Relaxation time of the one-dimensional symmetric zero range process with constant rate

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Abstract

We prove that the one-dimensional symmetric zero range dynamics, starting either with a periodic configuration or with a stationary exponential mixing probability distribution, converges to equilibrium faster than $\log t/t^{-1/2}$.

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1 Introduction

In this paper we present a sharp upper bound for the relaxation time to equilibrium of the one-dimensional symmetric zero range process with constant rate, starting either from a periodic configuration or from a stationary exponentially mixing probability distribution. We prove that the distance between the law of the process at time t and the invariant measure with the corresponding average number of particles decreases faster than $\log t/\sqrt{t}$.

The zero range process was introduced by F. Spitzer in [15]. The one-dimensional symmetric zero rage process with constant rate can be informally described as follows. Particles are distributed on $\mathbf{Z}$. Any site of $\mathbf{Z}$ may be occupied by any finite number of particles. Associated to the sites there are independent exponential clocks with parameter one. Each time a clock rings on a site, one of the particles on this site jumps to a nearest neighbor site chosen with probability $1/2$. In case the site is empty nothing happens.

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Ergodic properties of the zero range processes were studied by Andjel in [1]. In particular, in this paper it was proven that for the one dimensional symmetric case with constant rate the set of extremal invariant measures is $\{\mu_\rho : \rho \in (0,1]\}$, which are product measures with geometric number of particles, with parameter ρ, at each site.

Upper bound for the rate of convergence of spin-flip systems were obtained by several authors (see Holley [10], for instance, for a review for the Ising model). However for conservative systems the situation has been less studied. As far as we know, the only available results are the following. For the case with infinitely many independent random walks, a paper by Hoffman and Rosenthal [9] shows that with suitable initial configuration the rate is bounded above $(1/\sqrt{t})^d$ for all $d \geq 1$. For the case of the symmetric simple exclusion in any dimension Cancrini and Galves [4] proved that the rate of convergence is bounded above by $(\log t/\sqrt{t})^d$. Ours is the first result of the rate of convergence to equilibrium of a zero range process.

We should also mention the recent papers [5], [2], [3] and [12], which consider the L^2-decay of correlations for different types of processes in equilibrium. In particular in [12] it is proven, for a class of zero range processes in equilibrium, that the correlation in mean square decays faster than $(\log t)^{d+3}/t^{d/2}$. It should be stressed that these are equilibrium results which do not imply anything about the convergence rate for a process starting away from equilibrium.

Our proof has three main ingredients. The first one is Kesten's remark ([15], [11]) about the correspondence between the zero range process and the simple exclusion process as seen from a tagged particle. The second ingredient is the observation that at each fixed time the law of the exclusion process as seen from a tagged particle coincides with the law of the ordinary process given that there is a particle at the origin, when the process start with a stationary distribution (cf. [7], [14] and [6]). The third ingredient is the upper bound on the rate of convergence to equilibrium of the symmetric simple exclusion process presented in [4].

This paper is organized as follows. In section 2 we define the zero range process and state the theorem. In section 3 we recall some basic facts about the simple exclusion process which are used in the proof. The proof of the theorem is given in section 4.

2 Definitions and statement of the theorem

Let $\mathbf{Y} = \mathbf{N}^{\mathbf{Z}}$ be the state space of the zero range process. If ξ is an element of $\mathbf{Y}$ and $x \neq y$ are elements of $\mathbf{Z}$, let us denote by $\xi^{x,y}$ the element of $\mathbf{Y}$ defined as follows

$$\xi^{x,y}(z) = \begin{cases} \xi(x) - 1, & \text{if } z = x, \\ \xi(y) + 1, & \text{if } z = y. \\ \xi(z), & \text{otherwise.} \end{cases}$$

The generator of the one-dimensional symmetric zero range process with constant rate is defined by

$$Lf(\xi) = \sum_{x\in\mathbf{Z}} \mathbf{1}_{\xi(x)\geq 1} \left[f(\xi^{x,x+1}) - f(\xi)\right]$$

$$+ \sum_{x\in\mathbf{Z}} \mathbf{1}_{\xi(x)\geq 1}, \left[f(\xi^{x,x-1}) - f(\xi)\right]$$

where $f : \mathbf{Y} \to \mathbf{R}$ is any function depending on a finite number of coordinates.

For any $\rho \in (0, 1]$, the product measure μ_ρ is defined by

$$\mu_\rho(\xi \in \mathbf{Y} : \xi(x_i) = k_i, i = 1, ..., n) = \rho^n (1-\rho)^{\Sigma k_i}$$

for all $n, k_1, ..., k_n, \in \mathbf{N}$, $x_1, ..., x_n \in \mathbf{Z}$.

Let μ be a stationary probability measure on $\mathbf{Y}$. Let us denote by φ the average number of particles on site 0, *i.e.* $\varphi = \sum_{k=1}^{\infty} k\mu(\xi(0) = k)$. It will be convenient to define an associate parameter $\rho \in (0, 1]$ by

$$\varphi = \frac{1-\rho}{\rho}.$$

Definition 2.1 *Let μ be a stationary probability measure on $\mathbf{Y}$ with $\varphi = \frac{1-\rho}{\rho}$. We shall say that μ is exponentially mixing, if there exists $C > 0$ and $\gamma > 0$, such that for all n, $k_0, ..., k_{n-1} \in \mathbf{N}$*

$$\left|\mu\left(\xi(i) = k_i, i = 0, ..., n-1\right) - \rho^n (1-\rho)^{\Sigma_{i=0}^{n-1} k_i}\right| \leq Ce^{-\gamma(n+\Sigma_{i=0}^{n-1} k_i)}. \quad (1)$$

We shall call γ the mixing coefficient of μ.

Definition 2.2 *We shall call a configuration $\xi \in \mathbf{Y}$ periodic, if there exists a positive integer m such that $\xi(x) = \xi(x + km)$ for any integer k. The period of ξ is then the smallest positive integer m for which that property holds.*

For any periodic configuration ξ of period m, we define the parameter $\rho = \rho(\xi)$ as follows

$$\frac{1-\rho}{\rho} = \frac{1}{m}\sum_{x=1}^{m} \xi(x).$$

For each $n \geq 1$ and $k_1, ..., k_n$ we define

$$\mathbf{W}_{k_1,...,k_n} = \{\xi : \xi(i) = k_i, i = 1, ..., n\}.$$

and $N = n + \sum_{i=1}^{n} k_i$. **Theorem** *For any periodic configuration ξ and any stationary μ satisfying to the mixing condition (1), for all t large enough and all n and for all $k_1, k_2, ..., k_n \in \mathbf{N}$ we have*

$$\left|P\left\{\xi_t^* \in \mathbf{W}_{k_1,...,k_n}\right\} - \rho^n (1-\rho)^{k_1+\dots+k_n}\right| \leq C\, N \frac{\log t}{\sqrt{t}}, \quad (2)$$

where ξ_t^* means either the configuration at time t starting from ξ or from the initial measure μ. The constant C depends either on the period, or on the mixing constants, and ρ is the parameter corresponding either to the periodic configuration, or to the initial distribution.

3 Warming up

Let us first recall some basic facts about the simple exclusion process which will be needed in the proof. For some basic properties of the simple exclusion process we refer the reader to Liggett's chapter VIII [13]. Let

$$\mathbf{X} = \{0,1\}^{\mathbf{Z}}.$$

In what follows we shall denote by $(\eta_t^*)_t$ the usual simple symmetrical exclusion process on $\mathbf{X}$ starting either from a fixed configuration or from an initial measure, indicated by the upper index. The generator of the one-dimensional symmetric simple exclusion process is defined by

$$Gf(\eta) = \sum_{x\in\mathbf{Z}} \frac{1}{2}\left[f(\eta^{x,x+1}) - f(\eta)\right]$$

where $\eta^{x,y}$ is the element of $\mathbf{X}$ defined as follows

$$\eta^{x,y}(z) = \begin{cases} \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y. \\ \eta(z), & \text{otherwise.} \end{cases}$$

and $f : \mathbf{X} \to \mathbf{R}$ is any function depending on a finite number of coordinates.

We need also to introduce the dual process. It is a pure jump Markov process take values in $\mathcal{Z}$, the set of all finite subsets of $\mathbf{Z}$. Its generator is defined by

$$\tilde{G}f(A) = \sum_{x\in\mathbf{Z}} \frac{1}{2}\left[f(A^{x,x+1}) - f(A)\right]$$

where for any $A \in \mathcal{Z}$, $A^{x,y}$ is the element of $\mathcal{Z}$ defined as

$$A^{x,y} = A\triangle\{x,y\},$$

where $\triangle$ stands for the symmetric difference and $f : \mathcal{Z} \to \mathbf{R}$ is any bounded function. Let us denote by Z_t^A as the dual process at time t starting from set A. By a notational abuse we shall write Z_t^a instead of $Z_t^{\{a\}}$. We recall that, using Harris graphical construction, it is possible to construct a coupled family $\{(Z_t^x)_t, x \in \mathbf{Z}\}$, in such a way that, for any finite A, the following *additivity* property holds

$$Z_t^A = \bigcup_{a\in A} Z_t^a.$$

The symmetric simple exclusion process and its dual are related by the formula

$$\mathbf{P}\{\eta_t^\zeta(a) = 1, a \in A\} = \mathbf{P}\{\zeta(Z_t^a) = 1, a \in A\}. \tag{3}$$

For more details on duality and graphical representation we refer the reader to Harris seminal paper [8].

A key ingredient of our proof is the following theorem [4]

Theorem(Cancrini-Galves (1995)) *Let $\eta \in \mathbf{X}$ be any periodic configuration and let ν be any stationary distribution satisfying the mixing condition*

$$\nu(\eta \in \mathbf{X} : \eta(0) = \eta(x+1) = 1) - \rho^2 \leq Ce^{-\bar{\gamma}x}, \tag{4}$$

for any $x \geq 0$. Then for all t large enough and all $n \in \mathbf{N}$ we have

$$|P\{\eta_t^* \in \mathbf{\Lambda_n}\} - \rho^n| \leq C\, n \frac{\log t}{\sqrt{t}}, \tag{5}$$

where $\Lambda_n = \{\eta : \eta(x) = 1, x = 1, ..., n\}$ and η_t^* denotes the one-dimensional symmetric simple exclusion process at time t, starting either from η or from the initial measure ν. The constant C depends either on the period, or on the mixing constants, and ρ is the density of the initial distribution.

Actually in [4] the result is stated and proved for any dimension and the mixing condition is stated in a more restrictive way. However in the one-dimensional case it is clear that the simplified mixing condition (10) is sufficient.

Let us now recall the relation between the one-dimensional nearest-neighbor simple exclusion and the zero range process with constant rate. Let

$$\hat{\mathbf{X}} = \{\eta \in \mathbf{X} : \eta(0) = 1, \sum_{x \geq 0} \eta(x) = \infty, \sum_{x \leq 0} \eta(x) = \infty\}.$$

and let $(\widehat{\eta_t^*})_t$ be the simple symmetrical exclusion process as seen from the tagged particle on $\hat{\mathbf{X}}$. If ν is translation invariant measure on $\mathbf{X}$, let us denote by $\hat{\nu}$ its Palm measure which is a probability measure on $\hat{\mathbf{X}}$, defined by $\hat{\nu}(.) = \nu(.|\eta(0) = 1)$.

Let us define a map from $\mathbf{Y}$ to $\hat{\mathbf{X}}$ in the following way. For any $\xi \in \mathbf{Y}$, let us define $X_0 = 0$ and for any integer $i \geq 1$

$$X_i = \sum_{k=0}^{i-1} \xi(k) + i \text{ and } X_{-i} = -\sum_{k=1}^{i} \xi(k) - i.$$

Let us define the configuration $\hat{\eta} \in \hat{\mathbf{X}}$ corresponding to ξ as follows

$$\hat{\eta}(x) = \begin{cases} 1, & \text{if } x = X_i \text{ for some } i \in \mathbf{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

Given a stationary probability measure μ on $\mathbf{Y}$, let us denote by $\hat{\nu}$ the image of μ by this map. This is a notational abuse, as $\hat{\nu}$ is not define as a Palm measure. Actually, the additional hypothesis that μ has an average number of particles at site 0 equal to $\varphi = \frac{1-\rho}{\rho}$, assure the existence of a probability measure ν on $\mathbf{X}$ with density ρ and such that the above defined $\hat{\nu}$ is its Palm measure.

Given a one dimensional symmetric zero range process with constant rate $(\xi_t)_t$, let $(\hat{\eta}_t)_t$ be its image by this map. The process $(\hat{\eta}_t)_t$ is the symmetric simple exclusion process as seen from a tagged particle (cf. [6]).

For any stationary measure ν on $\mathbf{X}$ and any cylinder C, we have

$$\mathbf{P}\left\{\eta_t^\nu \in C | \eta_t^\nu(0)=1\right\} = \mathbf{P}\left\{\widehat{\eta_t^\nu} \in C\right\}, \tag{6}$$

(cf [7], [14] and [6]).

4 Proof of the theorem

Let us start with the proof of the mixing case. We first prove the result when the mixing coefficient γ is larger than $\log 2$.

Lemma 4.1 *Let μ be a stationary measure satisfying to the mixing condition (1) with parameter ρ and mixing coefficient $\gamma > \log 2$. Then for all t large enough, all n and for all $k_1, k_2, ..., k_n \in \mathbf{N}$ we have*

$$\left| P\left\{\xi_t^\mu \in \mathbf{W}_{k_1,\dots,k_n}\right\} - \rho^n(1-\rho)^{k_1+\dots+k_n} \right| \leq C\ N\frac{\log t}{\sqrt{t}}, \tag{7}$$

where ξ_t^μ means the configuration at time t starting from the initial measure μ. The constant c depends on the mixing constants.

Proof We first remark that the mixing condition (1) for the zero range process implies a good mixing condition for the associated simple exclusion process.

Let $\hat{\nu}$ be the probability measure on $\hat{\mathbf{X}}$ corresponding to the measure μ on $\mathbf{Y}$.

By definition

$$\hat{\nu}(\hat{\eta} \in \hat{\mathbf{X}} : \hat{\eta}(n+1) = 1) =$$
$$= \mu(\xi \in \mathbf{Y} : \xi(0) = n) + \sum_{k=1}^{n} \mu\left(\bigcup_{1\leq j_1 < \dots < j_k \leq n} A_{j_1,\dots,j_k}\right), \tag{8}$$

where

$$A_{j_1,\dots,j_k} = \{\xi \in \mathbf{Y} : \xi(0) = j_1 - 1, \xi(1) = j_2 - j_1 - 1, \dots$$
$$\dots, \xi(k-1) = j_k - j_{k-1} - 1, \xi(k) = n - j_k\}.$$

By the mixing condition (1), the right hand side of (8) is bounded above by

$$\rho\left[\sum_{k=0}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}\right] + \sum_{k=0}^{n}\binom{n}{k}Ce^{-\gamma n}.$$

A straightforward computation gives the upper bound.

$$\hat{\nu}(\hat{\eta} \in \hat{\mathbf{X}} : \hat{\eta}(n+1) = 1) \leq \rho + Ce^{-(\bar{\gamma}n)}, \tag{9}$$

where $\bar{\gamma} = \gamma - \log 2 > 0$. By definition

$$\hat{\nu}(\hat{\eta} \in \hat{\mathbf{X}} : \hat{\eta}(n+1) = 1) = \frac{\nu(\eta \in \mathbf{X} : \eta(0) = \eta(n+1) = 1)}{\nu(\eta \in \mathbf{X} : \eta(0) = 1)}.$$

By hypothesis $\nu(\eta \in \mathbf{X} : \eta(0) = 1) = \rho$. Therefore, we obtain the mixing condition

$$\nu(\eta \in \mathbf{X} : \eta(0) = \eta(n+1) = 1) - \rho^2 \leq Ce^{-\bar{\gamma}n}. \tag{10}$$

As recalled in section 3, condition (10) assures that the rate of convergence of the exclusion process is bounded above by $\log t/\sqrt{t}$ [4].

To apply this result for the zero range we first remark that

$$\mathbf{P}\left\{\xi_t^\mu \in \mathbf{W}_{k_1,\ldots,k_n}\right\} = \mathbf{P}\left\{\widehat{\eta_t^\nu} \in C_{x_0,\ldots,x_n}\right\}$$

where $x_0 = 0$,$x_1 = k_1 + 1$,...,$x_n = \sum_{i=1}^n k_i + n$,

$$C_{x_0,\ldots,x_n} = \{\eta \in \mathbf{X} : \eta(x_0) = \ldots = \eta(x_n) = 1, \eta(x) = 0, x \in I_{x_0,\ldots,x_n}\},$$

and $I_{x_0,\ldots,x_n} = \{y \in \mathbf{Z} : x_0 < y < x_n, y \neq x_i, i = 1, \ldots, n-1\}$.

By (6) we have

$$\mathbf{P}\left\{\widehat{\eta_t^\nu} \in C_{x_0,\ldots,x_n}\right\} = \mathbf{P}\left\{\widehat{\eta}_t^\nu \in C_{x_0,\ldots,x_n} | \eta_t^\nu(0) = 1\right\}.$$

Now the result follows by a direct computation. ■

The next lemma shows that the restriction on the mixing coefficient γ can be eliminated.

Lemma 4.2 *For any $0 < \gamma_0 \leq \log 2 < \gamma_1$, there exists $t_0 > 0$, such that for any stationary measure μ on $\mathbf{Y}$, satisfying the mixing condition (1) with mixing coefficient γ_0, the law of $\xi_{t_0}^\mu$ satisfies the mixing condition with mixing coefficient γ_1, where ξ_t^μ stands for the one-dimensional symmetric zero range process with constant rate starting from the initial measure μ.*

Proof To avoid tedious details we shall only write the proof for a cylinder of size two. The general case is done in exactly the same way.

Using again the map from $\mathbf{Y}$ to $\hat{\mathbf{X}}$, for any $k_1, k_2 \in \mathbf{N}$, we have

$$\mathbf{P}\{\xi_t^\mu \in \mathbf{W}_{k_1,k_2}\} = \mathbf{P}\left\{\widehat{\eta_t^\nu} \in C_{x_0,x_1,x_2}\right\} \tag{11}$$

By (6)

$$\mathbf{P}\left\{\widehat{\eta_t^\nu} \in C_{x_0,x_1,x_2}\right\} = \frac{\mathbf{P}\left\{\eta_t^\nu \in C_{x_0,x_1,x_2}\right\}}{\mathbf{P}\left\{\eta_t^\nu(0) = 1\right\}}$$

By hypothesis, $\mathbf{P}\left\{\eta_t^\nu(0) = 1\right\} = \rho$. By duality

$$\mathbf{P}\left\{\eta_t^\nu \in C_{x_0,x_1,x_2}\right\}$$

$$= \sum_{u,v,w} \mathbf{P}(Z_t^0 = u, Z_t^{k_0+1} = v, Z_t^{k_0+k_1+2} = w)\mathbf{P}\left\{\eta^\nu \in C_{u,v,w}\right\}, \tag{12}$$

where (Z_t^x) is the dual process starting at x. The right hand side of (12) can be bonded above by

$$\leq \sum_{u<v<w, |w-u|>d} \mathbf{P}(Z_t^0 = u, Z_t^{k_0+1} = v, Z_t^{k_0+k_1+2} = w)\mathbf{P}\{\eta^\nu \in C_{u,v,w}\}$$

$$+\mathbf{P}(|Z_t^0 - Z_t^{k_0+k_1+2}| < d). \tag{13}$$

Now we have all the elements to conclude the proof. We want an upper bound for

$$\left|\mathbf{P}\{\xi_t^\mu \in \mathbf{W}_{k_1,k_2}\} - \rho^2(1-\rho)^{k_0+k_1}\right|. \tag{14}$$

Using (11), (12) and (13), (14) is bounded above by

$$\sum_{u<v<w, |w-u|>d} \mathbf{P}(Z_t^0 = u, Z_t^{k_0+1} = v, Z_t^{k_0+k_1+2} = w)\times$$

$$\left|\frac{1}{\rho}\mathbf{P}\{\eta^\nu \in C_{u,v,w}\} - \rho^2(1-\rho)^{w-u-2}\right|$$

$$+\frac{1}{\rho}\mathbf{P}(|Z_t^0 - Z_t^{k_0+k_1+2}| < d). \tag{15}$$

Using again the usual map and the mixing hypothesis on μ the first term of the right hand side of (15) is bounded above by

$$Ce^{-\gamma_0(d-1)}.$$

Using Liggett's correlation inequality we obtain

$$\mathbf{P}(|X_t^0 - X_t^{k_0+k_1+2}| < d) \leq c\frac{d}{\sqrt{t}}.$$

So that

$$\left|\mu_t(\xi(0) = k_0, \xi(1) = k_1) - \rho^2(1-\rho)^{k_0+k_1}\right| \leq Ce^{-\gamma_0(d-1)} + c\frac{d}{\rho\sqrt{t}}.$$

To concludes the proof it is enough to take $d > k_0 + k_1 + 2$ and t_0 such that the second member of this inequality is bounded by

$$C'e^{\gamma_1(1+k_0+k_1)}.$$

■

This concludes the proof of the theorem in the mixing case.

Now lets turn to the proof of the periodic case.

If we start the process with a periodic configuration ξ in $\mathbf{Y}$ with period say m then the corresponding configuration in $\hat{\mathbf{X}}$ will be also periodic with period $m' = m + \xi(0) + ... + \xi(m-1)$. Then again by Cancrini-Galves's theorem [4] the symmetric simple exclusion process starting with that periodic configuration (with one particle at the origin) will converge to the equilibrium with speed bounded by $C \log t/\sqrt{t}$.

As at time t there will be a particle at the origin with a probability

$$\rho - C\frac{\log t}{t} \leq \mathbf{P}(\eta_t^{\eta_0}(0) = 1) \leq \rho + C\frac{\log t}{t},$$

so that it will also reach the equilibrium for the exclusion process as seen from the tagged particle with speed $C' \log t/\sqrt{t}$, where $C' > C$ only depends on C and ρ.

■

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