THE GINZBURG-LANDAU EQUATION III. VORTEX DYNAMICS *

by

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Abstract. In this paper we study the time-dependent Ginzburg-Landau equation of the Schrödinger type in two dimensions, with initial conditions describing several well separated vortices. Our task is to develop the non-linear adiabatic theory for this and similar equations. Using the methods of effective action and of local solvability we derive equations for the vortex dynamics and radiation. As examples we consider the two special cases of radiation by the vacuum and by two 1-vortices.

Keywords: Ginzburg-Landau equation, energy functional, vortex, action, dynamics, adiabatic theory, vortex Hamiltonian.

1. Introduction

The non-linear Schrödinger equation is one of the simplest equations which has particle-like solutions – solitons, kinks and vortices – as well as wave like excitations. The interaction of the particle-like solutions between themselves as well as with wave excitations leads to a rather complex dynamics of such solutions. It is the goal of this paper to study this dynamics for the non-linear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + (|\psi|^2 - 1)\psi \tag{1.1}$$

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on $\mathbb{R}^2_x \times \mathbb{R}^2_t$, where Δ is the Laplacian in the spatial variable $x \in \mathbb{R}^2$ and for each t, $\psi(\cdot,t)$ $\mathbb{R}^2 \to \mathbb{C}$, and satisfies the boundary condition

$$|\psi(x,t)| \to 1 \qquad \text{as } |x| \to \infty \ . \tag{1.2}$$

Eqn (1.1) is a time dependent version of the celebrated Ginzburg-Landau equation,

$$-\Delta\psi + (|\psi|^2 - 1)\psi \ = \ 0 \tag{1.3}$$

with the boundary condition

$$|\psi| \to 1 \qquad \text{as } |x| \to \infty \ , \tag{1.4}$$

which in the present context is the equation for stationary solutions. Let (r, θ) be the polar coordinates of x. For each integer n, the latter equation has a solution of the form (modulo translations and rotations)

$$\psi^{(n)}(x) \ = \ f^{(n)}(r)e^{in\theta} \ , \tag{1.5}$$

where $f^{(n)}(r)$ is a unique function monotonically increasing from 0 to 1 as r runs from 0 to ∞ (see e.g. [OSa]). Such solutions are called n-vortices.

In this paper we study Eqn (1.1) with initial conditions which describes roughly several vortices with well separated centers. A precise definition will be given later, now we just present an example of such an initial condition:

$$\psi_{(0)}(x) \ = \ \prod_{i=1}^{K} \psi^{(n_i)}(x - a_i) \tag{1.6}$$

with $r(\underline{a}) \equiv \min_{i \neq j} |a_i - a_j| \gg 1$. The goal of the present paper is to understand the dynamics of the vortices as described by the solution ψ to Eqns (1.1)–(1.2). We show that in the leading order in $r(\underline{a})^{-1}$ the vortex centers satisfy the Hamiltonian equation

$$\dot{\underline{a}} \ = \ J\nabla H(\underline{a}) \ , \tag{1.7}$$

where $\underline{a} = (a_1, \ldots, a_K)$, $a_j \in \mathbb{R}^2$, J is the block-diagonal matrix with the blocks $\frac{1}{\pi n_i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $i = 1, \ldots, K$, and $H(\underline{a})$ is the Hamiltonian given by

$$H(\underline{a}) \;=\; -\pi \sum_{i \neq j} n_i n_j \ln |a_{ij}| \;, \tag{1.8}$$

where $a_{ij} = a_i - a_j$. In the next order in $r(\underline{a})^{-1}$ there is radiation due to the vortex motion, which we investigate in the simplest case of two 1-vortices.

The fact that in the leading order the vortex centers move according to Eqn (1.7) was found in [CM, G] and derived using multiple scale expansion in [N]. Our derivation is different and, we believe, lends itself better to further analysis. The radiation phenomenon in the present context seems has not been observed previously.

2. Renormalized Ginzburg-Landau energy

Solutions of Eqns (1.1)–(1.2) (or (1.3)–(1.4)) are classified by the total index (winding number) of ψ, considered as a vector field on $\mathbb{R}^2$, at ∞, i.e.

$$\deg \psi \;:=\; \frac{1}{2\pi} \int_{|x|=R} d(\arg \psi)$$

for R sufficiently large. We call this index (as opposed to local indices of ψ considered below, see Eqn (2.8)) the *degree* (or *total vorticity*) of ψ.

It is a straightforward observation that Eqn (1.3) is the equation for critical points of the following functional

$$\mathcal{E}(\psi) \;=\; \frac{1}{2} \int \left(|\nabla \psi|^2 + \frac{1}{2}(|\psi|^2 - 1)^2 \right) . \tag{2.1}$$

Indeed, if we define the variational derivative, $\partial_\psi \mathcal{E}(\psi)$, of $\mathcal{E}$ by

$$\mathrm{Re} \int \xi \partial_\psi \mathcal{E}(\psi) \;=\; \frac{\partial}{\partial \lambda} \mathcal{E}(\psi_\lambda) \Big|_{\lambda=0} \tag{2.2}$$

for any path ψ_λ s.t. $\psi_0 = \psi$ and $\frac{\partial}{\partial \lambda} \psi_\lambda \big|_{\lambda=0} = \xi$, then the l.h.s. of Eqn (1.1) is equal to $\overline{\partial_\psi \mathcal{E}(\psi)} = \partial_{\bar{\psi}} \mathcal{E}(\psi)$ for $\mathcal{E}(\psi)$ given in (2.1).

(2.1) is the celebrated Ginzburg-Landau (free) energy. However, there is a problem with it in our context. It is shown in [OSa] that if ψ is an arbitrary C^1 vector field on $\mathbb{R}^2$ s.t. $|\psi| \to 1$ as $|x| \to \infty$ uniformly in $\hat{x} = \frac{x}{|x|}$ and $\deg \psi \neq 0$, then $\mathcal{E}(\psi) = \infty$.

We renormalize the Ginzburg-Landau energy functional as follows (see [OSa]). Let $\chi(x)$ be a smooth real function on $\mathbb{R}^2$ s.t.

$$\chi(x) \;=\; \begin{cases} 1 & \text{for } |x| \geq 2, \\ 0 & \text{for } |x| \leq 1 \end{cases} \tag{2.3}$$

Define

$$\mathcal{E}_{\text{ren}}(\psi) \;=\; \frac{1}{2}\int \Big(|\nabla\psi|^2 - \frac{(\deg\psi)^2}{r^2}\chi + F(|\psi|^2) \Big) d^2x \tag{2.4}$$

where

$$F(u) \;=\; \frac{1}{2}(u-1)^2 \ . \tag{2.5}$$

We list here the most important properties of $\mathcal{E}_{\text{ren}}(\psi)$ (see [OSa] for the proofs):

(a) $\partial_{\bar{\psi}}\mathcal{E}_{\text{ren}}(\psi) = -\Delta\psi + F'(|\psi|^2)\psi$.

(b) Given n let $M_n = \Big\{ \psi = fe^{i\varphi} \mid \int\limits_{|x|\geq 2} \frac{1}{r^2}|1-f^2| < \infty$, f is continuous and $f(0) = 0$, $\int |\nabla(\varphi - n\theta)| r^{-1} < \infty$ and $\int |\nabla(\varphi - n\theta)|^2 < \infty \Big\}$. Then $\mathcal{E}_{\text{ren}}(\psi) < \infty \ \forall \psi \in M_n$.

(c) We have the following bound from below

$$\mathcal{E}_{\text{ren}}(\psi) \;\geq\; \mathcal{E}_{B(0,2)}(\psi) + \frac{1}{2}\int_{|x|\geq 2} \Big(\big|\nabla|\psi|\big|^2 - \frac{1}{2}|\nabla\varphi|^4 \Big) d^2x \ , \tag{2.6}$$

where $\varphi = \arg\psi$, and for $\Omega \subset \mathbb{R}^2$,

$$\mathcal{E}_{\Omega}(\psi) \;=\; \frac{1}{2}\int_{\Omega} \Big(|\nabla\psi|^2 - \frac{(\deg\psi)^2}{r^2}\chi + F(|\psi|^2) \Big) d^2x \ . \tag{2.7}$$

It is known that $\psi^{(n)}$ are local minima for $n = 0, \pm 1$ and are saddle points for $|n| \geq 2$ (see [OSa, Ma, LL]). Moreover, $\psi^{(n)}$ with $n = 0, \pm 1$ are the only local minima of $\mathcal{E}_{\text{ren}}(\psi)$ [Mb, CK], while as far as saddle points are concerned one expects that there are other solutions breaking the rotational symmetry [OSc].

Consider once differentiable functions ψ: $\mathbb{R}^2 \to \mathbb{C}$ satisfying $|\psi| \to 1$ as $|x| \to \infty$. Let $\underline{a} = (a_1, \ldots, a_m)$ and $\underline{n} = (n_1, \ldots, n_m)$, where $a_j \in \mathbb{R}^2$ and $n_j \in \mathbb{Z}$, $j = 1, \ldots, m$. $\underline{c} = (\underline{a}, \underline{n})$

will be called the *vortex configuration*. We say that ψ has the vortex configuration $\underline{c}$ and write $\operatorname{conf}\psi = \underline{c}$, if ψ has zeros (only) at $a_1, \ldots, a_m$ with local indices $n_1, \ldots, n_m$, respectively, i.e.

$$\int_{\gamma_j} d(\arg\psi) \;=\; 2\pi n_j \tag{2.8}$$

for any contour γ_j containing a_j, but not the other zeros of ψ and for $j = 1, \ldots, m$. Now we define

$$E(\underline{c}) \;=\; \inf\left\{\mathcal{E}_{\mathrm{ren}}(\psi) \mid \operatorname{conf}\psi = \underline{c}\right\} . \tag{2.9}$$

By property (c) of section 2, $E(\underline{c}) > -\infty$. We call $E(\underline{c})$ the *energy of the vortex configuration* $\underline{c}$. It will play a central rôle in our analysis. Usually, $\underline{n}$ is fixed during our considerations and we write $E(\underline{a})$ for $E(\underline{c})$, $\underline{c} = (\underline{a}, \underline{n})$.

In [OSb] we demonstrate the following asymptotic expression

$$E(\underline{a}) \;=\; \sum_{i=1}^{K} E_{n_i} + H(\underline{a}) + O\Big(\frac{1}{r(\underline{a})}\Big) , \tag{2.10}$$

where $E_n = \mathcal{E}_{\mathrm{ren}}(\psi^{(n)})$, the self-energy of the n-vortex, and $H(\underline{a})$ is the function given in (1.8).

3. Non-linear adiabatic theory

Consider Eqns (1.1)–(1.2) with initial conditions $\psi\big|_{t=0} = \psi_0$ having a given vortex configuration $\underline{c}_0 = (\underline{a}_0, \underline{n}_0)$ with $r(\underline{a}) \gg 1$ and of low energy:

$$\mathcal{E}_{\mathrm{ren}}(\psi_0) \;\leq\; E(\underline{a}) + O\big(r(\underline{a})^{-1}\big) . \tag{3.1}$$

We can think about the solutions ψ as containing two types of motion: a fast motion around vortex centers (with velocities $\sim |\nabla\psi^{(n_i)}(x_0)|$, $|x_0| \approx 1$) and a slow motion of the vortex centers themselves (with velocities $|\nabla_{a_i} E(\underline{a})|$). This suggests to use an adiabatic approximation. In this approximation, on the first step the vortex centers are frozen and the lowest energy is found under this restriction. This energy is, of course, a function of the positions of vortex centres. On the second step, this energy is used as a Hamiltonian for the motion of the vortex centers.

Specifically, let $\psi_{\underline{a}}$ be a differentiable family (depending on $\underline{a}$) of approximate minimizers of variational problem (2.9), s.t. $\psi_{\underline{a}_0} = \psi_0$. Their energy is

$$H_1(\underline{a}) \;=\; \mathcal{E}_{\mathrm{ren}}(\psi_{\underline{a}}) \; . \tag{3.2}$$

Use this energy as the Hamiltonian for the vortex centers

$$\dot{\underline{a}}(t) \;=\; -J\nabla H_1\big(\underline{a}(t)\big) \;, \tag{3.3}$$

where J is the same symplectic matrix as in (1.7), with the initial condition

$$\underline{a}(0) \;=\; \underline{a}_0 \;, \tag{3.4}$$

the vortex configuration of the initial condition, ψ_0, for (1.1)–(1.2). Solve Eqns (3.3)–(3.4) and form the adiabatic orbit

$$\psi^{\mathrm{adiab}} \;=\; \psi_{\underline{a}(t)} \; . \tag{3.5}$$

We will show below that ψ^{adiab} solves Eqn (1.1) to the order $O\big(r(\underline{a})^{-2}\big)$. Moreover, we conjecture that

$$\sup_x \big|\psi(x,t) - \psi^{\mathrm{adiab}}(x,t)\big| \;\le\; Cr(\underline{a})^{-2} \tag{3.6}$$

and present below arguments in support of this conjecture. We argue also that this estimate cannot be improved. Indeed, the correction to ψ^{adiab}, which is of the order $O\Big(\frac{1}{r(\underline{a})^2}\Big)$ describes the process of radiation by the system of moving vortices and therefore cannot be incorporated into ψ^{adiab}.

Functions (3.5) are the main object of the adiabatic theory while the justification of the conjecture above is its main goal.

On first sight there is unwelcome arbitrariness in the construction above: the choice of the family $\psi_{\underline{a}}$ satisfying (3.1) is not unique. However, as with the variational principle, this arbitrariness allows us a certain degree of flexibility. Moreover, differences between different choices of ψ_a's are of a higher order in $r(\underline{a})^{-1}$, e.g. it is shown in [OSb] that

$$\begin{aligned} H_1(\underline{a}) &= H(\underline{a}) + O\Big(\frac{1}{r(\underline{a})}\Big) , \\ &= E(\underline{a}) - \sum E_{n_i} + O\Big(\frac{1}{r(\underline{a})}\Big) , \end{aligned} \tag{3.7}$$

where $H(\underline{a})$ is given in (1.8) and is independent of the choice of $\psi_{\underline{a}}$, and they are absorbed by radiative excitations (see Section 4).

Justification of the non-linear adiabatic algorithm outlined above will be given in the next two sections. Here we mention only that convenient choice of the family $\psi_{\underline{a}}$ used in (3.2) is provided by introducing an extra potential binding the vortices, i.e. by using the pinning phenomenon (see [OSb]).

4. Effective action

In this section we produce our first argument justifying the adiabatic approach to the problem of vortex dynamics outlined in the previous section (cf. [M,AH,S]). We consider the (renormalized) action functional for Schrödinger equation (1.1):

$$S(\psi) = \int \mathcal{L}_{\rm dyn}(\psi, \dot{\psi}) dt - \int \mathcal{E}_{\rm ren}(\psi) dt , \tag{4.1}$$

where we used the notation $\dot{\psi} = \frac{\partial \psi}{\partial t}$ and where $\mathcal{L}_{\rm dyn}(\psi, \dot{\psi})$ is the dynamic part of the Lagrangian given by

$$\mathcal{L}_{\rm dyn}(\psi, \dot{\psi}) = \frac{1}{2} {\rm Im} \int \psi \frac{\partial \bar{\psi}}{\partial t} . \tag{4.2}$$

The main point of our adiabatic approximation is that we compute the Lagrangian $\mathcal{L}(\psi, \dot{\psi}) = \mathcal{L}_{\rm dyn}(\psi, \dot{\psi}) - \mathcal{E}_{\rm ren}(\psi)$ for ψ's which *minimize* its *static part* $\mathcal{E}_{\rm ren}(\psi)$ for a given vortex configuration $\underline{c} = (\underline{a}, \underline{n})$ modulo $O\Big(\frac{1}{r(\underline{a})}\Big)$:

$$\mathcal{E}_{\rm ren}(\psi) = E(\underline{a}) + O\Big(\frac{1}{r(\underline{a})}\Big) , \tag{4.3}$$

where $r(\underline{a}) = \min_{i\neq j} |a_i - a_j|$, the intervortex distance. Now we compute $\mathcal{L}_{\rm dyn}(\psi, \dot{\psi})$ by restricting it first to the disc $D_R = \{x \in \mathbb{R}^2 \mid |x| \le R\}$ of sufficiently large radius R. Moreover, we restrict ourselves to functions of the form

$$\psi \;=\; e^{i\varphi_{(j)}}\psi_j + O\Big(\frac{1}{r(\underline{a})\cdot r}\Big) \qquad \text{for } |x-a_j| \ll r(\underline{a})\ \forall j \tag{4.4}$$

and

$$\psi \;=\; e^{i\varphi_0}\Big(1 + O\Big(\frac{1}{|x|^2}\Big)\Big) \qquad \text{for } |x-a_j| \gg 1\ \forall j\ . \tag{4.5}$$

Here $\psi_j(x) = \psi^{(n_j)}(x-a_j)$, the n_j-vortex shifted to a_j, $r = |x-a_j|$ and

$$\varphi_{(j)} = \sum_{k\neq j}\varphi_k \quad \text{and} \quad \varphi_0 = \sum_k \varphi_k, \quad \text{where } \varphi_k(x) = n_k\theta(x-a_k), \tag{4.6}$$

with $\theta(x)$ being the polar angle of x. It is shown in [OSb] that (4.3) holds for functions ψ satisfying (4.4) and (4.5). (We conjecture that all functions satisfying (4.3), or a bit stronger estimate, satisfy also (4.4)–(4.5).) To glean the origin of the class of functions (4.4)–(4.5), we observe that for large intervortex distances, $r(\underline{a}) \gg 1$, functions which minimize (2.9) are close to the function $\psi_{(0)}(x) = \prod_{j=1}^{m}\psi^{(n_j)}(x-a_j)$ describing m "independent" vortices. Since $\psi^{(n)}(x) = e^{i\varphi_n(x)}\big(1+O(|x|^{-2})\big)$, the function $\psi_{(0)}$ clearly satisfies (4.4)–(4.5).

Let $f = |\psi|$ and $\varphi = \arg\psi$ so that $\psi = fe^{i\varphi}$. Then

$$\begin{aligned}\mathcal{L}_{{\rm dyn},R} \;&\equiv\; -\frac{1}{2}\int_{D_R} f^2\frac{\partial\varphi}{\partial t}\\ &=\; -\frac{1}{2}\int_{D_R}\frac{\partial\varphi}{\partial t} + \frac{1}{2}\int_{D_R}(1-f^2)\frac{\partial\varphi}{\partial t}\ .\end{aligned} \tag{4.7}$$

Proceeding as with the estimation of $E(\underline{a})$ in [OSb] (by decomposing the integral over discs, D_j, of the radii r_0 around the a_j's and over the rest and using (4.4) in D_j and (4.5), in $D_R\backslash\bigcup_j D_j$), we obtain

$$\begin{aligned}\frac{1}{2}\int(1-f^2)\frac{\partial\varphi}{\partial t} \;&=\pi\sum_{j\neq k} n_k n_j^2\frac{da_{jk}}{dt}\cdot\frac{\hat{J}a_{jk}}{|a_{jk}|^2}\ln|a_{jk}|\\ &\quad+\sum O\Big(\frac{1}{r(\underline{a})}\frac{da_j}{dt}\Big)\ ,\end{aligned} \tag{4.8}$$

where $a_{jk} = a_j - a_k$ and $\hat{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the symplectic matrix on $\mathbb{R}^2$.

For the first term on the r.h.s. of (4.7), we compute

$$\frac{1}{2}\int_{D_R} \frac{\partial \varphi_0}{\partial t} \;=\; -\sum_j \frac{n_j}{2|a_j|}\frac{da_j}{dt}\hat{J}a_j \int_0^R I(r) r dr \;, \tag{4.9}$$

where $I(r) = \int_0^{2\pi} \frac{r\cos\theta - |a_j|}{r^2 + a_j^2 - 2r|a_j|\cos\theta} d\theta$. Next, we find

$$\begin{aligned} I(r) \;&=\; -\frac{1}{2i|a_j|}\Big\{2\pi i + \frac{r^2 - a_j^2}{|a_j| r}\oint \frac{dz}{\left(z - \frac{r}{|a_j|}\right)\left(z - \frac{|a_j|}{r}\right)}\Big\} \\ &=\; \begin{cases} 0 & \text{if } r > |a_j| \\ -\frac{2\pi}{|a_j|} & \text{if } r < |a_j| \end{cases} \;. \end{aligned} \tag{4.10}$$

The last two relations yield

$$\frac{1}{2}\int_{D_R} \frac{\partial \varphi_0}{\partial t} \;=\; \frac{\pi}{2}\sum_j n_j \frac{da_j}{dt} \cdot \hat{J}a_j \;. \tag{4.11}$$

Combining this with Eqns (4.7) and (4.8) and taking the $R \to \infty$ limit, we arrive at

$$\mathcal{L}_{\text{dyn}} \;=\; -\frac{\pi}{2}\sum_j n_j \frac{da_j}{dt} \cdot \hat{J}a_j \;, \tag{4.12}$$

modulo $\sum_{j\neq k} O\left(\frac{\ln|a_{jk}|}{|a_{jk}|}\frac{da_j}{dt}\right)$. This together with the fact $\mathcal{E}_{\text{ren}}(\psi) = E(\underline{a}) + O\left(\frac{1}{r(\underline{a})}\right)$, mentioned above, yields

$$\mathcal{S}(\psi) \;=\; -\frac{1}{2}\pi\sum n_j \int \frac{da_j}{dt} \cdot \hat{J}a_j dt - \int E(\underline{a}) dt \tag{4.13}$$

modulo $O\left(\frac{1}{r(\underline{a})}\right)$. The critical points of this action satisfy the (Euler-Lagrange) equation

$$\dot{\underline{a}} \;=\; J\nabla E(\underline{a}) \;, \tag{4.14}$$

which completes our justification of the equation of motion for the vortex centers in the leading order, i.e. modulo $O\left(\frac{1}{r(\underline{a})^2}\right)$. Eqns (4.14) and (2.10) imply that if $\nabla H(\underline{a}) \neq 0$, then Eqn (1.7) holds in the leading order in $r(\underline{a})^{-1}$.

To understand the correction to action (4.13) due to radiation, we consider the region $D_0 = \{x \in D_R \mid |x - a_j| \gg 1 \, \forall j\}$, which is away from all the vortices. In this region a low energy function ψ can be written in the form

$$\psi \;=\; e^{i\varphi_0 + i\chi}(1+\xi) \;, \tag{4.15}$$

where ξ and χ are real and satisfy $\nabla^n \xi = O\Big(\frac{1}{r(\underline{a})^{2+n}}\Big)$ and $\nabla^n \chi = O\Big(\frac{1}{r(\underline{a})^{1+n}}\Big)$ for $n = 0, 1, 2$. Let

$$\Delta\mathcal{L}(\psi, \dot\psi) \;=\; \frac{1}{2}\,\mathrm{Im}\int_{D_0} \psi \frac{\partial \bar\psi}{\partial t} - \mathcal{E}_{D_0}(\psi) \;, \tag{4.16}$$

where $\mathcal{E}_{D_0}(\psi)$ is the renormalized energy of ψ in the region D_0 (i.e. the integration in (2.4) is restricted to D_0). Substituting (4.15) into $\Delta\mathcal{L}(\psi, \dot\psi)$ and keeping only the terms of the order up to $O\Big(\frac{1}{r(\underline{a})^4}\Big)$, we obtain

$$\Delta\mathcal{L}(\psi, \dot\psi) \;=\; -\int_{D_0} \Big\{ \xi\dot\chi + \frac{1}{2}|\nabla\chi|^2 + \xi\dot\varphi_0 + \xi^2 \Big\} \;. \tag{4.17}$$

Here we dropped the terms $\int_{D_0} \dot\chi = \frac{\partial}{\partial t}\int_{D_0} \chi$ and $\int_{D_0} \nabla\varphi_0 \cdot \nabla\chi = \int_{\partial D_0} \frac{\partial \varphi_0}{\partial n}\chi$, which depend only on the boundary values of the fields and therefore lead to boundary conditions. The resulting expression does not contain derivatives of ξ. Equating its variation over ξ to 0, we obtain

$$\xi \;=\; -\frac{1}{2}(\dot\chi + \dot\varphi_0) \;. \tag{4.18}$$

This equation can be considered as a constraint. We use it to exclude ξ from (4.17), which yields

$$\Delta\mathcal{L}(\psi, \dot\psi) \;=\; \frac{1}{2}\int_{D_0} \Big\{ \frac{1}{2}\dot\chi^2 - |\nabla\chi|^2 + \dot\chi\dot\varphi_0 + \frac{1}{2}\dot\varphi_0^2 \Big\} \;. \tag{4.19}$$

This expression together with (4.13) gives the next approximation for the action

$$\begin{aligned} S(\psi) = &-\int \Big\{ \frac{\pi}{2}\sum_j n_j \dot a_j \cdot \hat J a_j \, dt + E(\underline{a}) \Big\} dt \\ &+ \frac{1}{2}\int\int_{D_0} \Big\{ \frac{1}{2}\dot\chi^2 - |\nabla\chi|^2 + \dot\chi\dot\varphi_0 + \frac{1}{2}\dot\varphi_0^2 \Big\} \;. \end{aligned} \tag{4.20}$$

Thus in this approximation the action $S(\psi)$ for Eqn (1.1) describes the point vortices with the Hamiltonian $E(\underline{a})$ interacting with a radiation field χ whose action is $\frac{1}{2}\int\left(\frac{1}{2}\dot{\chi}^2-|\nabla\chi|^2\right)$.

5. Method of geometric solvability

In this section we apply to Eqn (1.1) the method of local solvability (cf. [OSb]) which shows that in the leading approximation in $r(\underline{a})^{-1}$ the vortices in ψ move according to Eqns (1.7)–(1.8). The idea of this method is that near the j-th vortex we look for the solution $\psi \equiv \psi(x,t)$ in the form

$$\psi \;=\; e^{i\varphi_{(j)}}(\psi_j+\xi_j) \qquad \text{for } |x-a_j| \ll r(\underline{a}) \;, \tag{5.1}$$

where $\varphi_{(j)}$ is the phase due to all the other vortices (see (4.6)), ψ_j is the n_j-vortex shifted to the point a_j:

$$\psi_j(x) \;=\; \psi^{(n_j)}(x-a_j) \tag{5.2}$$

and ξ_j is some small function, and away from all the vortices we look for ψ in the form

$$\psi \;=\; e^{i\varphi_0}(1+\xi_0) \qquad \text{for } |x-a_j| \gg 1 \;\forall j \;, \tag{5.3}$$

where φ_0 is the sum of phases for all the vortices (see (4.6)), and ξ_0 is a small function. Since $\varphi_0 = \varphi_j + \varphi_{(j)}$, the r.h. sides of Eqns (5.1) and (5.3) match on the common domains. On the next step we linearize the equations for ξ_i's, $i = 0,\ldots,K$, and *drop out terms which are of higher orders in the corresponding regions.* As a result we obtain $K+1$ linear equations,

$$L_i(\xi_i) \;=\; F_i \;, \tag{5.4}$$

$i = 0,\ldots,K$. We write out the solvability conditions for these equations

$$\int \eta F_i \;=\; 0 \qquad \forall \eta \in \mathrm{Ker}\, L_i \;, \tag{5.5}$$

where the integrals are tacitly understood to extend over the corresponding regions in which the equations are valid. These solvability conditions plus the requirement that F_i

are at most $O\big(r(\underline{a})^{-2}\big)$ yield equations (1.7)–(1.8) for the vortex dynamics. Finally, we check that the $K+1$ solutions match on the overlaps of their regions. Of course, once the ground work above is done, one can proceed by the perturbation theory to finding the solution to (1.1) to arbitrary order in $r(\underline{a})^{-1}$. We skip the last step in the present paper.

Now we write out the corresponding equations. We assume a priori that $\nabla^n \xi_j = O\big(r(\underline{a})^{-1-n}\big)$, $n = 0, 1, 2$, and $a_j = \Big(\frac{1}{r(\underline{a})}\Big)$ so that $\dot{\xi}_j = O\big(r(\underline{a})^{-2}\big)$, $\dot{\varphi}_{(j)} = O\big(r(\underline{a})^{-2}\big)$, etc. In linearizing around $e^{i\varphi_{(j)}}\psi_j$ in the region $|x - a_j| \ll r(\underline{a})$, we drop, besides the terms non-linear in ξ_j, the terms $\dot{\xi}_j$, $\dot{\varphi}_j \xi_j$, $\dot{\varphi}_{(j)}\psi_j$, $\nabla\varphi_{(j)} \cdot \nabla\xi_j$, $|\nabla\varphi_{(j)}|^2 \xi_j$ and $|\nabla\varphi_{(j)}|^2 \psi_j$, which are of the order $O\big(r(\underline{a})^{-2}\big)$, at least (the latter term can also be removed by deforming the vortex, see below), to obtain Eqn (5.4) with

$$\begin{aligned} L_j(\xi) &\equiv L_{\psi_j}(\xi) \\ &\equiv -\Delta\xi + (2|\psi_j|^2 - 1)\xi + \psi_j^2 \bar{\xi} \end{aligned} \tag{5.6}$$

and

$$F_j = 2i\nabla\varphi_{(j)} \cdot \nabla\psi_j + i\dot{\psi}_j \tag{5.7}$$

for $j = 1, \dots, K$. Now we note that

$$\dot{\psi}_j = -\dot{a}_j \nabla\psi_j \tag{5.8}$$

so that

$$F_j = i(-\dot{a}_j + 2\nabla\varphi_{(j)}) \cdot \nabla\psi_j \ . \tag{5.9}$$

Let $J_j = \frac{1}{\pi n_j}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Taking

$$\dot{a}_j = 2\nabla\varphi_{(j)}(a_j) \equiv J_j \nabla_{a_j} H(\underline{a}) \ , \tag{5.10}$$

and using that

$$\nabla\varphi_{(j)}(x) - \nabla\varphi_{(j)}(a_j) = O\Big(\frac{1}{r(\underline{a})^2}\Big) \ , \tag{5.11}$$

we arrive at the equation

$$L_{\psi_j}(\xi_j) \;=\; O\big(r(\underline{a})^{-2}\big)\ . \tag{5.12}$$

On the next step of the perturbation theory we modify the equation $\dot{a}_j = 2\nabla\varphi_{(j)}(a_j)$ by adding an $O\Big(\frac{1}{r(\underline{a})^2}\Big)$ term to the r.h.s. so that to make the r.h.s. of (5.12) (see (5.4)) orthogonal to the zero modes of L_{ψ_j} (see (5.5)) and so on. As a result we see that Eqn (5.4) has a solution and this solution is $O\Big(\frac{1}{r(\underline{a})^2}\Big)$, i.e. even smaller than we assumed. Thus Eqns (1.7)–(1.8) hold to the order $O\Big(\frac{1}{r(\underline{a})^2}\Big)$.

Note that the leading order equation for $\underline{a}$, Eqn (5.10), yields that the free term F_j is zero in the leading order $O\big(r(\underline{a})^{-1}\big)$. This is a luck which we do not expect to repeat in higher orders. However, condition (5.5) can always be achieved. To verify this condition one has to know the zero modes of L_{ψ_j}. It is shown in [OSa] that the only zero modes of the operator L_{ψ_j} are those due to breaking the gauge symmetry, $\eta_0 = i\psi_j$, and translational symmetry $\eta_1 = \partial_{x_i}\psi_j$ and $\eta_2 = \partial_{x_2}\psi_j$. Since η_0 does not decay at ∞ it does not enter into condition (5.7).

The equation $L_0(\xi_0) = F_0$ in the region $|x - a_j| \gg 1\ \forall j$ is obtained and the matching problem investigated similarly to the corresponding stationary equation in [OSb] and we omit this analysis here.

Finally we show how to remove the term $|\nabla\varphi_{(j)}|^2\psi_j$ from the r.h.s. of the equation of ξ_j. To this end instead of linearizing Eqn (1.1) around the function $e^{i\varphi_{(j)}}\psi_j$ we linearize it around the deformed n_j-vortex $e^{i\varphi_{(j)}}\psi_j^{\mathrm{def}}$, where

$$\psi_j^{\mathrm{def}} \;:=\; e^{i\varphi_j}\sqrt{1-|\nabla\varphi_{(j)}|^2}\,|\psi_j|\big(\sqrt{1-|\nabla\varphi_{(j)}|^2}\,x\big)\ . \tag{5.17}$$

A straightforward computation shows that the new equation for ξ_j does not contain the term $|\nabla\varphi_{(j)}|^2\psi_j^{\mathrm{def}}$.

6. Radiation

We have shown in Section 4 that, in general, adiabatically moving vortices radiate. In other words the solution of Eqn (1.1) with an initial condition representing well separating vortices can be written in the form

$$\psi \ = \ \psi^{\rm adiab} + \psi^{\rm rad} \ , \tag{6.1}$$

where $\psi^{\rm rad}$ describe the radiation emitted by moving vortices. In this section we find the wave equation for $\psi^{\rm rad}$ and present an elementary discussion of this equation. In the next two sections we consider two simple examples.

Plugging (6.1) into (1.1) yields an equation for $\psi^{\rm rad}$. A straightforward analysis of this equation shows that $|\psi^{\rm rad}|$ is slowly varying and is much smaller than fast varying $\arg\psi^{\rm rad}$. So we concentrate on the latter. Let $f = |\psi|$ and $\varphi = \arg\psi$ so that $\psi = fe^{i\varphi}$. Then Eqn (1.1) implies the following equations for f and φ

$$-\frac{\partial\varphi}{\partial t} \ = \ -\frac{\Delta f}{f} + |\nabla\varphi|^2 + f^2 - 1 \tag{6.2}$$

and

$$\frac{\partial f}{\partial t} \ = \ -\Delta\varphi f - 2\nabla\varphi\cdot\nabla f \ . \tag{6.3}$$

Solving the first of these equations for f^2 as

$$f^2 \ = \ 1 - |\nabla\varphi|^2 - \frac{\partial\varphi}{\partial t} + \frac{1}{f}\Delta f \tag{6.4}$$

and substituting the result into the second equation, we find

$$\frac{\partial^2\varphi}{\partial t^2} - 2\Delta\varphi \ = \ F \ , \tag{6.5}$$

where

$$\begin{aligned} F \ = \ & -2\nabla\varphi\cdot\nabla\Big(|\nabla\varphi|^2 + \frac{\partial\varphi}{\partial t} - \frac{1}{f}\Delta f\Big) \\ & -2(1-f^2)\Delta\varphi - \frac{\partial}{\partial t}\Big(|\nabla\varphi|^2 - \frac{1}{f}\Delta f\Big) \ . \end{aligned} \tag{6.6}$$

We solve Eqn (6.5) by perturbation theory in the r.h.s.. Hence on the first step (in the zero approximation) we set the r.h.s. to zero,

$$\frac{\partial^2\varphi}{\partial t^2} - 2\Delta\varphi \;=\; 0\;, \tag{6.7}$$

and look for φ in the form

$$\varphi = \varphi^{\mathrm{adiab}} + \chi\,, \quad \text{where } \varphi^{\mathrm{adiab}} = \sum_j n_j\theta\big(x - a_j(t)\big) \tag{6.8}$$

with $\theta(x)$, the polar angle of $x \in \mathbb{R}^2$ and $\underline{a}(t)$, the solution to (4.14) with the initial condition $\underline{a}(0)$ given by the vortex configuration of the initial condition for the Schrödinger equation (1.1) (see Eqns (3.5) and (4.4)–(4.6)). Since $\Delta\varphi^{\mathrm{adiab}} = 0$, we have in the leading order in $r(\underline{a})^{-1}$

$$\frac{\partial^2\chi}{\partial t^2} - 2\Delta\chi \;=\; -\frac{\partial^2\varphi^{\mathrm{adiab}}}{\partial t^2} \tag{6.9}$$

with zero initial conditions. Note that this equation can be also obtained by varying action (4.20) w.r. to the field χ. Applying to this equation Green's function method, we find

$$\chi \;=\; -G * \frac{\partial^2\varphi^{\mathrm{adiab}}}{\partial t^2}\;, \tag{6.10}$$

where $*$ stands for the convolution in x and t, and G solves the equation

$$\Big(\frac{\partial^2}{\partial t^2} - 2\Delta\Big)G \;=\; \delta(t)\delta(x)$$

with the causality condition that $G = 0$ for $t < 0$. It can be found explicitly:

$$G(t,x) \;=\; \frac{\theta(\sqrt{2}t - r)}{2\pi\sqrt{2}\sqrt{2t^2 - r^2}}\;, \tag{6.11}$$

where $r = |x|$ and $\theta(t) = 1$ if $t \geq 0$ and $= 0$ if $t < 0$. Since $\frac{\partial^2\varphi^{\mathrm{adiab}}}{\partial t^2} = O\big(r(\underline{a})^{-2}r^{-2}\big)$, one can deduce from (6.10) that

$$\chi \;=\; O\big(r(\underline{a})^{-2}\big)\;, \tag{6.12}$$

which agree with our analysis in Section 5 (see the paragraph after Eqn (5.14)). On the other hand, Eqns (6.4) and (6.8) imply that

$$1 - f^2 \;=\; O\big(r(\underline{a})^{-4}\big) \;\ll\; |\chi| \;, \tag{6.13}$$

which justifies our idea the phase plays a dominant rôle in the radiation process.

Remark. Equation (6.5) (or Eqns (6.2)–(6.3)) has, in a leading order in $\frac{1}{r(\underline{a})}$, also a solitary wave solution. Indeed, retaining in (6.6) only leading terms, we obtain

$$\Big(\frac{\partial^2}{\partial t^2} - 2\Delta\Big)\varphi \;=\; -2\frac{\partial}{\partial t}|\nabla\varphi|^2 - 2\frac{\partial\varphi}{\partial t}\Delta\varphi - \frac{1}{2}\frac{\partial^2}{\partial t^2}\Delta\varphi \;. \tag{6.14}$$

This equation has a solution of the form $\varphi(x_1 - vt)$, where $\alpha(z) = \frac{\partial\varphi}{\partial z}$ satisfies the equation

$$(v^2 - 2)\alpha \;=\; 3v\alpha^2 - \frac{v^2}{2}\alpha'' \;. \tag{6.15}$$

This equation with the boundary conditions $\alpha \to 0$ as $z \to \pm\infty$ is equivalent to the equation

$$(v^2 - 2)\alpha^2 - 2v\alpha^3 + \frac{v^2}{2}(\alpha')^2 \;=\; 0 \;, \tag{6.16}$$

whose solution is

$$\alpha \;=\; -\frac{A}{ch^2(\beta z)} \;, \tag{6.17}$$

where

$$\beta^2 = \frac{2 - v^2}{2v^2} \quad \text{and} \quad A \;=\; \frac{2 - v^2}{2v} \;. \tag{6.18}$$

Equations (6.14) and (6.16) are valid only for

$$0 < \sqrt{2} - v \ll 1 \;.$$

7. Radiation by a perturbation of the vacuum

We consider initial conditions, ψ_0, for Eqn (1.1) with no local vorticities and consequently of $\deg\psi_0 = 0$ and which are small perturbations of the vacuum solution $\psi^{(0)} \equiv 1$:

$$\psi_0 \;=\; e^{i\chi_0}(1 + \xi_0) \;, \tag{7.1}$$

where χ_0 and ξ_0 are real and small as well. We look for the solution to Eqn (1.1) in the form

$$\psi \;=\; e^{i\chi}(1+\xi)\;, \tag{7.2}$$

where χ and ξ are real and small. Thus in this case $\psi^{\mathrm{adiab}} \equiv 1$. Plugging (7.2) into Eqn (1.1) and dropping the non-linear terms, we obtain the equation for χ and ξ

$$-\frac{\partial\chi}{\partial t} \;=\; (-\Delta+2)\xi \quad \text{and} \quad \frac{\partial\xi}{\partial t} \;=\; -\Delta\chi \tag{7.3}$$

which yield the equations

$$\frac{\partial^2\chi}{\partial t^2} - (-\Delta+2)\Delta\chi \;=\; 0 \quad \text{and} \quad \xi \;=\; -(-\Delta+2)^{-1}\frac{\partial\chi}{\partial t} \;\approx\; -\frac{1}{2}\frac{\partial\chi}{\partial t}\;, \tag{7.4}$$

which agree with Eqn (6.7) and with our a priori assumption above that the corrections to the modulus are much smaller than the corrections to the phase.

The general solution of Eqns (7.4) is a superposition of the plane waves

$$\chi \;=\; \int A(k)\cos\big(\omega t - k\cdot x + \varphi_0(k)\big)\,d^2k \tag{7.5}$$

and

$$\xi \;\approx\; \int \frac{\omega}{2}A(k)\sin\big(\omega t - k\cdot x + \varphi_0(k)\big)\,d^2k\;, \tag{7.6}$$

where the frequency ω and the wave vector $k \in \mathbb{R}^2$ are related by the dispersion law

$$\omega \;=\; \sqrt{k^2(2+k^2)} \;\approx\; \sqrt{2}|k| \tag{7.7}$$

(we are interested in large distances and small $|k|$) and the functions $A(k)$ and $\varphi_0(k)$ are determined (in the leading order) by initial condition (7.1):

$$\int A(k)\cos\big(k\cdot x - \varphi_0(k)\big)\,d^2k \;=\; \chi_0(x) \tag{7.8}$$

and

$$-\int \frac{\omega}{2}A(k)\sin\big(k\cdot x - \varphi_0(k)\big)\,d^2k \;=\; \xi_0(x)\;. \tag{7.9}$$

8. Radiation by two 1-vortices

Now we consider Eqn (1.1) with an initial condition ψ_0 describing two 1-vortices at the distance $2a \gg 1$ from one another. Then in the adiabatic approximation, Eqn (1.7), the vortices rotate around their "center-of-vorticity", $\frac{1}{2}(1 \cdot a_1 + 1 \cdot a_2) = \frac{1}{2}(a_1 + a_2)$, with the frequency

$$\omega \;=\; \frac{1}{a^2}\;. \tag{8.1}$$

Since the center-of-vorticity is conceived,

$$\sum n_j a_j(t) \;=\; \text{const}\;, \tag{8.2}$$

under the evolution (1.7), we can place it into the origin, so that $a_1 = \pm a_2$ and can be written, modulo a constant phase shift due to the initial conditions, as

$$a_j \;=\; \pm a(\cos \omega t, \sin \omega t), \qquad j = 1, 2\;. \tag{8.3}$$

Using this equation and definition (6.8) of φ^{adiab}, we obtain

$$\frac{\partial \varphi^{\text{adiab}}}{\partial t} \;=\; 2a^2\omega \frac{a^2 - r^2 \cos 2(\theta - \omega t)}{a^4 + r^4 - 2r^2 a^2 \cos 2(\theta - \omega t)}\;. \tag{8.4}$$

Thus the phase χ satisfies wave equation (6.9) with a source in the r.h.s. which depends on r, a, θ and t only in the combinations $\frac{a^2}{r^2}$ and $\cos 2(\theta - \omega t)$ and is a periodic and even function of the second variable. Such a source generates a wave which for $r \gg 1$ has the form (see [OSe])

$$\chi \;=\; -\frac{(\pi\omega)^{1/2}\omega a^2}{2^{3/4} r^{1/2}} \cos\left[\sqrt{2}\omega(r - \sqrt{2}t) + 2\theta - \frac{\pi}{4}\right] \tag{8.5}$$

plus higher harmonics, with the n-th harmonic being of the order $O\Big(\sqrt{\frac{\omega}{2}}(\omega a)^{2n}\Big)$. Then the energy conservation law implies that the distance, $2a$, between vortices must increase in time.

In order to find the dependence of $a = a(t)$ on t we compute the average $\langle S_\psi \rangle$ of the energy flux density,

$$S_\psi \;=\; -\mathrm{Re}(\dot{\psi}\nabla\bar{\psi}) \;=\; -\dot{\chi}\nabla\chi \tag{8.6}$$

over the period, $\frac{\pi}{\omega}$, of the adiabatic motion of the vortices. Using (8.5), we find

$$\langle S_\psi \rangle \;=\; \frac{\pi}{2a^6}\frac{x}{|x|^2} \;. \tag{8.7}$$

Then the energy conservation law

$$\frac{\partial}{\partial t}\int_\Omega e(\psi) \;=\; \int_{\partial\Omega} S_\psi \;, \tag{8.8}$$

where $e(\psi) = \frac{1}{2}|\nabla\psi|^2 + \frac{1}{4}(|\psi|^2 - 1)^2$, the energy density, implies that

$$\frac{da}{dt} \;=\; \frac{\pi}{2a^5} \;. \tag{8.9}$$

Integrating this equation we find

$$a(t)^6 \;=\; a(0)^6 + 3\pi(t - t_0) \;, \tag{8.10}$$

i.e. $a(t)$ grows with time as $(3\pi t)^{\frac{1}{6}}$. See [OSe] for more details.

References

[AH] M. Atiyah and N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles,* Princeton Univ. Press, 1988.

[BBH] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg-Landau Vortices,* Birkhäuser, Basel, 1994.

[CK] S. Chanillo and M. Kiesling, Symmetry of solutions of Ginzburg-Landau equations, *Compt. Rend. Acad. Sci. Paris,* t. 327, Série I (1995) 1023–1026.

[CM] J. Creswick and N. Morrison, On the dynamics of quantum vortices, *Phys. Lett. A* **76** (1980), 267.

[G] E. Gross, Dynamics of interacting bosons, in *Physics of Many Particle Systems*, ed. E. Meeron, Gordon and Breach, NY, 1966, 268.

[H] P. Hagan, Spiral waves in reaction diffusion equations, *SIAM J. Applied Math.* **42** (1982), 762–786.

[JT] A. Jaffe and C. Taubes, *Vortices and Monopoles,* Birkhäuser, 1980.

[Ma] N.S. Manton, A remark on scattering of BPS monopoles, *Phys. Letters* **110**B, N1 (1981), 54–56.

[LL] E.M. Lieb and M. Loss, Symmetry of the Ginzburg-Landau minimizers in a disc.

[M1] P. Mironescu, On the stability of radial solutions of the Ginzburg-Landau equation, *J. Funct. Anal.* **130** (1995), 334–344.

[M2] P. Mironescu, Les minimiseurs locaux pour l'équation de Ginzburg-Landau sont à symmétrie radiale, preprint, 1996.

[N] J. Neu, Vortices in complex scalar fields, *Physica D* **43** (1990), 385–406.

[OSa] Yu.N. Ovchinnikov and I.M. Sigal, Ginzburg-Landau equation I. General discussion, preprint 1995, to appear in *P.D.E.'s and their Applications* (L. Seco et al., eds.).

[OSb] Yu.N. Ovchinnikov and I.M. Sigal, The Ginzburg-Landau equation II. The energy of vortex configurations, preprint, 1996.

[OSc] Yu.N. Ovchinnikov and I.M. Sigal, Symmetry breaking in the Ginzburg-Landau equation, preprint, 1996.

[OSe] Yu.N. Ovchinnikov and I.M. Sigal, Long-time behaviour of Ginzburg-Landau vortices, in preparation.

[S] I. Shafrir, Remarks on solutions of $-\Delta u = (1-|u|^2)u$ in $\mathbb{R}^2$, *C.R. Acad. Sci. Paris,* t. 318, Série I (1994), 327–331.

[[Sh] P.A. Shah, Vortex scattering at near critical coupling, *Nucl. Phys. B* **429** (1994), 259–276.

[Str] M. Struwe, *Variational Methods,* Springer-Verlag, 1990.

[Stu] D. Stuart, *Comm. Math. Phys.* **159** (1994), 51.

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