Entropy for Translation-Invariant Random-Cluster Measures

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Abstract. We study translation-invariant random-cluster measures with techniques from large deviation theory and convex analysis. In particular, we prove a large deviation principle with rate function given by a specific entropy, and a DLR variational principle that characterizes translation-invariant random-cluster measures as the solutions of the variational equation for free energy. Consequences of these theorems include results about edge and cluster densities of translation-invariant random-cluster measures.

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1. Introduction. The random-cluster model was introduced around 1970 by Fortuin and Kasteleyn as a family of models that contains the Bernoulli bond percolation model, the Ising model, and the q-state Potts models as special cases. Since then it has developed into a standard tool for studying the Ising and Potts models. More recently, properties of the random-cluster measures themselves have become an object of study. The starting point for our paper is Grimmett's [Gr3] systematic development of elements of a thermodynamic formalism for this model, beginning with a definition of infinite-volume random-cluster measures in terms of a DLR equation (Dobrushin-Lanford-Ruelle). These ideas were also touched on but not pursued in the papers [ACCN, BC]. As for Gibbsian systems, the validity of a large deviation principle and a variational characterization of infinite-volume measures are natural next questions. These we answer in the present paper, and derive some consequences.

Here is a brief overview of the DLR approach of [Gr3]. We study random-cluster measures on the graph formed by putting edges between nearest-neighbor vertices of the d-dimensional hypercubic lattice $\mathbb{Z}^d$. The edge that connects two vertices x and y that are ℓ^1-distance one apart is denoted by $\langle x, y\rangle$. $\mathbb{E}$ stands for the set of all edges, while $\mathbb{E}_\Lambda$ denotes the set of edges $\langle x, y\rangle$ such that $x, y \in \Lambda$ for a subset $\Lambda \subseteq \mathbb{Z}^d$. $\Omega = \{0,1\}^{\mathbb{E}}$ is the space of edge configurations $\omega = (\omega(e))_{e\in\mathbb{E}}$. The value $\omega(e) = 1$ indicates that edge e is present, $\omega(e) = 0$ that edge e is absent. The set of edges of a configuration ω is $\eta(\omega) = \{e \in \mathbb{E} : \omega(e) = 1\}$.

Fix two parameters $p \in (0, 1)$ and $q > 0$. For $\xi \in \Omega$ and a finite rectangle $\Lambda \subseteq \mathbb{Z}^d$, Ω^ξ_Λ is the subset of configurations $\omega \in \Omega$ that satisfy $\omega(e) = \xi(e)$ for all edges $e \notin \mathbb{E}_\Lambda$. Let $k(\omega, \Lambda)$ denote the number of connected components in the graph $(\mathbb{Z}^d, \eta(\omega))$ that intersect Λ. The *finite-volume random-cluster measure* on $\mathbb{E}_\Lambda$ with boundary condition ξ is the probability measure $\phi^\xi_{\Lambda,p,q}$ supported on Ω^ξ_Λ and defined by

$$\phi^\xi_{\Lambda,p,q}(\omega) = \frac{1}{Z^\xi_{\Lambda,p,q}}\left\{\prod_{e\in\mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\}q^{k(\omega,\Lambda)}, \qquad \omega \in \Omega^\xi_\Lambda, \tag{1.1}$$

with the normalizing factor

$$Z^\xi_{\Lambda,p,q} = \sum_{\omega\in\Omega^\xi_\Lambda}\left\{\prod_{e\in\mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\}q^{k(\omega,\Lambda)}. \tag{1.2}$$

The measures (1.1) satisfy the consistency condition

$$\int \phi^\zeta_{\Delta,p,q}(\omega)\, \phi^\xi_{\Lambda,p,q}(d\zeta) = \phi^\xi_{\Lambda,p,q}(\omega)$$

for $\omega \in \Omega^\xi_\Lambda$, whenever $\Delta \subseteq \Lambda$. Thus the family of probability measures $\{\phi^\xi_{\Lambda,p,q}\}$ can be regarded as a *specification* in the statistical mechanical sense [Ge, p. 16]. An *infinite-volume* random-cluster measure is then any probability measure ψ on Ω that satisfies

$$\psi(A) = \int \phi^\xi_{\Lambda,p,q}(A)\, \psi(d\xi) \tag{1.3}$$

for all measurable sets $A \subseteq \Omega$. The class of all such probability measures is denoted by $\mathcal{R}_{p,q}$. Equation (1.3) is the DLR definition of infinite-volume measures.

These are some of the basic known results that the reader can find in [Gr3]: $\mathcal{R}_{p,q}$ is a nonempty convex set of probability measures, and its extreme boundary consists of limits of the finite-volume measures (1.1) as $\Lambda \to \mathbb{Z}^d$. As for all statistical mechanical lattice models, the interesting question is the possibility of *phase transition.* This is the situation where $\mathcal{R}_{p,q}$ contains more than one measure. It follows from general convexity arguments that for a fixed q there are at most countably many values of p for which this nonuniqueness of the infinite-volume measure may occur. In dimension $d = 2$ one can use duality to show that phase transition cannot happen for any other value of p except $p = \sqrt{q}/(1+\sqrt{q}\,)$. Phase transition is known to happen for large enough q in all dimensions higher than one [LM].

Under the natural partial ordering of configurations ($\omega \leq \omega'$ if $\omega(e) \leq \omega'(e)$ for all $e \in \mathbb{E}$) there are two extreme boundary conditions $\xi \equiv 0$ and $\xi \equiv 1$, corresponding to having all edges absent or all edges present outside $\mathbb{E}_\Lambda$. The measure $\phi^0_{\Lambda,p,q}$ is the *free* random-cluster measure, while $\phi^1_{\Lambda,p,q}$ is the *wired.* The terminology comes from the connections imposed on the boundary vertices of Λ. If $q > 1$ one can use the FKG inequality to show that the finite-volume measures $\phi^0_{\Lambda,p,q}$ and $\phi^1_{\Lambda,p,q}$ converge to the elements $\phi^0_{p,q}$ and $\phi^1_{p,q}$ of $\mathcal{R}_{p,q}$, as $\Lambda \to \mathbb{Z}^d$. These measures are extreme infinite-volume random-cluster measures in the sense that the inequalities $\phi^0_{p,q} \leq \psi \leq \phi^1_{p,q}$ hold for all $\psi \in \mathcal{R}_{p,q}$. (Inequalities between measures are interpreted in the standard sense of stochastic dominance, see for example Sect. II.2 in [Lg].) Consequently the occurrence of a phase-transition is equivalent to $\phi^0_{p,q} \neq \phi^1_{p,q}$, and it can be characterized in terms of the edge densities and percolation probabilities of $\phi^0_{p,q}$ and $\phi^1_{p,q}$.

In this paper we study translation-invariant random-cluster measures from the point of view of the convex duality of free energy and entropy. After some preliminary definitions in Section 2, Section 3 introduces the notions of entropy and relative entropy for these models, and shows how entropy appears in a variational characterization of free energy and also as the rate function of the large deviation principle of the empirical measure under random-cluster measures. This picture is then completed by the DLR variational principle that characterizes translation-invariant random-cluster measures as precisely those measures that solve the variational equation for free energy.

In that first part of the paper, the edge density and cluster density of a measure are identified as the variables dual to the parameters (p, q) of the random-cluster model. In Section 4 we study the properties of the free energy and its convex dual, which can be regarded as a combinatorially defined entropy. This enables us to describe the edge and cluster density combinations that are possible for translation-invariant random cluster measures. The FKG inequality is not needed in Sections 3 and 4, so the results are valid for all values of q.

In Section 5 we extend some earlier characterizations of the parameter values that have a unique random-cluster measure, in terms of the differential properties of the free energy, and prove some results about the dependence of edge and cluster density on the parameters. The FKG inequality becomes an important tool so most of the results are for $q > 1$ only. Section 6 strengthens one result for the special case of dimension two.

In Section 7 we look at the case where q is an integer ≥ 2. In this case free and wired random-cluster measures can be viewed as expectations, under Potts measures, of certain random, nonstationary product measures on the edges. This brings us into contact with two recent themes of large deviation theory: Nonstationary processes and mixtures of processes. From this perspective we derive an alternative formula for the entropy that appeared in Section 3. As an easy corollary we see that the translation-invariant random-cluster measures are precisely the expectations of the above-mentioned nonstationary product measures under translation-invariant Potts measures.

The original work of Fortuin and Kasteleyn was published in references [Fo1, Fo2, FK]. Applications of random-cluster measures to the study of Potts models and other interacting systems can be found in [ACCN, BC, Hä2]. For general overviews of random-cluster measures and for more references to past work we refer the reader to [Gr1, Gr2, Gr3].

2. Basic definitions. Here are some further notational conventions. In general we follow the notation of Grimmett's paper [Gr3], except for replacing (p, q) by new parameters (π, α), see (2.1) below.

Throughout, the dimension d satisfies $d \geq 2$. For any subset $\Lambda \subseteq \mathbb{Z}^d$, $|\Lambda|$ and $|\mathbb{E}_\Lambda|$ denote the number of elements in Λ and $\mathbb{E}_\Lambda$, respectively. The boundary $\partial\Lambda$ is the set of vertices in Λ that are adjacent to the complement Λ^c. A limit as $\Lambda \to \mathbb{Z}^d$ means that Λ increases along a sequence of nested rectangles to eventually contain any finite subset of $\mathbb{Z}^d$. Vertices x of $\mathbb{Z}^d$ act on edge configurations $\omega \in \Omega$ by translations τ_x:

$$\tau_x \omega(\langle u, v\rangle) = \omega(\langle u + x, v + x\rangle) \text{ for all edges } \langle u, v\rangle \in \mathbb{E}.$$

For finite $\Lambda \subseteq \mathbb{Z}^d$, $\Omega_\Lambda = \{0, 1\}^{\mathbb{E}_\Lambda}$ is the space of edge configurations among the nearest-neighbor vertices in Λ. Sometimes we write $\mathbb{E}(\Lambda)$ for $\mathbb{E}_\Lambda$ to avoid multiple subscripts. If $\zeta \in \Omega_\Lambda$ for some finite set Λ, then $[\zeta]$ stands for the cylinder subset of Ω generated by ζ, in other words,

$$[\zeta] = \{\omega \in \Omega : \omega(e) = \zeta(e) \text{ for } e \in \mathbb{E}_\Lambda\}.$$

If ω is a configuration defined on a larger set than $\mathbb{E}_\Lambda$, $\omega(\mathbb{E}_\Lambda)$ denotes its restriction to an element of Ω_Λ. For $\omega, \xi \in \Omega$ and $\Lambda \subseteq \mathbb{Z}^d$, ω^ξ_Λ denotes the configuration that satisfies

$$\omega^\xi_\Lambda(e) = \begin{cases} \omega(e), & e \in \mathbb{E}_\Lambda \\ \xi(e), & e \notin \mathbb{E}_\Lambda. \end{cases}$$

When the rectangle Λ is understood from the context, we let $\omega^\xi = \omega^\xi_\Lambda$. 0 and 1 denote the configurations $\omega \equiv 0$ and $\omega \equiv 1$, respectively. So ω^0 has no edges outside $\mathbb{E}_\Lambda$, and ω^1 has all potential edges outside $\mathbb{E}_\Lambda$ present. For any set A of edges, $\mathcal{F}_A$ is the σ-field in Ω generated by the coordinate variables $(\omega(e))_{e \in A}$.

The space of probability measures on Ω is denoted by $\mathcal{M}$. It is compact under the usual weak topology, defined by declaring that $\mu_n \to \mu$ if $\mu_n([\zeta]) \to \mu([\zeta])$ for all

$\zeta \in \Omega_\Lambda$ and all finite $\Lambda \subseteq \mathbb{Z}^d$. (See [Bi] for a standard treatment of this topology.) A probability measure μ is translation-invariant if $\mu \circ \tau_x = \mu$ for all x, and the set of such probability measures is denoted by $\mathcal{M}^\tau$, a compact subspace of $\mathcal{M}$. The integral of a function f against a measure μ is denoted by $\mu(f)$. We shall write ω also for the (identity) random variable defined on Ω, so the integral $\mu(f)$ will also appear in the form $\mu(f(\omega))$. The indicator function of an event A appears as I_A.

For the random-cluster measures we find it useful to replace the standard parameters (p, q) with the parameters (π, α) defined by

$$\pi = \log \frac{p}{1-p} \qquad \text{and} \qquad \alpha = d^{-1} \log q, \tag{2.1}$$

so that both π and α range over $(-\infty, +\infty)$. Bernoulli percolation corresponds to the case $\alpha = 0$, and $(\pi, \alpha) = (0, 0)$ is Bernoulli percolation with edge density $1/2$. Except for Section 7, everything will be expressed in terms of (π, α) rather than (p, q). The finite-volume random-cluster measures of (1.1) become

$$\phi^\xi_{\Lambda,\pi,\alpha}(\omega) = \frac{1}{Y^\xi_{\Lambda,\pi,\alpha}} \exp\bigl\{\pi|\eta(\omega) \cap \mathbb{E}_\Lambda| + d\alpha k(\omega, \Lambda)\bigr\}, \qquad \omega \in \Omega^\xi_\Lambda, \tag{2.2}$$

with the normalizing factor

$$Y^\xi_{\Lambda,\pi,\alpha} = \sum_{\omega \in \Omega^\xi_\Lambda} \exp\bigl\{\pi|\eta(\omega) \cap \mathbb{E}_\Lambda| + d\alpha k(\omega, \Lambda)\bigr\} = (1-p)^{-|\mathbb{E}_\Lambda|} Z^\xi_{\Lambda,\pi,\alpha}. \tag{2.3}$$

The class $\mathcal{R}_{\pi,\alpha}$ of random-cluster measures is the set of probability measures $\psi \in \mathcal{M}$ that satisfy

$$\psi(A) = \int \phi^\xi_{\Lambda,\pi,\alpha}(A) \psi(d\xi) \tag{2.4}$$

for all measurable sets $A \subseteq \Omega$ and all finite rectangles $\Lambda \subseteq \mathbb{Z}^d$. $\mathcal{R}^\tau_{\pi,\alpha} = \mathcal{R}_{\pi,\alpha} \cap \mathcal{M}^\tau$ is the subclass of translation-invariant random-cluster measures. By Thm. 3.2(a) in [Gr3], $\mathcal{R}^\tau_{\pi,\alpha}$ is a nonempty convex set for each value $(\pi, \alpha) \in \mathbb{R}^2$. It follows from Remark 3.12 below that $\mathcal{R}^\tau_{\pi,\alpha}$ is also compact.

The *free energy* $f(\pi, \alpha)$ is a finite, convex function of $(\pi, \alpha) \in \mathbb{R}^2$, defined by

$$f(\pi, \alpha) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\mathbb{E}_\Lambda|} \log Y^\xi_{\Lambda,\pi,\alpha}. \tag{2.5}$$

For the existence of the limit and its independence of ξ, see Thm. 4.1 in [Gr3].

For $\omega \in \Omega$, we define the following two bounded, continuous functions:

$$u(\omega) = d^{-1} \sum_{i=1}^d \omega(\hat{e}_i)$$

where $\hat{e}_i$ is the "ith basic unit vector", or the edge between the origin and the site $(0, \dots, 0, 1, 0, \dots, 0)$ with the unique 1 in the ith slot. And

$$\kappa(\omega) = \frac{1}{|C(\omega)|}$$

where $C(\omega)$ is the connected cluster of vertices containing the origin. For any $\mu \in \mathcal{M}^\tau$,

$$\lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\mathbb{E}_\Lambda|}\mu\bigl(|\eta(\omega)\cap\mathbb{E}_\Lambda|\bigr) = \mu(u) \tag{2.6}$$

and

$$\lim_{\Lambda\to\mathbb{Z}^d} \frac{d}{|\mathbb{E}_\Lambda|}\mu\bigl(k(\omega,\Lambda)\bigr) = \mu(\kappa). \tag{2.7}$$

The quantity $\mu(u)$ is the edge density of the measure μ, and $\mu(\kappa)$ the cluster density. A general translation-invariant measure may have d distinct edge densities in different coordinate directions, hence the need for the d-fold average in the definition of $u(\omega)$. It is clear from the form of the exponential factors in (2.2) that, in the thermodynamic formalism of the model, the pair $(\mu(u),\mu(\kappa))$ will be the conjugate variable of (π,α), in the sense of convex duality.

3. Large deviations and the variational principle. We start with some familiar definitions of entropy. Let μ be a translation-invariant probability measure on the configuration space Ω. For finite rectangles Λ, the *entropy* $H_\Lambda(\mu)$ of μ on the finite space Ω_Λ is defined by

$$H_\Lambda(\mu) = -\sum_{\omega\in\Omega_\Lambda}\mu([\omega])\log\mu([\omega]).$$

$H_\Lambda(\mu)$ is a number between 0 and $|\mathbb{E}_\Lambda|\log 2$. The infinite volume limit is given by

$$h(\mu) = \lim_{\Lambda\to\mathbb{Z}^d}\frac{1}{|\mathbb{E}_\Lambda|}H_\Lambda(\mu). \tag{3.1}$$

We shall argue later that this limit exists and the function $h(\mu)$ is upper semicontinuous.

The *relative entropy* or *Kullback-Leibler distance* $H(Q|P)$ of two probability measures P and Q defined on a common measurable space is defined by

$$H(Q|P) = \int \log\frac{dQ}{dP}\,dQ \tag{3.2}$$

if $Q \ll P$, and $H(Q|P)=\infty$ otherwise. As a function of the pair (Q,P), $H(Q|P)$ is convex and lower semicontinuous in the product of weak topologies of probability measures. $H(Q|P)\geq 0$, and $H(Q|P)=0$ if and only if $Q=P$. Working with relative entropy is made convenient by the variational formula

$$H(Q|P) = \sup_g\{Q(g)-\log P(e^g)\} \tag{3.3}$$

where the supremum is over all bounded, measurable functions g. This formula is originally due to Donsker and Varadhan [DV]. Often it is desired to restrict Q and

P to a sub-σ-field $\mathcal{A}$ of the original σ-field for the purpose of computing entropy. Then we write $H_{\mathcal{A}}(Q|P) = H(Q_{\mathcal{A}}|P_{\mathcal{A}})$ where now $Q_{\mathcal{A}}$ and $P_{\mathcal{A}}$ are the appropriate restrictions. Proofs of the basic properties of $H(Q|P)$ can be found in references [DeZ, DS, Va].

In particular, let $\mu, \nu \in \mathcal{M}^\tau$. For finite rectangles Λ we have the relative entropy $H_{\mathbb{E}_\Lambda}(\mu|\nu)$ of the restrictions $\mu_{\mathbb{E}_\Lambda}$ and $\nu_{\mathbb{E}_\Lambda}$ to the σ-field $\mathcal{F}_{\mathbb{E}_\Lambda}$. The *specific relative entropy* of μ and ν is defined by

$$h(\mu|\nu) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\mathbb{E}_\Lambda|} H_{\mathbb{E}_\Lambda}(\mu|\nu) \tag{3.4}$$

whenever this limit exists. For any choice $(\xi^{(\Lambda)})$ of boundary configurations, possibly depending on the rectangles Λ, we define an entropy $I_{\pi,\alpha}(\mu)$ relative to the random-cluster specification as the limit

$$I_{\pi,\alpha}(\mu) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\mathbb{E}_\Lambda|} H(\mu_\Lambda | \phi^{\xi^{(\Lambda)}}_{\Lambda,\pi,\alpha}), \tag{3.5}$$

provided this limit exists.

Theorem 1. *Let μ be a translation-invariant probability measure on Ω.*

(a) *The entropy $I_{\pi,\alpha}(\mu)$ defined by* (3.5) *exists and is independent of the choice of $(\xi^{(\Lambda)})$. It satisfies*

$$I_{\pi,\alpha}(\mu) = f(\pi, \alpha) - h(\mu) - \pi\mu(u) - \alpha\mu(\kappa). \tag{3.6}$$

$I_{\pi,\alpha}$ is a nonnegative, lower semicontinuous, affine function on $\mathcal{M}^\tau$.

(b) *For any translation-invariant random-cluster measure $\psi \in \mathcal{R}^\tau_{\pi,\alpha}$, the specific relative entropy $h(\mu|\psi)$ defined by* (3.4) *exists and $h(\mu|\psi) = I_{\pi,\alpha}(\mu)$. In particular, $h(\mu|\psi)$ does not depend upon the choice of the member ψ of $\mathcal{R}^\tau_{\pi,\alpha}$, and $h(\phi|\psi) = 0$ for $\phi, \psi \in \mathcal{R}^\tau_{\pi,\alpha}$.*

(c) *The entropy $h(\mu)$ and the free energy $f(\pi, \alpha)$ satisfy the variational formula*

$$f(\pi, \alpha) = \sup_{\mu \in \mathcal{M}^\tau} \{h(\mu) + \pi\mu(u) + \alpha\mu(\kappa)\}. \tag{3.7}$$

The entropy $I_{\pi,\alpha}(\mu)$ appears as the rate function in the large deviation bounds of the empirical measure $M_\Lambda(\omega)$, defined for finite rectangles $\Lambda \subseteq \mathbb{Z}^d$ by

$$M_\Lambda(\omega) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \delta_{\tau_x \omega}. \tag{3.8}$$

The integral of a function g on Ω against $M_\Lambda(\omega)$ is given by

$$M_\Lambda(\omega, g) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} g(\tau_x \omega).$$

Extend the function $I_{\pi,\alpha}(\mu)$ to all of $\mathcal{M}$ by declaring $I_{\pi,\alpha}(\mu) = +\infty$ for $\mu \in \mathcal{M} \setminus \mathcal{M}^\tau$. Let $\operatorname{int} A$ and $\operatorname{cl} A$ denote the weak interior and closure, respectively, of a subset $A \subseteq \mathcal{M}$.

Theorem 2. *For any measurable set $A \subseteq \mathcal{M}$,*

$$
\begin{aligned}
(3.9)\qquad -\inf_{\mu \in \mathrm{int}\, A} I_{\pi,\alpha}(\mu) &\le \liminf_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\mathbb{E}_\Lambda|} \log \left[\inf_{\xi \in \Omega} \phi^{\xi}_{\Lambda,\pi,\alpha}(M_\Lambda \in A) \right] \\
&\le \limsup_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\mathbb{E}_\Lambda|} \log \left[\sup_{\xi \in \Omega} \phi^{\xi}_{\Lambda,\pi,\alpha}(M_\Lambda \in A) \right] \\
&\le -\inf_{\mu \in \mathrm{cl}\, A} I_{\pi,\alpha}(\mu).
\end{aligned}
$$

Naturally, since the bounds are uniform in ξ, they are also valid for any infinite-volume measure $\psi \in \mathcal{R}^{\tau}_{\pi,\alpha}$. Next comes the Dobrushin-Lanford-Ruelle variational principle that characterizes membership in $\mathcal{R}^{\tau}_{\pi,\alpha}$ in terms of entropy.

Theorem 3. *Let $\psi \in \mathcal{M}^{\tau}$. The following statements are equivalent.*

(i) $\psi \in \mathcal{R}^{\tau}_{\pi,\alpha}$.
(ii) $I_{\pi,\alpha}(\psi) = 0$.
(iii) $h(\psi) + \pi\psi(u) + \alpha\psi(\kappa) = f(\pi,\alpha) = \sup_{\mu \in \mathcal{M}^{\tau}} \{h(\mu) + \pi\mu(u) + \alpha\mu(\kappa)\}$.

As mentioned in the introduction, when $\alpha > 0$ (equivalently, $q > 1$) the limiting measures

$$
(3.10)\qquad \phi^0_{\pi,\alpha} = \lim_{\Lambda \to \mathbb{Z}^d} \phi^0_{\Lambda,\pi,\alpha} \qquad \text{and} \qquad \phi^1_{\pi,\alpha} = \lim_{\Lambda \to \mathbb{Z}^d} \phi^1_{\Lambda,\pi,\alpha}
$$

exist and belong to $\mathcal{R}^{\tau}_{\pi,\alpha}$. (Proofs can be found in [BC, Gr3].) The measures $\phi^0_{\Lambda,\pi,\alpha}$ and $\phi^1_{\Lambda,\pi,\alpha}$ are invariant under permutations of the coordinate axes, and hence so are $\phi^0_{\pi,\alpha}$ and $\phi^1_{\pi,\alpha}$. Consequently for $b = 0, 1$, $\phi^b_{\pi,\alpha}(u) = \phi^b_{\pi,\alpha}(\omega(e))$, the *edge density* which is independent of $e \in \mathbb{E}$. The statement in Theorem 3(iii) can then be written

$$
(3.11)\qquad f(\pi,\alpha) = h(\phi^b_{\pi,\alpha}) + \pi\phi^b_{\pi,\alpha}(\omega(e)) + \alpha\phi^b_{\pi,\alpha}(\kappa), \qquad b = 0, 1.
$$

Remark 3.12. A continuity property of random-cluster measures comes as an easy consequence of the variational principle: Suppose $(\pi_n, \alpha_n) \to (\pi, \alpha)$ and $\psi_n \in \mathcal{R}^{\tau}_{\pi_n,\alpha_n}$. Then all limit points of $\{\psi_n\}$ lie in $\mathcal{R}^{\tau}_{\pi,\alpha}$. For suppose $\psi_n \to \psi$. Then $f(\pi_n, \alpha_n) \to f(\pi, \alpha)$ (f is continuous by virtue of finiteness and convexity), $\psi_n(u) \to \psi(u)$, and $\psi_n(\kappa) \to \psi(\kappa)$, and $h(\psi) \ge \limsup h(\psi_n)$ because h is upper semicontinuous. So the first equality of Thm. 3(iii) becomes in the limit

$$
h(\psi) + \pi\psi(u) + \alpha\psi(\kappa) \ge f(\pi,\alpha),
$$

which implies that $\psi \in \mathcal{R}^{\tau}_{\pi,\alpha}$ and $h(\psi_n) \to h(\psi)$.

Proof of Theorem 1. To see why the limit in (3.1) exists, write the variational formula (3.3) in the form

$$
(3.13)\qquad H_\Lambda(\mu) = \inf_g \{\log \sum_{\omega \in \Omega_\Lambda} e^{g(\omega)} - \mu(g)\}
$$

where the infimum is over all functions g on the finite space Ω_Λ. Proof of (3.13) utilizes Jensen's inequality and the fact that the infimum is achieved at $g(\omega) = \log \mu(\omega)$ if $\mu(\omega) > 0$ for all ω, otherwise one approximates $\log \mu(\omega)$ with finite functions. See Lemma 3.2.13 in [DS]. Equation (3.13) implies that $H_\Lambda(\mu)$ is subadditive in Λ, in other words, if $\Lambda_1, \ldots, \Lambda_n$ are disjoint rectangles contained in the rectangle Λ, then

$$H_\Lambda(\mu) \leq H_{\Lambda_1}(\mu) + \cdots + H_{\Lambda_n}(\mu).$$

It follows that the limit in (3.1) exists and

$$h(\mu) = \inf_\Lambda \frac{1}{|\mathbb{E}_\Lambda|} H_\Lambda(\mu). \tag{3.14}$$

The existence of the limit $I_{\pi,\alpha}(\mu)$ in (3.5), its independence of the $(\xi^{(\Lambda)})$, and equation (3.6) are all consequences of the equality

$$H(\mu_\Lambda | \phi^\xi_{\Lambda,\pi,\alpha}) = \log Y^\xi_{\Lambda,\pi,\alpha} - H_\Lambda(\mu) - \pi\mu(|\eta(\omega^\xi) \cap \mathbb{E}_\Lambda|) - d\alpha\mu(k(\omega^\xi, \Lambda)),$$

the limits in (2.5)–(2.7) and (3.1), and the observation that

$$0 \leq k(\omega^0, \Lambda) - k(\omega^\xi, \Lambda) \leq |\partial\Lambda| \tag{3.15}$$

for $\omega \in \Omega_\Lambda$ and $\xi \in \Omega$.

Equation (3.13) implies that $H_\Lambda(\mu)$ is a concave function of μ, hence so is $h(\mu)$. Convexity of $h(\mu)$ follows from the inequality

$$(tx + (1-t)y)\log(tx + (1-t)y) \geq tx \log x + (1-t)y \log y - |x-y|/e,$$

valid for $x, y \geq 0$ and $t \in (0,1)$ (see Exercise 4.4.41 in [DS]). Thus $h(\mu)$ is affine in μ, and the same follows for $I_{\pi,\alpha}(\mu)$ by virtue of equation (3.6). The lower semicontinuity of $I_{\pi,\alpha}(\mu)$ follows similarly: Equations (3.13) and (3.14) imply that $h(\mu)$ is upper semicontinuous, and since u and κ are bounded continuous functions on Ω, (3.6) implies the lower semicontinuity of $I_{\pi,\alpha}(\mu)$.

The existence of the specific relative entropy and the equality $h(\mu|\psi) = I_{\pi,\alpha}(\mu)$ for $\psi \in \mathcal{R}^\tau_{\pi,\alpha}$ follow from

$$\log \phi^\xi_{\Lambda,\pi,\alpha}([\omega]) - \log \phi^0_{\Lambda,\pi,\alpha}([\omega]) = O(|\partial\Lambda|),$$

valid uniformly in $\xi \in \Omega$ and $\omega \in \Omega_\Lambda$.

We postpone the proof of part (c) till after the proof of Theorem 2. $\square$

Proof of Theorem 2. We appeal to standard arguments from large deviation theory. First, it is well-known that the large deviation bounds (3.9) hold if $\phi^\xi_{\Lambda,\pi,\alpha}$ is replaced by $B_{1/2}$, the i.i.d. Bernoulli measure on the edges of $\mathbb{E}$ with edge density 1/2, and $I_{\pi,\alpha}(\mu)$ is replaced by the specific relative entropy $h(\mu|B_{1/2})$ which, in fact, equals $I_{0,0}(\mu)$. See for example [FO, Ol]. Next, note that

$$\phi^\xi_{\Lambda,0,0}(M_\Lambda(\omega) \in A) = B_{1/2}(M_\Lambda(\omega^\xi) \in A)$$

and that $M_\Lambda(\omega)$ and $M_\Lambda(\omega^\xi)$ come close in the weak topology, as $\Lambda \to \mathbb{Z}^d$, uniformly in both ω and ξ. This, the compactness of $\mathcal{M}$, and the lower semicontinuity of $I_{0,0}(\mu)$ allow one to derive the uniform bounds (3.9) for the case $(\pi, \alpha) = (0, 0)$.

Now let (π, α) be general, but before obtaining $\phi^\xi_{\Lambda,\pi,\alpha}$ we need one more intermediate step: Define the probability measure γ^ξ_Λ for $\omega \in \Omega^\xi_\Lambda$ by

$$\gamma^\xi_\Lambda(\omega) = \frac{1}{W^\xi_\Lambda} \exp\big\{|\mathbb{E}_\Lambda| M_\Lambda(\omega, \pi u + \alpha\kappa)\big\} \phi^\xi_{\Lambda,0,0}(\omega),$$

where W^ξ_Λ is the appropriate normalizing factor. An explicit calculation shows that

$$\lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\mathbb{E}_\Lambda|} \log W^\xi_\Lambda = f(\pi, \alpha) - \log 2, \tag{3.16}$$

uniformly in ξ. Following Exercise 2.1.24 in [DS] we see that the uniform large deviation bounds (3.9) hold for $\gamma^\xi_\Lambda(M_\Lambda \in A)$ with rate function

$$\mu \mapsto I_{0,0}(\mu) - \pi\mu(u) - \alpha\mu(\kappa) + f(\pi, \alpha) - \log 2$$

which equals $I_{\pi,\alpha}(\mu)$ by (3.6) and because

$$I_{0,0}(\mu) = \log 2 - h(\mu). \tag{3.17}$$

Finally, we can replace γ^ξ_Λ by $\phi^\xi_{\Lambda,\pi,\alpha}$ on account of

$$|\mathbb{E}_\Lambda| M_\Lambda(\omega, \pi u + \alpha\kappa) = \pi|\eta(\omega) \cap \mathbb{E}_\Lambda| + d\alpha k(\omega, \Lambda) + O(|\partial\Lambda|),$$

which is valid uniformly in $\omega \in \Omega$. $\square$

Proof of Theorem 1, part (c). This is a consequence of Varadhan's theorem [DS, Thm. 2.1.10] applied to the limit in (3.16), together with the large deviation bounds (3.9) for $(\pi, \alpha) = (0, 0)$ and equation (3.17). $\square$

As preparation for the proof of Thm. 3, we first argue that a measure ψ with $I_{\pi,\alpha}(\psi) = 0$ satisfies the so-called *finite energy condition* which guarantees the a.s. uniqueness of an infinite cluster.

Lemma 3.18. *Suppose $\psi \in \mathcal{M}^\tau$ satisfies $I_{\pi,\alpha}(\psi) = 0$. Let Δ be any finite set of vertices of $\mathbb{Z}^d$ and, as before, $\mathbb{E}_\Delta = \{\langle x, y\rangle : x, y \in \Delta\}$. Then there is a constant $c_0 > 0$ such that $\psi([\zeta]|\mathcal{F}_{\mathbb{E}(\Delta)^c}) \geq c_0$ ψ-a.s. for all $\zeta \in \{0, 1\}^{\mathbb{E}_\Delta}$.*

Proof. Pick and fix $\zeta \in \{0, 1\}^{\mathbb{E}_\Delta}$. Let K be any event in $\{0, 1\}^{\mathbb{E}^c_\Delta}$ that depends on only finitely many edges. Pick a cube Λ centered at the origin and large enough so that $\Delta \subseteq \Lambda$ and K is $\mathcal{F}_{\mathbb{E}_\Lambda}$-measurable. Let m be a large positive integer, and let Λ_1 denote the cube obtained as a disjoint union of m^d copies of Λ, translated by the vertices $x_1, \ldots, x_{m^d}$:

$$\Lambda_1 = \bigcup_{i=1}^{m^d} (x_i + \Lambda). \tag{3.19}$$

Let $\Gamma_0 = \emptyset$ and

$$\Gamma_k = (x_1 + \Lambda) \cup \cdots \cup (x_k + \Lambda), \qquad k = 1, 2, \ldots, m^d. \tag{3.20}$$

Finally, let A_k be a the set of edges that satisfies

$$\tau_{x_k} A_k = \mathbb{E}_{\Gamma_k} \smallsetminus \mathbb{E}_{(x_k+\Delta)} \supseteq \mathbb{E}_{\Gamma_{k-1}}, \qquad k = 1, 2, \ldots, m^d. \tag{3.21}$$

Next a calculation with entropy. Since there is a measure $\phi \in \mathcal{R}^{\tau}_{\pi,\alpha}$, by Theorem 1 we can obtain $I_{\pi,\alpha}(\psi)$ by calculating $h(\psi|\phi)$.

$$\begin{aligned}
&H_{\mathbb{E}(\Lambda_1)}(\psi|\phi) \\
&= \sum_{k=1}^{m^d} \Big\{ H_{\mathbb{E}(\Gamma_k)}(\psi|\phi) - H_{\mathbb{E}(\Gamma_{k-1})}(\psi|\phi) \Big\} \\
&\geq \sum_{k=1}^{m^d} \Big\{ H_{\mathbb{E}(\Gamma_k - x_k)}(\psi|\phi) - H_{A_k}(\psi|\phi) \Big\} \\
&= \sum_{k=1}^{m^d} \int H_{\mathbb{E}(\Gamma_k - x_k)}\big(\psi(\,\cdot\,|\mathcal{F}_{A_k}) \big| \phi(\,\cdot\,|\mathcal{F}_{A_k})\big)\, d\psi \\
&\geq \sum_{k=1}^{m^d} \int_K \{ -\delta\psi([\zeta]|\mathcal{F}_{A_k}) - \log\phi(\exp(-\delta \mathrm{I}_{[\zeta]})|\mathcal{F}_{A_k}) \}\, d\psi.
\end{aligned} \tag{3.22}$$

The first inequality above comes from translation invariance and the monotonicity of entropy, together with the inclusion relation in (3.21). The second equality comes from the conditional entropy formula, Lemma 4.4.7 in [DS]. Here $\psi(\,\cdot\,|\mathcal{F}_{A_k})$ is the conditional probability of ψ, given the σ-field $\mathcal{F}_{A_k}$. The last inequality comes from two steps: First restrict the integration to the event K. Then apply the variational formula (3.3) to the function $f(\omega) = -\delta \mathrm{I}_{[\zeta]}(\omega)$ where δ is a positive constant. The inclusion $\Gamma_k - x_k \supseteq \Lambda \supseteq \Delta$, valid for all $k = 1, \ldots, m^d$, is needed here.

Now notice that, by the form (2.2) of the finite-volume specification and the DLR condition (2.4) on ϕ, there is a constant $c_0 > 0$ such that

$$\phi([\zeta]|\mathcal{F}_{A_k}) \geq c_0$$

holds ϕ-a.s. for all choices of $\Lambda \supseteq \Delta$, all $k = 1, \ldots, m^d$, and for all m. Hence

$$\begin{aligned}
&\phi(\exp[-\delta \mathrm{I}_{[\zeta]}]|\mathcal{F}_{A_k}) \\
&= e^{-\delta}\phi([\zeta]|\mathcal{F}_{A_k}) + \phi([\zeta]^c|\mathcal{F}_{A_k}) \\
&\leq 1 - (1 - e^{-\delta})c_0
\end{aligned}$$

is true ϕ-a.s. and consequently also ψ-a.s. because ϕ charges all the atoms of the finite σ-field $\mathcal{F}_{A_k}$. Continuing from the calculation in (3.22), we get

$$\begin{aligned}
&H_{\mathbb{E}(\Lambda_1)}(\psi|\phi) \\
&\geq \sum_{k=1}^{m^d} (-\delta) \int_K \psi([\zeta]|\mathcal{F}_{A_k})\, d\psi - m^d \psi(K) \log(1 - (1 - e^{-\delta})c_0) \\
&= -\,\delta m^d \psi(K \cap [\zeta]) - m^d \psi(K) \log(1 - (1 - e^{-\delta})c_0).
\end{aligned} \tag{3.23}$$

Now divide through by m^d, and let $m \to \infty$ (equivalently, $\Lambda_1 \to \mathbb{Z}^d$) while holding Λ fixed, and use the equality $h(\psi|\phi) = 0$:

$$0 \geq -\delta\psi(K \cap [\zeta]) - \psi(K)\log(1 - (1 - e^{-\delta})c_0).$$

Divide through by δ and let $\delta \searrow 0$ to get

$$\psi(K \cap [\zeta]) \geq c_0\psi(K).$$

Since K was an arbitrary finite-dimensional event, the conclusion follows. □

Proof of Theorem 3. The equivalence (ii)$\Longleftrightarrow$(iii) and the implication (i)$\Longrightarrow$(ii) are contained in Theorem 1, so it only remains to prove (ii)$\Longrightarrow$(i). As has been observed by many authors, the uniqueness of the infinite component for random-cluster measures plays a role similar to that of quasilocality for Gibbsian lattice systems. From the version of the Burton-Keane [BK] uniqueness theorem given as Thm. 1 in [GKN] and from Lemma 3.18, it follows that if $I_{\pi,\alpha}(\psi) = 0$ then ψ-a.s. there is at most one infinite component. Pick and fix an element $\phi \in \mathcal{R}^\tau_{\pi,\alpha}$. As remarked earlier, Thm. 3.2(a) in [Gr3] allows us to do so. The implication (ii)$\Longrightarrow$(i) will follow from proving this statement:

$$\text{(3.24)} \qquad \begin{array}{l} \text{If there is at most one infinite component } \psi\text{-a.s.} \\ \text{and } \psi \notin \mathcal{R}^\tau_{\pi,\alpha}, \text{ then } h(\psi|\phi) > 0. \end{array}$$

The rest of our proof is similar to earlier arguments given for Gibbsian systems [Pr, DSZ].

Since $\psi \notin \mathcal{R}^\tau_{\pi,\alpha}$, there is a finite rectangle Δ such that the probability measures ψ and

$$\widetilde{\psi} \equiv \int \phi^\xi_{\Delta,\pi,\alpha}(\cdot)\psi(d\xi)$$

are distinct. Consequently they differ on $\mathcal{F}_{\mathbb{E}_\Gamma}$ for some finite rectangle $\Gamma \supseteq \Delta$ centered at the origin, and then there is an $\varepsilon > 0$ such that

$$\text{(3.25)} \qquad \begin{aligned} 0 < \varepsilon &\leq H_{\mathbb{E}(\Gamma)}\bigl(\psi|\widetilde{\psi}\bigr) \\ &= \int H_{\mathbb{E}(\Delta)}\bigl(\psi(\,\cdot\,|\mathcal{F}_{\mathbb{E}(\Gamma)\smallsetminus\mathbb{E}(\Delta)})\big|\widetilde{\psi}(\,\cdot\,|\mathcal{F}_{\mathbb{E}(\Gamma)\smallsetminus\mathbb{E}(\Delta)})\bigr)d\psi. \end{aligned}$$

The equality follows from the conditional entropy formula because ψ and $\widetilde{\psi}$ coincide on the σ-field $\mathcal{F}_{\mathbb{E}(\Gamma)\smallsetminus\mathbb{E}(\Delta)}$. We shall write this as

$$\text{(3.26)} \qquad \varepsilon \leq \int H_{\mathbb{E}(\Delta)}\bigl(\psi(\,\cdot\,|\mathcal{F}_{\mathbb{E}(\Gamma)\smallsetminus\mathbb{E}(\Delta)})\big|\phi(\,\cdot\,|\mathcal{F}_{\mathbb{E}(\Gamma)\smallsetminus\mathbb{E}(\Delta)})\bigr)d\psi + \int \log Q_{\Delta,\Gamma}d\psi,$$

where

$$Q_{\Delta,\Gamma}(\omega) = \frac{\phi([\omega(\mathbb{E}_\Delta)]|\omega(\mathbb{E}_\Gamma \smallsetminus \mathbb{E}_\Delta))}{\widetilde{\psi}([\omega(\mathbb{E}_\Delta)]|\omega(\mathbb{E}_\Gamma \smallsetminus \mathbb{E}_\Delta))}.$$

The above notation means the conditional probability of the cylinder event $[\omega(\mathbb{E}_\Delta)]$, given the realization $\omega(\mathbb{E}_\Gamma \smallsetminus \mathbb{E}_\Delta)$. At this point we insert a technical lemma:

Lemma 3.27. *Given the rectangle Δ and $\varepsilon > 0$, there exists a rectangle Λ such that $\Lambda \supseteq \Delta$ and*

$$\int \log Q_{\Delta,\Gamma}\, d\psi \le \frac{\varepsilon}{2} \tag{3.28}$$

for all rectangles $\Gamma \supseteq \Lambda$.

Proof. To prove the lemma, rewrite $Q_{\Delta,\Gamma}(\omega)$ as

$$Q_{\Delta,\Gamma}(\omega) = \frac{\int \phi^{\omega^{\xi}_{\Gamma}}_{\Delta,\pi,\alpha}([\omega(\mathbb{E}_\Delta)])\, \phi(d\xi\, |\omega(\mathbb{E}_\Gamma \setminus \mathbb{E}_\Delta))}{\int \phi^{\omega^{\xi'}_{\Gamma}}_{\Delta,\pi,\alpha}([\omega(\mathbb{E}_\Delta)])\, \psi(d\xi' |\omega(\mathbb{E}_\Gamma \setminus \mathbb{E}_\Delta))}\,.$$

Now suppose that, for this fixed ω, Γ is chosen so large that no finite connected component of the graph $(\mathbb{Z}^d, \eta(\omega))$ that intersects Δ is connected to a boundary vertex of Γ, and so that if two vertices $x, y \in \Delta$ are connected in $(\mathbb{Z}^d, \eta(\omega))$, they are also connected in $(\mathbb{Z}^d, \eta(\omega) \cap \mathbb{E}_\Gamma)$. If ω has at most one infinite component, it follows that $k(\omega^{\xi}_{\Gamma}, \Delta) = k(\omega^{\xi'}_{\Gamma}, \Delta)$ for all $\xi, \xi' \in \Omega$, so

$$\phi^{\omega^{\xi}_{\Gamma}}_{\Delta,\pi,\alpha}([\omega(\mathbb{E}_\Delta)]) = \phi^{\omega^{\xi'}_{\Gamma}}_{\Delta,\pi,\alpha}([\omega(\mathbb{E}_\Delta)]) \text{ for all } \xi, \xi' \in \Omega,$$

and consequently $Q_{\Delta,\Gamma}(\omega) = 1$. Thus ψ-a.s. $\log Q_{\Delta,\Gamma}(\omega)$ converges to zero as $\Gamma \to \mathbb{Z}^d$. The lemma follows from the bound

$$|\log Q_{\Delta,\Gamma}(\omega)| \le O(|\partial\Delta|), \text{ uniformly in } \Gamma \text{ and } \omega,$$

a consequence of inequalities (3.15). $\square$

Returning to the proof of statement (3.24), choose a cube Λ, centered at the origin, large enough so that $\Lambda \supseteq \Delta$ and both (3.25) and (3.28) hold for $\Gamma \supseteq \Lambda$. (By the monotonicity of entropy, the validity of (3.28) is not jeopardized by increasing Γ.)

Exactly as in the proof of Lemma 3.18, let Λ_1 and $\Gamma_0, \ldots, \Gamma_{m^d}$ be as in (3.19)–(3.20). Let $\Delta_k = x_k + \Delta$ be the translate of Δ. Then inequalities (3.26) and (3.28) give, for each k,

$$\begin{aligned}
\varepsilon &\le \int H_{\mathbb{E}(\Delta_k)}\big(\psi(\,\cdot\,|\mathcal{F}_{\mathbb{E}(\Gamma_k)\setminus\mathbb{E}(\Delta_k)})\big|\phi(\,\cdot\,|\mathcal{F}_{\mathbb{E}(\Gamma_k)\setminus\mathbb{E}(\Delta_k)})\big)\,d\psi + \frac{\varepsilon}{2} \\
&= H_{\mathbb{E}(\Gamma_k)}(\psi|\phi) - H_{\mathbb{E}(\Gamma_k)\setminus\mathbb{E}(\Delta_k)}(\psi|\phi) + \frac{\varepsilon}{2} \\
&\le H_{\mathbb{E}(\Gamma_k)}(\psi|\phi) - H_{\mathbb{E}(\Gamma_{k-1})}(\psi|\phi) + \frac{\varepsilon}{2}\,,
\end{aligned}$$

where we used again both the conditional entropy formula and the monotonicity of entropy. Adding up these inequalities gives

$$\begin{aligned}
H_{\mathbb{E}(\Lambda_1)}(\psi|\phi) &= \sum_{k=1}^{m^d} \Big\{ H_{\mathbb{E}(\Gamma_k)}(\psi|\phi) - H_{\mathbb{E}(\Gamma_{k-1}}(\psi|\phi) \Big\} \\
&\ge m^d \varepsilon/2,
\end{aligned}$$

from which, by dividing by $|\mathbb{E}_{\Lambda_1}|$ and by letting $m \to \infty$, $h(\psi|\phi) \ge \varepsilon/(2|\mathbb{E}_\Lambda|) > 0$. This completes the proof of Theorem 3. $\square$

4. The convex conjugate of free energy. For a point $(s,t) \in [0,1]^2$, let

$$J(s,t) = -\sup\{h(\mu) : \mu \in \mathcal{M}^\tau, \mu(u) = s, \mu(\kappa) = t\}, \tag{4.1}$$

where as usual the supremum of an empty set is equal to $+\infty$. Since $0 \le h(\mu) \le \log 2$, we have that either $-\log 2 \le J(s,t) \le 0$ or $J(s,t) = +\infty$. The former happens iff $(\mu(u), \mu(\kappa)) = (s,t)$ for some $\mu \in \mathcal{M}^\tau$, and then the supremum in the definition is attained at some measure μ, by the upper semicontinuity of $h(\mu)$ and the compactness of the space $\mathcal{M}^\tau$. From this and the affinity of $h(\mu)$ it follows that J is a lower semicontinuous convex function. Extend J to all of $\mathbb{R}^2$ by declaring $J(s,t) = +\infty$ for $(s,t) \notin [0,1]^2$. Then in the language of Rockafellar [Rf], J is a closed, proper convex function.

The *effective domain* $\operatorname{dom} J$ of J is given by

$$\operatorname{dom} J \equiv \{(s,t) \in \mathbb{R}^2 : J(s,t) < \infty\} = \{(\mu(u), \mu(\kappa)) : \mu \in \mathcal{M}^\tau\}. \tag{4.2}$$

It is a compact convex subset of $[0,1]^2$. Perhaps surprisingly, elementary combinatorial estimates suffice for a complete description of $\operatorname{dom} J$, given in the next theorem. Let $\mathcal{R}^\tau$ be the set of all translation-invariant random-cluster measures, that is, the union of all collections $\mathcal{R}^\tau_{\pi,\alpha}$ over $(\pi,\alpha) \in \mathbb{R}^2$.

Theorem 4. *The image of $\mathcal{M}^\tau$ under the map $\mu \mapsto (\mu(u), \mu(\kappa))$ is the closed triangle with vertices $(1/d, 0)$, $(1,0)$, and $(0,1)$. The set $\mathcal{R}^\tau$ maps onto the interior of this triangle, in other words,*

$$\operatorname{int}(\operatorname{dom} J) = \{(\psi(u), \psi(\kappa)) : \psi \in \mathcal{R}^\tau\}. \tag{4.3}$$

The next theorem contains the convex duality of J and f and some properties related to it. The boundary $\operatorname{bd} A$ of a set is the difference between the closure $\operatorname{cl} A$ and the interior $\operatorname{int} A$. The *subdifferential* $\partial J(s,t)$ of the convex function J is the set of *subgradients* (π,α) that satisfy

$$J(s',t') - J(s,t) \ge \pi(s'-s) + \alpha(t'-t) \tag{4.4}$$

for all $(s',t') \in \mathbb{R}^2$. J is differentiable at (s,t) iff $\partial J(s,t)$ is a singleton set, and then the gradient $\nabla J(s,t)$ is the unique subgradient. The reader is referred to Rockafellar's monograph [Rf] for the concepts and results from convex analysis used here.

Theorem 5. (a) *J and the free energy f are convex conjugates in the sense that*

$$f(\pi,\alpha) = \sup_{(s,t)\in\mathbb{R}^2} \{\pi s + \alpha t - J(s,t)\} \tag{4.5}$$

and

$$J(s,t) = \sup_{(\pi,\alpha)\in\mathbb{R}^2} \{\pi s + \alpha t - f(\pi,\alpha)\}. \tag{4.6}$$

(b) *J is continuously differentiable on* $\operatorname{int}(\operatorname{dom} J)$*, and the gradient* ∇J *maps* $\operatorname{int}(\operatorname{dom} J)$ *onto* $\mathbb{R}^2$ *according to this rule:* $\nabla J(s,t) = (\pi, \alpha)$ *iff* $(s,t) = (\psi(u), \psi(\kappa))$ *for some* $\psi \in \mathcal{R}^{\tau}_{\pi,\alpha}$*. The subdifferential* $\partial J(s,t)$ *is empty for* $(s,t) \in \operatorname{bd}(\operatorname{dom} J)$*.*

(c) *For each* $(\pi, \alpha) \in \mathbb{R}^2$*, the subdifferential* $\partial f(\pi, \alpha)$ *is nonempty compact, convex subset of the open triangle* $\operatorname{int}(\operatorname{dom} J)$*. The sets* $\partial f(\pi, \alpha)$ *are mutually disjoint, and*

$$\partial f(\pi, \alpha) = \{(\psi(u), \psi(\kappa)) : \psi \in \mathcal{R}^{\tau}_{\pi,\alpha}\}. \tag{4.7}$$

There is also a combinatorial definition of J: For a configuration $\xi \in \Omega$ and $\varepsilon > 0$, let $N^{\xi}_{\Lambda}(s,t,\varepsilon)$ be the number of configurations $\omega \in \Omega^{\xi}_{\Lambda}$ that simultaneously satisfy

$$\frac{|\eta(\omega) \cap \mathbb{E}_\Lambda|}{|\mathbb{E}_\Lambda|} \in (s - \varepsilon, s + \varepsilon) \qquad \text{and} \qquad \frac{k(\omega, \Lambda)}{|\Lambda|} \in (t - \varepsilon, t + \varepsilon). \tag{4.8}$$

Then

$$J(s,t) = -\lim_{\varepsilon \searrow 0} \limsup_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \log N^{\xi}_{\Lambda}(s,t,\varepsilon), \tag{4.9}$$

as can be verified with the help of Theorem 2 and equation (3.17).

Finally, some remarks about the measures that map into the boundary of the triangle $\operatorname{dom} J$ via $\mu \mapsto (\mu(u), \mu(\kappa))$. Obviously, the vertices $(0,1)$ and $(1,0)$ are given by the point masses on the configurations $\omega \equiv 0$ and $\omega \equiv 1$, respectively. The vertex $(1/d, 0)$ is the image of the uniform spanning tree measure μ^U of Pemantle [Pe]. This measure has edge density $1/d$ because it is a limit of uniform measures on spanning trees of finite graphs $(\Lambda, \mathbb{E}_\Lambda)$, and each such tree has $|\Lambda| - 1$ edges, or approximately $|\mathbb{E}_\Lambda|/d$. Häggström [Hä1] proved that as $\pi \to -\infty$, $\alpha \to -\infty$, and $\pi - d\alpha \to +\infty$, elements of $\mathcal{R}_{\pi,\alpha}$ converge to μ^U. The boundary points $(s, 0)$ for $1/d < s < 1$ are of course images of convex combinations of μ^U and δ_1, but also of these ergodic measures: Let B_r be the Bernoulli measure with edge density r, and let μ be the image of the ergodic product measure $\mu^U \otimes B_r$ under the map $(\omega, \omega') \mapsto \omega \vee \omega'$. Then μ itself is ergodic, and satisfies $(\mu(u), \mu(\kappa)) = (1/d + r(1 - 1/d), 0)$. (In other words, to pick a configuration under μ, first pick a configuration under μ^U and then add missing edges by flipping independent coins.)

We start the proofs with Theorem 5. As a preliminary observation we record the following consequence of the variational principle:

Lemma 4.10. *Suppose* $(\pi, \alpha) \neq (\pi', \alpha')$*, and we have two random-cluster measures* $\psi \in \mathcal{R}^{\tau}_{\pi,\alpha}$ *and* $\psi' \in \mathcal{R}^{\tau}_{\pi',\alpha'}$*. Then*

$$0 < (\pi' - \pi)[\psi'(u) - \psi(u)] + (\alpha' - \alpha)[\psi'(\kappa) - \psi(\kappa)]. \tag{4.11}$$

In particular, such random-cluster measures for distinct parameter values cannot satisfy $(\psi(u), \psi(\kappa)) = (\psi'(u), \psi'(\kappa))$*.*

Proof. First note that the classes $\mathcal{R}_{\pi,\alpha}$ are mutually disjoint. For suppose $\psi \in \mathcal{R}_{\pi,\alpha} \cap \mathcal{R}_{\pi',\alpha'}$. Then by the DLR equation (2.4) we have

$$\int \phi^{\xi}_{\Lambda,\pi,\alpha}([\zeta]) g(\xi)\, \psi(d\xi) = \int \phi^{\xi}_{\Lambda,\pi',\alpha'}([\zeta]) g(\xi)\, \psi(d\xi)$$

for all $\zeta \in \Omega_\Lambda$ and bounded, $\mathcal{F}_{\mathbb{E}(\Lambda)^c}$-measurable g. From this follows

$$\phi^\xi_{\Lambda,\pi,\alpha}([\zeta]) = \phi^\xi_{\Lambda,\pi',\alpha'}([\zeta]) \qquad \text{for } \psi\text{-a.e. } \xi,$$

which forces $(\pi,\alpha) = (\pi',\alpha')$.

Since $\psi' \notin \mathcal{R}^\tau_{\pi,\alpha}$ and $\psi \notin \mathcal{R}^\tau_{\pi',\alpha'}$, the variational principle gives us the strict inequalities

$$h(\psi) + \pi\psi(u) + \alpha\psi(\kappa) > h(\psi') + \pi\psi'(u) + \alpha\psi'(\kappa)$$

and

$$h(\psi') + \pi'\psi'(u) + \alpha'\psi'(\kappa) > h(\psi) + \pi'\psi(u) + \alpha'\psi(\kappa).$$

A rearrangement of these gives (4.11). □

Proof of Theorem 5. Part (a): Equation (4.5) is a direct consequence of equation (3.7) and the definition (4.1) of J. Equation (4.6) then follows from Thm. 12.2 in [Rf].

Parts (b) and (c) are proved in stages:

Claim 1. Suppose $(\pi,\alpha) \in \partial J(s,t)$. Then $J(s,t) = -h(\psi)$ for some $\psi \in \mathcal{R}^\tau_{\pi,\alpha}$ that satisfies $(\psi(u),\psi(\kappa)) = (s,t)$. Conversely, if $\psi \in \mathcal{R}^\tau_{\pi,\alpha}$, then $J(\psi(u),\psi(\kappa)) = -h(\psi)$ and $(\pi,\alpha) \in \partial J(\psi(u),\psi(\kappa))$.

Since $\partial J(s,t)$ is nonempty, (s,t) must lie in $\operatorname{dom} J$, and so there exists a measure $\psi \in \mathcal{M}^\tau$ that satisfies $(\psi(u),\psi(\kappa)) = (s,t)$ and $J(s,t) = -h(\psi)$. $(\pi,\alpha) \in \partial J(s,t)$ implies, by [Rf, Thm. 23.5], that

$$\begin{aligned} f(\pi,\alpha) &= \pi s + \alpha t - J(s,t) \\ &= h(\psi) + \pi\psi(u) + \alpha\psi(\kappa), \end{aligned}$$

and then $\psi \in \mathcal{R}^\tau_{\pi,\alpha}$ follows from Theorem 3, the variational principle.

For the converse part, the variational principle implies that $h(\psi) \geq h(\mu)$ for all $\mu \in \mathcal{M}^\tau$ that satisfy $(\mu(u),\mu(\kappa)) = (\psi(u),\psi(\kappa))$, from which follows $J(\psi(u),\psi(\kappa)) = -h(\psi)$. The inclusion $(\pi,\alpha) \in \partial J(\psi(u),\psi(\kappa))$ follows from

$$f(\pi,\alpha) = \pi\psi(u) + \alpha\psi(\kappa) - J(\psi(u),\psi(\kappa))$$

and [Rf, Thm. 23.5]. This proves Claim 1.

Claim 2. $\partial J(s,t)$ contains at most one element, and so J is differentiable at (s,t) whenever $\partial J(s,t)$ is nonempty.

If $\partial J(s,t)$ contains two distinct elements (π,α) and (π',α'), then by Claim 1 there are measures $\psi \in \mathcal{R}^\tau_{\pi,\alpha}$ and $\psi' \in \mathcal{R}^\tau_{\pi',\alpha'}$ that satisfy $(s,t) = (\psi(u),\psi(\kappa)) = (\psi'(u),\psi'(\kappa))$ and thereby violate Lemma 4.10. The differentiability of J follows from [Rf, Thm. 25.1].

We can now complete the proof of part (b). By [Rf, Thm. 23.4], $(s,t) \in \operatorname{int}(\operatorname{dom} J)$ implies that $\partial J(s,t)$ is nonempty, so by Claim 2 J is differentiable on int (dom J). From [Rf, Cor. 25.5.1] follows the continuous differentiability of J. Suppose $(s,t) \in \operatorname{bd}(\operatorname{dom} J)$. If $\partial J(s,t)$ were nonempty, then by [Rf, Thm. 23.4] it would have to be unbounded, contradicting Claim 2. Together this shows that int (dom J) is precisely the set on which J is subdifferentiable, and by Claim 1 this is exactly $\{(\psi(u), \psi(\kappa)) : \psi \in \mathcal{R}^\tau\}$. (This proves equation (4.3), which is needed in the next paragraph.) The map $(s,t) \mapsto (\pi, \alpha) = \nabla J(s,t)$ is well-defined on int (dom J), and maps onto $\mathbb{R}^2$ by Claims 1 and 2.

Part (c): A subdifferential is always a closed convex set [Rf, p. 215]. By [Rf, Thm. 23.5] and Claim 1, $(s,t) \in \partial f(\pi, \alpha)$ iff $(\pi, \alpha) \in \partial J(s,t)$ iff $(s,t) = (\psi(u), \psi(\kappa))$ for some $\psi \in \mathcal{R}^\tau_{\pi,\alpha}$. This gives equation (4.7). $\partial f(\pi, \alpha) \subseteq \operatorname{int}(\operatorname{dom} J)$ follows from (4.7) and (4.3), and the mutual disjointness of the sets $\partial f(\pi, \alpha)$ from (4.7) and Lemma 4.10. □

For the proof of Theorem 4, first a combinatorial lemma.

Lemma 4.12. *Suppose (V, E) is a finite connected subgraph of $(\mathbb{Z}^d, \mathbb{E})$. Then $|E| \leq d(|V| - 1)$.*

Proof. The statement of the lemma holds if $|V| = 1$ or 2. Let V be arbitrary, and assume by induction that the statement holds for any connected subgraph (V', E') of $(\mathbb{Z}^d, \mathbb{E})$ with $|V'| \leq |V| - 1$.

Let v_0 be the minimal element of V under lexicographic ordering. Then the degree j of v_0 satisfies $j \leq d$ because the edges incident to v_0 must be toward increasing directions along coordinate axes. Remove v_0 and its j incident edges. The original graph (V, E) splits into ℓ connected graphs (V_1, E_1),..., (V_ℓ, E_ℓ), $1 \leq \ell \leq j$, with

$$|V| = 1 + \sum_{i=1}^{\ell} |V_i| \qquad \text{and} \qquad |E| = j + \sum_{i=1}^{\ell} |E_i|. \tag{4.13}$$

By induction $|E_i| \leq d(|V_i| - 1)$ for $i = 1, \ldots, \ell$, which together with $1 \leq \ell \leq j \leq d$ and equation (4.13) implies $|E| \leq d(|V| - 1)$. □

Corollary 4.14. *For all $\omega \in \Omega$ and finite rectangles $\Lambda \subseteq \mathbb{Z}^d$,*

$$k(\omega, \Lambda) \leq |\Lambda| - \frac{1}{d} |\eta(\omega) \cap \mathbb{E}_\Lambda|. \tag{4.15}$$

Proof. Since $k(\omega^0, \Lambda) \geq k(\omega, \Lambda)$ for all $\omega \in \Omega$, it suffices to derive the result for ω^0. Let (V_i, E_i), $i = 1, \ldots, k(\omega^0, \Lambda)$, be the connected components of the graph $(\Lambda, \eta(\omega) \cap \mathbb{E}_\Lambda)$. Then by the lemma,

$$\begin{aligned}
|\eta(\omega) \cap \mathbb{E}_\Lambda| &= \sum_{i=1}^{k(\omega^0,\Lambda)} |E_i| \\
&\leq d \sum_{i=1}^{k(\omega^0,\Lambda)} |V_i| - d k(\omega^0, \Lambda) \\
&= d|\Lambda| - d k(\omega^0, \Lambda),
\end{aligned}$$

and the conclusion follows. □

Proof of Theorem 4. The paragraph after equation (4.9) indicates that $\operatorname{dom} J$ contains the points $(1/d, 0)$, $(1, 0)$, and $(0, 1)$. Since $\operatorname{dom} J$ is closed and convex, it follows that the triangle spanned by these points is contained in $\operatorname{dom} J$. To prove that $\operatorname{dom} J$ is contained in this triangle we show that $(s, t) \in \operatorname{dom} J$ implies

$$t \geq 1 - ds \tag{4.16}$$

and

$$t \leq 1 - s. \tag{4.17}$$

To prove (4.16), draw in the edges of $\eta(\omega) \cap \mathbb{E}_\Lambda$ one by one. Initially $k(0, \Lambda) = |\Lambda|$ and all vertices of the rectangle Λ are isolated. Since each edge reduces the number of components by 0 or 1, we get

$$k(\omega^0, \Lambda) \geq |\Lambda| - |\eta(\omega) \cap \mathbb{E}_\Lambda|$$

and then by (3.15)

$$k(\omega, \Lambda) \geq |\Lambda| - |\eta(\omega) \cap \mathbb{E}_\Lambda| - |\partial\Lambda|. \tag{4.18}$$

Integrate (4.18) against an arbitrary $\mu \in \mathcal{M}^\tau$, divide by $|\Lambda|$, let $\Lambda \to \mathbb{Z}^d$, and use (2.6)–(2.7). This gives (4.16). Inequality (4.17) comes similarly from (4.15).

Finally, equation (4.3) was already argued in the proof of Theorem 5(b). □

5. The phase transition. The next theorem extends the characterizations in [Gr3] of the set of (π, α) for which there is a unique random-cluster measure. The measures $\phi^b_{\pi,\alpha}$, $b = 0, 1$, are defined as in (3.10). $\{0 \leftrightarrow \infty\}$ denotes the event that the connected cluster at the origin is infinite, equivalently that $\kappa(\omega) = 0$. Differentiability of f at (π, α) has its standard meaning, namely that

$$f(\pi', \alpha') - f(\pi, \alpha) = \nabla f(\pi, \alpha) \cdot (\pi' - \pi, \alpha' - \alpha) + o(|(\pi', \alpha') - (\pi, \alpha)|).$$

However, for convex functions the existence of all partial derivatives is equivalent to differentiability [Rf, Thm. 25.2].

Theorem 6. *Suppose $\alpha > 0$ ($q > 1$). The following statements are all equivalent to $\phi^0_{\pi,\alpha} = \phi^1_{\pi,\alpha}$ which in turn is equivalent to $|\mathcal{R}_{\pi,\alpha}| = 1$.*

(i) *f is differentiable at (π, α).*

(ii) *$\dfrac{\partial f}{\partial \pi}(\pi, \alpha)$ exists.*

(iii) *$\dfrac{\partial f}{\partial \alpha}(\pi, \alpha)$ exists.*

(iv) *$\phi^0_{\pi,\alpha}(\omega(e)) = \phi^1_{\pi,\alpha}(\omega(e))$.*

(v) *$\phi^0_{\pi,\alpha}(\kappa) = \phi^1_{\pi,\alpha}(\kappa)$.*

(vi) *$\phi^0_{\pi,\alpha}(0 \leftrightarrow \infty) = \phi^1_{\pi,\alpha}(0 \leftrightarrow \infty)$.*

Underlying the theorem are these formulas for the left and right derivatives of f: For $\alpha \geq 0$ and all $\psi \in \mathcal{R}^{\tau}_{\pi,\alpha}$,

$$\frac{\partial f}{\partial \pi^-}(\pi,\alpha) = \phi^0_{\pi,\alpha}(\omega(e)) \leq \psi(u) \leq \phi^1_{\pi,\alpha}(\omega(e)) = \frac{\partial f}{\partial \pi^+}(\pi,\alpha) \tag{5.1}$$

and

$$\frac{\partial f}{\partial \alpha^-}(\pi,\alpha) = \phi^1_{\pi,\alpha}(\kappa) \leq \psi(\kappa) \leq \phi^0_{\pi,\alpha}(\kappa) = \frac{\partial f}{\partial \alpha^+}(\pi,\alpha). \tag{5.2}$$

Not much of Theorem 6 can be salvaged for $\alpha < 0$, due to the lack of a comparison principle for random-cluster measures. We cannot even prove that the nonuniqueness of the random-cluster measure is equivalent to the nondifferentiability of f. Presently we have this result:

Theorem 7. *For arbitrary α, the following holds:*

(a) *If f is not differentiable at (π,α), then $\mathcal{R}^{\tau}_{\pi,\alpha}$ cannot be a singleton.*

(b) *If $\partial f/\partial\pi$ exists at (π,α), then $\psi(u)$ is a constant over $\psi \in \mathcal{R}^{\tau}_{\pi,\alpha}$, and similarly for $\partial f/\partial\alpha$ and $\psi(\kappa)$.*

The first part of statement (b) appeared as Thm. 4.5(a) in [Gr3].

Let $\mathcal{D}$ be the set of values (π,α) at which ∇f does not exist. For $\alpha \geq 0$ Theorem 6 implies that $(\pi,\alpha) \notin \mathcal{D}$ if either $\partial f/\partial\pi$ or $\partial f/\partial\alpha$ exists. Consequently, by general convex analytic facts, the sections

$$\mathcal{D}_\pi = \{\alpha : (\pi,\alpha) \in \mathcal{D}\} \qquad \text{and} \qquad \mathcal{D}_\alpha = \{\pi : (\pi,\alpha) \in \mathcal{D}\}$$

can be at most countable in the half-space $\alpha \geq 0$. The same countability claim holds for the points of non-existence of the partial $\partial f/\partial\pi$ along horizontal lines (fixed α), and for $\partial f/\partial\alpha$ along vertical lines (fixed π), for all α. But for $\alpha < 0$ we do not know whether the existence of one partial implies the existence of the other and thereby the differentiability of f.

Another interesting question is whether, in the event of a phase transition, all members of $\mathcal{R}^{\tau}_{\pi,\alpha}$ are convex combinations of $\phi^0_{\pi,\alpha}$ and $\phi^1_{\pi,\alpha}$. (Speaking now about the case $\alpha > 0$.) This question has been answered for large enough α: There is a unique critical $\pi_c(\alpha)$ such that $\mathcal{R}_{\pi,\alpha}$ is a singleton for $\pi \neq \pi_c(\alpha)$, while at $\pi = \pi_c(\alpha)$, $\mathcal{R}^{\tau}_{\pi_c(\alpha),\alpha}$ is the convex hull of the distinct extreme measures $\phi^0_{\pi_c(\alpha),\alpha}$ and $\phi^1_{\pi_c(\alpha),\alpha}$ [LM].

This question can be asked in two stages: First, does $\partial f(\pi,\alpha)$ contain anything else besides the line segment from (s^0,t^0) to (s^1,t^1)? Here we have written

$$(s^b,t^b) = \big(s^b(\pi,\alpha), t^b(\pi,\alpha)\big) = (\phi^b_{\pi,\alpha}(\omega(e)), \phi^b_{\pi,\alpha}(\kappa)), \qquad b = 0,1. \tag{5.3}$$

If the answer is yes, then not all random-cluster measures are convex combinations of $\phi^0_{\pi,\alpha}$ and $\phi^1_{\pi,\alpha}$, by equation (4.7). But if the answer is no, we have to ask further whether it is possible to have distinct $\psi, \psi' \in \mathcal{R}^{\tau}_{\pi,\alpha}$ such that $(\psi(u),\psi(\kappa)) = (\psi'(u),\psi'(\kappa))$. (Lemma 4.10 precludes such a possibility if ψ and ψ' are random-cluster measures for different parameters.)

The next section gives a partial answer to the first question for two dimensions. The next theorem gives bounds on $\partial f(\pi,\alpha)$ that follow from stochastic monotonicity. Note that $s^1 \geq s^0$, $t^0 \geq t^1$, and also $ds^0 + t^0 \leq ds^1 + t^1$ because $|\eta(\omega) \cap \mathbb{E}_\Lambda| + k(\omega^0,\Lambda)$ is an increasing function of ω.

Theorem 8. *Suppose $\alpha > 0$ ($q > 1$). On the (s,t)-plane, the compact convex set $\partial f(\pi, \alpha)$ contains the line segment from (s^0, t^0) to (s^1, t^1), and is contained in the closed parallelogram with vertices (s^0, t^0), $(s^1 - d^{-1}(t^0 - t^1), t^0)$, (s^1, t^1), and $(s^0 + d^{-1}(t^0 - t^1), t^1)$ (in clockwise order from the upper left corner). Except for the corners (s^0, t^0) and (s^1, t^1), $\partial f(\pi, \alpha)$ does not contain any other points on the horizontal lines $t = t^0$ and $t = t^1$.*

Finally we address the issue of how the subdifferentials $\partial f(\pi, \alpha)$ are situated relative to each other as (π, α) varies. Let H_e denote the event that the endpoints of the edge e are in distinct connected clusters. Below, we write $\psi_{\pi,\alpha}$ for an arbitrary translation-invariant random-cluster measure for parameters (π, α).

Theorem 9. (a) *The following holds for all α: For a fixed α, the edge density $\psi_{\pi,\alpha}(u)$ depends on π in a strictly increasing fashion, in other words if $\pi' > \pi$, then $\psi_{\pi',\alpha}(u) > \psi_{\pi,\alpha}(u)$ for all choices $\psi_{\pi,\alpha} \in \mathcal{R}^\tau_{\pi,\alpha}$ and $\psi_{\pi',\alpha} \in \mathcal{R}^\tau_{\pi',\alpha}$. Similarly, the cluster density $\psi_{\pi,\alpha}(\kappa)$ is strictly increasing in its dependence on α as π is held fixed.*

(b) *For $\alpha \geq 0$, further monotonicities hold: $\psi_{\pi,\alpha}(u)$ is strictly decreasing in its dependence on α, and $\psi_{\pi,\alpha}(\kappa)$ is strictly decreasing in its dependence on π. More precisely, suppose $\pi_0 \leq \pi \leq \pi' \leq \pi_1$ and $0 \leq \alpha_0 \leq \alpha \leq \alpha' \leq \alpha_1$ and let $A = \phi^1_{\pi_1,\alpha_0}(H_e)\phi^0_{\pi_0,\alpha_1}(\omega(e)) > 0$. Then*

$$\psi_{\pi,\alpha'}(u) - \psi_{\pi,\alpha}(u) \leq -Ad(\alpha' - \alpha) \qquad \text{if } \alpha' > \alpha, \tag{5.4}$$

and

$$\psi_{\pi',\alpha}(\kappa) - \psi_{\pi,\alpha}(\kappa) \leq -Ad(\pi' - \pi) \qquad \text{if } \pi' > \pi. \tag{5.5}$$

Part (a) is an immediate consequence of Lemma 4.10, and is really the simplest constraint on the edge and cluster densities that inequality (4.11) yields. A weaker version of the first statement of part (a) appeared as Thm. 4.2(b) in [Gr3]. Part (b) is proved by an estimate of the mixed second partial derivative of $f(\pi, \alpha)$ in finite volume that we prove with the help of the FKG inequality, hence the need for $\alpha \geq 0$. For $\alpha \geq 0$, inequality (4.11) can be combined with the standard comparison inequalities [Gr3, Thm. 2.2] to obtain further constraints on the edge and cluster densities' dependence on (π, α).

We begin the proofs with a technical lemma and then argue Theorem 6.

Lemma 5.6. *Assume the following: $\psi \in \mathcal{R}^\tau_{\pi,\alpha}$, $\psi' \in \mathcal{R}^\tau_{\pi',\alpha'}$, $\pi' \leq \pi$, $\psi \leq \psi'$, and $\psi(\kappa) = \psi'(\kappa)$. Then $\psi = \psi'$.*

Proof. Let P be the standard coupling of ψ and ψ' on the space $\{(\omega, \zeta) \in \Omega^2 : \omega \leq \zeta\}$ as given for example by [Lg, Thm. II.2.4]. P can be assumed translation-invariant; for if it is not, replace it with any limit point of the averages of P under translations. Let $C(x, \omega)$ be the connected cluster containing vertex x in the graph $(\mathbb{Z}^d, \eta(\omega))$, and correspondingly for $C(x, \zeta)$. By the ordering, $C(x, \omega) \subseteq C(x, \zeta)$ for all pairs (ω, ζ). But $\psi(\kappa) = \psi'(\kappa)$ and translation-invariance imply that $E[|C(x, \omega)|^{-1}] = E[|C(x, \zeta)|^{-1}]$ and consequently $C(x, \omega) = C(x, \zeta)$ P-a.s.

Pick and fix an edge $e = \langle x, y\rangle$. Let $A = \{x \leftrightarrow y\}$ be the event that x and y are connected, and $B = \{x \leftrightarrow y \text{ off } e\}$ the event that x and y are connected in the graph $(\mathbb{Z}^d, \eta(\omega) \setminus \{e\})$. By the conclusion of the first paragraph,

$$\psi(A) = \psi'(A). \tag{5.7}$$

By the dominance assumption $\psi \le \psi'$,

$$\psi(\omega(e) = 1) \le \psi'(\omega(e) = 1). \tag{5.8}$$

For $\Lambda = \{x, y\}$ and $\xi \in B$, the specification (2.2) gives

$$\phi^{\xi}_{\Lambda,\pi,\alpha}(\omega(e) = 1) = \frac{e^{\pi}}{1 + e^{\pi}}.$$

By the DLR equation and the assumption $\pi' \le \pi$,

$$\begin{aligned} \psi(\omega(e) = 0|B) &= \frac{1}{\psi(B)} \int_B \phi^{\xi}_{\Lambda,\pi,\alpha}(\omega(e) = 0)\, \psi(d\xi) = \frac{1}{1 + e^{\pi}} \\ &\le \frac{1}{1 + e^{\pi'}} = \psi'(\omega(e) = 0|B). \end{aligned} \tag{5.9}$$

Again by dominance $\psi(B) \le \psi'(B)$, so

$$\begin{aligned} \psi(\{\omega(e) = 0\} \cap A) = \psi(\{\omega(e) = 0\} \cap B) &= \psi(B)\psi(\omega(e) = 0|B) \\ &\le \psi'(B)\psi'(\omega(e) = 0|B) = \psi'(\{\omega(e) = 0\} \cap B) \\ &= \psi'(\{\omega(e) = 0\} \cap A). \end{aligned} \tag{5.10}$$

Since $A = \{\omega(e) = 1\} \cup \big(\{\omega(e) = 0\} \cap A\big)$, combining (5.7), (5.8), and (5.10) gives $\psi(\omega(e) = 1) = \psi'(\omega(e) = 1)$, and thereby $\psi = \psi'$. $\square$

Proof of Theorem 6. Theorems 4.2 and 5.3 in [Gr3] give the equivalences

$$\phi^0_{\pi,\alpha} = \phi^1_{\pi,\alpha} \iff \text{(ii)} \iff \text{(iv)} \iff \text{(vi)}.$$

Lemma 5.6 gives $\phi^0_{\pi,\alpha} = \phi^1_{\pi,\alpha} \iff$ (v). (Take $\psi = \phi^0_{\pi,\alpha}$, $\psi' = \phi^1_{\pi,\alpha}$, and $\pi' = \pi$.) Consequently, due to equation (4.7), (v) $\Longrightarrow$ (i) because $\phi^0_{\pi,\alpha} = \phi^1_{\pi,\alpha}$ implies that there is a unique random-cluster measure. (i) $\Longrightarrow$ (iii) trivially, and the proof of (ii) $\Longrightarrow$ (iv) given on p. 1479–1480 of [Gr3] can be adapted to prove (iii) $\Longrightarrow$ (v). $\square$

Proof of Theorem 7. (a) If f is not differentiable at (π, α), then by [Rf, Thm. 25.1] the set of subgradients in (4.7) cannot be a singleton, and consequently $\mathcal{R}^{\tau}_{\pi,\alpha}$ cannot be a singleton.

(b) As mentioned, the first part appeared as Thm. 4.5(a) in [Gr3], and the proof on p. 1481 of [Gr3] adapts to prove the second part too. $\square$

Proof of Theorem 8. The containment of $\partial f(\pi, \alpha)$ in the parallelogram follows from the inequalities $\phi^0_{\pi,\alpha} \le \psi \le \phi^1_{\pi,\alpha}$, valid for all $\psi \in \mathcal{R}^{\tau}_{\pi,\alpha}$, and the remarks preceding the statement of the Theorem. That there cannot be a measure $\psi \in \mathcal{R}^{\tau}_{\pi,\alpha}$ such that $\psi(\kappa) = \phi^b_{\pi,\alpha}(\kappa)$ for either $b = 0$ or $b = 1$ follows from Lemma 5.6. $\square$

Define the finite volume free energy by

$$f^{\xi}_{\Lambda}(\pi, \alpha) = \frac{1}{|\mathbb{E}_{\Lambda}|} \log Y^{\xi}_{\Lambda,\pi,\alpha}\,, \tag{5.11}$$

with $Y^{\xi}_{\Lambda,\pi,\alpha}$ as in equation (2.3). Recall that H_e is the event that the endpoints of the edge e are in distinct connected clusters.

Lemma 5.12. *For $\alpha \geq 0$, any finite rectangle Λ and boundary condition ξ,*

$$\frac{\partial^2}{\partial\pi\partial\alpha} f^{\xi}_{\Lambda}(\pi,\alpha) \leq -\frac{d}{|\mathbb{E}_\Lambda|}\sum_{e\in\mathbb{E}_\Lambda} \phi^1_{\Lambda,\pi,\alpha}(H_e)\phi^0_{\Lambda,\pi,\alpha}(\omega(e)). \tag{5.13}$$

Proof. The left-hand side of (5.13) equals

$$\begin{aligned} &\frac{d}{|\mathbb{E}_\Lambda|}\sum_{e\in\mathbb{E}_\Lambda} \phi^{\xi}_{\Lambda,\pi,\alpha}\big(\omega(e)k(\omega,\Lambda)\big) \\ &\qquad - \frac{d}{|\mathbb{E}_\Lambda|}\phi^{\xi}_{\Lambda,\pi,\alpha}\Big(\sum_{e\in\mathbb{E}_\Lambda}\omega(e)\Big)\phi^{\xi}_{\Lambda,\pi,\alpha}\big(k(\omega,\Lambda)\big). \end{aligned} \tag{5.14}$$

For a given ω, let $\mathcal{C}=\mathcal{C}(\omega)$ be the collection of subsets of Λ that form the connected clusters counted by $k(\omega,\Lambda)$. An element of $\mathcal{C}(\omega)$ is not necessarily connected in the graph $(\Lambda,\eta(\omega)\cap\mathbb{E}_\Lambda)$ due to the effect of edges outside $\mathbb{E}_\Lambda$. For a vertex x, $B(x)$ is the element of $\mathcal{C}$ that contains x and $E(x)$ is the set of edges in $\eta(\omega)$ with both endpoints in $B(x)$. $B(x)^c$ is the set of vertices of Λ not connected to x, and then $\mathbb{E}(B(x)^c)$ is the set of edges in $\mathbb{E}_\Lambda$ both of whose endpoints lie in $B(x)^c$. The σ-field $\mathcal{B}_x$ contains the information about $B(x)$ and $E(x)$. Notice in particular that, given $\mathcal{B}_x$, $\omega(e)=0$ for all edges e that connect $B(x)$ and $B(x)^c$. Using

$$k(\omega,\Lambda)=\sum_{x\in\Lambda}\frac{1}{|B(x)|} \tag{5.15}$$

we rewrite the first term of (5.14) as

$$\begin{aligned} &\frac{d}{|\mathbb{E}_\Lambda|}\sum_{e\in\mathbb{E}_\Lambda} \phi^{\xi}_{\Lambda,\pi,\alpha}\big(\omega(e)k(\omega,\Lambda)\big) \\ &= \frac{d}{|\mathbb{E}_\Lambda|}\sum_{x\in\Lambda}\phi^{\xi}_{\Lambda,\pi,\alpha}\Big(\frac{1}{|B(x)|}\sum_{e\in\mathbb{E}_\Lambda}\phi^{\xi}_{\Lambda,\pi,\alpha}\{\omega(e)|\mathcal{B}_x\}\Big) \\ &= \frac{d}{|\mathbb{E}_\Lambda|}\Big\{\phi^{\xi}_{\Lambda,\pi,\alpha}\Big(\sum_{x\in\Lambda}\frac{|E(x)|}{|B(x)|}\Big) \\ &\qquad + \sum_{x\in\Lambda}\phi^{\xi}_{\Lambda,\pi,\alpha}\Big(\frac{1}{|B(x)|}\sum_{e\in\mathbb{E}(B(x)^c)}\phi^{\xi}_{\Lambda,\pi,\alpha}\{\omega(e)|\mathcal{B}_x\}\Big)\Big\} \\ &= \frac{d}{|\mathbb{E}_\Lambda|}\Big\{\phi^{\xi}_{\Lambda,\pi,\alpha}\Big(\sum_{e\in\mathbb{E}_\Lambda}\omega(e)\Big) \\ &\qquad + \phi^{\xi}_{\Lambda,\pi,\alpha}\Big(\sum_{B\in\mathcal{C}}\sum_{e\in\mathbb{E}(B^c)}\phi^{\xi}_{\Lambda,\pi,\alpha}\{\omega(e)|\omega(\mathbb{E}^+_B)\equiv 0\}\Big)\Big\}. \end{aligned} \tag{5.16}$$

In the last line, the symbol $\mathbb{E}^+_B$ denotes the set of edges in $\mathbb{E}_\Lambda$ that are incident to the vertex set B. We used the fact that if e is not incident to B, then

$$\phi^{\xi}_{\Lambda,\pi,\alpha}\{\omega(e)|B(x)=B\}=\phi^{\xi}_{\Lambda,\pi,\alpha}\{\omega(e)|\omega(\mathbb{E}^+_B)\equiv 0\} \tag{5.17}$$

because B must be disconnected from the rest of Λ, and the effect on e is the same as with all edges incident to B set equal to zero.

Now switch attention to the second term in (5.14). First note the equality

$$k(\omega,\Lambda)=|\mathcal{C}(\omega)|=\sum_{B\in\mathcal{C}(\omega)}\mathrm{I}_{\{e\in\mathbb{E}(B^c)\}}(\omega)+1+\mathrm{I}_{H_e}(\omega), \tag{5.18}$$

valid for any fixed edge $e=\langle x,y\rangle$ and all ω. (Proof: The first sum on the right misses one or two elements of $\mathcal{C}(\omega)$ depending on whether the endpoints of e are in the same cluster or not.) By (5.18), the second term in (5.14) can be written as

$$\begin{aligned}
&\frac{d}{|\mathbb{E}_\Lambda|}\phi^\xi_{\Lambda,\pi,\alpha}\Big(\sum_{e\in\mathbb{E}_\Lambda}\phi^\xi_{\Lambda,\pi,\alpha}\big(\omega(e)\big)k(\omega,\Lambda)\Big)\\
&=\frac{d}{|\mathbb{E}_\Lambda|}\phi^\xi_{\Lambda,\pi,\alpha}\Big(\sum_{e\in\mathbb{E}_\Lambda}\phi^\xi_{\Lambda,\pi,\alpha}\big(\omega(e)\big)\Big[\sum_{B\in\mathcal{C}}\mathrm{I}_{\{e\in\mathbb{E}(B^c)\}}+1+\mathrm{I}_{H_e}\Big]\Big)\\
&=\frac{d}{|\mathbb{E}_\Lambda|}\Big\{\phi^\xi_{\Lambda,\pi,\alpha}\Big(\sum_{e\in\mathbb{E}_\Lambda}\omega(e)\Big)+\phi^\xi_{\Lambda,\pi,\alpha}\Big(\sum_{B\in\mathcal{C}}\sum_{e\in\mathbb{E}(B^c)}\phi^\xi_{\Lambda,\pi,\alpha}\big(\omega(e)\big)\Big)\\
&\qquad+\sum_{e\in\mathbb{E}_\Lambda}\phi^\xi_{\Lambda,\pi,\alpha}(H_e)\phi^\xi_{\Lambda,\pi,\alpha}\big(\omega(e)\big)\Big\}.
\end{aligned} \tag{5.19}$$

Now subtract the last line of (5.19) from the last line of (5.16). By the FKG inequality

$$\phi^\xi_{\Lambda,\pi,\alpha}\{\omega(e)|\omega(\mathbb{E}^+_B)\equiv 0\}-\phi^\xi_{\Lambda,\pi,\alpha}\big(\omega(e)\big)\le 0,$$

and consequently the quantity in (5.14) is at most

$$-\frac{d}{|\mathbb{E}_\Lambda|}\sum_{e\in\mathbb{E}_\Lambda}\phi^\xi_{\Lambda,\pi,\alpha}(H_e)\phi^\xi_{\Lambda,\pi,\alpha}\big(\omega(e)\big).$$

The statement (5.13) follows from stochastic dominance, since H_e is decreasing and $\omega(e)$ increasing, as functions of ω. $\square$

Proof of Theorem 9. As indicated, part (a) follows from Lemma 4.10. For part (b) we prove (5.4) and leave the analogous proof of (5.5) to the reader. Utilizing (5.1) and stochastic dominance, the goal is to prove

$$\frac{\partial f}{\partial\pi^+}(\pi,\alpha')-\frac{\partial f}{\partial\pi^-}(\pi,\alpha)\le -A(\pi,\alpha,\alpha')d(\alpha'-\alpha) \tag{5.20}$$

for $\alpha'>\alpha\ge 0$, with

$$A(\pi,\alpha,\alpha')=\phi^1_{\pi,\alpha}(H_e)\phi^0_{\pi,\alpha'}(\omega(e)).$$

Let us recall the most basic comparison inequality [Gr3, Thm. 2.2]:

$$\phi^\xi_{\Lambda,\pi_0,\alpha_0}\le\phi^\xi_{\Lambda,\pi_1,\alpha_1}\ \text{if}\ \alpha_1\le\alpha_0,\ \alpha_0\ge 0,\ \text{and}\ \pi_1\ge\pi_0. \tag{5.21}$$

From Lemma 5.12 and (5.21), we get

$$\frac{\partial f_\Lambda^\xi}{\partial \pi}(\pi,\alpha') - \frac{\partial f_\Lambda^\xi}{\partial \pi}(\pi,\alpha) \le -\frac{d(\alpha'-\alpha)}{|\mathbb{E}_\Lambda|}\sum_{e\in\mathbb{E}_\Lambda} \phi^1_{\Lambda,\pi,\alpha}(H_e)\phi^0_{\Lambda,\pi,\alpha'}(\omega(e)). \tag{5.22}$$

Suppose first that $\partial f/\partial\pi$ exists at (π,α) and (π,α'). By a convex-analytic lemma [E, Lemma IV.6.3] the left-hand side of (5.22) converges to the left-hand side of (5.20) as $\Lambda \to \mathbb{Z}^d$. For a fixed edge e, the quantity $\phi^1_{\Lambda,\pi,\alpha}(H_e)$ increases to $\phi^1_{\pi,\alpha}(H_e)$ as $\Lambda\to\mathbb{Z}^d$ because $\phi^1_{\Lambda,\pi,\alpha}\searrow\phi^1_{\pi,\alpha}$ and the event H_e is decreasing. Similarly $\phi^0_{\Lambda,\pi,\alpha'}(\omega(e))$ increases to $\phi^0_{\pi,\alpha'}(\omega(e))$, and (5.20) follows.

Now for the general case. Recall from Theorem 6 that on each horizontal and vertical line in the $\{\alpha\ge 0\}$ half-space of the (π,α)-plane, both partial derivatives of f exist except for at most countably many exceptional points. First pick $\alpha_0\in(\alpha,\alpha')$ such that $\partial f/\partial\pi(\pi,\alpha_0)$ exists. Then pick $\pi_0<\pi<\pi_1$ so that $\partial f/\partial\pi$ exists at (π_i,α), (π_i,α'), and (π_i,α_0) for $i=0,1$. By the case proved above,

$$\begin{aligned}\frac{\partial f}{\partial\pi}(\pi_1,\alpha') &\le \frac{\partial f}{\partial\pi}(\pi_1,\alpha_0) - A(\pi_1,\alpha_0,\alpha')d(\alpha'-\alpha_0)\\ &= \frac{\partial f}{\partial\pi}(\pi_0,\alpha_0) - A(\pi_1,\alpha_0,\alpha')d(\alpha'-\alpha_0)\\ &\qquad + \left[\frac{\partial f}{\partial\pi}(\pi_1,\alpha_0) - \frac{\partial f}{\partial\pi}(\pi_0,\alpha_0)\right].\end{aligned}$$

The inequality $\partial f/\partial\pi(\pi_0,\alpha_0)\le\partial f/\partial\pi(\pi_0,\alpha)$ follows from (5.1) and (5.21) because the derivatives exist. Thus we get

$$\begin{aligned}\frac{\partial f}{\partial\pi}(\pi_1,\alpha') &\le \frac{\partial f}{\partial\pi}(\pi_0,\alpha) - A(\pi_1,\alpha_0,\alpha')d(\alpha'-\alpha_0)\\ &\qquad + \left[\frac{\partial f}{\partial\pi}(\pi_1,\alpha_0) - \frac{\partial f}{\partial\pi}(\pi_0,\alpha_0)\right].\end{aligned}\tag{5.23}$$

Now let $\pi_0\nearrow\pi$ and $\pi_1\searrow\pi$. First note that

$$\frac{\partial f}{\partial\pi}(\pi_1,\alpha')\to\frac{\partial f}{\partial\pi^+}(\pi,\alpha') \qquad\text{and}\qquad \frac{\partial f}{\partial\pi}(\pi_0,\alpha)\to\frac{\partial f}{\partial\pi^-}(\pi,\alpha).$$

Secondly, since $\partial f/\partial\pi(\pi,\alpha_0)$ exists,

$$\frac{\partial f}{\partial\pi}(\pi_1,\alpha_0) - \frac{\partial f}{\partial\pi}(\pi_0,\alpha_0) \;\to\; \frac{\partial f}{\partial\pi}(\pi,\alpha_0) - \frac{\partial f}{\partial\pi}(\pi,\alpha_0) = 0.$$

Thirdly, we argue that

$$\lim_{\pi_1\searrow\pi} A(\pi_1,\alpha_0,\alpha') \ge A(\pi,\alpha_0,\alpha'). \tag{5.24}$$

This follows from

$$\lim_{\pi_1\searrow\pi}\phi^0_{\pi_1,\alpha_0}(\omega(e)) = \phi^1_{\pi,\alpha_0}(\omega(e)) \ge \phi^0_{\pi,\alpha_0}(\omega(e)), \tag{5.25}$$

and

$$\lim_{\pi_1 \searrow \pi} \phi^1_{\pi_1,\alpha'}(H_e) = \phi^1_{\pi,\alpha'}(H_e). \tag{5.26}$$

Equation (5.25) is a part of equation (4.14) in [Gr3]. For (5.26), note first that $\psi(\mathrm{bd}\, H_e) = 0$ for all translation-invariant random-cluster measures ψ because $\omega \in \mathrm{bd}\, H_e$ implies that the endpoints of e lie in distinct infinite clusters. Hence (5.26) is equivalent to

$$\lim_{\pi_1 \searrow \pi} \phi^1_{\pi_1,\alpha'}(K) = \phi^1_{\pi,\alpha'}(K) \tag{5.27}$$

for the closure K of the complement of H_e. H_e is a decreasing event, and so K is a closed increasing event, and then (5.27) follows from Prop. 4.4(b) in [Gr3].

Finally, let $\alpha_0 \searrow \alpha$, and observe that $A(\pi, \alpha_0, \alpha') \geq A(\pi, \alpha, \alpha')$ due to (5.21) and the decreasing nature of H_e. Now inequality (5.23) has turned into inequality (5.20). $\square$

6. The two-dimensional case. Here we strengthen the conclusion of Theorem 8 for the case $d = 2$ by a convex-analytic observation. For $d = 2$ and $\alpha \geq 0$, duality can be used to show that $|\mathcal{R}^\tau_{\pi,\alpha}| > 1$ can happen only if $\pi = \alpha$ (equivalently, $p = \sqrt{q}/(1+\sqrt{q})$, see Thm. 4.3 in [Gr3]). Consequently, $\partial f(\pi, \alpha)$ is a singleton unless $\pi = \alpha$.

Theorem 10. *For all but countably many $\alpha \in \mathbb{R}$, $\partial f(\alpha, \alpha)$ is a line segment with slope -1 on the (s,t)-plane.*

The theorem itself does not require $d = 2$, but we do not know whether the statement has any relevance in higher dimensions. In the case $\alpha \geq 0$ it does not imply that all random-cluster measures are convex combinations of $\phi^0_{\alpha,\alpha}$ and $\phi^1_{\alpha,\alpha}$, only that all $\psi \in \mathcal{R}^\tau_{\alpha,\alpha} \setminus \{\phi^0_{\alpha,\alpha}, \phi^1_{\alpha,\alpha}\}$ have a number $\theta \in (0,1)$ such that

$$\psi(u) = \theta\phi^0_{\alpha,\alpha}(\omega(e)) + (1-\theta)\phi^1_{\alpha,\alpha}(\omega(e)) \quad \text{and} \quad \psi(\kappa) = \theta\phi^0_{\alpha,\alpha}(\kappa) + (1-\theta)\phi^1_{\alpha,\alpha}(\kappa).$$

In light of this result, the known results for large α of Laanait et al. [LM], and conjectures about the continuity of the phase transition (eqn. (1.5) in [Gr3]), the following conjecture appears natural: In two dimensions, $\mathcal{R}_{\alpha,\alpha}$ is a singleton for $\alpha < \alpha_0$, and $\mathcal{R}_{\alpha,\alpha}$ is the convex hull of the two distinct measures $\phi^0_{\alpha,\alpha}$ and $\phi^1_{\alpha,\alpha}$ for $\alpha > \alpha_0$. The Aizenman-Higuchi theorem would then imply that $\alpha_0 \geq (\log 2)/2$ (see Remark 7.9 in the next section), and indeed it is believed that $\alpha_0 = \log 2$ in two dimensions (eqn. (1.6) in [Gr3]).

Proof of Theorem 10. The function $g(\alpha) = f(\alpha, \alpha)$ is convex, and $(s,t) \in \partial f(\alpha, \alpha)$ implies $s+t \in \partial g(\alpha)$. Since g is differentiable at all but countably many α, it follows that $s + t$ is constant over $(s,t) \in \partial f(\alpha, \alpha)$ for all but countably many α. $\square$

7. The case of positive integral q. As mentioned in the introduction, originally much of the interest in random-cluster measures centered on their relationship with Ising and Potts models. This connection involves the random-cluster measures with parameter values $q = 2, 3, 4, \ldots$ In this section we specialize to this case the results of Section 3 on entropy. Fix such a q and let $p \in (0, 1)$ and the dimension $d \geq 2$.

To describe the Potts model, let $\Sigma = \{1, \ldots, q\}^{\mathbb{Z}^d}$ be the space of spin configurations. The Ising model is the case $q = 2$. Translations act as usual, $(\tau_x \sigma)(y) = \sigma(x + y)$ for $x, y \in \mathbb{Z}^d$. $\mathcal{M}(\Sigma)$ denotes the space of probability measures on Σ and $\mathcal{M}^\tau(\Sigma)$ the subspace of translation-invariant probability measures. In this notation Σ is replaced by Ω when the measures are on the edges instead of spins. For finite $\Lambda \subseteq \mathbb{Z}^d$ and a boundary condition $\zeta \in \Sigma$, a probability measure $\gamma^\zeta_{\Lambda,\beta,q}$ on $\Sigma_\Lambda = \{1, \ldots, q\}^\Lambda$ is defined by

$$\gamma^\zeta_{\Lambda,\beta,q}(\sigma) = \frac{1}{Z_{\Lambda,\beta,q}} \exp\Bigg\{-\beta \sum_{\langle x,y\rangle \in \mathbb{E}_\Lambda} \mathrm{I}_{\{\sigma(x)\neq\sigma(y)\}} - \beta \sum_{\substack{\langle x,y\rangle \in \mathbb{E} \\ x\in\Lambda, y\in\Lambda^c}} \mathrm{I}_{\{\sigma(x)\neq\zeta(y)\}}\Bigg\}. \tag{7.1}$$

The parameter $\beta > 0$ is the inverse temperature. We write $\zeta = j$ for the boundary condition when $\zeta(y) = j$ for all vertices $y \notin \Lambda$, for a fixed $j \in \{1, \ldots, q\}$. The *free boundary condition* is denoted by $\zeta = 0$. This means that there is no influence from outside Λ and the second sum in the exponent of (7.1) disappears.

The measures (7.1) form a specification. The set $\mathcal{G}^\tau_{\beta,q}$ of translation-invariant infinite-volume Potts measures contains those measures $\gamma \in \mathcal{M}^\tau(\Sigma)$ that satisfy

$$\gamma(A \times B) = \int_B \gamma^\zeta_{\Lambda,\beta,q}(A)\, \gamma(d\zeta)$$

for all finite $\Lambda \subseteq \mathbb{Z}^d$ and measurable sets $A \subseteq \Sigma_\Lambda$ and $B \subseteq \Sigma_{\Lambda^c}$.

It is well-known that the specific relative entropy

$$h(\rho|\gamma) = \lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|} H_\Lambda(\rho|\gamma) = \lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|} H(\rho_\Lambda|\gamma^0_{\Lambda,\beta,q}) \tag{7.2}$$

exists for any $\rho \in \mathcal{M}^\tau(\Sigma)$ and $\gamma \in \mathcal{G}^\tau_{\beta,q}$. The finite-volume entropy $H_\Lambda(\mu|\rho)$ is now over the σ-field generated by the spins $\sigma(x)$ in Λ instead of edges, compare with (3.4). Furthermore, the quantity $h(\rho|\gamma)$ is independent of the choice of $\gamma \in \mathcal{G}^\tau_{\beta,q}$, and $h(\rho|\gamma) = 0$ if and only if $\rho \in \mathcal{G}^\tau_{\beta,q}$ (see [DSZ]).

For a spin configuration σ that is defined on the endpoints x, y of the edge $e = \langle x, y\rangle$, define the measure $\nu^\sigma_e = \nu_e^{\sigma(x),\sigma(y)}$ on the edge $\omega(e)$ by

$$\nu^\sigma_e(\omega(e)) = \mathrm{I}_{\{\sigma(x)\neq\sigma(y)\}}\delta_0(\omega(e)) + \mathrm{I}_{\{\sigma(x)=\sigma(y)\}}\big[p\delta_1(\omega(e)) + (1-p)\delta_0(\omega(e))\big].$$

In other words, two nearest-neighbor spins in agreement are connected with an edge with probability p, and otherwise edge e is absent. When σ is defined on a rectangle Λ or on the entire lattice $\mathbb{Z}^d$, let ν^σ_Λ and ν^σ denote the product measures

$$\nu^\sigma_\Lambda(\omega) = \bigotimes_{e\in\mathbb{E}_\Lambda} \nu^\sigma_e(\omega(e)) \qquad \text{and} \qquad \nu^\sigma(d\omega) = \bigotimes_{e\in\mathbb{E}} \nu^\sigma_e(d\omega(e)).$$

Under ν^σ, the edge process $(\omega(e))_{e\in\mathbb{E}}$ is independent but nonstationary. If p and β satisfy $p = 1 - e^{-\beta}$, then for $\omega \in \Omega^0_\Lambda$,

$$\phi^0_{\Lambda,p,q}(\omega) = \int \nu^\sigma_\Lambda(\omega)\, \gamma^0_{\Lambda,\beta,q}(d\sigma). \tag{7.3}$$

Underlying this formula is a coupling of the random-cluster measure and the Potts measure, introduced by Swendsen and Wang [SW] and Edwards and Sokal [ES].

Given translation-invariant probability measures ρ on Σ and μ on Ω, a specific entropy of μ relative to the random measure ν^σ is defined as follows:

$$R_\Lambda(\mu|\rho) = \sup_g \bigl\{\mu(g) - \int_{\Sigma_\Lambda} \log \nu^\sigma_\Lambda(e^g)\, \rho_\Lambda(d\sigma)\bigr\} \tag{7.4}$$

and

$$r(\mu|\rho) = \lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\mathbb{E}_\Lambda|} R_\Lambda(\mu|\rho) = \sup_\Lambda \frac{1}{|\mathbb{E}_\Lambda|} R_\Lambda(\mu|\rho). \tag{7.5}$$

As before, the supremum in (7.4) is over functions g on Ω_Λ. The existence of the limit and the second equality in (7.5) are proved in [Se1].

The relevance of the entropy $r(\mu|\rho)$ is the following: If the spin configuration σ is *generic* for ρ, meaning that the empirical measure

$$M_\Lambda(\sigma) = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \delta_{\tau_x\sigma}$$

converges to ρ weakly as $\Lambda \to \mathbb{Z}^d$, then the probabilities $\nu^\sigma(M_\Lambda(\omega) \in A)$ satisfy large deviation estimates of the type stated in Theorem 3, with rate function $\mu \mapsto r(\mu|\rho)$. This can be proved by the argument of [Se1]. (The only reason we cannot directly quote a theorem from [Se1] is that the results in [Se1] are for spins instead of edges.) Secondly, this entropy appears in the variational characterization of disordered Gibbs measures in [Se2].

From the point of view of the measures ν^σ, Theorem 3 gives large deviations for the mixtures

$$\int \nu^\sigma_\Lambda(M_\Lambda(\omega) \in A)\, \gamma^0_{\Lambda,\beta,q}(d\sigma). \tag{7.6}$$

A natural question is whether the rate function $I_{\pi,\alpha}(\mu)$ of Theorem 3 is related to $r(\mu|\rho)$, and this is answered by the next theorem. Write $I_{p,q}(\mu)$ for $I_{\pi,\alpha}(\mu)$ when p, q, π, and α are related by equation (2.1).

Theorem 11. *Suppose $q \geq 2$ is a positive integer. Pick any translation-invariant Potts measure $\gamma \in \mathcal{G}^\tau_{\beta,q}$ for the inverse temperature β that satisfies $p = 1 - e^{-\beta}$. Then for $\mu \in \mathcal{M}^\tau(\Omega)$,*

$$I_{p,q}(\mu) = \inf_{\rho\in\mathcal{M}^\tau(\Sigma)} \{r(\mu|\rho) + d^{-1}h(\rho|\gamma)\}. \tag{7.7}$$

Corollary 1. *For positive integral $q \geq 2$ and translation-invariant measures ψ on Ω, $\psi \in \mathcal{R}^\tau_{p,q}$ if and only if $\psi = \int \nu^\sigma(\cdot)\gamma(d\sigma)$ for some $\gamma \in \mathcal{G}^\tau_{\beta,q}$ with $p = 1 - e^{-\beta}$.*

A further corollary to this statement is that all translation-invariant random-cluster measures are limits of averages of translates of product measures on the edges. Namely, if σ is generic for $\gamma \in \mathcal{G}^\tau_{\beta,q}$ and $\psi = \int \nu^\sigma(\cdot)\gamma(d\sigma)$, then by the continuity of $\sigma \mapsto \nu^\sigma$,

$$\psi = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \nu^{\tau_x \sigma} \tag{7.8}$$

in the weak topology of $\mathcal{M}(\Omega)$. There is also exponentially fast convergence of empirical averages in ν^σ-probability: For any weak neighborhood U of ψ,

$$\nu^\sigma(M_\Lambda(\omega) \notin U) \leq e^{-c_0|\Lambda|}$$

for some constant $c_0 > 0$. This follows from the large deviation estimate for $\nu^\sigma(M_\Lambda(\omega) \notin U)$ mentioned above, and the fact that $r(\mu|\gamma) = 0$ if and only if $\mu = \int \nu^\sigma(\cdot)\gamma(d\sigma)$ [Se1, Thm. 2.8].

Remark 7.9. Let $\gamma^j_{\beta,q} = \lim_{\Lambda \to \mathbb{Z}^d} \gamma^j_{\Lambda,\beta,q}$, $j \in \{0, 1, \ldots, q\}$. These limiting Potts measures exist and are elements of $\mathcal{G}^\tau_{\beta,q}$ (see [BC] and p. 67–69 in [Ge]). For the free and wired boundary conditions, the statement of Corollary 1 specializes to these equalities:

$$\phi^0_{p,q} = \int \nu^\sigma(\cdot)\, \gamma^0_{\beta,q}(d\sigma) \quad \text{and} \quad \phi^1_{p,q} = \int \nu^\sigma(\cdot)\, \gamma^j_{\beta,q}(d\sigma),\ 1 \leq j \leq q. \tag{7.10}$$

The phase transition of the random-cluster model can be characterized in terms of Potts measures in this way: $\phi^0_{p,q} = \phi^1_{p,q}$ if and only if $\gamma^0_{\beta,q}$ is in the convex hull of $\{\gamma^j_{\beta,q} : 1 \leq j \leq q\}$, which in turn is equivalent to $\gamma^0_{\beta,q} = q^{-1}\sum_{j=1}^q \gamma^j_{\beta,q}$. According to the Aizenman-Higuchi theorem [Ai, Hi], in the case $d = q = 2$ (the two-dimensional ferromagnetic Ising model) the set $\mathcal{G}^\tau_{\beta,2}$ equals the convex hull of the pair $\{\gamma^1_{\beta,2}, \gamma^2_{\beta,2}\}$. Thus there is no phase transition for the random-cluster model for $d = q = 2$. For the sake of completeness, we outline a proof of (7.10) and the equivalence stated above at the end of the section.

To prove Theorem 11, we introduce periodized configurations of both edges and spins. From now on we use the fixed sequence of cubes $\Lambda_n = \{-n, \ldots, n\}^d$ centered at the origin. Let us write edges $e = \langle x, y\rangle$ with the convention that $y_i \geq x_i$ for each $i = 1, \ldots, d$, so that, in fact, $y_i = x_i$ for all but one coordinate j for which $y_j = x_j + 1$. The vertex $x^{(n)}$, "x modulo Λ_n", is defined as the unique vertex of the set $\{x + (2n+1)y : y \in \mathbb{Z}^d\}$ that lies in Λ_n. For $\omega \in \Omega$ and $\sigma \in \Sigma$, the periodized configurations are defined by

$$\omega^{(n)}(\langle x, y\rangle) = \omega(\langle x^{(n)}, x^{(n)} + y - x\rangle) \qquad \text{and} \qquad \sigma^{(n)}(x) = \sigma(x^{(n)}).$$

Notice that $\omega^{(n)}(e) = \omega(e)$ for all edges e whose lower endpoint lies in Λ_n. Next we consider the periodized empirical measures $M_{\Lambda_n}(\omega^{(n)})$ and $M_{\Lambda_n}(\sigma^{(n)})$. They are translation-invariant measures. Let

$$\mathcal{M}^\tau_{\Lambda_n} = \{M_{\Lambda_n}(\sigma^{(n)}) : \sigma \in \Sigma\},$$

a closed subset of $\mathcal{M}^\tau(\Sigma)$. The benefit of periodization is that the distributions in the mixture (7.6) then depend on σ only through the empirical measure, as stated in the next lemma.

Lemma 7.11. *Suppose the empirical measures $M_{\Lambda_n}(\sigma^{(n)})$ and $M_{\Lambda_n}(\zeta^{(n)})$ of the periodized configurations $\sigma^{(n)}$ and $\zeta^{(n)}$ coincide. Then the distributions of $M_{\Lambda_n}(\omega^{(n)})$ under $\nu^{\sigma^{(n)}}$ and $\nu^{\zeta^{(n)}}$ coincide. There is a continuous $\mathcal{M}\big(\mathcal{M}^\tau(\Omega)\big)$-valued map $\rho \mapsto \lambda_{\Lambda_n}(\rho, \cdot)$ defined on $\mathcal{M}^\tau_{\Lambda_n}$ that satisfies*

$$\nu^{\sigma^{(n)}}(M_{\Lambda_n}(\omega^{(n)}) \in A) = \lambda_{\Lambda_n}(M_{\Lambda_n}(\sigma^{(n)}), A) \tag{7.12}$$

for measurable sets $A \subseteq \mathcal{M}(\Omega)$.

Proof. Periodicity implies that $\omega^{(n)}(\langle x+z, y+z\rangle) = \omega^{(n)}(\langle x+z^{(n)}, y+z^{(n)}\rangle)$ for all $e = \langle x, y\rangle \in \mathbb{E}$ and $z \in \mathbb{Z}^d$. From this follows $\tau_x\omega^{(n)} = \tau_{x^{(n)}}\omega^{(n)}$, and then

$$M_{\Lambda_n}(\tau_x\omega^{(n)}) = M_{\Lambda_n}(\omega^{(n)}). \tag{7.13}$$

We do a calculation to show that

$$\nu^{\sigma^{(n)}}\big[G(\tau_z\omega^{(n)})\big] = \nu^{\tau_z\sigma^{(n)}}\big[G(\omega^{(n)})\big] \quad \text{for any } z \in \mathbb{Z}^d, \tag{7.14}$$

and for any bounded measurable function G on Ω. Let E'_n be the set of edges with lower endpoint in Λ_n. By the periodicity and general measure-theoretic facts, it suffices to consider a function of the type

$$G(\omega^{(n)}) = \prod_{\langle x,y\rangle \in E'_n} g_{\langle x,y\rangle}\big(\omega^{(n)}(\langle x, y\rangle)\big),$$

where the $g_{\langle x,y\rangle}$'s are functions on $\{0,1\}$. Then the expectation on the left-hand side of equation (7.14) becomes

$$\begin{aligned}
&\nu^{\sigma^{(n)}}\Big[\prod_{\langle x,y\rangle \in E'_n} g_{\langle x,y\rangle}\big(\omega^{(n)}(\langle x+z, y+z\rangle)\big)\Big] \\
&= \nu^{\sigma^{(n)}}\Big[\prod_{\langle x,y\rangle \in E'_n} g_{\langle x,y\rangle}\big(\omega^{(n)}(\langle (x+z)^{(n)}, (x+z)^{(n)} + y - x\rangle)\big)\Big] \\
&= \prod_{\langle x,y\rangle \in E'_n} \nu_{\langle x,y\rangle}\big\{\sigma^{(n)}((x+z)^{(n)}), \sigma^{(n)}((x+z)^{(n)} + y - x)\big\}\big[g_{\langle x,y\rangle}\big] \\
&= \prod_{\langle x,y\rangle \in E'_n} \nu_{\langle x,y\rangle}\big\{\tau_z\sigma^{(n)}(x), \tau_z\sigma^{(n)}(y)\big\}\big[g_{\langle x,y\rangle}\big] \\
&= \nu^{\tau_z\sigma^{(n)}}\Big[\prod_{\langle x,y\rangle \in E'_n} g_{\langle x,y\rangle}\big(\omega^{(n)}(\langle x, y\rangle)\big)\Big].
\end{aligned}$$

This proves (7.14). Above we wrote $\nu_{\langle x,y\rangle}\{\sigma(x), \sigma(y)\} = \nu^{\sigma(x),\sigma(y)}_{\langle x,y\rangle}$ to avoid complicated superscripts.

Let G be any bounded measurable function on $\mathcal{M}^\tau(\Omega)$, and let σ and ζ be as in the statement of the lemma. Since an empirical measure assigns mass to only finitely many points, it follows that $\zeta^{(n)} = \tau_z\sigma^{(n)}$ for some $z \in \Lambda$. Consequently by (7.13) and (7.14),

$$\nu^{\sigma^{(n)}}\big[G\big(M_{\Lambda_n}(\omega^{(n)})\big)\big] = \nu^{\sigma^{(n)}}\big[G\big(M_{\Lambda_n}(\tau_z\omega^{(n)})\big)\big]$$
$$= \nu^{\tau_z\sigma^{(n)}}\big[G\big(M_{\Lambda_n}(\omega^{(n)})\big)\big] = \nu^{\zeta^{(n)}}\big[G\big(M_{\Lambda_n}(\omega^{(n)})\big)\big].$$

This implies that the measure $\lambda_{\Lambda_n}(\rho,\,\cdot\,)$ can be defined by equation (7.12). The continuity of $\rho \mapsto \lambda_{\Lambda_n}(\rho,\,\cdot\,)$ follows from the continuity of $\sigma \mapsto \nu^\sigma$ and the compactness of Σ. $\square$

Following the proof in [Se1] shows that the large deviation theorem of the probabilities $\nu^{\sigma^{(n)}}(M_{\Lambda_n}(\omega^{(n)}) \in A)$ is completely driven by the convergence of the empirical measure of σ. The result can now be stated in this way: Let $\rho_n \in \mathcal{M}^\tau_{\Lambda_n}$ be any sequence of translation-invariant empirical measures that converge to $\rho \in \mathcal{M}^\tau(\Sigma)$ as $n \to \infty$. Then for any measurable set $A \subseteq \mathcal{M}(\Omega)$,

$$\begin{aligned} -\inf_{\mu\in\mathrm{int}\,A} r(\mu|\rho) &\le \liminf_{n\to\infty} \frac{1}{|\mathbb{E}_{\Lambda_n}|}\log\lambda_{\Lambda_n}(\rho_n, A) \\ &\le \limsup_{n\to\infty} \frac{1}{|\mathbb{E}_{\Lambda_n}|}\log\lambda_{\Lambda_n}(\rho_n, A) \\ &\le -\inf_{\mu\in\mathrm{cl}\,A} r(\mu|\rho). \end{aligned} \tag{7.15}$$

This points to an alternative derivation of Theorem 3 for integral q. The large deviation principle

$$\begin{aligned} -\inf_{\rho\in\mathrm{int}\,A} h(\rho|\gamma) &\le \liminf_{n\to\infty} \frac{1}{|\Lambda_n|}\log\gamma^0_{\Lambda_n,\beta,q}(M_{\Lambda_n}(\sigma^{(n)}) \in A) \\ &\le \limsup_{n\to\infty} \frac{1}{|\Lambda_n|}\log\gamma^0_{\Lambda_n,\beta,q}(M_{\Lambda_n}(\sigma^{(n)}) \in A) \\ &\le -\inf_{\rho\in\mathrm{cl}\,A} h(\rho|\gamma) \end{aligned} \tag{7.16}$$

is well-known [DSZ,FO,Ol]. By Lemma 7.11 the distribution

$$\gamma^0_{\Lambda_n,\beta,q}(M_{\Lambda_n}(\sigma^{(n)}) \in d\rho)$$

serves as the mixing measure for the periodized version of the probability in (7.6), and equation (7.15) gives large deviation principles for the components in the mixture. From these ingredients a recent result of Grunwald's [Gw, Thm. 2.3] gives a large deviation principle for the mixture with rate function exactly as given on the right-hand side of (7.7).

Proof of Theorem 11. Let $I(\mu) = \inf_{\rho\in\mathcal{M}^\tau(\Sigma)}\{r(\mu|\rho) + d^{-1}h(\rho|\gamma)\}$ denote the right-hand side of (7.7). Let g be any function on Ω_{Λ_n} and ρ any translation-invariant probability measure on Σ. Then, by (7.3), Jensen's inequality, and (7.4),

$$\begin{aligned}
&\mu(g) - \log \phi^0_{\Lambda_n,p,q}(e^g) \\
&\le \mu(g) - \log \int_{\Sigma_{\Lambda_n}} \nu^\sigma_{\Lambda_n}(e^g)\left(\frac{d\rho_{\Lambda_n}}{d\gamma^0_{\Lambda_n,\beta,q}}(\sigma)\right)^{-1} \rho_{\Lambda_n}(d\sigma) \\
&\le \mu(g) - \int_{\Sigma_{\Lambda_n}} \log \nu^\sigma_{\Lambda_n}(e^g)\rho_{\Lambda_n}(d\sigma) + \int_{\Sigma_{\Lambda_n}} \log \frac{d\rho_{\Lambda_n}}{d\gamma^0_{\Lambda_n,\beta,q}}\, d\rho_{\Lambda_n} \\
&\le R_{\Lambda_n}(\mu|\rho) + H(\rho_{\Lambda_n}|\gamma^0_{\Lambda_n,\beta,q}).
\end{aligned}$$

Taking supremum over g gives

$$H(\mu_{\Lambda_n}|\phi^0_{\Lambda_n,p,q}) \le R_{\Lambda_n}(\mu|\rho) + H(\rho_{\Lambda_n}|\gamma^0_{\Lambda_n,\beta,q}),$$

and dividing by $\mathbb{E}_{\Lambda_n}$ and letting $n \to \infty$ gives $I_{p,q}(\mu) \le I(\mu)$.

To prove the opposite inequality we need the preliminaries laid down above. Fix m and a bounded $\mathcal{F}_{\mathbb{E}(\Lambda_m)}$-measurable function g. By our Theorem 2 and Varadhan's theorem [DS, Thm. 2.1.10], the convex dual $I^*_{p,q}$ of $I_{p,q}$ satisfies

$$I^*_{p,q}(g) = \lim_{n\to\infty} \frac{1}{|\mathbb{E}_{\Lambda_n}|} \log \phi^0_{\Lambda_n,p,q}\big(\exp\{|\mathbb{E}_{\Lambda_n}|M_{\Lambda_n}(\omega,g)\}\big).$$

We shall show that

$$I^*_{p,q}(g) \le I^*(g) \equiv \sup_{\mu\in\mathcal{M}^\tau(\Omega)} \{\mu(g) - I(\mu)\}. \tag{7.17}$$

This suffices to prove the theorem. First observe that

$$\begin{aligned}
&\nu^\sigma\big(\exp\{|\mathbb{E}_{\Lambda_n}|M_{\Lambda_n}(\omega,g)\}\big) \\
&= \nu^{\sigma^{(n)}}\big(\exp\{|\mathbb{E}_{\Lambda_n}|M_{\Lambda_n}(\omega^{(n)},g)\}\big) \cdot \exp\{O(|\Lambda_n| - |\Lambda_{n-m}|)\} \\
&= \int_{\mathcal{M}^\tau(\Omega)} \exp\{|\mathbb{E}_{\Lambda_n}|\mu(g)\}\, \lambda_{\Lambda_n}(M_{\Lambda_n}(\sigma^{(n)}), d\mu) \\
&\qquad \times \exp\{O(|\Lambda_n| - |\Lambda_{n-m}|)\},
\end{aligned} \tag{7.18}$$

by ignoring those $x \in \Lambda_n$ for which $x + \Lambda_m \not\subseteq \Lambda_n$, and by (7.12). By Varadhan's theorem and the large deviation principle (7.15),

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{|\mathbb{E}_{\Lambda_n}|} \log \int_{\mathcal{M}^\tau(\Omega)} \exp\{|\mathbb{E}_{\Lambda_n}|\mu(g)\}\, \lambda_{\Lambda_n}(\rho_n, d\mu) \\
&= G(\rho) \equiv \sup_{\mu\in\mathcal{M}^\tau(\Omega)} \{\mu(g) - r(\mu|\rho)\}
\end{aligned} \tag{7.19}$$

whenever $\mathcal{M}^\tau_{\Lambda_n} \ni \rho_n \to \rho$. It follows that, given $\varepsilon > 0$, every $\rho \in \mathcal{M}^\tau(\Sigma)$ has a closed neighborhood U_ρ and an integer $n(\rho)$ such that whenever $n \ge n(\rho)$ and $\rho' \in \mathcal{M}^\tau_{\Lambda_n} \cap U_\rho$,

$$\int_{\mathcal{M}^\tau(\Omega)} \exp\{|\mathbb{E}_{\Lambda_n}|\mu(g)\}\, \lambda_{\Lambda_n}(\rho', d\mu) \le \exp\{|\mathbb{E}_{\Lambda_n}|(G(\rho) + \varepsilon)\}, \tag{7.20}$$

for otherwise (7.19) can be contradicted. (This observation was earlier used in a similar context by Dinwoodie and Zabell [DiZ].) Shrink the neighborhoods further so that

$$h(\rho'|\gamma) \geq h(\rho|\gamma) - \varepsilon \quad \text{for } \rho' \in U_\rho. \tag{7.21}$$

This is possible by the lower semicontinuity of $\rho \mapsto h(\rho|\mu)$. Cover the compact space $\mathcal{M}^\tau(\Sigma)$ with finitely many such neighborhoods $U_{\rho^1}, \ldots, U_{\rho^k}$. Then by (7.18) and (7.20),

$$\begin{aligned}
&\frac{1}{|\mathbb{E}_{\Lambda_n}|} \log \phi^0_{\Lambda_n,p,q}\big(\exp\{|\mathbb{E}_{\Lambda_n}| M_{\Lambda_n}(\omega, g)\}\big) \\
&= \frac{1}{|\mathbb{E}_{\Lambda_n}|} \log \iint \exp\{|\mathbb{E}_{\Lambda_n}| \mu(g)\}\, \lambda_{\Lambda_n}(M_{\Lambda_n}(\sigma^{(n)}), d\mu)\, \gamma^0_{\Lambda_n,\beta,q}(d\sigma) \\
&\qquad + O\big(|\mathbb{E}_{\Lambda_n}|^{-1}(|\Lambda_n| - |\Lambda_{n-m}|)\big) \\
&= \frac{1}{|\mathbb{E}_{\Lambda_n}|} \log \Bigg[\sum_{i=1}^k \gamma^0_{\Lambda_n,\beta,q}(M_{\Lambda_n}(\sigma^{(n)}) \in U_{\rho^i}) \exp\{|\mathbb{E}_{\Lambda_n}|(G(\rho^i) + \varepsilon)\}\Bigg] \\
&\qquad + O\big(|\mathbb{E}_{\Lambda_n}|^{-1}(|\Lambda_n| - |\Lambda_{n-m}|)\big).
\end{aligned}$$

Letting $n \to \infty$ and using the large deviation principle (7.16) together with (7.19) and (7.21) gives

$$\begin{aligned}
I^*_{p,q}(g) &\leq \max_{1\leq i\leq k} \big\{-d^{-1} \inf_{\rho \in U_{\rho^i}} h(\rho|\gamma) + G(\rho^i) + \varepsilon\big\} \\
&\leq \sup_{\mu \in \mathcal{M}^\tau(\Omega)} \max_{1\leq i\leq k} \big\{\mu(g) - \big[r(\mu|\rho^i) + d^{-1} h(\rho^i|\gamma)\big]\big\} + 2\varepsilon \\
&\leq I^*(g) + 2\varepsilon.
\end{aligned}$$

This proves Theorem 11. □

Proof of Corollary 1. The infimum on the right-hand side of (7.7) is achieved due to the compactness of $\mathcal{M}^\tau(\Sigma)$ and the lower semicontinuity of the functions $\rho \mapsto r(\psi|\rho)$ and $\rho \mapsto h(\rho|\gamma)$. Thus by Theorems 3 and 11, $\psi \in \mathcal{R}^\tau_{p,q}$ if and only if $r(\psi|\rho) = h(\rho|\gamma) = 0$ for some $\rho \in \mathcal{M}^\tau(\Sigma)$. But $h(\rho|\gamma) = 0$ if and only if $\rho \in \mathcal{G}^\tau_{\beta,q}$ and $r(\mu|\rho) = 0$ if and only if $\mu = \int \nu^\sigma(\cdot)\rho(d\sigma)$ [Se1, Thm. 2.8]. □

Proof of Remark 7.9. The first half of (7.10) comes by passing to the limit in (7.3). To get the corresponding equality for $\phi^1_{p,q}$, we consider these edge sets: For a finite rectangle $\Lambda \subseteq \mathbb{Z}^d$, let E^+ be the set of edges $e \in \mathbb{E}$ that have one endpoint in Λ, and $\overline{\Lambda}$ the set of vertices adjacent to a vertex in Λ. The probability measures

$$\phi^1_{\Lambda^+,p,q}(\omega) = \frac{1}{Z^1_{\Lambda^+,p,q}} \Big\{\prod_{e \in E^+} p^{\omega(e)}(1-p)^{1-\omega(e)}\Big\} q^{k(\omega,\overline{\Lambda})}$$

are well-defined for ω that satisfy $\omega(e) = 1$ for $e \notin E^+$. Due to the inequalities

$$\phi^1_{\overline{\Lambda},p,q} \leq \phi^1_{\Lambda^+,p,q} \leq \phi^1_{\Lambda,p,q}$$

the convergence $\phi^1_{p,q} = \lim_{\Lambda\to\mathbb{Z}^d}\phi^1_{\Lambda^+,p,q}$ holds. The point is that, for any $j \in \{1,\dots,q\}$,

$$\phi^1_{\Lambda^+,p,q}(\omega) = \int \bigotimes_{e\in E^+} \nu^\sigma_e(\omega(e))\, \gamma^j_{\Lambda,\beta,q}(d\sigma), \tag{7.22}$$

where we take $\sigma \equiv j$ outside Λ in the definition of ν^σ_e for $e \in E^+ \setminus \mathbb{E}_\Lambda$. Passing to the $\Lambda \to \mathbb{Z}^d$ limit in (7.22) gives the second half of (7.10). From this it is clear that $\phi^0_{p,q} = \phi^1_{p,q}$ holds if $\gamma^0_{\beta,q}$ lies in the convex hull of $\{\gamma^j_{\beta,q} : 1 \le j \le q\}$.

The converse comes from turning this argument around: Fix a rectangle Λ_m and a spin configuration $\sigma \in \Sigma_{\Lambda_m}$. For any $n > m$, define the cylinder event generated by σ by

$$[\sigma] = \{\zeta \in \Sigma_{\Lambda_n} : \zeta(x) = \sigma(x) \text{ for } x \in \Lambda_m\}.$$

Let H be the set of $\omega \in \Omega$ for which $x, y \in \Lambda_m$ lie in distinct components whenever $\sigma(x) \ne \sigma(y)$. Then for $n > m$,

$$\gamma^0_{\Lambda_n,\beta,q}([\sigma]) = \int q^{-k(\omega,\Lambda_m)} \mathrm{I}_H(\omega)\, \phi^0_{\Lambda_n,p,q}(d\omega) \tag{7.23}$$

and

$$\frac{1}{q}\sum_{j=1}^q \gamma^j_{\Lambda_n,\beta,q}([\sigma]) = \int q^{-k(\omega,\Lambda_m)} \mathrm{I}_H(\omega)\, \phi^1_{\Lambda_n^+,p,q}(d\omega). \tag{7.24}$$

These formulas come from the fact that, to produce a Potts measure, one picks an edge configuration according to a random-cluster measure and then assigns spin values at random to the connected components. This is straightforward for free boundary conditions (equation (7.23)). For $\gamma^j_{\Lambda_n,\beta,q}$, $j \in \{1,\dots,q\}$, one picks ω according to $\phi^1_{\Lambda_n^+,p,q}$ and lets the boundary condition j determine the spins at vertices connected to Λ_n^c. In formula (7.24) this effect is averaged out by the sum over j.

The function $g(\omega) = q^{-k(\omega,\Lambda_m)} \mathrm{I}_H(\omega)$ is $\phi^b_{p,q}$-a.s. continuous ($b = 0, 1$) due to the a.s. uniqueness of an infinite cluster, hence we can let $n \to \infty$ in (7.23)–(7.24) to get

$$\gamma^0_{\beta,q}([\sigma]) = \int g\, d\phi^0_{p,q}$$

and

$$\frac{1}{q}\sum_{j=1}^q \gamma^j_{\beta,q}([\sigma]) = \int g\, d\phi^1_{p,q}.$$

This implies that $\gamma^0_{\beta,q} = q^{-1}\sum_{j=1}^q \gamma^j_{\beta,q}$ whenever $\phi^0_{p,q} = \phi^1_{p,q}$. $\square$

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