathcal{E}), \tag{3.24}$$

where $\mathcal{E}^p/S_p$ denotes the set of orbits. (3.23) and (3.24) should be compared with the definition of the Fock space (4.30).

If e is a generic notation for an element of $\mathcal{E}$, then a generic element of $\Gamma(\mathcal{E})$ will be sometimes denoted $\underline{e}$

Note that if $|\mathcal{E}| = 1$, then $\Gamma(\mathcal{E}) \simeq \mathbf{N}$.

If $f : \mathcal{E}_1 \to \mathcal{E}_2$ is a map, than we have a natural map $\Gamma(f) : \Gamma(\mathcal{E}_1) \to \Gamma(\mathcal{E}_2)$.

3.2 "Third quantization" of a set

$\Gamma(\Gamma(\mathcal{E}))$ will be denoted for shortnesss $\Gamma^{\rm tq}(\mathcal{E})$, where tq stands for "third quantizaton". It is the family of sequences $k = (k_m)_{m\in\Gamma(\mathcal{E})}$ with values in $\mathbf{N}$ with all but a finite number of elements vanishing.

If $n \in \Gamma(\mathcal{E})$, then we set

$$\Gamma^{\rm tq}_n(\mathcal{E}) := \left\{ k \in \Gamma^{\rm tq}(\mathcal{E}) \ : \ n = \sum_{m\in\Gamma(\mathcal{E})} k_m m \right\}.$$

Clearly,

$$\Gamma^{\rm tq}(\mathcal{E}) = \bigsqcup_{n\in\Gamma(\mathcal{E})} \Gamma^{\rm tq}_n(\mathcal{E}).$$

We define the map

$$j^{\mathrm{tq}} : \Gamma^{\mathrm{tq}}(\mathcal{E}) \to \Gamma(\mathcal{E})$$

by setting $j^{\mathrm{tq}}(k) := n$ if $k \in \Gamma_n(\mathcal{E})$.

3.3 Permutations that preserve species

Suppose that the set $\mathcal{E}$ is equipped with a total order $<$. Let $p \in \mathbf{N}$ and $n \in \Gamma_p(\mathcal{E})$. We define

$$\delta_n : \{1, \dots, p\} \to \mathcal{E}$$

by the following conditions:

$$(1) \qquad \left|\delta_n^{-1}\{e\}\right| = n_e, \qquad e \in \mathcal{E};$$

$$(2) \qquad 1 \le q < q' \le p \;\Rightarrow\; \delta_n q \le \delta_n q'.$$

Let $\sigma \in S_p$. We say that $\sigma \in S_n$ if

$$\sigma\left(\delta_n^{-1}\{e\}\right) = \delta_n^{-1}\{e\}, \quad e \in \mathcal{E}.$$

Clearly, S_n is a subgroup of S^p. Note that

$$S_n \simeq \mathop{\times}_{e \in \mathcal{E}} S_{n_e},$$

$$|S_n| = \prod_{e \in \mathcal{E}} n_e! =: n!. \tag{3.25}$$

Treating δ_n as a "labelling" of numbers in $\{1, \dots, p\}$ with "species" we can say that S_n is the subgroup of permutations that preserve species.

Assume that $\{\mathcal{X}_e\}$, $e \in \mathcal{E}$, is a family of sets and $\{\mathcal{H}_e\}$, $e \in \mathcal{E}$, is a family of Hilbert spaces. Then for $p \in \mathbf{N}$ and $n \in \Gamma_p(\mathcal{E})$, we define

$$\mathcal{X}_n := \mathop{\times}_{e \in \mathcal{E}} \mathcal{X}_e^{n_e} = \mathop{\times}_{q=1}^{p} \mathcal{X}_{\delta_n q}, \qquad \mathcal{H}_n := \mathop{\otimes}_{e \in \mathcal{E}} \mathcal{H}_e^{\otimes n_e} = \mathop{\otimes}_{q=1}^{p} \mathcal{H}_{\delta_n q}.$$

3.4 Clusters with composition $n \in \Gamma(\mathcal{E})$

Now suppose that $\mathcal{J}$ is a set and we fix a map $\mathcal{J} \to \mathcal{E}$. In other words, we choose a partition of $\mathcal{J}$ labelled by elements of $\mathcal{E}$:

$$\mathcal{J} = \bigsqcup_{e \in \mathcal{E}} \mathcal{J}_e. \tag{3.26}$$

Recall from Subsection 1.4 that $\mathcal{C}$ denotes the family of clusters. Let $n \in \Gamma(\mathcal{E})$. Then we set

$$\mathcal{C}_n := \{c \in \mathcal{C} \;:\; |c \cap \mathcal{J}_e| = n_e, \; e \in \mathcal{E}\}.$$

($\mathcal{C}_n$ is the family of clusters with composition n). Clearly,

$$\mathcal{C} = \bigsqcup_{n \in \Gamma(\mathcal{E})} \mathcal{C}_n.$$

We fix a total order $<$ in $\mathcal{J}$. We can assume that, for $e, e' \in \mathcal{E}$,

$$j \in \mathcal{J}_e,\ j' \in \mathcal{J}_{e'},\ e < e' \ \Rightarrow\ j < j'.$$

Recall from Subsection 1.4, that we have the map

$$\pi_c : \{1, \ldots, p\} \to c.$$

Note that if $c \in \mathcal{C}_n$, then

$$\pi_c q \in \mathcal{J}_e \ \Leftrightarrow\ \delta_n q = e.$$

Suppose now that $\{\mathcal{X}_j\}$, $j \in \mathcal{J}$, is a family of sets and $\{\mathcal{H}_j\}$, $j \in \mathcal{J}$, is a family of Hilbert spaces. Recall that for any $c \in \mathcal{C}$ we defined in Subsection 1.5 the set $\mathcal{X}_c$ and the Hilbert space $\mathcal{H}_c$.

Assume now that we have

$$\mathcal{X}_j = \mathcal{X}_e, \quad \mathcal{H}_j = \mathcal{H}_e, \quad j \in \mathcal{J}_e, \tag{3.27}$$

Let $n \in \Gamma(\mathcal{E})$ and $c \in \mathcal{C}_n$. Then

$$\mathcal{X}_c = \mathcal{X}_n, \quad \mathcal{H}_c = \mathcal{H}_n.$$

3.5 Permutations that preserve species and a clustering

Let $p \in \mathbf{N}$, and let $\tilde{\mathcal{A}}_p$ denote the family of clusterings (partitions into disjoins subsets) of $\{1, \ldots, p\}$. The group S_p acts on $\tilde{\mathcal{A}}_p$. We denote by $\sigma\alpha \in \tilde{\mathcal{A}}_p$ the action of $\sigma \in S_p$ on $\alpha \in \tilde{\mathcal{A}}_p$.

First note that given an order $<$ in $\mathcal{E}$, we can endow $\Gamma(\mathcal{E})$ and $\Gamma(\mathcal{E}) \times \mathbf{N}$ with certain natural total orders.

In fact, for $n, n' \in \Gamma(\mathcal{E})$, we will write $n < n'$ if for all $e \in \mathcal{E}$

$$n_{e'} = n'_{e'},\ \ e' < e \ \Rightarrow\ \ n_e \leq n'_e.$$

For $(m, i), (m', i') \in \Gamma(\mathcal{E}) \times \mathbf{N}$, we will write $(m, i) < (m', i')$ if

$$m < m' \quad \text{or} \quad (m = m' \quad \text{and} \quad i < i')\,.$$

Let $p \in \mathbf{N}$, $n \in \Gamma_p(\mathcal{E})$, $k \in \Gamma_n^{\mathrm{tq}}(\mathcal{E})$. Next we would like to define a certain natural clustering $\alpha_k \in \tilde{\mathcal{A}}_p$ and a labelling $\delta_k : \{1, \ldots, p\} \to \mathcal{E}$.

First we define a map

$$\tilde{\delta}_k : \{1,\ldots,p\} \to \Gamma(\mathcal{E}) \times \mathbf{N}$$

by the following three conditions:

$$(1) \quad 1 \le q < q' \le p \Rightarrow \tilde{\delta}_k q \le \tilde{\delta}_k q',$$

$$(2) \quad \tilde{\delta}_k\{1,\ldots,p\} = \bigcup_{m\in\Gamma(\mathcal{E})} m \times \{1,\ldots,k_m\},$$

$$(3) \quad \text{if } q \in \mathbf{N}, \quad m \in \Gamma_q(\mathcal{E}), \quad \text{then} \quad \left|\tilde{\delta}_k^{-1}(m \times \{i\})\right| = q.$$

Let

$$\alpha_k := \left\{\tilde{\delta}_k^{-1}(m \times \{i\}) \ : \ m \in \Gamma(\mathcal{E}), \quad 1 \le i \le k_m\right\}$$

be the clustering of $\{1,\ldots,p\}$ associated with k.

Next we define

$$\delta_k : \{1,\ldots,p\} \to \mathcal{E}$$

as follows. Let $q \in \mathbf{N}$, $m \in \Gamma_q(\mathcal{E})$ and $0 \le i \le k_m$ Then for some $0 \le q_0 \le p-q$ we have

$$\tilde{\delta}_k^{-1}(m \times \{i\}) = \{q_0+1,\ldots,q_0+q\} \in \alpha_k.$$

We set

$$\delta_k(q_0+s) := \delta_m s, \quad s = 1,\ldots,q,$$

where δ_m was defined in Subsection 3.3. This ends the definition of δ_k.

Let $\sigma \in S_p$. We say that $\sigma \in S_k$ if the following two conditions hold:

$$(1) \quad \sigma\left(\delta_k^{-1}\{e\}\right) = \delta_k^{-1}\{e\}, \quad e \in \mathcal{E},$$

$$(2) \quad \sigma\alpha_k = \alpha_k.$$

Clearly, S_k is a subgroup of S_p. Note that

$$S_k \simeq \mathop{\times}_{m\in\Gamma(\mathcal{E})} \left(\mathop{\times}_{e\in\mathcal{E}} S_{m_e}\right)^{k_m} \times S_{k_m},$$

$$|S_k| = \prod_{m\in\Gamma(\mathcal{E})} \left(\prod_{e\in\mathcal{E}} m_e!\right)^{k_m} k_m! =: k!^{\mathrm{tq}}. \tag{3.28}$$

S_k has the interpretation of the subgroup of permutations that do not change species given by the labelling δ_k and do not change the clustering α_k. Note that in general S_k is not a subgroup of S_n, because the labellings δ_n and δ_k are in general different.

Assume again that $\{\mathcal{X}_e\}$, $e \in \mathcal{E}$, is a family of sets and $\{\mathcal{H}_e\}$, $e \in \mathcal{E}$, is a family of Hilbert spaces. We set

$$\mathcal{X}_k := \mathop{\times}_{m\in\Gamma(\mathcal{E})} \left(\mathop{\times}_{e\in\mathcal{E}} \mathcal{X}_e^{m_e}\right)^{k_m} = \mathop{\times}_{q=1}^{p} \mathcal{X}_{\delta_k q},$$

$$\mathcal{H}_k := \bigotimes_{m\in\Gamma(\mathcal{E})} \left(\bigotimes_{e\in\mathcal{E}} \mathcal{H}_e^{\otimes m_e}\right)^{\otimes k_m} = \bigotimes_{q=1}^{p} \mathcal{H}_{\delta_k q}$$

Note that if $\sigma \in S_k$, then $\theta(\sigma)$ acts on $\mathcal{X}_k$ and $\Theta(\sigma)$ acts on $\mathcal{H}_k$.

3.6 Clusterings associated with $k \in \Gamma^{\mathrm{tq}}(\mathcal{E})$

Suppose that $\mathcal{J}$ is a set satisfying (3.26). Let $\mathcal{A}$ denote the family of clusterings (see Subsection 1.6).

If $k \in \Gamma^{\mathrm{tq}}(\mathcal{E})$, then we define

$$\mathcal{A}_k := \{a \in \mathcal{A} \ : \ |\{b \in a \ : \ b \in \mathcal{C}_m\}| = k_m, \ m \in \Gamma(\mathcal{E})\}$$

(clusterings containing exactly k_m clusters of composition m for each $m \in \Gamma(\mathcal{E})$). Clearly,

$$\mathcal{A} = \bigsqcup_{k \in \Gamma^{\mathrm{tq}}(\mathcal{E})} \mathcal{A}_k.$$

Recall that $\mathcal{E}$ and $\Gamma(\mathcal{E})$ are endowed with total orders. We assume that every $a \in \mathcal{A}$ is endowed with a total order $<$ satisfying

$$b, b' \in a, \ n, n' \in \Gamma(\mathcal{E}), \ b \in \mathcal{C}_n, \ b' \in \mathcal{C}_{n'} \ n < n' \ \Rightarrow \quad b < b'.$$

Recall that in Subsection 1.6 we defined a map

$$\pi_a : \{1, \dots, p\} \to c.$$

Note that if $a \in \mathcal{A}_k$, then

$$\pi_a q \in \mathcal{J}_e \Leftrightarrow \delta_k q = e.$$

Suppose now that $\{\mathcal{X}_j\}$, $j \in \mathcal{J}$, is a family of sets and $\{\mathcal{H}_j\}$, $j \in \mathcal{J}$, is a family of Hilbert spaces. Recall that for any $a \in \mathcal{A}$ we defined in Subsection 1.7 the set $\mathcal{X}_a$ and the Hilbert space $\mathcal{H}_a$.

If we assume (3.27) and $a \in \mathcal{A}_k$, then we have

$$\mathcal{X}_a = \mathcal{X}_k, \qquad \mathcal{H}_a = \mathcal{H}_k.$$

3.7 Identification operators

Let $p \in \mathbf{N}$. In this subsection we concentrate our attention on the "cluster" $\{1, \dots, p\}$ and its clusterings. We will introduce some notation, which will essentially mimick notation of Subsections 1.6 and 3.6 adapted to the case of $\{1, \dots, p\}$

Let $\tilde{\mathcal{C}}^p$ denote the family of subsets of $\{1, \dots, p\}$. Let $n \in \Gamma_p(\mathcal{E})$, $m \in \Gamma(\mathcal{E})$ and $\gamma \in \tilde{\mathcal{C}}^p$. We write $\gamma \in \tilde{\mathcal{C}}^n_m$ if

$$\left|\gamma \cap \delta_n^{-1}\{e\}\right| = m_e, \qquad e \in \mathcal{E}.$$

Recall that $\tilde{\mathcal{A}}_p$ denotes the family of clusterings of $\{1, \dots, p\}$. If $k \in \Gamma^{\mathrm{tq}}_n(\mathcal{E})$ and $\alpha \in \tilde{\mathcal{A}}_p$, then we write $\alpha \in \tilde{\mathcal{A}}_k$ if

$$\left|\alpha \cap \tilde{\mathcal{C}}^n_m\right| = k_m.$$

Clearly,

$$\tilde{\mathcal{A}}_p = \bigsqcup_{k \in \Gamma_n^{\mathrm{tq}}(\mathcal{E})} \tilde{\mathcal{A}}_k.$$

Let $p \in \mathbf{N}$, $n \in \Gamma_p(\mathcal{E})$ and $k \in \Gamma_n^{\mathrm{tq}}(\mathcal{E})$. In any $\alpha \in \tilde{\mathcal{A}}_p$ we fix a total order $<$ such that

$$\gamma, \gamma' \in \alpha, \quad m, m' \in \Gamma(\mathcal{E}),\ \gamma \in \mathcal{C}_m^n,\ \gamma' \in \mathcal{C}_{m'}^n,\ m < m' \ \Rightarrow\ \gamma < \gamma'.$$

As in Subsection 1.6 we then define $\pi_\alpha \in S_p$. We define also $\sigma_\alpha \in S_p$ by setting

$$\sigma_\alpha := \pi_\alpha^{-1}.$$

Note the following facts:

Proposition 3.1 *Let $p \in \mathbf{N}$, $n \in \Gamma_p(\mathcal{E})$, $k \in \Gamma_n^{\mathrm{tq}}(\mathcal{E})$ and $\alpha \in \tilde{\mathcal{A}}_k$, Then*
(0) $\sigma_\alpha\left(\delta_k^{-1}\{e\}\right) = \delta_n^{-1}\{e\}$, $e \in \mathcal{E}$.
(1) $\sigma_\alpha \alpha_k = \alpha$. (2) The map

$$S_n \ni \sigma \mapsto (\sigma_{\sigma\alpha}^{-1} \sigma \sigma_\alpha, \sigma\alpha) \in S_k \times \tilde{\mathcal{A}}_k$$

is bijective.
(3) Let $\alpha_0 \in \tilde{\mathcal{A}}_k$. Then $\sigma_{\alpha_0} S_k \sigma_{\alpha_0}^{-1}$ is a subgroup of S_n fixing the clustering α_0. For any $\alpha \in \tilde{\mathcal{A}}_k$ we have $\sigma_\alpha \sigma_{\alpha_0}^{-1} \in S_n$ and the map

$$\tilde{\mathcal{A}}_k \ni \alpha \mapsto \left[\sigma_\alpha \sigma_{\alpha_0}^{-1}\right] \in S_n / \sigma_{\alpha_0} S_k \sigma_{\alpha_0}^{-1}$$

is bijective.

4 Second quantization in the category of $\mathbf{Z}_2$-Hilbert spaces

4.1 Fock spaces

A pair $(\mathcal{H}, \epsilon)$ where $\mathcal{H}$ is a Hilbert space and ϵ is a unitary involution in $\mathcal{H}$ is called a $\mathbf{Z}_2$-Hilbert space. We define the bosonic and fermionic part of $\mathcal{H}$:

$$\mathcal{H}^{\mathrm{b}} = \ker(\epsilon - \mathbf{1}), \qquad \mathcal{H}^{\mathrm{f}} := \ker(\epsilon + \mathbf{1}).$$

Clearly, $\mathcal{H} = \mathcal{H}^{\mathrm{b}} \oplus \mathcal{H}^{\mathrm{f}}$.

Let $(\mathcal{H}_1, \epsilon_1)$, $(\mathcal{H}_2, \epsilon_2)$ be $\mathbf{Z}_2$-Hilbert spaces. Then a map $A : \mathcal{H}_1 \to \mathcal{H}_2$ is said to be $\mathbf{Z}_2$-linear if

$$A\epsilon_1 = \epsilon_2 A.$$

Let $p \in \mathbf{N}$. We introduce a twisted representation of the group of permutations S_p on $\mathcal{H}^{\otimes p}$. Let $\sigma \in S_p$ and $I \subset \{1, \ldots, p\}$ with $|I| = q$. Let π_1 be the unique order preserving bijection of $\{1, \ldots, q\}$ onto I and π_2 be the unique order preserving bijection of $\{1, \ldots, q\}$ onto $\sigma(I)$. We define a permutation σ_I in S_q by setting

$$\sigma_I := \pi_2^{-1} \sigma \pi_1.$$

If $\phi_i \in \mathcal{H}^{\mathrm{f}}$ for $i \in I$ and $\phi_i \in \mathcal{H}^{\mathrm{b}}$ for $i \in \{1, \ldots, p\} \backslash I$, then

$$\Theta^{\epsilon}(\sigma) \phi_1 \otimes \ldots \otimes \phi_p := \mathrm{sgn}(\sigma_I) \Theta(\sigma) \phi_1 \otimes \ldots \otimes \phi_p.$$

Note that if $\epsilon = 1$, then $\Theta^{\epsilon}(\sigma) = \Theta(\sigma)$ and if $\epsilon = -1$, then $\Theta^{\epsilon}(\sigma) = \mathrm{sgn}(\sigma) \Theta(\sigma)$.

We also set

$$\mathrm{sgn}^{\epsilon}(\sigma) := \Theta^{\epsilon}(\sigma) \Theta(\sigma^{-1}). \tag{4.29}$$

We define the projections

$$P_p^{\epsilon} := \frac{1}{p!} \sum_{\sigma \in S_p} \Theta^{\epsilon}(\sigma),$$

and the Hilbert spaces

$$\Gamma_p(\mathcal{H}) = P_p^{\epsilon} \mathcal{H}^{\otimes p}, \qquad \Gamma(\mathcal{H}) = \mathop{\oplus}_{p=0}^{\infty} \Gamma_p(\mathcal{H}). \tag{4.30}$$

$\Gamma(\mathcal{H})$ is called the Fock space over $(\mathcal{H}, \epsilon)$. We set

$$\Gamma_{\mathrm{fin}} := \mathop{\oplus}_{\mathrm{fin}, p=0}^{\infty} \Gamma_p(\mathcal{H}).$$

Obviously, $\mathbf{C} = \Gamma_0(\mathcal{H})$. The vector $1 \in \mathbf{C}$ will be denoted by ω and will be called the vacuum.

Clearly, we have a natural embedding

$$\mathcal{H} = \Gamma_1(\mathcal{H}) \subset \Gamma(\mathcal{H}).$$

Let $(\mathcal{H}_1, \epsilon_1)$, $(\mathcal{H}_2, \epsilon_2)$ be $\mathbf{Z}_2$-Hilbert spaces. Then so is $(\mathcal{H}_1 \oplus \mathcal{H}_2, \epsilon_1 \oplus \epsilon_2)$. We have a natural isomorhism

$$\Gamma(\mathcal{H}_1 \oplus \mathcal{H}_2) \simeq \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2).$$

(See Subsection 4.8)

In particular,

$$\Gamma(\mathcal{H}) \simeq \Gamma^{\mathrm{b}}(\mathcal{H}^{\mathrm{b}}) \otimes \Gamma^{\mathrm{f}}(\mathcal{H}^{\mathrm{f}}),$$

where $\Gamma^{\mathrm{b}}(\mathcal{H}^{\mathrm{b}})$ is the bosonic Fock space over $\mathcal{H}^{\mathrm{b}}$ and $\Gamma^{\mathrm{f}}(\mathcal{H}^{\mathrm{f}})$ is the fermionic Fock space over $\mathcal{H}^{\mathrm{f}}$.

4.2 Operators on Fock spaces – functor Γ

If $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a $\mathbf{Z}_2$-linear map, then we define the operator

$$\Gamma_p(A) : \Gamma_p(\mathcal{H}_1) \to \Gamma_p(\mathcal{H}_2)$$

by setting

$$\Gamma_p(A) := A^{\otimes p}\Big|_{\Gamma_p(\mathcal{H}_1)}$$

We also define

$$\Gamma(A) : \Gamma(\mathcal{H}_1) \to \Gamma(\mathcal{H}_2)$$

as

$$\Gamma(A) := \underset{p\in\mathbf{N}}{\oplus} \Gamma_p(A).$$

Note that

$$\Gamma(A)\Big|_{\mathcal{H}_1} = A.$$

Note that if $A_1 : \mathcal{H}_1 \to \mathcal{H}_2$, $A_2 : \mathcal{H}_2 \to \mathcal{H}_3$, then

$$\Gamma(A_2)\Gamma(A_1) = \Gamma(A_2A_1).$$

One of important examples of the use of the functor Γ is the statistics operator $\Gamma(\epsilon)$.

4.3 Operators on Fock spaces – functor $\mathrm{d}\Gamma$

If B is a $\mathbf{Z}_2$-linear operator on $\mathcal{H}$, then we define

$$\mathrm{d}\Gamma_p(B) : \Gamma_p(\mathcal{H}) \to \Gamma_p(\mathcal{H})$$

as

$$\mathrm{d}\Gamma_p(B) := \sum_{q=1}^{p} \mathbf{1}^{\otimes q-1} \otimes B \otimes \mathbf{1}^{\otimes p-q}\Big|_{\Gamma_p(\mathcal{H})}.$$

We also define

$$\mathrm{d}\Gamma(B) : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$$

as

$$\mathrm{d}\Gamma(B) := \overset{\infty}{\underset{p=0}{\oplus}} \mathrm{d}\Gamma_p(B).$$

Note that

$$\mathrm{d}\Gamma(A)\Big|_{\mathcal{H}} = A,$$
$$[\mathrm{d}\Gamma(A), \mathrm{d}\Gamma(B)] = \mathrm{d}\Gamma([A,B]),$$
$$\mathrm{e}^{\mathrm{d}\Gamma(A)} = \Gamma\left(\mathrm{e}^{A}\right).$$

4.4 Fock space as an associative algebra

For $\phi_i \in \Gamma_{p_i}(\mathcal{H})$, $i = 1, 2$, we define the product

$$\phi_1 \Box \phi_2 := \sqrt{\frac{(p_1 + p_2)!}{p_1! p_2!}} P^{\epsilon}_{p_1+p_2} \phi_1 \otimes \phi_2.$$

We extend $\Box$ to the whole $\Gamma_{\mathrm{fin}}(\mathcal{H})$ by linearity.

Using

$$P^{\epsilon}_{p_1+p_2} P^{\epsilon}_{p_2} \otimes \mathbf{1}^{\otimes p_1} = P^{\epsilon}_{p_2+p_1},$$

we easily see that the operation $\Box$ is associative.

Remark 4.1 *Note that, in the case of a fermionic Fock space, the product $\Box$ is different from the usual exterior product $\wedge$. In fact,*

$$\phi_1 \wedge \phi_2 := \frac{(p_1 + p_2)!}{p_1! p_2!} P^{\epsilon}_{p_1+p_2} \phi_1 \otimes \phi_2.$$

Let us note the following formula for a multiple $\Box$-product. Let $\phi_i \in \Gamma_{r_i}(\mathcal{H})$, $i = 1, \dots, q$, $p = \sum_{i=1}^{q} r_i$. Then

$$\mathop{\Box}_{i=1}^{q} \phi_i = \sqrt{\frac{p!}{\Pi_{i=1}^{q} r_i!}} P^{\epsilon}_{p} \mathop{\otimes}_{i=1}^{q} \phi_i.$$

4.5 Creation and annihilation operators

For $\phi \in \mathcal{H} = \Gamma_1(\mathcal{H})$ we define the creation and annihilation operators

$$a^*(\phi)\psi := \phi \Box \psi, \quad \psi \in \Gamma_{\mathrm{fin}}(\mathcal{H}),$$

$$a(\phi) := (a^*(\phi))^*.$$

They satisfy the well known commutation/anticommutation relations

$$a(\phi_1)a^*(\phi_2) - a^*(\epsilon\phi_2)a(\phi_1) = (\phi_1|\phi_2),$$

$$a(\phi_1)a(\phi_2) - a(\epsilon\phi_2)a(\phi_1) = 0,$$

$$a(\phi_1)a(\phi_2) - a(\epsilon\phi_2)a(\phi_1) = 0, \qquad \phi_1, \phi_2 \in \mathcal{H}.$$

One can also introduce the so called multiple creation/annihilation operators, by setting for $\phi \in \Gamma_{\mathrm{fin}}(\mathcal{H})$

$$a^*(\phi)\psi := \phi \Box \psi, \quad \psi \in \Gamma_{\mathrm{fin}}(\mathcal{H}),$$

Remark 4.2 *In the literature one usually assumes that the one-particle space is $\mathcal{H} = L^2(X, \mathrm{d}x)$, where X is a space with a measure $\mathrm{d}x$. One introduces "operator valued measures" $a^*(x)$, $a(x)$ and for $\phi \in \mathcal{H}$ one writes*

$$\int a^*(x)\phi(x)\mathrm{d}x = a^*(\phi), \qquad \int a(x)\phi(x)\mathrm{d}x = a(\phi).$$

Then if $\phi \in \Gamma_p(\mathcal{H})$, $\psi \in \Gamma(\mathcal{H})$, we have

$$\phi \Box \psi = (p!)^{-1/2} \int_{X^p} \phi(x_1, \ldots, x_p) a^*(x_1) \cdots a^*(x_p) \psi \mathrm{d}x_1 \ldots \mathrm{d}x_p.$$

4.6 Operators on Fock spaces – functor $\mathrm{d}\Gamma_{(q)}$

Let $q \in \mathbf{N}$ and $B \in B(\Gamma_q(\mathcal{H}))$. We define

$$\mathrm{d}\Gamma_{(q)p}(B) : \Gamma_p(\mathcal{H}) \to \Gamma_p(\mathcal{H})$$

by setting

$$\mathrm{d}\Gamma_{(q)p}(B) := \begin{cases} 0, & p < q, \\ \frac{p!}{(p-q)!} P_p^\epsilon B \otimes \mathbf{1}^{\otimes p-q}\Big|_{\Gamma_p(\mathcal{H})}, & p \geq q. \end{cases}$$

We also define

$$\mathrm{d}\Gamma_{(q)}(B) : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$$

as

$$\mathrm{d}\Gamma_{(q)}(B) := \mathop{\oplus}_{p=0}^{\infty} \mathrm{d}\Gamma_{(q)p}(B).$$

Clearly, if B is a $\mathbf{Z}_2$-linear operator on $\mathcal{H}$, then $\mathrm{d}\Gamma_{(1)}(B) = \mathrm{d}\Gamma(B)$.

Remark 4.3 *If $B \in B(\Gamma_q(L^2(X, \mathrm{d}x)))$ and the integral kernel of B equals*

$$b(x_1, \ldots, x_q, x'_1, \ldots, x'_q),$$

then

$$\begin{aligned} \mathrm{d}\Gamma_{(q)}(B) \ &:= \textstyle\int_{X^{2q}} b(x_1, \ldots, x_q, x'_1, \ldots, x'_q) \\ &\quad \times a^*(x_1) \ldots a^*(x_{q_2}) a(x'_{q_1}) \ldots a(x'_1) \mathrm{d}x'_1 \ldots \mathrm{d}x'_{q_1} \mathrm{d}x_1 \ldots \mathrm{d}x_{q_2}. \end{aligned}$$

4.7 "Third quantization" of a $\mathbf{Z}_2$-Hilbert space I

Let $(\mathcal{H}, \epsilon)$ be a $\mathbf{Z}_2$-Hilbert space. Let $\Gamma(\mathcal{H})$ be the corresponding Fock space with the vacuum ω and the product $\Box$.

Clearly, $(\Gamma(\mathcal{H}), \Gamma(\epsilon))$ is also a $\mathbf{Z}_2$-Hilbert space. So we may define "the third quantization" of $\mathcal{H}$, that is $\Gamma^{\mathrm{tq}}(\mathcal{H}) := \Gamma(\Gamma(\mathcal{H}))$. Its vacuum will be denoted ω^{tq} and its product $\Box^{\mathrm{tq}}$.

We define a map

$$J^{\mathrm{tq}} : \Gamma^{\mathrm{tq}}(\mathcal{H}) \to \Gamma(\mathcal{H})$$

by setting

$$J^{\mathrm{tq}}\phi_1 \Box^{\mathrm{tq}} \dots \Box^{\mathrm{tq}} \phi_q := \phi_1 \Box \dots \Box \phi_q, \quad \phi_1, \dots, \phi_q \in \Gamma(\mathcal{H}).$$

The above formula defines J^{tq} on a total set in $\Gamma_q(\Gamma(\mathcal{H}))$. It can be extended by linearity to a bounded operator on $\Gamma_q(\Gamma(\mathcal{H}))$. Then it can be extended to an unbounded operator on a dense domain in $\Gamma^{\mathrm{tq}}(\mathcal{H})$.

An alternative definition of J^{tq} will be given in Subsection 4.10.

4.8 Fock space over a direct sum I

Let $(\mathcal{H}, \epsilon)$ be a $\mathbf{Z}_2$-Hilbert space. Let $\Gamma(\mathcal{H})$ be the corresponding Fock space with the vacuum ω and the product $\Box$.

Let $\mathcal{E}$ be a totally ordered set. Suppose that $(\mathcal{H}_e, \epsilon_e)$, $e \in \mathcal{E}$, are $\mathbf{Z}_2$-Hilbert spaces. Let $\Gamma(\mathcal{H}_e)$ be the corresponding Fock spaces with the vacua ω_e and the product $\Box_e$. We assume that

$$\mathcal{H} = \underset{e\in\mathcal{E}}{\oplus} \mathcal{H}_e, \qquad \epsilon = \underset{e\in\mathcal{E}}{\oplus} \epsilon_e. \tag{4.31}$$

We would like to discuss a certain natural unitary operator

$$U : \underset{e\in\mathcal{E}}{\otimes} \Gamma(\mathcal{H}_e) \to \Gamma(\mathcal{H}). \tag{4.32}$$

If $\mathcal{E}$ is infinite, then the tensor product on the left of (4.32) is understood as the so-called restricted tensor product associated with the sequence of vectors $\omega_e \in \mathcal{H}_e$.

For a sequence $\phi_e \in \Gamma(\mathcal{H}_e)$, $e \in \mathcal{E}$, with all but a finite number of ϕ_e's different from ω_e, the operator U is defined as

$$U \underset{e\in\mathcal{E}}{\otimes} \phi_e := \underset{e\in\mathcal{E}}{\Box} \phi_e.$$

4.9 Fock space over a direct sum II

Next we would like to describe an equivalent characterization of the map U, which does not use the $\Box$-product.

Let $p \in \mathbf{N}$ and $n \in \Gamma_p(\mathcal{E})$. We define

$$\Gamma_n(\mathcal{H}) := \underset{e\in\mathcal{E}}{\otimes} \Gamma_{n_e}(\mathcal{H}_e).$$

We define an operator acting on $\mathcal{H}^{\otimes p}$:

$$P_n^\epsilon := \frac{1}{n!} \sum_{\sigma \in S_n} \Theta^\epsilon(\sigma).$$

where $n!$ was defined in 3.25. Clearly, P_n^ϵ is a projection and

$$P_n^\epsilon \mathcal{H}_n = \Gamma_n(\mathcal{H}),$$

where $\mathcal{H}_n$ was defined in Subsection 3.3. Clearly,

$$\underset{e\in\mathcal{E}}{\otimes} \Gamma(\mathcal{H}_e) = \underset{n\in\Gamma(\mathcal{E})}{\oplus} \Gamma_n(\mathcal{H}).$$

For any $p \in \mathbf{N}$ and $n \in \Gamma(\mathcal{E})$, we have

$$U\Big|_{\Gamma_n(\mathcal{H})} = \sqrt{\frac{p!}{n!}} P_p^\epsilon\Big|_{\Gamma_n(\mathcal{H})} = \sqrt{\frac{n!}{p!}} \sum_{[\sigma]\in S_p/S_n} \Theta^\epsilon(\sigma)\Big|_{\Gamma_n(\mathcal{H})},$$

where we take a single representant in each class in S_p/S_n. We also have

$$U\left(\underset{n\in\Gamma_p(\mathcal{E})}{\oplus} \Gamma_n(\mathcal{H})\right) = \Gamma_p(\mathcal{H}).$$

We will often use the identifications

$$\Gamma(\mathcal{H}) \simeq \underset{n\in\Gamma(\mathcal{E})}{\oplus} \Gamma_n(\mathcal{H}), \qquad \Gamma_p(\mathcal{H}) \simeq \underset{n\in\Gamma_p(\mathcal{E})}{\oplus} \Gamma_n(\mathcal{H}).$$

Using these identifications we can write the following formula for the $\Box$-product. Let $n \in \Gamma(\mathcal{E})$, $k \in \Gamma_n^{\mathrm{tq}}(\mathcal{E})$. Choose an arbitrary clustering $\alpha \in \tilde{\mathcal{A}}_k$ and vectors $\phi_{m,i} \in \Gamma(\mathcal{H})$, $i = 1, \dots, k_m$, $m \in \Gamma(\mathcal{E})$. Then

$$\underset{m\in\Gamma(\mathcal{E})}{\Box} \overset{k_m}{\underset{i=1}{\Box}} \phi_{m,i} = \prod_{e\in\mathcal{E}} \sqrt{\frac{n_e!}{\prod_{m\in\Gamma(\mathcal{E})}(m_e!)^{k_m}}} P_n^\epsilon \Theta^\epsilon(\sigma_\alpha) \underset{m\in\Gamma(\mathcal{E})}{\otimes} \overset{k_m}{\underset{i=1}{\otimes}} \phi_{m,i}.$$

For $\sigma \in S_p$ we define

$$\mathrm{sgn}_n^\epsilon : \mathcal{H}_n \to \mathcal{H}_n$$

by setting

$$\mathrm{sgn}_n^\epsilon(\sigma) := \mathrm{sgn}^\epsilon(\sigma)\Big|_{\mathcal{H}_n}.$$

Clearly, if we assume that $\epsilon_e = \pm 1$, $e \in \mathcal{E}$, then $\mathrm{sgn}_n(\sigma) = \pm 1$.

4.10 "Third quantization" of a Fock space II

Next we would like to give an equivalent alternative characterization of the map J^{tq} that does not use the $\Box$-product.

For $r \in \Gamma(\mathbf{N})$ we set

$$\Gamma_r^{\mathrm{tq}}(\mathcal{H}) := \underset{p\in\mathbf{N}}{\otimes} \Gamma_{r_p}\left(\Gamma_p(\mathcal{H})\right).$$

We have

$$\Gamma(\mathcal{H}) = \underset{p\in\mathbf{N}}{\oplus} \Gamma_p(\mathcal{H}).$$

Hence by Subsection 4.9 with $\mathcal{E} = \mathbf{N}$, we have a natural identification

$$\Gamma^{\mathrm{tq}}(\mathcal{H}) \simeq \underset{r\in\Gamma(\mathbf{N})}{\oplus} \Gamma^{\mathrm{tq}}_r(\mathcal{H}).$$

Let $p \in \mathbf{N}$ and $r \in \Gamma_p(\mathbf{N})$. Set

$$J^{\mathrm{tq}}_r := J^{\mathrm{tq}}\Big|_{\Gamma^{\mathrm{tq}}_r(\mathcal{H})}.$$

Then

$$J^{\mathrm{tq}}_r : \Gamma^{\mathrm{tq}}_r(\mathcal{H}) \to \Gamma_p(\mathcal{H})$$

and we have the identity

$$J^{\mathrm{tq}}_r := \sqrt{\frac{p!}{r!^{\mathrm{tq}}}} P^{\epsilon}_p\Big|_{\Gamma^{\mathrm{tq}}_r(\mathcal{H})} = \sqrt{\frac{r!^{\mathrm{tq}}}{p!}} \sum_{\alpha\in\tilde{\mathcal{A}}_r} \Theta^{\epsilon}(\sigma_\alpha)\Big|_{\Gamma^{\mathrm{tq}}_r(\mathcal{H})}.$$

where $p!$ is the usual factorial,

$$\prod_{q\in\mathbf{N}} (q!)^{r_q} r_q! =: r!^{\mathrm{tq}},$$

and $\tilde{\mathcal{A}}_r$ is the family of clusterings of $\{1, \dots, p\}$ having r_q q-element clusters.

4.11 "Third quantization" of a direct sum

We return to the notation and assumptions of Subsection 4.9. Let $k \in \Gamma^{\mathrm{tq}}(\mathcal{E})$. We define

$$\Gamma^{\mathrm{tq}}_k(\mathcal{H}) := \underset{m\in\Gamma(\mathcal{E})}{\otimes} \Gamma_{k_m}(\Gamma_m(\mathcal{H})) = \underset{m\in\Gamma(\mathcal{E})}{\otimes} \Gamma_{k_m}\left(\underset{e\in\mathcal{E}}{\otimes} \Gamma_{m_e}(\mathcal{H}_e)\right),$$

Set

$$P^{\epsilon}_k := \frac{1}{k!^{\mathrm{tq}}} \sum_{\sigma\in S_k} \Theta^{\epsilon}(\sigma).$$

Clearly,

$$P^{\epsilon}_k \mathcal{H}_k = \Gamma^{\mathrm{tq}}_k(\mathcal{H}).$$

We have the identifications:

$$\Gamma(\mathcal{H}) \simeq \underset{n\in\Gamma(\mathcal{E})}{\oplus} \Gamma_n(\mathcal{H}), \qquad \Gamma^{\mathrm{tq}}(\mathcal{H}) \simeq \underset{k\in\Gamma^{\mathrm{tq}}(\mathcal{E})}{\oplus} \Gamma^{\mathrm{tq}}_k(\mathcal{H})$$

Fix $n \in \Gamma(\mathcal{E})$ and $k \in \Gamma^{\mathrm{tq}}_n(\mathcal{E})$. Set

$$J^{\mathrm{tq}}_k := J^{\mathrm{tq}}\Big|_{\Gamma^{\mathrm{tq}}_k(\mathcal{H})},$$

where $J^{\rm tq}$ was defined in Subsection 4.7. Then

$$J_k^{\rm tq} : \Gamma_k^{\rm tq}(\mathcal{H}) \to \Gamma_n(\mathcal{H})$$

and

$$J_k^{\rm tq} = \sqrt{\frac{n!}{k!^{\rm tq}}} P_n^\epsilon \Theta^\epsilon(\sigma_{\alpha_0})\Big|_{\Gamma_k^{\rm tq}(\mathcal{H})} = \sqrt{\frac{k!^{\rm tq}}{n!}} \sum_{\alpha \in \tilde{\mathcal{A}}_k} \Theta^\epsilon(\sigma_\alpha)\Big|_{\Gamma_k^{\rm tq}(\mathcal{H})}. \tag{4.33}$$

where $n!$ was defined in 3.25 and $k!^{\rm tq}$ was defined in 3.28 and we have chosen an arbitrary clustering $\alpha_0 \in \tilde{\mathcal{A}}$.

4.12 Algebras $C_\infty(\Gamma(\mathcal{X}))$

Let $\mathcal{X}$ be a locally compact set. Then so is $\mathcal{X}^p$, $\Gamma_p(\mathcal{X})$ and $\Gamma(\mathcal{X})$.

Let $\mathcal{H}$ be a Hilbert space. Let

$$C_\infty(\mathcal{X}) \ni f \to \gamma(f) \in B(\mathcal{H}) \tag{4.34}$$

be a $*$-representation. For any $p \in \mathbf{N}$ this representation generates a natural representation from $C_\infty(\mathcal{X}^p)$ into $B(\mathcal{H}^{\otimes p})$. It is covariant wrt the action of S_p on $\mathcal{X}^p$. Hence we obtain a representation

$$\Gamma_p(\gamma) : C_\infty(\Gamma_p(\mathcal{X})) \to B(\Gamma_p(\mathcal{H})).$$

This leads to a representation

$$\Gamma(\gamma) : C_\infty(\Gamma(\mathcal{X})) \to B(\Gamma(\mathcal{H})). \tag{4.35}$$

defined as

$$\Gamma(\gamma) := \mathop{\oplus}_{p \in \mathbf{N}} \Gamma_p(\gamma).$$

If instead of the notation involving γ we use the notation

$$f(A) = \gamma(f), \quad f \in C_\infty(\mathcal{X}),$$

then $\Gamma(\gamma)$ will be denoted

$$F(\underline{A}) = \Gamma(\gamma)(F), \quad F \in C_\infty(\Gamma(\mathcal{X})).$$

5 Identical particles

5.1 Elementary particles

Let X be a finite dimensional Euclidean space – the configuration space of a single particle. x denotes the position operator in $L^2(X)$ and $D := \frac{1}{i}\nabla_x$ the momentum.

Let (h,ϵ) be a $\mathbf{Z}^2$-Hilbert space, describing various kinds of particles. Let m be a $\mathbf{Z}_2$-linear strictly positive operator on h, called the one-particle mass operator. In the case of Coulomb systems, let z be a $\mathbf{Z}^2$-linear self-adjoint operator on h, called the charge operator. We assume that

$$[m,z]=0.$$

The full one-particle space is defined as

$$\mathcal{H}:=L^2(X)\otimes h.$$

For any $x\in X$, let $v(x)$ be a $\mathbf{Z}_2$-linear operator on $h\otimes h$. If $\tau\in S_2$ is a transposition, then we assume that

$$\tau v(x)\tau=v(-x),$$
$$[v(x),m\otimes\mathbf{1}]=[v(x),z\otimes\mathbf{1}]=0.$$

The operator $v(x\otimes\mathbf{1}-\mathbf{1}\otimes x)$ acts on $\mathcal{H}\otimes\mathcal{H}$ and $\Gamma_2(\mathcal{H})$ is its invariant subspace. It can be used two define the two-particle potential, denoted also v, as the linear operator

$$v:\Gamma_2(\mathcal{H})\to\Gamma_2(\mathcal{H}),\qquad v:=v(x\otimes\mathbf{1}-\mathbf{1}\otimes x)\Big|_{\Gamma_2(\mathcal{H})}.$$

Suppose that $e\in\mathcal{E}$ labels the species of particles. For convenience we fix a total order in $\mathcal{E}$. Let h_e be the Hilbert space describing the internal degrees of freedom of the particles of the eth species. Let $m_e\in\mathbf{R}_+$ be its mass, $\epsilon_e=\pm1$ its statistics and

$$\mathcal{H}_e:=L^2(X)\otimes h_e$$

its one-particle Hilbert space. In the case of Coulomb systems, let $z_e\in\mathbf{R}$ be its charge. We asume that

$$h=\underset{e\in\mathcal{E}}{\oplus}h_e,\quad\text{and hence,}\qquad\mathcal{H}=\underset{e\in\mathcal{E}}{\oplus}\mathcal{H}_e.$$

With respect to these decompositions we can write

$$\epsilon=\underset{e\in\mathcal{E}}{\oplus}\epsilon_e.\quad m=\underset{e\in\mathcal{E}}{\oplus}m_e,\quad z=\underset{e\in\mathcal{E}}{\oplus}z_e.$$

$$v(x)=\underset{e,e'\in\mathcal{E}}{\oplus}v_{ee'}(x)$$

where for every $e,e'\in\mathcal{E}$ we assume that

$$v_{ee'}(x)(1+D^2)^{-1}\text{ is compact on }L^2(X)\otimes h_e\otimes h_{e'}.\tag{5.36}$$

$\Gamma(\mathcal{H})$ is the physical Fock space. The full kinetic energy is defined as the operator on $\Gamma(\mathcal{H})$:

$$K:=\mathrm{d}\Gamma\left(\frac{D^2}{2m}\right).$$

The full potential is defined as

$$V := \mathrm{d}\Gamma_{(2)}(v).$$

V is relatively bounded wrt K with an infinitesimal bound. The Hamiltonian is defined as the self-adjoint operator on $\Gamma(\mathcal{H})$

$$H := K + V.$$

On $\Gamma(\mathcal{H}) \ominus \Gamma_0(\mathcal{H})$ (the orthogonal complement of the vacuum) we define the operators of the center-of-mass momentum and the center-of-mass position:

$$D^{\mathrm{cm}} := \mathrm{d}\Gamma(D), \qquad x^{\mathrm{cm}} := (\mathrm{d}\Gamma(m))^{-1}\,\mathrm{d}\Gamma(xm)$$

We have

$$[x^{\mathrm{cm}}, D^{\mathrm{cm}}] = i\mathbf{1}.$$

Hence, for $X^{\mathrm{cm}} = X$,

$$\Gamma(\mathcal{H}) \ominus \Gamma_0(\mathcal{H}) \simeq L^2(X^{\mathrm{cm}}) \otimes \Gamma(\mathcal{H})^{\mathrm{red}}$$

and the operators x^{cm}, D^{cm} act only on $L^2(X^{\mathrm{cm}})$.

We have

$$[x^{\mathrm{cm}}, K] = (\mathrm{d}\Gamma(m))^{-1} D^{\mathrm{cm}}, \qquad [D^{\mathrm{cm}}, K] = 0.$$

Hence $K - (2\mathrm{d}\Gamma(m))^{-1}(D^{\mathrm{cm}})^2$ commutes with x^{cm} and D^{cm}. Therefore, on $\Gamma(\mathcal{H}) \ominus \Gamma_0(\mathcal{H})$

$$K = (2\mathrm{d}\Gamma(m))^{-1}(D^{\mathrm{cm}})^2 + \mathbf{1} \otimes K^{\mathrm{red}}.$$

V commutes with x^{cm} and D^{cm}. Hence on $\Gamma(\mathcal{H}) \ominus \Gamma_0(\mathcal{H})$

$$V = \mathbf{1} \otimes V^{\mathrm{red}}.$$

We define the reduced Hamiltonian as the self-adjoint operator on $\Gamma(\mathcal{H})^{\mathrm{red}}$

$$H^{\mathrm{red}} := K^{\mathrm{red}} + V^{\mathrm{red}}.$$

Obviously, on $\Gamma(\mathcal{H}) \ominus \Gamma_0(\mathcal{H})$

$$H = (2\mathrm{d}\Gamma(m))^{-1}(D^{\mathrm{cm}})^2 + \mathbf{1} \otimes H^{\mathrm{red}}.$$

Note that

$$[H, \mathbf{1} \otimes H^{\mathrm{red}}] = 0.$$

Let us define the free region in $\Gamma(X)$:

$$\Gamma(X)^{\mathrm{free}} := \left\{ (n_x)_{x\in X} \in \Gamma(X) \ : \ n_x \in \{0,1\},\ x \in X \right\},$$

(all the particles are at different points) and the center-of-mass region in $\Gamma(X)$:

$$\Gamma(X)^{\mathrm{cm}} := \left\{ (n_x)_{x\in X} \in \Gamma(X) \ : \ |\{x \ : \ n_x > 0\}| = 1 \right\},$$

(all the particles are at the same point).

We set $\mathcal{X} := X \times \mathcal{E}$. We have the obvious map

$$\mathcal{X} \ni (x,e) \mapsto x \in X.$$

It induces the map

$$\Gamma(\mathcal{X}) \to \Gamma(X). \tag{5.37}$$

We define $\Gamma(\mathcal{X})^{\mathrm{free}}$ and $\Gamma(\mathcal{X})^{\mathrm{cm}}$ as the preimages of $\Gamma(X)^{\mathrm{free}}$ and $\Gamma(X)^{\mathrm{cm}}$ with respect to (5.37). Note that we have the natural identifications

$$\Gamma(X)^{\mathrm{cm}} \simeq X \times \mathbf{N}, \qquad \Gamma(\mathcal{X})^{\mathrm{cm}} \simeq X \times \Gamma(\mathcal{E}).$$

The following $*$-homomorphisms have an obvious definition (see Subsection 1.2)

$$C_\infty(\mathcal{E}) \ni f \mapsto f(\hat{e}) := \underset{e\in\mathcal{E}}{\oplus} f(e) \in B(h),$$

$$C_\infty(X) \ni f \mapsto f(x) \in B(L^2(X)),$$
$$C_\infty(X) \ni f \mapsto f(D) \in B(L^2(X)),$$

We also have the tensor products of these homomorphisms:

$$C_\infty(\mathcal{X}) \ni f \to f(x,\hat{e}) \in B(\mathcal{H}),$$

$$C_\infty(\mathcal{X}) \ni f \to f(D,\hat{e}) \in B(\mathcal{H}).$$

Sometimes we will use their second quantized extensions (see Subsection 4.12):

$$C_\infty(\Gamma(\mathcal{X})) \ni F \to F(\underline{x,\hat{e}}) \in B(\Gamma(\mathcal{H})),$$

$$C_\infty(\Gamma(\mathcal{X})) \ni F \to F(\underline{D,\hat{e}}) \in B(\Gamma(\mathcal{H})),$$

Recall that in Subsection 4.8 we defined

$$\Gamma_n(\mathcal{H}) := \underset{e\in\mathcal{E}}{\otimes} \Gamma_{n_e}(\mathcal{H}_e).$$

Clearly, we can introduce $\Gamma_n(\mathcal{H})^{\mathrm{red}}$, by writing

$$\Gamma_n(\mathcal{H}) = L^2(X^{\mathrm{cm}}) \otimes \Gamma_n(\mathcal{H})^{\mathrm{red}}.$$

We have the identifications

$$\Gamma(\mathcal{H}) \simeq \underset{n\in\Gamma(\mathcal{E})}{\oplus} \Gamma_n(\mathcal{H}), \qquad \Gamma(\mathcal{H})^{\mathrm{red}} \simeq \underset{n\in\Gamma(\mathcal{E})}{\oplus} \Gamma_n(\mathcal{H})^{\mathrm{red}}.$$

Using these identifications we can decompose the mass, charge and statistics opertors, as well as the Hamiltonians H and H^{red}:

$$\mathrm{d}\Gamma(m) = \underset{n\in\Gamma(\mathcal{E})}{\oplus} m_n, \qquad \mathrm{d}\Gamma(z) = \underset{n\in\Gamma(\mathcal{E})}{\oplus} z_n, \qquad \Gamma(\epsilon) = \underset{n\in\Gamma(\mathcal{E})}{\oplus} \epsilon_n,$$

$$H = \underset{n\in\Gamma(\mathcal{E})}{\oplus} H_n, \qquad H^{\mathrm{red}} = \underset{n\in\Gamma(\mathcal{E})}{\oplus} H_n^{\mathrm{red}}.$$

5.2 Asymptotic Particles

Define the asymptotic one-particle space and the reduced asymptotic one-partcle space:

$$\mathcal{H}^{\mathrm{as}} := \mathrm{Ran}\mathbf{1}^{\mathrm{pp}}(\mathbf{1}\otimes H^{\mathrm{red}}) \subset \Gamma(\mathcal{H}), \qquad h^{\mathrm{as}} := \mathrm{Ran}\mathbf{1}^{\mathrm{pp}}(H^{\mathrm{red}}) \subset \Gamma(\mathcal{H})^{\mathrm{red}}.$$

We will write x^{as}, D^{as}, m^{as}, z^{as}, ϵ^{as} instead of $x^{\mathrm{cm}}\Big|_{\mathcal{H}^{\mathrm{as}}}$, $D^{\mathrm{cm}}\Big|_{\mathcal{H}^{\mathrm{as}}}$, $\mathrm{d}\Gamma(m)\Big|_{\mathcal{H}^{\mathrm{as}}}$, $\mathrm{d}\Gamma(z)\Big|_{\mathcal{H}^{\mathrm{as}}}$, $\Gamma(\epsilon)\Big|_{\mathcal{H}^{\mathrm{as}}}$. We will write X^{as} instead of X^{cm}. Clearly,

$$\mathcal{H}^{\mathrm{as}} \simeq L^2(X^{\mathrm{as}})\otimes h^{\mathrm{as}}.$$

Define

$$k^{\mathrm{as}} := H\Big|_{h^{\mathrm{as}}}, \qquad \mu^{\mathrm{as}} := H^{\mathrm{red}}\Big|_{h^{\mathrm{as}}}.$$

Clearly,

$$k^{\mathrm{as}} = (2m^{\mathrm{as}})^{-1}(D^{\mathrm{as}})^2 + \mathbf{1}\otimes\mu^{\mathrm{as}}.$$

The family of asymptotic particles is defined as

$$\mathcal{E}^{\mathrm{as}} := \{n\in\Gamma(\mathcal{E}) \ : \ \mathcal{H}^{\mathrm{as}}\cap\Gamma_n(\mathcal{H}) \neq \{0\}\}.$$

For $n\in\mathcal{E}^{\mathrm{as}}$ we set

$$\Gamma_n(\mathcal{E}^{\mathrm{as}}) := \Gamma_n^{\mathrm{tq}}(\mathcal{E})\cap\Gamma(\mathcal{E}^{\mathrm{as}}).$$

We set

$$\mathcal{H}_n^{\mathrm{as}} := \mathcal{H}^{\mathrm{as}}\cap\Gamma_n(\mathcal{H}), \qquad h_n^{\mathrm{as}} := h^{\mathrm{as}}\cap\Gamma_n(\mathcal{H})^{\mathrm{red}}.$$

Clearly, we can write the decompositions

$$\mathcal{H}^{\mathrm{as}} \simeq \bigoplus_{n\in\mathcal{E}^{\mathrm{as}}}\mathcal{H}_n^{\mathrm{as}}, \qquad h^{\mathrm{as}} \simeq \bigoplus_{n\in\mathcal{E}^{\mathrm{as}}} h_n^{\mathrm{as}}.$$

We can decompose the asymptotic one-particle operators as follows:

$$m^{\mathrm{as}} = \bigoplus_{n\in\mathcal{E}^{\mathrm{as}}} m_n, \qquad z^{\mathrm{as}} = \bigoplus_{n\in\mathcal{E}^{\mathrm{as}}} z_n, \qquad \epsilon^{\mathrm{as}} = \bigoplus_{n\in\mathcal{E}^{\mathrm{as}}}\epsilon_n.$$

$$k^{\mathrm{as}} = \bigoplus_{n\in\mathcal{E}^{\mathrm{as}}} k_n, \qquad \mu^{\mathrm{as}} = \bigoplus_{n\in\mathcal{E}^{\mathrm{as}}}\mu_n.$$

The asymptotic Fock space is $\Gamma(\mathcal{H}^{\mathrm{as}})$. The $\Box$-product in $\Gamma(\mathcal{H}^{\mathrm{as}})$ is denoted $\Box^{\mathrm{as}}$; creation and annihilation operators are denoted $a^{\mathrm{as}*}(\phi)$, $a^{\mathrm{as}}(\phi)$ for $\phi\in\mathcal{H}^{\mathrm{as}}$. The asymptotic Hamiltonian is

$$K^{\mathrm{as}} := \mathrm{d}\Gamma\left(k^{\mathrm{as}}\right).$$

We define

$$\mathcal{X}^{\mathrm{as}} := X^{\mathrm{as}}\times\mathcal{E}^{\mathrm{as}}.$$

We define $\Gamma(\mathcal{X}^{\mathrm{as}})^{\mathrm{free}}$ similarly, as $\Gamma(\mathcal{X}^{\mathrm{as}})^{\mathrm{free}}$ with X^{as}, $\mathcal{X}^{\mathrm{as}}$, $\mathcal{E}^{\mathrm{as}}$ replacing X, $\mathcal{X}$, $\mathcal{E}$.

The following $*$-homomorphisms have an obvious definition

$$C_\infty(\mathcal{E}^{\mathrm{as}}) \ni f \mapsto f(\hat{e}^{\mathrm{as}}) := \underset{n\in\mathcal{E}^{\mathrm{as}}}{\oplus} f(n) \in B(h^{\mathrm{as}}),$$

$$C_\infty(X^{\mathrm{as}}) \ni f \mapsto f(x^{\mathrm{as}}) \in B(L^2(X^{\mathrm{as}})),$$
$$C_\infty(X^{\mathrm{as}}) \ni f \mapsto f(D^{\mathrm{as}}) \in B(L^2(X^{\mathrm{as}})),$$

We also have the tensor products of these homomorphisms:

$$C_\infty(\mathcal{X}^{\mathrm{as}}) \ni f \to f(x^{\mathrm{as}}, \hat{e}^{\mathrm{as}}) \in B(\mathcal{H}^{\mathrm{as}}),$$

$$C_\infty(\mathcal{X}^{\mathrm{as}}) \ni f \to f(D^{\mathrm{as}}, \hat{e}^{\mathrm{as}}) \in B(\mathcal{H}^{\mathrm{as}}).$$

Sometimes we will use their second quantized extensions

$$C_\infty(\Gamma(\mathcal{X}^{\mathrm{as}})) \ni F \to F(\underline{x^{\mathrm{as}}, \hat{e}^{\mathrm{as}}}) \in B(\Gamma(\mathcal{H}^{\mathrm{as}})),$$

$$C_\infty(\Gamma(\mathcal{X}^{\mathrm{as}})) \ni F \to F(\underline{D^{\mathrm{as}}, \hat{e}^{\mathrm{as}}}) \in B(\Gamma(\mathcal{H}^{\mathrm{as}})),$$

For $k \in \Gamma(\mathcal{E}^{\mathrm{as}})$ we set

$$\Gamma_k(\mathcal{H}^{\mathrm{as}}) := \underset{m\in\mathcal{E}^{\mathrm{as}}}{\otimes} \Gamma_{k_m}(\mathcal{H}^{\mathrm{as}}_m).$$

Clearly, we have the decomposition

$$\Gamma(\mathcal{H}^{\mathrm{as}}) \simeq \underset{k\in\Gamma(\mathcal{E}^{\mathrm{as}})}{\oplus} \Gamma_k(\mathcal{H}^{\mathrm{as}}).$$

We can decompose various operators, for example

$$K^{\mathrm{as}} \simeq \underset{k\in\Gamma(\mathcal{E}^{\mathrm{as}})}{\oplus} K^{\mathrm{as}}_k.$$

5.3 Identification operators

Recall that in Subsection 4.7 we defined the maps

$$J^{\mathrm{tq}} : \Gamma^{\mathrm{tq}}(\mathcal{H}) \to \Gamma(\mathcal{H}), \qquad j^{\mathrm{tq}} : \Gamma^{\mathrm{tq}}(\mathcal{X}) \to \Gamma(\mathcal{X}).$$

Recall that $\mathcal{H}^{\mathrm{as}} \subset \Gamma(\mathcal{H})$ and $\mathcal{X}^{\mathrm{as}} \subset \Gamma(\mathcal{X})$. Hence $\Gamma(\mathcal{H}^{\mathrm{as}}) \subset \Gamma^{\mathrm{tq}}(\mathcal{H})$, $\Gamma(\mathcal{X}^{\mathrm{as}}) \subset \Gamma^{\mathrm{tq}}(\mathcal{X})$ and we can define the identification operators

$$J^{\mathrm{as}} : \Gamma(\mathcal{H}^{\mathrm{as}}) \to \Gamma(\mathcal{H}), \qquad j^{\mathrm{as}} : \Gamma(\mathcal{X}^{\mathrm{as}}) \to \Gamma(\mathcal{X})$$

as

$$J^{\mathrm{as}} := J^{\mathrm{tq}}\Big|_{\Gamma(\mathcal{H}^{\mathrm{as}})}, \qquad j^{\mathrm{as}} := j^{\mathrm{tq}}\Big|_{\Gamma(\mathcal{X}^{\mathrm{as}})}.$$

Let $n \in \mathcal{E}^{\mathrm{as}}$ and $k \in \Gamma_n(\mathcal{E}^{\mathrm{as}})$. Recall that $\Gamma_k(\mathcal{H}^{\mathrm{as}}) \subset \Gamma^{\mathrm{tq}}_k(\mathcal{H})$ $\Gamma_k(\mathcal{X}^{\mathrm{as}}) \subset \Gamma^{\mathrm{tq}}_k(\mathcal{X})$ We set

$$J^{\mathrm{as}}_k := J^{\mathrm{as}}\Big|_{\Gamma_k(\mathcal{H}^{\mathrm{as}})} = J^{\mathrm{tq}}_k\Big|_{\Gamma_k(\mathcal{H}^{\mathrm{as}})}, \qquad j^{\mathrm{as}}_k := j^{\mathrm{as}}\Big|_{\Gamma_k(\mathcal{X}^{\mathrm{as}})} = j^{\mathrm{tq}}_k\Big|_{\Gamma_k(\mathcal{X}^{\mathrm{as}})}.$$

Clearly,

$$J^{\mathrm{as}}_k : \Gamma_k(\mathcal{H}^{\mathrm{as}}) \to \Gamma_n(\mathcal{H}), \qquad j^{\mathrm{as}}_k : \Gamma_k(\mathcal{X}^{\mathrm{as}}) \to \Gamma_n(\mathcal{X}).$$

5.4 Asymptotic velocity

Theorem 5.1 *For every $e, e' \in \mathcal{E}$ assume*

$$(1+D^2)^{-1}\nabla_x v_{ee'}(x)\mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R})(1+D^2)^{-1} \in L^1(\mathrm{d}R). \tag{5.38}$$

Then for every $F \in C_\infty(\Gamma(\mathcal{X}))$ there exists the limit

$$\mathrm{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} F\left(\underline{\tfrac{x}{t}, \hat{e}}\right) \mathrm{e}^{-\mathrm{i}tH}. \tag{5.39}$$

The limit (5.39) defines a nondegenerate homomorphism denoted by

$$C_\infty(\Gamma(\mathcal{X})) \ni F \mapsto F(P^+) \in B(\Gamma(\mathcal{H})).$$

We have

$$[F(P^+), H] = 0,$$
$$\mathbf{1}_{\Gamma(\mathcal{X})^{\mathrm{cm}}}(P^+) = \mathcal{H}^{\mathrm{as}}.$$

If $V = 0$, then $F(P^+) = F(\underline{D, \hat{e}})$.

5.5 Short-range case

Let us state the main result of our paper in the short-range case.

Theorem 5.2 *For every $e, e' \in \mathcal{E}$ assume*

$$(1+D^2)^{-1} v_{ee'}(x)\mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R})(1+D^2)^{-1/2} \in L^1(\mathrm{d}R). \tag{5.40}$$

Then

$$\Omega^+ := \mathrm{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} J^{\mathrm{as}} \mathrm{e}^{-\mathrm{i}tK^{\mathrm{as}}}$$

exists and is unitary from $\Gamma(\mathcal{H}^{\mathrm{as}})$ to $\Gamma(\mathcal{H})$. It satisfies

$$\Omega^+ K^{\mathrm{as}} = H\Omega^+, \tag{5.41}$$

$$F(P^+)\Omega^+ = \Omega^+ F \circ j^{\mathrm{as}}(\underline{D^{\mathrm{as}}, \hat{e}^{\mathrm{as}}}), \tag{5.42}$$

for $F \in C_\infty(\Gamma(\mathcal{X}))$.

The proof of this theorem will be given in Subsections 5.9, 5.10 and 5.11.

Remark 5.3 *It is possible and instructive to describe the result of the above theorem using creation/annihilation operators, in the spirit of the Haag-Ruelle theory. Let $\phi \in \mathcal{H}^{\mathrm{as}} \cap \Gamma_{\mathrm{fin}}(\mathcal{H})$. Then for any $\psi \in \Gamma_{\mathrm{fin}}(\mathcal{H})$ there exist*

$$\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} a^*(\mathrm{e}^{-\mathrm{i}tH}\phi)\mathrm{e}^{-\mathrm{i}tH}\psi =: a^{+*}(\phi)\psi,$$

$$\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} a(\mathrm{e}^{-\mathrm{i}tH}\phi)\mathrm{e}^{-\mathrm{i}tH}\psi =: a^+(\phi)\psi,$$

(where $a^*(\phi)$, $a(\phi)$ denote multiple creation/annihillation operators). We have

$$a^+(\phi) = \Omega^+ a^{\mathrm{as}}(\phi)\Omega^{+*}$$

$$a^{+*}(\phi) = \Omega^+ a^{\mathrm{as}*}(\phi)\Omega^{+*}.$$

Hence

$$[a^+(\phi_1), a^{+*}(\phi_2)] = [a^+(\phi_1), a^{+*}(\phi_2)] = 0, \quad \phi_1, \phi_2 \in \mathcal{H}^{\mathrm{as}},$$

$$[a^+(\phi_1), a^{+*}(\phi_2)] = (\phi_2|\phi_1), \quad \phi_1, \phi_2 \in \mathcal{H}^{\mathrm{as}},$$

$$a(\phi)\omega = 0, \quad \phi \in \mathcal{H}^{\mathrm{as}}.$$

Moreover, the set

$$a^{+*}(\phi_1)\cdots a^{+*}(\phi_q)\omega, \quad \phi_1, \dots, \phi_q \in \mathcal{H}^{\mathrm{as}},\ q = 0, 1, \dots$$

spans the whole $\Gamma(\mathcal{H})$.

5.6 Long-range case – free region

¿From now on in the long-range case we will always assume that, for any $e, e' \in \mathcal{E}$,

$$v_{ee'}(x) = v^{\mathrm{s}}_{ee'}(x) + v^{\mathrm{l}}_{ee'}(x),$$

$v^{\mathrm{l}}_{ee'}(x) \in \mathbf{R}$ and $v^{\mathrm{s}}_{ee'}(x) \in B(h_e \otimes h_{e'})$. We will set as operators on $h \otimes h$

$$v^{\mathrm{s}}(x) := \mathop{\oplus}_{e,e'\in\mathcal{E}} v^{\mathrm{s}}_{ee'}(x), \qquad v^{\mathrm{l}}(x) := \mathop{\oplus}_{e,e'\in\mathcal{E}} v^{\mathrm{l}}_{ee'}(x)$$

and as operators on $\Gamma_2(\mathcal{H})$:

$$v^{\mathrm{s}} := v^{\mathrm{s}}\left(x \otimes \mathbf{1} - \mathbf{1} \otimes x\right)\Big|_{\Gamma_2(\mathcal{H})}, \qquad v^{\mathrm{l}} := v^{\mathrm{l}}\left(x \otimes \mathbf{1} - \mathbf{1} \otimes x\right)\Big|_{\Gamma_2(\mathcal{H})}.$$

In the case of $v^{\mathrm{l}}_{ee'}(x)$ we will consider two types of assumptions. Either we will assume that it is the Coulomb potential

$$v^{\mathrm{l}}_{ee'}(x) := \frac{z_e z_{e'}}{|x|},$$

in other words

$$v^{\mathrm{l}} = \frac{z \otimes z}{|x \otimes \mathbf{1} - \mathbf{1} \otimes x|}\Big|_{\Gamma_2(\mathcal{H})},$$

or we will give more general decay conditions.

The following theorem gives a description of scattering in the asymptotically free region. Note that the decay conditions on potentials are quite general.

Theorem 5.4 *(1) For any $e, e' \in \mathcal{E}$, assume that*

$$
\begin{aligned}
&(1+D^2)^{-1} v^{\mathrm{s}}_{ee'}(x) \mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R}) \in L^1(\mathrm{d}R), \\
&\lim_{|x|\to\infty} v^{\mathrm{l}}_{ee'}(x) = 0, \\
&\nabla_x^n v^{\mathrm{l}}_{ee'}(x) \mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R}) \in \langle R\rangle^{1-p} L^1([1,\infty[, \mathrm{d}R), \qquad p = 1, 2.
\end{aligned}
\tag{5.43}
$$

Then we can find a real function $[0,\infty[\times\Gamma(\mathcal{X}) \ni (t, \underline{\xi, e}) \to S(t, \underline{\xi, e})$ such that there exists

$$\Omega^{\mathrm{free},+} := \mathrm{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tS(t,\underline{D,\hat{e}})}.$$

The operator $\Omega^{\mathrm{free},+}$ is an isometry from $\Gamma(\mathcal{H}^{\mathrm{as}})$ to $\Gamma(\mathcal{H})$ and it satisfies

$$\Omega^{\mathrm{free},+} K = H \Omega^{\mathrm{free},+},$$

$$\Omega^{\mathrm{free},+} F(\underline{D, \hat{e}}) = F(P^+) \Omega^{\mathrm{free},+},$$

$$\mathrm{Ran}\Omega^{\mathrm{free},+} = \mathrm{Ran}\mathbf{1}^{\mathrm{free},+},$$

where

$$\mathbf{1}^{\mathrm{free},+} := \mathbf{1}_{\Gamma(\mathcal{X})^{\mathrm{free}}}(P^+).$$

(2) If we replace the third line of assumption (5.43) by

$$\nabla_x v^{\mathrm{l}}_{ee'}(x) \mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R}) \langle R\rangle^{1/2} \in L^1([1,\infty[, \mathrm{d}R),$$

then we can set

$$S(t, D) := \mathrm{d}\Gamma((2m)^{-1}D^2) + \mathrm{d}\Gamma_{(2)}(g(t)),$$

where

$$g(t) = \int_0^t v^{\mathrm{l}} \left(t' \frac{D}{m} \otimes \mathbf{1} - t' \mathbf{1} \otimes \frac{D}{m} \right) \Big|_{\Gamma_2(\mathcal{H})} \mathrm{d}t'.$$

(3) In the Coulomb case we can set

$$g(t) := \frac{z \otimes z \log t}{\left| \frac{D}{m} \otimes \mathbf{1} - \mathbf{1} \otimes \frac{D}{m} \right|} \Big|_{\Gamma_2(\mathcal{H})}.$$

(4) In the short-range case, that is under the condition 5.40 we can set

$$g(t) = 0.$$

5.7 Long-range case – asymptotic interacting Hamiltonian

Before we describe scattering for identical particles in the long-range case we need to introduce some additional definitions.

For any $n, n' \in \Gamma(\mathcal{E})$ we set

$$v^{\mathrm{l}}_{nn'}(x) = \sum_{e,e' \in \mathcal{E}} n_e n'_{e'} v^{\mathrm{l}}_{ee'}(x).$$

We have

$$h^{\mathrm{as}} \otimes h^{\mathrm{as}} = \underset{n,n' \in \mathcal{E}^{\mathrm{as}}}{\oplus} h^{\mathrm{as}}_n \otimes h^{\mathrm{as}}_{n'}.$$

We define the asymptotic potential on $h^{\mathrm{as}} \otimes h^{\mathrm{as}}$

$$v^{\mathrm{as}}(x) := \underset{n,n' \in \mathcal{E}^{\mathrm{as}}}{\oplus} v^{\mathrm{l}}_{nn'}(x).$$

In the case of Coulomb potentials we set

$$v^{\mathrm{l}}_{nn'}(x) = \frac{z_n z_{n'}}{|x|},$$

and hence

$$v^{\mathrm{as}}(x) := \frac{z \otimes z}{|x|}.$$

Now on $\Gamma_2(\mathcal{H}^{\mathrm{as}})$ we can set

$$v^{\mathrm{as}} := v^{\mathrm{as}}\left(x \otimes \mathbf{1} - \mathbf{1} \otimes x\right)\Big|_{\Gamma_2(\mathcal{H}^{\mathrm{as}})}.$$

We define the second-quantized asymptotic potential and asymptotic Hamiltonian as operators on $\Gamma(\mathcal{H}^{\mathrm{as}})$:

$$\begin{aligned} V^{\mathrm{as}} &:= \mathrm{d}\Gamma_{(2)}(v^{\mathrm{as}}), \\ H^{\mathrm{as}} &:= K^{\mathrm{as}} + V^{\mathrm{as}}. \end{aligned}$$

Theorem 5.5 *Let $\mu = \sqrt{3} - 1$ and for any $e, e' \in \mathcal{E}$,*

$$\begin{aligned} &(1 + D^2)^{-1} v^{\mathrm{s}}_{ee'}(x) \mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R}) \in L^1(\mathrm{d}R), \\ &\lim_{|x| \to \infty} v^{\mathrm{l}}_{ee'}(x) = 0, \\ &\nabla_x v^{\mathrm{l}}_{ee'}(x) \mathbf{1}_{[1,\infty[}(\tfrac{|x|}{R}) \langle R \rangle^{\mu} \in L^1([1, \infty[, \mathrm{d}R). \end{aligned} \tag{5.44}$$

Let $P^{\mathrm{as},+}$ be the asymptotic velocity and $\mathbf{1}^{\mathrm{as,free},+}$ be the projection on the asymptotically free states defined using the Hamiltonian H^{as}. Then there exists

$$\Omega^+ := \mathrm{s-}\lim_{t \to \infty} \mathrm{e}^{\mathrm{i}tH} J^{\mathrm{as}} \mathrm{e}^{-\mathrm{i}tH^{\mathrm{as}}} \mathbf{1}^{\mathrm{as,free},+}.$$

Moreover, Ω^+ is unitary from $\mathrm{Ran}\mathbf{1}^{\mathrm{as,free},+} \subset \Gamma(\mathcal{H}^{\mathrm{as}})$ *to* $\Gamma(\mathcal{H})$ *and*

$$\Omega^+ H^{\mathrm{as}} = H\Omega^+,$$

$$\Omega^+ F \circ j^{\mathrm{as}}(P^{\mathrm{as},+}) = F(P^+)\Omega^+.$$

Remark 5.6 *Note that in the short-range case*

$$H^{\mathrm{as}} = K^{\mathrm{as}}, \qquad P^{\mathrm{as},+} = \underline{D^{\mathrm{as}}, \hat{e}^{\mathrm{as}}}, \qquad \mathbf{1}^{\mathrm{as,free},+} = \mathbf{1},$$

and Theorem 5.5 reduces to Theorem 5.2.

5.8 Long-range case – modified wave operators

Define the operator on $\mathcal{H}^{\mathrm{as}} \otimes \mathcal{H}^{\mathrm{as}}$:

$$g^{\mathrm{as}}(t) := \int_0^t v^{\mathrm{as}} \left(t' \frac{D^{\mathrm{as}}}{m^{\mathrm{as}}} \otimes \mathbf{1} - t' \mathbf{1} \otimes \frac{D^{\mathrm{as}}}{m^{\mathrm{as}}} \right) \Big|_{\Gamma_2(\mathcal{H})} \mathrm{d}t'.$$

In the Coulomb case set

$$g^{\mathrm{as}}(t) := \frac{z^{\mathrm{as}} \otimes z^{\mathrm{as}} \log t}{\left| \frac{D^{\mathrm{as}}}{m^{\mathrm{as}}} \otimes \mathbf{1} - \mathbf{1} \otimes \frac{D^{\mathrm{as}}}{m^{\mathrm{as}}} \right|} \Bigg|_{\Gamma_2(\mathcal{H})}.$$

The second quantized Dollard modifier equals

$$G^{\mathrm{as}}(t) := \mathrm{d}\Gamma_{(2)}(g^{\mathrm{as}}(t)).$$

Theorem 5.7 *Suppose that the assumptions of Theorem 5.5 hold. Then there exists*

$$\mathrm{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} J^{\mathrm{as}} \mathrm{e}^{-\mathrm{i}tK^{\mathrm{as}} - \mathrm{i}G^{\mathrm{as}}(t)} =: \Omega^{\mathrm{mod},+}, \tag{5.45}$$

$\Omega^{\mathrm{mod},+}$ *is a unitary operator from* $\Gamma(\mathcal{H}^{\mathrm{as}})$ *to* $\Gamma(\mathcal{H})$ *and*

$$\Omega^{\mathrm{mod},+} K^{\mathrm{as}} = H\Omega^{\mathrm{mod},+}.$$

$$\Omega^{\mathrm{mod},+} F \circ j^{\mathrm{as}}(\underline{D^{\mathrm{as}}, \hat{e}^{\mathrm{as}}}) = F(P^+)\Omega^{\mathrm{mod},+}.$$

Clearly, Theorem 5.4 applied to H^{as} and Theorem 5.5 imply Theorem 5.7.

5.9 Distinguishable particles revisited

This subsection serves as a preparation for the proof of Theorems 5.5 and 5.7. In this subsection we recall and rephrase constructions and the results of Section 2 on distinguishable particles in a convenient manner.

Our starting point is a system of bosonic and fermionic particles described in Subsections 5.1 and 5.2 labelled by $\mathcal{E}$. We set $\mathcal{J} := \mathcal{E} \times \mathbf{N}$, we define the set of clusters $\mathcal{C}$ and of clusterings $\mathcal{A}$ as in Section 1. For $(e,i) \in \mathcal{J}$ we set $h_{(e,i)} := h_e$, $X_{(e,i)} := X_e$, $m_{(e,i)} := m_e$, etc. For $(e,i),(e',i') \in \mathcal{J}$ we set $v_{(e,i)(e'i')}(x) := v_{ee'}(x)$. Using the formalism of Section 2 we introduce a system of distinguishable particles labelled by the set $\mathcal{J} = \mathcal{E} \times \mathbf{N}$ with the above defined internal Hilbert spaces, configuration spaces, masses, potentials, etc.

Let $n \in \Gamma(\mathcal{E})$. Recall that in Subsection 3.3 we defined the Hilbert space

$$\mathcal{H}_n := \underset{e\in\mathcal{E}}{\otimes} \mathcal{H}_e^{\otimes n_e}.$$

Recall also that $\mathcal{H}_n$ equals $\mathcal{H}_c$ for all $c \in \mathcal{C}_n$.

The Hamiltonian H_c, defined in Subsection 2.2 for $c \in \mathcal{C}$, acts on $\mathcal{H}_c$. In particular, if $c \in \mathcal{C}_n$, it acts on $\mathcal{H}_n$. The Hamiltonian H_c does not depend on c as long as $c \in \mathcal{C}_n$. It will be denoted $\tilde{H}_n$. It is a self-adjoint operator acting on $\mathcal{H}_n$. Clearly, it satisfies

$$[\Theta(\sigma), \tilde{H}_n] = [\Theta^\epsilon(\sigma), \tilde{H}_n] = 0, \qquad \sigma \in S_n.$$

(We use the tilde in $\tilde{H}_n$ to avoid confusion with H_n defined in Subsection 5.1).

Let $n \in \Gamma(\mathcal{E})$ and $k \in \Gamma^{\mathrm{tq}}(\mathcal{E})$. In Subsection 3.5 we defined

$$\mathcal{H}_k := \underset{m\in\Gamma(\mathcal{E})}{\otimes} (\mathcal{H}_m)^{\otimes k_m}.$$

It equals $\mathcal{H}_a$ for $a \in \mathcal{A}_k$.

Define also the Hilbert space

$$\mathcal{H}_k^{\mathrm{as}} := \underset{m\in\mathcal{E}^{\mathrm{as}}}{\otimes} (\mathcal{H}_m^{\mathrm{as}})^{\otimes k_m}.$$

It equals $\mathcal{H}_a^{\mathrm{as}}$ for $a \in \mathcal{A}_k$.

The Hamiltonian K_a^{as}, defined in Subsection 2.3 for $a \in \mathcal{A}$, acts on $\mathcal{H}_a^{\mathrm{as}}$. It does not depend on a as long as $a \in \mathcal{A}_k$. We denote it by $\tilde{K}_k^{\mathrm{as}}$. $\tilde{K}_k^{\mathrm{as}}$ acts on the Hilbert space $\mathcal{H}_k^{\mathrm{as}}$. We have

$$[\Theta(\sigma), \tilde{K}_k^{\mathrm{as}}] = [\Theta^\epsilon(\sigma), \tilde{K}_k^{\mathrm{as}}] = 0, \quad \sigma \in S_k.$$

(We use the tilde in $\tilde{K}_k^{\mathrm{as}}$ to avoid confusion with the Hamiltonian H_k defined in Subsection 5.2).

Let $n \in \Gamma(\mathcal{E})$. $k \in \Gamma_n^{\mathrm{tq}}(\mathcal{E})$ and $\alpha \in \tilde{\mathcal{A}}_k$. We have the maps

$$\Theta(\sigma_\alpha) : \mathcal{H}_k \to \mathcal{H}_n$$

$$J_\alpha^{\mathrm{as}} := \Theta(\sigma_\alpha)\Big|_{\mathcal{H}_k^{\mathrm{as}}} : \mathcal{H}_k^{\mathrm{as}} \to \mathcal{H}_n.$$

The following theorem follows immediately from Theorem 2.2

Theorem 5.8 *Let $n \in \mathcal{E}^{\rm as}$, $k \in \Gamma(\mathcal{E}^{\rm as})$. Then for any $\alpha \in \tilde{\mathcal{A}}_k$ there exists*

$$\Omega_\alpha^+ := \text{s}-\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}t\tilde{H}_n} J_\alpha^{\rm as} \mathrm{e}^{-\mathrm{i}t\tilde{K}_k^{\rm as}}.$$

Moreover,

$$\mathcal{H}_n = \mathop{\oplus}_{k\in\Gamma_n(\mathcal{E}^{\rm as})} \mathop{\oplus}_{\alpha\in\tilde{\mathcal{A}}_k} \mathrm{Ran}\Omega_\alpha^+. \tag{5.46}$$

Let us note also the following property of wave operators:

Lemma 5.9 *Let $n \in \mathcal{E}^{\rm as}$, $k \in \Gamma(\mathcal{E}^{\rm as})$, $\alpha \in \tilde{\mathcal{A}}_k$ and $\sigma \in S_n$. Then*

$$\Theta(\sigma)\Omega_\alpha^+ = \Omega_{\sigma\alpha}^+ \Theta(\sigma_{\sigma\alpha}^{-1}\sigma\sigma_\alpha).$$

Proof. Note the following facts:
(1) $\sigma \in S_n$, hence $\Theta(\sigma)$ commutes with $\tilde{H}_n$.
(2) $\sigma_{\sigma\alpha}^{-1}\sigma\sigma_\alpha \in S_k$, hence $\Theta(\sigma_{\sigma\alpha}^{-1}\sigma\sigma_\alpha)$ commutes with $\tilde{K}_k^{\rm as}$.
(3) $\Theta(\sigma)J_\alpha^{\rm as} = J_{\sigma\alpha}^{\rm as}\Theta(\sigma_{\sigma\alpha}^{-1}\sigma\sigma_\alpha)$.
Therefore,

$$\begin{aligned}\Theta(\sigma)\Omega_\alpha^+ &= \text{s}-\lim_{t\to\infty} \Theta(\sigma)\mathrm{e}^{\mathrm{i}t\tilde{H}_n} J_\alpha^{\rm as} \mathrm{e}^{-\mathrm{i}t\tilde{K}_k^{\rm as}} \\ &= \text{s}-\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}t\tilde{H}_n} J_{\sigma\alpha}^{\rm as} \mathrm{e}^{-\mathrm{i}t\tilde{K}_k^{\rm as}} \Theta(\sigma_{\sigma\alpha}^{-1}\sigma\sigma_\alpha) = \Omega_{\sigma\alpha}^+\Theta(\sigma_{\sigma\alpha}^{-1}\sigma\sigma_\alpha).\end{aligned}$$

5.10 Embedding identical particles into distinguishable particles

Recall that, for any $n \in \Gamma(\mathcal{E})$, we can embed the space $\Gamma_n(\mathcal{H})$ in $\mathcal{H}_n$:

$$\Gamma_n(\mathcal{H}) = P_n^\epsilon \mathcal{H}_n.$$

Moreover, $\Gamma_n(\mathcal{H})$ is an invariant subspace for $\tilde{H}_n$ and

$$H_n = \tilde{H}_n\Big|_{\Gamma_n(\mathcal{H})}.$$

Likewise, for $k \in \Gamma(\mathcal{E}^{\rm as})$, we can embed the space $\Gamma_k(\mathcal{H}^{\rm as})$ in $\mathcal{H}_k^{\rm as}$:

$$\Gamma_k(\mathcal{H}^{\rm as}) = P_k^\epsilon \mathcal{H}_k^{\rm as}.$$

Moreover, $\Gamma_k(\mathcal{H}^{\rm as})$ is an invariant subspace for $\tilde{K}_k^{\rm as}$ and

$$K_k^{\rm as} = \tilde{K}_k^{\rm as}\Big|_{\Gamma_k(\mathcal{H}^{\rm as})}.$$

Let $n \in \mathcal{E}^{\mathrm{as}}$ and $k \in \Gamma(\mathcal{E}^{\mathrm{as}})$. Recall that the identification operator

$$J_k^{\mathrm{as}} : \Gamma_k(\mathcal{H}^{\mathrm{as}}) \to \Gamma(\mathcal{H})$$

equals

$$\begin{aligned} J_k^{\mathrm{as}} &= \sqrt{\tfrac{k!^{\mathrm{tq}}}{n!}} \sum_{\alpha \in \tilde{\mathcal{A}}_k} \Theta^\epsilon(\sigma_\alpha)\Big|_{\Gamma_k(\mathcal{H}^{\mathrm{as}})} \\ &= \sqrt{\tfrac{k!^{\mathrm{tq}}}{n!}} \sum_{\alpha \in \tilde{\mathcal{A}}_k} \mathrm{sgn}_n^\epsilon(\sigma_\alpha) J_\alpha^{\mathrm{as}}. \end{aligned}$$

We will show the following statement, which will imply Theorem 5.5:

Theorem 5.10 *Assume the hypotheses of Theorem 5.5. Let $n \in \mathcal{E}^{\mathrm{as}}$ and $k \in \Gamma(\mathcal{E}^{\mathrm{as}})$. Then*

$$\mathrm{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH_n} J_k^{\mathrm{as}} \mathrm{e}^{-\mathrm{i}tK_k^{\mathrm{as}}} =: \Omega_k^+$$

exists, is isometric and

$$\bigoplus_{k \in \Gamma_n(\mathcal{E}^{\mathrm{as}})} \mathrm{Ran}\Omega_k^+ = \Gamma_n(\mathcal{H}).$$

We will also see that the wave operator for identical particles can be expressed in terms of wave operators for distinguishable particles as follows:

$$\Omega_k^+ = \sqrt{\tfrac{k!^{\mathrm{tq}}}{n!}} \sum_{\alpha \in \tilde{\mathcal{A}}_k} \mathrm{sgn}_n^\epsilon(\sigma_\alpha) \Omega_\alpha^+ \Big|_{\Gamma_k(\mathcal{H}^{\mathrm{as}})} \tag{5.47}$$

5.11 Existence and completeness of wave operators

In this subsection we are going to prove Theorem 5.10.

Let us prove the existence of wave operators. Let $\phi_k \in \Gamma_k(\mathcal{H}^{\mathrm{as}})$. We check that the limit

$$\begin{aligned} \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH_n} J_k^{\mathrm{as}} \mathrm{e}^{-\mathrm{i}tK_k^{\mathrm{as}}} \phi_k &= \sqrt{\tfrac{k!^{\mathrm{tq}}}{n!}} \sum_{\alpha \in \tilde{\mathcal{A}}_k} \mathrm{sgn}_n^\epsilon(\sigma_\alpha) \mathrm{e}^{\mathrm{i}t\tilde{H}_n} J_\alpha^{\mathrm{as}} \mathrm{e}^{-\mathrm{i}t\tilde{K}^{\mathrm{as}}} \phi_k \\ &= \sqrt{\tfrac{k!^{\mathrm{tq}}}{n!}} \sum_{\alpha \in \tilde{\mathcal{A}}_k} \mathrm{sgn}_n^\epsilon(\sigma_\alpha) \Omega_\alpha^+ \phi_k \end{aligned} \tag{5.48}$$

exists. The vectors in the above sum are orthogonal and have the same norm $\sqrt{\frac{k!^{\mathrm{tq}}}{n!}}$. Moreover, $|\tilde{\mathcal{A}}_k| = \frac{n!}{k!^{\mathrm{tq}}}$. Hence the norm of (5.48) is $\|\phi_k\|$. Hence Ω_k^+ exists, is isometric and (5.47) is true.

Let us prove now the asymptotic completeness. Let $\psi_n \in \Gamma_n(\mathcal{H})$. By the asymptotic completeness (5.46), we know that there exists a unique family of vectors $\tilde{\phi}_\alpha$, $\alpha \in \tilde{\mathcal{A}}_p$, such that if $k \in \Gamma_n(\mathcal{E}^{\mathrm{as}})$ and $\alpha \in \tilde{\mathcal{A}}_k$, then $\tilde{\phi}_\alpha \in \mathcal{H}_k^{\mathrm{as}}$ and

$$\psi_n = \sum_{\alpha \in \tilde{\mathcal{A}}_p} \Omega_\alpha^+ \tilde{\phi}_\alpha. \tag{5.49}$$

By setting

$$\tilde{\phi}_\alpha = \mathrm{sgn}_n^\epsilon(\sigma_\alpha)\phi_\alpha,$$

we can rewrite (5.49) as

$$\psi_n = \sum_{\alpha\in\tilde{\mathcal{A}}_p} \mathrm{sgn}_n^\epsilon(\sigma_\alpha)\Omega_\alpha^+\phi_a.$$

Next we note that $P_n^\epsilon\psi_n = \psi_n$. Hence

$$\begin{aligned}
\psi_n &= \tfrac{1}{n!}\sum_{\sigma\in S_n}\Theta^\epsilon(\sigma)\sum_{\alpha\in\tilde{\mathcal{A}}_p}\mathrm{sgn}_n^\epsilon(\sigma_\alpha)\Omega_\alpha^+\phi_\alpha \\
&= \tfrac{1}{n!}\sum_{\sigma\in S_n}\sum_{\alpha\in\tilde{\mathcal{A}}_p}\mathrm{sgn}_n^\epsilon(\sigma_{\sigma\alpha})\Omega_{\sigma\alpha}^+\Theta^\epsilon(\sigma_{\sigma\alpha}^{-1}\sigma\sigma_\alpha)\phi_\alpha \\
&= \tfrac{1}{n!}\sum_{k\in\Gamma_n(\mathcal{E}^{\mathrm{as}})}\sum_{\beta,\alpha\in\tilde{\mathcal{A}}_k}\sum_{\sigma'\in S_k}\mathrm{sgn}_n^\epsilon(\sigma_\beta)\Omega_\beta^+\Theta^\epsilon(\sigma')\phi_\alpha \\
&= \sum_{k\in\Gamma_n(\mathcal{E}^{\mathrm{as}})}\tfrac{k!^{\mathrm{tq}}}{n!}\sum_{\beta,\alpha\in\tilde{\mathcal{A}}_k}\mathrm{sgn}_n^\epsilon(\sigma_\beta)\Omega_\beta^+P_k^\epsilon\psi_\alpha \\
&= \sum_{k\in\Gamma_n(\mathcal{E}^{\mathrm{as}})}\Omega_k^+\phi_k,
\end{aligned}$$

where

$$\phi_k = \sqrt{\frac{k!^{\mathrm{tq}}}{n!}}\sum_{\alpha\in\tilde{\mathcal{A}}_k}P_k^\epsilon\phi_\alpha. \tag{5.50}$$

One may also remark that, using the uniqueness of $\tilde{\phi}_\alpha$, we have $P_k^\epsilon\phi_\alpha = \phi_\alpha \in \Gamma_k(\mathcal{H}^{\mathrm{as}})$ and $\phi_\alpha = \phi_{\alpha'}$ for $\alpha,\alpha' \in \tilde{\mathcal{A}}_k$. Therefore, we can rewrite (5.50) in a simpler fashion: for any $\alpha \in \tilde{\mathcal{A}}_k$, we have

$$\phi_k = \sqrt{\frac{n!}{k!^{\mathrm{tq}}}}\phi_\alpha.$$

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