On Sinai-Bowen-Ruelle measures on horocycles of 3-D Anosov flows

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Abstract

Let ϕ^t be a topologically mixing Anosov flow on a 3-D compact manifolds M. Every unstable fiber (horocycle) of such a flow is dense in M. Sinai proved in 1992 that the one-dimensional SBR measures on long segments of unstable fibers converge uniformly to the SBR measure of the flow. We establish an explicit bound on the rate of convergence in terms of integrals of Hölder continuous functions on M.

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1 Introduction

Let $\phi^t : M \to M$ be a C^2 Anosov flow on a smooth compact 3-D Riemannian manifold M. This means that ϕ^t has no fixed points, and at every $x \in M$ there is a $D\phi^t$-invariant splitting of the tangent space

$$\mathcal{T}_x M = E_x^s \oplus E_x^u \oplus E_x^\phi \tag{1}$$

into stable, unstable and neutral (parallel to the direction of the flow) one-dimensional subspaces. We assume that ϕ^t is topologically mixing.

Let μ be the Sinai-Bowen-Ruelle (SBR) measure for ϕ^t. For topologically mixing flows, μ is the only invariant measure whose conditional distributions on unstable fibers are absolutely continuous with respect to the Riemannian length [7]. Also [3], the SBR measure is the weak limit of the measure $\phi^t\mu_0$, as $t \to \infty$, for any smooth measure μ_0 on M. The SBR measure μ is a Gibbs measure [7] and Bernoulli [6].

The measure μ can be approximated by its conditional measures on one-dimensional unstable fibers as follows [8]. Let $x \in M$ and $R > 0$. Denote by $W^u_{x,R}$ the segment of the unstable fiber through x of length R on which x is the central point (equidistant from the endpoints). Now, for any $t > 0$ we take the normalized Lebesgue measure on the curve $\phi^{-t} W^u_{x,R}$ and pull this measure back onto $W^u_{x,R}$ under the action of ϕ^t. We get a probability measure on $W^u_{x,R}$ denoted by $\nu^u_{x,R,t}$. The weak limit

$$\nu^u_{x,R} = \lim_{t \to \infty} \nu^u_{x,R,t} \tag{2}$$

exists and is a smooth probability measure on $W^u_{x,R}$. It is the SBR measure μ conditioned on the curve $W^u_{x,R}$.

The measures $\nu^u_{x,R}$ are invariant under the flow ϕ^t in the following sense: if $\phi^t W^u_{x,R} \subset W^u_{y,S}$ for some $y \in M$ and $S > 0$, then the measure $\nu^u_{y,S}$ conditioned on $\phi^t W^u_{x,R}$ and pulled back under ϕ^{-t} will coincide with $\nu^u_{x,R}$. In particular, if $W^u_{x,R} \subset W^u_{y,S}$, then $\nu^u_{y,S}$ conditioned on $W^u_{x,R}$ coincides with $\nu^u_{x,R}$. The density $f_{x,R}(y)$, $y \in W^u_{x,R}$ of the measure $\nu^u_{x,R}$ with respect to the Riemannian length satisfies the Anosov-Sinai formula [1, 8, 4]

$$\frac{f_{x,R}(y_1)}{f_{x,R}(y_2)} = \lim_{t \to -\infty} \frac{\Lambda^u_t(y_1)}{\Lambda^u_t(y_2)} \tag{3}$$

where $\Lambda^u_t(y)$ is the Jacobian of the linear map $D\phi^t : E^u_y \to E^u_{\phi^t y}$.

The topological mixing of ϕ^t means by definition that every global unstable fiber $\Gamma^u_x = \cup_R W^u_{x,R}$ is dense in M. The K-mixing property of μ implies [8] that the weak limit of $\nu^u_{x,R}$, as $R \to \infty$, exists and coincides with the SBR measure μ for a.e. point x. Sinai has proved [8] that the weak convergence $\nu^u_{x,R} \to \mu$, as $R \to \infty$, occurs for *every* point $x \in M$ and is uniform in x in the following sense.

Theorem 1 (Sinai, [8]) *For any continuous function $F(x)$ on M and any $\varepsilon > 0$ there is an $R_\varepsilon > 0$ such that*

$$\left| \int_{W^u_{x,R}} F \, d\nu^u_{x,R} - \int_M F \, d\mu \right| \le \varepsilon \tag{4}$$

for all $x \in M$ and $R > R_\varepsilon$.

Technically, Sinai proved (4) for geodesic flows on surfaces of negative curvature, but his proof works for the all Anosov flows discussed here without any changes. Sinai termed the property (4) the *uniform distribution* of horocycles (this is the name for unstable and stable fibers of geodesic flows). As Sinai pointed out [8], for geodesic flows on noncompact surfaces of finite area the property (4) fails – every such surface has a finite number of families of closed (!) horocycles. Nonclosed horocycles are still dense and uniformly distributed [8]. Sinai's theorem was motivated by a remark made by Zagier [9], according to which the Riemannian hypothesis for zeroes of ζ-function could be derived from the properties of closed horocycles on the modular surface.

We are going to estimate the rate of the weak convergence $\nu^u_{x,R} \to \mu$. Our main result is the following.

Theorem 2 *Let $\phi^t : M \to M$ be a topologically mixing C^2 Anosov flow on a 3-D manifold, under the UNF assumption stated below. Let F be a Hölder continuous function with Hölder exponent $\alpha > 0$ on M. Then for all $x \in M$ and $R > 1$ we have*

$$\left| \int_{W^u_{x,R}} F \, d\nu^u_{x,R} - \int_M F \, d\mu \right| \le C_{\phi,F} \cdot \exp\left[-\alpha d_\phi (\ln R)^{1/2} \right] \tag{5}$$

where the factor d_ϕ depends on the flow ϕ^t alone, and $C_{\phi,F}$ depends on both ϕ^t and F, but both factors are independent of R and x.

We now formulate the assumption we call UNF - uniform nonintegrability of foliations. We denote by $W^{u,s}_x$ the (one-dimensional) local stable and unstable fibers through $x \in M$. We denote by

$$W^{wu,ws}_x = \phi^{[-\varepsilon,\varepsilon]} W^{u,s}_x$$

the weak local unstable and stable (two-dimensional) leaves through x.

Let $U \subset M$ be an open set, small enough so that both families of local stable and unstable fibers in U are orientable. Fix some orientations of those families in U. Let $y \in U$ and $\delta > 0$ a small number. On the local unstable and stable fibers W^u_y and W^s_y we take two positively oriented segments of length δ starting at y and terminating at some points $y_1 \in W^u_y$ and $y_2 \in W^s_y$, respectively. It is clear that the two points $y' = W^s_{y_1} \cap W^{wu}_{y_2}$ and $y'' = W^u_{y_2} \cap W^{ws}_{y_1}$ lie on the same orbit of the flow, i.e. $y' = \phi^\tau y''$ for some small number $\tau = \tau_y(\delta)$. We call this τ the *temporal distance* between the local fibers $W^s_{y_1}$ and $W^u_{y_2}$, see also [4].

The foliations by local stable and unstable fibers are said to be *jointly integrable* [5] in U if $\tau_y(\delta) = 0$ for all $y \in U$ and small $\delta > 0$. In that case those are subfoliations of the same C^1 foliation of U by surfaces. Plante's [5] results imply that the flow ϕ^t is topologically mixing iff there is an open domain $U \subset M$ where the stable and unstable foliations are *not* jointly integrable. Motivated by this, we call the next assumption the *uniform nonitegrability* of stable and unstable foliations.

Assumption UNF. There are $\delta_0 > 0$ and an open domain $U \subset M$ where both families of stable and unstable fibers are orientable, and for some orientation we have, at every $y \in U$ and all $0 < \delta < \delta_0$,

$$0 < \underline{d} < \frac{\tau_y(\delta)}{\delta^2} < \overline{d} < \infty \tag{6}$$

where $\underline{d}$ and $\overline{d}$ do not depend on y.

This assumption was first introduced in [4]. Based on it, a stretched exponential bound on correlation functions for the flow ϕ^t was established. It was also shown there that this assumption is always satisfied for 3-D contact Anosov flows and, in particular, for all geodesic flows on compact surfaces of (constant or variable) negative curvature.

Corollary 3 *Let $\phi^t : M \to M$ be a 3-D contact Anosov flow. Then the statement of Theorem 2 holds true. In particular, it holds for geodesic flows on compact surfaces of negative curvature.*

2 Markov partitions into boxes

Our proof of Theorem 2 is based on Markov approximations to Anosov flows developed in [4]. We will often refer to that paper, and for the reader's convenience we use here the same notations. One can thus consider this paper as a continuation of [4].

The main construction of the paper [4] is an increasing sequence of partitions of the manifold M into small boxes, which enjoy special Markov properties. We describe those partitions below, in a slightly modified form adjusted to our current needs.

First, let $\mathcal{R} = \{R_1, \dots, R_I\}$ be a Markov family of rectangles in M defined by Bowen [2]. Every rectangle is a small C^2 compact surface in M transversal to the flow ϕ^t. The boundary of each rectangle consists of four curves, two lying on local unstable leaves of ϕ^t and two others on local stable leaves. Rectangles need not be foliated by stable or unstable fibers. However, local stable and unstable leaves intersect a rectangle in smooth curves that we call *induced* stable and unstable fibers on the rectangle.

Let $\Omega = \cup R_i$. The surface Ω is a cross-section for the flow ϕ^t, see [2]. Denote by $T : \Omega \to \Omega$ the first return map and by $l(x) > 0$, $x \in \Omega$, the first return time. The flow ϕ^t is then isomorphic to a suspension flow built over the map $T : \Omega \to \Omega$ under the ceiling function $l(x)$.

The function $l(x)$ is piecewise C^2 smooth with discontinuities along a finite collection of induced stable fibers in the rectangles of $\mathcal{R}$. The function $l(x)$ on Ω is bounded away from 0 and ∞. The map $T : \Omega \to \Omega$ is piecewise C^2 smooth and hyperbolic. The above mentioned induced fibers on rectangles are just stable and unstable fibers for T. The partition of Ω into rectangles $R_i \in \mathcal{R}$ is a Markov partition for T.

We fix the Markov family $\mathcal{R}$, assuming its rectangles be small enough, see [2, 4]. In what follows, we denote by a_i, c_i, d_i, for $i = 1, 2, \dots$ and z, t_0 various positive constants determined only by the flow ϕ^t and the family $\mathcal{R}$.

Now let $m \geq 0$ be an integer-valued parameter. A refinement $\mathcal{A}_m$ of the Markov partition $\mathcal{R}$ was constructed in [4], with the following properties. First, $\mathcal{A}_m$ is a Markov partition itself, its atoms are small subrectangles in the rectangles $R_i \in \mathcal{R}$. For every rectangle $A \in \mathcal{A}_m$ we can find two integers $n_+(A) > 0$ and $n_-(A) > 0$ such that the images $T^{n_+(A)}A$ and $T^{-n_-(A)}A$ are subrectangles in some rectangles $R \in \mathcal{R}$, stretching across those rectangles completely (from one boundary curve to the opposite one). One can call the numbers $n_\pm(A)$ the *ranks* of the rectangle $A \in \mathcal{A}_m$. In the constructions made in [4], the ranks $n_\pm(A)$ depended on $A \in \mathcal{A}_m$, but they were bounded as follows:

$$d_1 m \leq n_\pm(A) \leq d_2 m$$

with some constants $0 < d_1 < d_2 < \infty$. Therefore, the sizes and measures of all rectangles $A \in \mathcal{A}_m$ decrease exponentially in m, see [4] for precise estimates.

Now, put $\delta = c_1 e^{-a_1 m}$. For every rectangle $A \in \mathcal{A}_m$ we take sets

$$X_i(A) = \phi^{[i\delta,(i+1)\delta]} A \tag{7}$$

for all $i = 0, 1, \dots, i_A$, where i_A is the smallest positive integer such that $X_{i_A+1}(A) \cap (\operatorname{int} \Omega) \neq \emptyset$. We call the sets (7) boxes. Every box is a domain bounded by two rectangles (top and bottom), two unstable leaves and two stable leaves. Boxes can only intersect one another in boundary points. Denote by $\hat{\mathcal{A}}_m$ the collection of all these boxes, and by $\hat{M}_m$ their union. The measure μ conditioned on $\hat{M}_m$ is denoted by $\hat{\mu}_m$. The complement $M \setminus \hat{M}_m$ is not empty, but it consists of tiny gaps between the rectangles $R_i \in \mathcal{R}$ and the boxes facing those rectangles 'from underneath'. The size (measured in the direction of the flow) and the μ-measure of those gaps are exponentially small in m.

We now restrict the flow ϕ^t onto $\hat{M}_m$ as follows. The new flow, $\hat{\phi}^t_m : \hat{M}_m \to \hat{M}_m$, will act just like ϕ^t on the interior of $\hat{M}_m$. As a trajectory $\hat{\phi}^t_m x$ reaches $\partial \hat{M}_m$, i.e. is about to enter a gap between two components of $\hat{M}_m$, it instantaneously jumps forward across that gap, but stays on the same trajectory of the original flow ϕ^t, and then continues in $\hat{M}_m$. Thus, for any $x \in \hat{M}_m$ and $t > 0$ we have $\hat{\phi}^t_m x = \phi^{t'} x$ for some $t' \geq t$. It was shown in [4] that

$$|t' - t| \leq c_2 t e^{-a_2 m} \tag{8}$$

so that the points $\hat{\phi}^t_m x$ and $\phi^t x$ are close for relatively small t.

Obviously, the flow $\hat{\phi}^t_m$ preserves the measure $\hat{\mu}_m$. It was shown in [4] that $\hat{\phi}^t_m$ is a hyperbolic flow with singularities. Its stable and unstable fibers are time-shifts of stable and unstable fibers of the map T, which we termed induced fibers.

The dynamics of the flow $\hat{\phi}^t_m$ has a clearly pronounced "discrete" or "quantum" character. Let $\hat{T}_m = \hat{\phi}^\delta_m$ (one can think of δ as a quantum of time). The map $\hat{T}_m$ moves every box forward onto the next one in $\hat{\mathcal{A}}_m$, except for the boxes whose top faces are on the border of a component of $\hat{M}_m$. Those border boxes are moved by $\hat{T}_m$ across the gaps and their images will fill the boxes $X \in \hat{\mathcal{A}}_m$ adjacent to the surface Ω on the other side of the gap. Due to these properties of the dynamics $\hat{\phi}^t_m$ it can be well approximated by a Markov chain, as it was shown in [4].

We now summarize some of the results of [4]. Take an arbitrary box $X_0 \in \hat{\mathcal{A}}_m$, and let $t > 0$. Consider the conditional distribution $\hat{\mu}_m(\cdot/\hat{\phi}^t_m X_0)$ on the boxes $X \in \hat{\mathcal{A}}_m$ defined by

$$\hat{\mu}_m(X/\hat{\phi}^t_m X_0) = \hat{\mu}_m(X \cap \hat{\phi}^t_m X_0) \cdot [\hat{\mu}_m(X_0)]^{-1}$$

For a function F on M, we define the average of F with respect to the above distribution by

$$\langle F/\hat{\phi}^t_m X_0 \rangle = \sum_{X \in \hat{\mathcal{A}}_m} \bar{F}_m(X) \cdot \hat{\mu}_m(X/\hat{\phi}^t_m X_0)$$

where

$$\bar{F}_m(X) = [\hat{\mu}_m(X)]^{-1} \int_X F(x)\, d\hat{\mu}_m(x)$$

Theorem 4 *There are constants $z > 0$ and $t_0 > 0$ such that for every $t > t_0$ one can take $m = [z\sqrt{t}]$ and then for any Hölder continuous function F on M one has*

$$\left| \int_{\hat{M}_m} F(x)\, d\hat{\mu}_m(x) - \langle F/\hat{\phi}^t_m X \rangle \right| \leq C_{F,1} \cdot c_3 \cdot \exp(-\alpha a_3 m) \tag{9}$$

for every box $X \in \hat{\mathcal{A}}_m$. Here α is the Hölder exponent of F, and the factor $C_{F,1} > 0$ depends on the function F alone.

This theorem actually says that the image $\hat{\phi}^t_m X$ is pretty much uniformly distributed over the space $\hat{M}_m$. In other words, the conditional measure on $\hat{\phi}^t_m X$ reproduces the invariant measure $\hat{\mu}_m$ of the flow $\hat{\phi}^t_m$ very accurately, as specified by (9). We emphasize the important relation $m = [z\sqrt{t}]$ between m and t.

This theorem follows immediately from the results of [4], but we should give a warning. The bounds on correlation functions developed in [4] required the above 'uniformity' of the distribution of $\hat{\phi}^t_m X$ *on the average* over the boxes $X \in \hat{\mathcal{A}}_m$, rather than for every single box X. So, all the main estimates in [4] are given by averaging over those boxes. Fortunately, the paper [4] also contains a proof of the uniformity of the distribution of $\hat{\phi}^t_m X$ for *every* box $X \in \hat{\mathcal{A}}_m$, see remarks in the end of Section 16 and Theorem 6.1 in [4].

By using the smallness of $\mu(M \setminus \hat{M}_m)$ and (8) we get

Corollary 5 *Under the conditions of Theorem 4 we have*

$$\left| \int_M F(x)\, d\mu(x) - (\mu(X))^{-1} \int_{\phi^t X} F(x)\, d\mu(x) \right| \leq C_{F,2} \cdot c_4 \cdot \exp(-\alpha a_4 m) \qquad (10)$$

for every box $X \in \hat{\mathcal{A}}_m$. Here α is the Hölder exponent of F, and the factor $C_{F,2} > 0$ depends on the function F alone.

Moreover, we can modify the box X here without harming the property (10) so that the new box will be foliated by unstable fibers in the following way. Let $x \in X$ and let $W^s_x(X)$ be the smallest segment of the local stable fiber through x that terminates on the local unstable leaves bounding the box X (of course, $W^s_x(X)$ may go out of the box X and then terminate on the continuation of the unstable leaf which contains a face of ∂X). For every point $y \in W^s_x(X)$ we take a segment of the local unstable fiber $W^u_y(X)$ that, in the same way as before, terminates on two local stable leaves bounding X (or their continuation beyond ∂X). The surface

$$B^u_x(X) = \cup_{y \in W^s_x(X)} W^u_y(X)$$

is foliated by unstable fibers. Sinai [8] called such surfaces u-cells. We now take $Y_x = \phi^{[0,\delta]} B^u_x(X)$. This is a domain, bounded by two surfaces $B^u_x(X)$, $\phi^\delta B^u_x(X)$, two local stable leaves, and two local unstable leaves. So we can call Y_x a box. This is our modification of the box X. The box Y_x is obviously foliated by unstable fibers of the flow ϕ^t. There is a natural one-to-one smooth correspondence $S : X \to Y_x$ which preserves the measure, leaves every point $x \in X$ on its trajectory and moves points no farther than by $c_5 e^{-a_5 m}$. The map S sends the bottom of the box X onto the surface $B^u_x(X)$. We then immediately obtain

Corollary 6 *Under the conditions of Theorem 4 we have*

$$\left| \int_M F(x)\, d\mu(x) - (\mu(Y_x))^{-1} \int_{\phi^t Y_x} F(x)\, d\mu(x) \right| \leq C_{F,3} c_6 \cdot \exp(-\alpha a_6 m) \qquad (11)$$

for the modification Y_x of every box $X \in \hat{\mathcal{A}}_m$. Here α is the Hölder exponent of F, and the factor $C_{F,3} > 0$ depends on the function F alone.

3 Proof of the main theorem

The proof of Theorem 2 is based on Corollary 6 and a few relatively simple arguments.

First, we recall the notion of the holonomy map. For any two close unstable fibers $W_1^u, W_2^u \in M$ the map $H : W_1^u \to W_2^u$ defined by $H(y) = W_y^{ws} \cap W_2^u$ is called canonical isomorphism, or holonomy map [1, 4]. Its Jacobian, DH, with respect to the Riemannian length on the curves W_1^u, W_2^u is bounded away from 0 and ∞. Moreover, it is close to one if the fibers are close enough to each other [4]. This property is commonly known as the absolute continuity of stable and unstable foliations [1, 4].

The following lemma gives a specific bound on the Jacobian DH:

Lemma 7 *There are constants $a, c > 0$, determined by the flow ϕ^t, such that*

$$\exp(-c\varepsilon^a) \le DH \le \exp(c\varepsilon^a) \tag{12}$$

where $\varepsilon = \mathrm{dist}(y, H(y))$.

Proof. Put $y_* = W_y^s \cap W_{H(y)}^{wu}$. Denote by H_* the holonomy map $W_1^u \to W_{y_*}^u$. There is a small τ_* such that $\phi^{\tau_*} W_{y_*}^u = W_2^u$. Then we have $DH(y) = DH_*(y) \cdot \Lambda_{\tau_*}^u(y_*)$. For the Jacobian $DH_*(y)$ an analog of Anosov-Sinai formula [1, 4] holds, which says that $DH_*(y) = \lim_{t\to\infty} \Lambda_t^u(y)/\Lambda_t^u(y_*)$. The existence of this limit and its closeness to one required by (12) follows from the facts that $y_* \in W_y^s$ and the function $\Lambda_t^u(\cdot)$ is Hölder continuous on M for any t, see [4]. Lemma 7 is proved.

We now turn to the proof of Theorem 2. Let $t > t_0$ and $m = [z\sqrt{t}]$, as in Corollary 6. Take an arbitrary box $X \in \hat{\mathcal{A}}_m$. For any $x \in X$ the modified box Y_x is foliated by unstable fibers. Its image $Y_{x,t} = \phi^t Y_x$ is also a domain foliated by unstable fibers. We denote the partition of $Y_{x,t}$ into unstable fibers by $\xi^u(Y_{x,t})$. Since $t \sim m^2$, the length of unstable fibers $W^u \in \xi^u(Y_{x,t})$ grows exponentially in m^2:

$$c_7 e^{a_7 m^2} \le \mathrm{length}(W^u) \le c_8 e^{a_8 m^2} \tag{13}$$

It is easy to see that all the unstable fibers in the partition $\xi^u(Y_{x,t})$ are canonically isomorphic. For any two fibers $W_1^u, W_2^u \in \xi^u(Y_{x,t})$ and any point $y \in W_1^u$ we have

$$\mathrm{dist}(y, H(y)) \le c_9 e^{-a_9 m}$$

Therefore, due to Lemma 7,

$$\exp(-c_{10} e^{-a_{10} m}) \le DH \le \exp(c_{10} e^{-a_{10} m}) \tag{14}$$

We now condition the measure μ on the domain $Y_{x,t}$ with respect to the partition $\xi^u(Y_{x,t})$. It induces smooth probability measures on the fibers $W^u \in \xi^u(Y_{x,t})$, which we denote by $\nu_{W^u}^u$. The density $f_{W^u}(y)$, $y \in W^u$, of the measure $\nu_{W^u}^u$ with respect to the Riemannian length satisfies the ratio formula (3):

$$\frac{f_{W^u}(y_1)}{f_{W^u}(y_2)} = \kappa_{W^u}(y_1, y_2) = \lim_{\tau\to-\infty} \frac{\Lambda_\tau^u(y_1)}{\Lambda_\tau^u(y_2)} \tag{15}$$

The function κ satisfies the rule $\kappa_{W^u}(y_1,y_2)\cdot\kappa_{W^u}(y_2,y_3)=\kappa_{W^u}(y_1,y_3)$. The density f_{W^u} can be computed by

$$f_{W^u}(y)=\frac{\kappa_{W^u}(y,y_0)}{\int_{W^u}\kappa_{W^u}(y,y_0)\,dy} \tag{16}$$

where $y_0\in W^u$ is any point, and the integration is performed with respect to the Riemannian length on W^u.

Let $W_1^u,W_2^u\in\xi(Y_x)$ be two unstable fibers, and $y_1,y_2\in W_1^u$. We will show that

$$\frac{\kappa_{W_1^u}(y_1,y_2)}{\kappa_{W_2^u}(H(y_1),H(y_2))}=e^{\varepsilon} \tag{17}$$

for some $|\varepsilon|\le c_{11}e^{-a_{11}m}$. Indeed,

$$\frac{\kappa_{W_1^u}(y_1,y_2)}{\kappa_{W_2^u}(H(y_1),H(y_2))}=\frac{\Lambda_{-t}^u(y_1)}{\Lambda_{-t}^u(H(y_1))}\,\frac{\Lambda_{-t}^u(H(y_2))}{\Lambda_{-t}^u(y_2)}\cdot e^{\varepsilon'} \tag{18}$$

for some $|\varepsilon|\le c_{11}e^{-a_{11}m}$, because all four points $\phi^{-t}y_i,\phi^{-t}H(y_i)$, $i=1,2$, lie in the small box Y_x whose size decrease exponentially in m. To estimate the two fractions on the right-hand side of (18), it is enough to note that the points $\phi^{-t}y_1,\phi^{-t}H(y_1)$ belong in one stable leaf in Y_x, as well as the points $\phi^{-t}y_2,\phi^{-t}H(y_2)$, and use again the Hölder continuity of the function $\Lambda_s^u(\cdot)$ for any s. So, we get (17).

Combining (14), (16) and (17) gives the bound

$$\exp(-c_{12}e^{-a_{12}m})\le D_*H\le\exp(c_{12}e^{-a_{12}m}) \tag{19}$$

where

$$D_*H(y)=\frac{f_{W_2^u}(H(y))}{f_{W_1^u}(y)}\cdot DH$$

is the Jacobian of the holonomy map now measured with respect to the induced measures $\nu_{W_1^u}^u,\nu_{W_2^u}^u$ on the unstable fibers $W_1^u,W_2^u\in\xi^u(Y_{x,t})$, rather than their Riemannian length.

Corollary 6 can be now reformulated:

Corollary 8 *Under the conditions of Theorem 4, for any unstable fiber $W^u\in\xi^u(Y_{x,t})$ we have*

$$\left|\int_M F(x)\,d\mu(x)-\int_{W^u}F\,d\nu_{W^u}^u\right|\le C_{F,4}\cdot c_{13}\cdot\exp(-\alpha a_{13}m) \tag{20}$$

Here α is the Hölder exponent of F, and the factor $C_{F,4}>0$ depends on the function F alone.

This corollary, along with the bounds (13) complete the proof of Theorem 2 provided the unstable fiber $W_{x,R}^u$ entering that theorem belongs in $\xi^u(Y_{x,t})$ for some box $X\in\mathcal{A}_m$ for some $m\ge 1$.

For a generic unstable fiber $W^u_{x,R}$ we first find a $t > 0$ such that the preimage $\phi^{-t}W^u_{x,R}$ has a length of order one (independently of R) and is located some positive distance apart from the rectangles of the Markov family $\mathcal{R}$. Obviously,

$$d_3 \ln R \leq t \leq d_4 \ln R$$

Then we take $m = [z\sqrt{t}]$, and the corresponding partition $\hat{\mathcal{A}}_m$ of $\hat{M}_m$ into boxes. The stable leaves bounding those boxes will partition the curve $\phi^{-t}W^u_{x,R}$ into short unstable fibers. Every short fiber, except for the two on both ends of $\phi^{-t}W^u_{x,R}$, will belong in some modified box defined in the end of the previous section. Its image under ϕ^t will then satisfy the bound (20). These images constitute nonoverlapping parts of the given fiber $W^u_{x,R}$. The two short fibers at the ends of $\phi^{-t}W^u_{x,R}$, which we left out, can be neglected since their relative lengths are less than $c_{14}e^{-a_{14}m}$.

The proof of Theorem 2 is accomplished.

4 Concluding remarks

1. The definition of the uniformity of the distribution of unstable manifolds can be extended to multidimensional Anosov flows as follows, see [8]. Let $x \in M$ and Γ^u_x be a (global) unstable manifold through x. Consider a sequence of open subsets $U_j \subset \Gamma^u_x$ of finite diameter (in the inner metric on Γ^u_x) such that

(a) $U_1 \subset U_2 \subset \cdots$ and $\cup_j U_j = \Gamma^u_x$;

(b) for any $R > 0$ let $U_j(R) \subset U_j$ be the subset of points whose distance to ∂U_j is greater than R; then

$$\lim_{j\to\infty} \nu^u_{U_j}(U_j(R)) = 1$$

where $\nu^u_{U_j}$ is the induced probability measure on the manifold U_j defined in the same way as in (2).

Definition [8]. The unstable manifold Γ^u_x is said to be uniformly distributed if for any continuous function $F(x)$ on M we have

$$\lim_{j\to\infty} \int_{U_j} F\, d\nu^u_{U_j} = \int_M F\, d\mu$$

for any sequence $U_j \subset \Gamma^u_x$ specified above.

Sinai proved [8] that for geodesic flows on compact manifolds of negative sectional curvature the horocycles (unstable manifolds) are uniformly distributed. The extension of Theorem 2 to multidimensional Anosov (or geodesic) flows is likely to be true but not available yet, because the results of [4] which we used here are completed only for 3-D flows.

2. The rate of convergence established by Theorem 2 does not seem to be optimal. It is natural to assume that the square root can be removed in (5), see remarks in the

introduction to [4]. Then one gets an algebraic bound const$\cdot R^{-a}$, $a > 0$, which looks more like an optimal one.

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References

[1] Anosov D.V. and Sinai Ya.G., Some smooth ergodic systems, *Russ. Math. Surveys* **22** (1967), 103–167.

[2] Bowen R., Symbolic dynamics for hyperbolic flows, *Amer. J. Math.*, **95** (1973), 429–459.

[3] Bowen R. and Ruelle D., The ergodic theory of Axiom A flows, *Inventiones math.*, **29** (1975), 181–202.

[4] Chernov N.I., Markov approximations and decay of correlations for Anosov flows, manuscript (1995).

[5] Plante J., Anosov flows, *Amer. J. Math.*, **94** (1972) 729–754.

[6] Ratner M., Anosov flows with Gibbs measures are also Bernoullian, *Israel J. Math.* **17** (1974) 380–391.

[7] Sinai Ya.G., Gibbs measures in ergodic theory, *Russ. Math. Surveys*, **27**, (1972), 21–69.

[8] Sinai Ya.G., Geodesic flows on manifolds of negative curvature, Lectures at Summer School, Trieste, 1992. In: *Algorithms, Fractals and Dynamics*, Ed. by Y. Takahashi, Plenum, Tokyo, 1995, pp. 199–213.

[9] Zagier D., *Eisenstein Series and the Rieman Zeta-Function, Automorphic Forms, Representation Theory and Arithmetic*, Tata Institute of Fund. Res., Bombay, 1979, Springer-Verlag, Heidelberg, New York, 1981, pp. 275–301.