Lifetime of the Wannier-Stark resonances and perturbation theory

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Abstract. We consider the small field asymptotics of the lifetime of metastable states in Wannier-Stark problems. Assuming that at zero field we have Bloch operators with only the first gap open and using the regular perturbation theory, we prove that the behavior of the lifetime computed by means of the Fermi Golden Rule is proportional to the correct one with the factor $\left(\frac{\pi}{3}\right)^2$. The connection with adiabatic problems is briefly discussed.

1 Introduction

The Wannier-Stark (or Stark-Wannier or Stark-Bloch) problem is actually older than it can be guessed from its name and it has been studied at least since 1934 by Zener [32]. In fact, the existence of metastable states is a consequence of the Zener barriers between the tilted bands produced by an external homogeneous electric field superimposed to a one-dimensional periodic crystal. By means of a natural and simple idea Wannier [30] suggested the restriction of the full operator to the single-band space of the Bloch operator so that bound states displaced on ladders (called Wannier ladders) turn out explicitly. Actually, because the experimental and the theoretical studies were not able to found such bound states in the real problem, some doubts on their very existence was raised [31].

Only in the eighties the existence of metastable states, associated to resonances (hereafter called Wannier-Stark states), for the Wannier-Stark problem was universally accepted because of accurate numerical [4] and analytical studies [2]. Furthermore, experiments on heterostructures gave the first experimental evidence of existence of metastable states (see, for a review, [3]). Maioli and ourselves [16], considering particular models with a finite number of bands, proved the existence of the Stark-Wannier states by means of the Wannier approximation and the regular perturbation theory for weak electric field; it was also given the asymptotics of the width of the resonances by means of the Fermi Golden Rule approximation, i.e. the second perturbation order. The numerical coefficient of the leading term of the width of the resonances is turned out to be $\frac{1}{2}C^2$ with $C = \frac{1}{3}\pi$ for the two bands model with crystal period 2π. The same value of C appears also in old perturbative calculations [20,21,24] and Kane and Blount [22] suggested that this value of C is not the exact one which should be equal to 1. Recently, in the same class of models Buslaev and Dmitrieva [10], using an adiabatic approximation in the x representation, have obtained $C = 1$.

As a continuation and completion of the paper [16], here we rigorously compute (Theorem 1) the asymptotics of the width of the resonances in two bands models satisfying a technical hypothesis (Hypothesis 2). We use this hypothesis in order to control the error in (15). Actually, without this hypothesis we are in the same condition of Buslaev and Dmitrieva [10] and we cannot improve their result. This hypothesis is verified in some cases (see Remark 2 and 3).

We prove that any term of the convergent perturbation series contributes to give the correct numerical coefficient obtaining

$$C = 2\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{6^{2n+1}(2n+1)!} = 2\sin\tfrac{1}{6}\pi = 1.$$

Our proof is based on the regular perturbation theory, on previous computations of perturbation integrals obtained by Berry [7] and on the proof that the self-coupling term of the second infinite band is irrelevant at the leading order. In such a way we justify the Fermi Golden Rule for the Wannier-Stark problem with $C = 1$ instead of $C = \frac{1}{3}\pi$. Heuristic considerations explaining this result are also discussed in Remarks 6 and 7. In particular, in Remark 6 it is shown that C could be computed by means of the Borel transform of the asymptotic expansion of the eigenfunction obtained by the Adam-Wannier method [8, 9, 19]:

$$C = \tfrac{1}{3}\pi F(\tfrac{1}{6}, -\tfrac{1}{6}; 1; 1) = \tfrac{1}{3}\pi \lim_{\beta\uparrow 1} \sum_{n=0}^{\infty} \frac{\Gamma\left(n-\frac{1}{6}\right)\Gamma\left(n+\frac{1}{6}\right)}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(-\frac{1}{6}\right)(n!)^2}\beta^n.$$

The above result can be extended to any model with the first n gaps open, $n \geq 1$, and the others closed. More precisely, one obtains an upper and a lower bound of the width of the resonances. Upper bound is given by (15) where now $\rho_Z(f)$ is the Agmon length of the Zener barrier created by the n-th tilted gap. As for the lower bound one obtains $\ln\left(|\Im E_{j,m}(f)|\right) \geq -2\sum_{\ell=1}^{n} \rho_\ell(f)$ where $E_{j,m}(f)$ is the j-th resonance of the m-th ladder, $m = 1, 2, ..., n$ and $j \in \mathsf{Z}$, and where $\rho_\ell(f)$ is the Agmon length of the Zener barrier created by the ℓ-th tilted gap. In order to perform a more detailed analysis one must consider the crossing phenomenon between ladders of resonances as made in [17]. Let us quote a recent paper by Asch and Briet [1] where they give a lower bound of the imaginary part of Stark-Wannier states.

This paper is organized as follows: in Section 2 we discuss the models and we state our main result (Theorem 1); in Section 3 we give the proof of Theorem 1 and in Section 4 we collect some final remarks, in particular we briefly show (Remark 7) that a Wannier-Stark problem can be associated with an adiabatic problem.

2 Models and results

We consider the self-adjoint operator H_f formally defined on $L^2(\mathsf{R}, \mathrm{d}x)$ as

$$H_f = H_B + fx \qquad H_B = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \qquad f > 0 \tag{1}$$

where $V(x)$ is a periodic real-valued function with period T.

We assume that:

Hypothesis 1: *The Bloch operator H_B has the first gap open and the others closed:*

$$\sigma(H_B) = [E_1^b, E_1^t] \cup [E_2^b, +\infty), \qquad E_1^t < E_2^b. \tag{2}$$

For instance let $V(x)$ be the Lamé potential [12]

$$V(x) \equiv V(x;t) = 2t^2 \mathrm{sn}^2\,(x;t)\,, \qquad 0 < t < 1, \tag{3}$$

where $\mathrm{sn}(x;t)$ is the Jacobian elliptic function with modulus t; the period of $V(x;t)$ is $T = 2K(t)$ where

$$K(t) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - t^2 \sin^2\theta}}. \tag{4}$$

In such a case we have that

$$\sigma(H_B) = [t^2, 1] \cup [1 + t^2, +\infty).$$

From Hypothesis 1 it follows that $V(x)$ is an analytic function in the strip $|\Im x| < \lambda_0$ for some $\lambda_0 > 0$ (see Theorem XIII.91d in [28]). Then, the spectrum of H_f is purely absolutely continuous [5]:

$$\sigma(H_f) = \sigma_{\mathrm{ac}}(H_f) = \mathsf{R}. \tag{5}$$

In order to define resonances we perform the analytic translation [18] $x \to x + \lambda$, $-\lambda_0 < \Im\lambda < 0$, and we consider the crystal momentum representation of the translated operator (called extended crystal momentum representation or simply ECMR; see, for a review, Appendix I in [17]). In such a representation the Wannier-Stark operator H_f takes the form

$$H_f^\lambda = H_f^{\mathrm{DB},\lambda} + fX^\lambda. \tag{6}$$

It acts on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1 = L^2(\mathcal{B}, \mathrm{d}k/L)$ with periodic boundary condition, $\mathcal{B} = \mathsf{R}/L\mathsf{Z} = (-\frac{1}{2}L, +\frac{1}{2}L]$ is the Brillouin zone and $L = 2\pi/T$, and $\mathcal{H}_2 = L^2(\mathsf{R}, \mathrm{d}p)$; k denotes the crystal momentum (or quasimomentum) variable in the finite band and p denotes the crystal momentum variable in the infinite band (sometimes are both denoted by p for the sake of simplicity).

The operator $H_f^{\mathrm{DB},\lambda}$ is the decoupled band approximation of H_f^λ and it is defined as:

$$H_f^{\mathrm{DB},\lambda} = \mathrm{diag}(H_1, H_2^\lambda) \tag{7}$$

where

$$H_1 = if\mathrm{d}_k + \mathcal{E}_1(k) + fX_{1,1}(k), \qquad \mathrm{d}_k := \frac{\mathrm{d}}{\mathrm{d}k} \tag{8}$$

and

$$H_2^\lambda = if\mathrm{d}_p + \mathcal{E}_2(p) + f\lambda, \qquad \mathrm{d}_p := \frac{\mathrm{d}}{\mathrm{d}p}. \tag{9}$$

Here $\mathcal{E}_1(k)$ denotes the first band function and $\mathcal{E}_2(p)$ the second one. $\mathcal{E}_1(k)$ is a periodic function with period L while $\mathcal{E}_2(p)$ has the asymptotic behavior $\mathcal{E}_2(p) = p^2(1 + \mathcal{O}(p^{-1}))$ as p goes to infinity (see Figure 1). They are real-analytic functions and they have branch points of square root type [25] at $k_c = \frac{1}{2}L + ir_c$ and $\bar{k}_c$, $r_c > 0$:

$$\Lambda(k_c) = \Lambda(\bar{k}_c) = 0, \qquad \Lambda(p) := \mathcal{E}_2(p) - \mathcal{E}_1(p).$$

In the following we neglect the intraband term $X_{1,1}(k)$. This attitude is justified because this term can be taken constant by a gauge choice of the first Bloch function and, in particular, it is exactly zero if the crystal is symmetrical (see §II in [16]).

For the sake of definiteness let us assume that the period T is 2π and that the crystal is symmetrical, i.e.:

$$V(-x) = V(x) = V(x + 2\pi), \qquad \forall x \in \mathsf{R}. \tag{10}$$

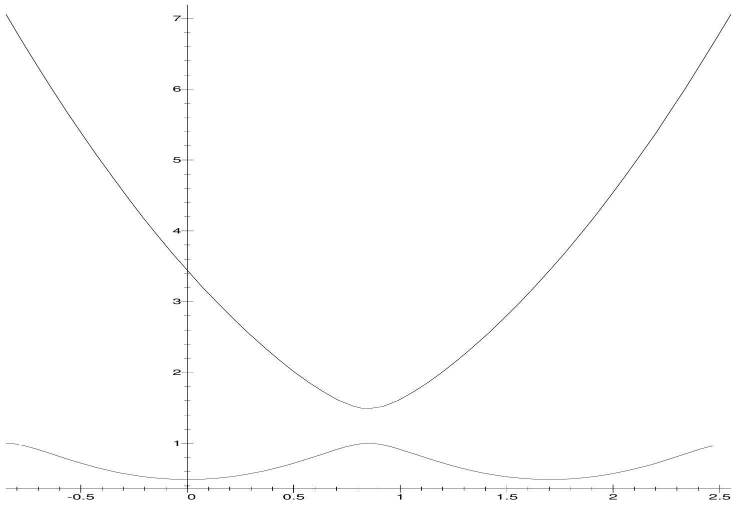

Figure 1: Graph of the band functions $\mathcal{E}_1(p)$ and $\mathcal{E}_2(p)$. At the branch point $k_c = \frac{1}{2}L + ir_c$ (and $\bar{k}_c$) the two band functions coincide: $\mathcal{E}_1(k_c) = \mathcal{E}_2(k_c) = E_G$. For the Lamé potential with $t = 0.7$ we have $E_G = 1.2245$, $L = 1.7021$ and $r_c = 0.1434$.

The spectrum of $H_f^{\mathrm{DB},\lambda}$ is given by:

$$\sigma(H_f^{\mathrm{DB},\lambda}) \quad = \quad \{E_j = E_0 + j2\pi f,\ j \in \mathsf{Z}\} \cup \{z \in \mathsf{C}\ :\ \Im z = f \Im \lambda\}\,.$$

Indeed, the operator H_1, defined on $\mathcal{H}_1$, has purely discrete spectrum given by the Wannier ladder of real eigenvalues E_j, $E_0 = \int_{\mathcal{B}} \mathcal{E}_1(k)\mathrm{d}k$ is the mean value of the first band function, and the operator H_2^λ, defined on $\mathcal{H}_2$, has purely essential spectrum on the line $f\lambda + \mathsf{R}$.

Recalling that the term X^λ which couples the two bands is a bounded operator (Theorem 3 in [16]), taking the decoupled band approximation $H_f^{\mathrm{DB},\lambda}$ as the unperturbed operator and the coupling term X^λ as a bounded perturbation, then we consider the problem of existence and computation of the resonances of H_f in the framework of the regular perturbation theory for $f > 0$ small enough. In [16] was proved that the resonances of H_f are the discrete eigenvalues of H_f^λ in the strip $f \Im \lambda < \Im z < 0$ and they consist in one ladder $\{E_j(f) = E_0(f) + j2\pi f,\ j \in \mathbb{Z}\}$. Moreover, for f small enough it was obtained $E_0(f)$ by means of the convergent series

$$E_0(f) = \sum_{n=0}^{\infty} f^n E_0^n(f) \tag{11}$$

where $E_0^0(f) = E_0$, $E_1(f) = 0$ and $E_0^2(f)$ is such that (see §IV in [16] where we set $T = 2\pi$)

$$\Im E_0^2(f) = -\tfrac{1}{2} C^2 f^{-1} e^{-2\rho_Z(f)} \left(1 + \mathcal{O}(f^{2/3})\right) \qquad \text{as } f \to 0 \tag{12}$$

with $C = \frac{1}{3}\pi$. $\rho_Z(f)$ is the Agmon length of the Zener barrier between the two bands created by the tilted gap:

$$\begin{aligned} \rho_Z(f) &= -\frac{i}{f}\int_{\Re k_c}^{k_c} \Lambda(p)\mathrm{d}p = -\frac{i}{2f}\oint_{\mathcal{C}} \mathcal{E}(p)\mathrm{d}p = \frac{1}{f}\int_{\text{gap}} |\Im p(E)|\mathrm{d}E \\ &= \int_{\text{barrier}} |\Im p(E_1^t - fx)|\mathrm{d}x, \end{aligned}$$

$\mathcal{C}$ is the clockwise regular contour around the cut directly linking the two branch points k_c and $\bar{k}_c$, $\mathcal{E}(p)$ is the energy function defined by $\mathcal{E}(0) = \mathcal{E}_1(0)$ on the Riemann sheet with the above cut and $p(E) = \mathcal{E}^{-1}(E)$. Let us stress that $\rho_Z(f)$ can be computed in terms of the periodic potential V; indeed, let $\langle V \rangle$ be the mean value of the periodic potential, then from formula (2.12) in [26] it follows that

$$\rho_Z(f) = \frac{1}{f}\int_{\text{gap}} |\Im p(E)|\mathrm{d}E = \frac{1}{8f}\int_0^{2\pi} [V(x) - \langle V \rangle]^2 \mathrm{d}x. \tag{13}$$

From (12) and from the Fermi Golden Rule we stated in [17] the conjecture that the imaginary part of the resonance $E_0(f)$ has asymptotic behavior given by (12) (multiplied by f^2) for some value of the positive constant C.

We stress that, although the validity of the Fermi Golden Rule has been proved for the case of the Auger states in helium [28], its validity for the Wannier-Stark problem is not evident. Indeed, in the Auger case we have a quadratic behavior of the width of resonances with respect to the perturbative parameter and so higher order terms cannot contribute to the leading behavior. In contrast, in the Wannier-Stark case we expect an exponentially small width and so the dominance of the second term does not immediately follow. In fact, each term $E_0^n(f)$ of the Rayleigh-Schrödinger series contributes to the leading term of the imaginary part of the resonance.

The main result of this paper consists in proving the above conjecture. In order to do this let us define here the anti-Stokes lines starting from $\bar{k}_c$ as:

$$\Im\left[S(p)\right] = 0, \qquad S(p) := \int_{\bar{k}_c}^{p} \Lambda(\tau)\mathrm{d}\tau. \tag{14}$$

The angles between the real axis and the limit directions of the anti-Stokes lines at $\bar{k}_c$ are $\frac{1}{6}\pi$, $\frac{5}{6}\pi$ and $-\frac{1}{2}\pi$ because $\Lambda(p)$ has a square root branch point at $\bar{k}_c$ and it is a real and positive function when restricted to $p = \frac{1}{2} + ir$, $-r_c < r < +r_c$ [13].

We assume that:

Hypothesis 2: *The two anti-Stokes lines starting from $\bar{k}_c$ with directions $\frac{1}{6}\pi$ and $\frac{5}{6}\pi$ belong to $\mathcal{S}_0 \cup \{\bar{k}_c\}$ where $\mathcal{S}_\delta$ denotes the complex strip $\{p \in C : |\Im p| < r_c - \delta\}$.*

Let us stress that one can check the validity of this hypothesis for a class of models (see Remarks 2 and 3).

We note that $\Lambda(p)$ is real and positive for real p and that $\Lambda(p) \sim p^2$ as p goes to infinity in the strip $\mathcal{S}_0$, so the above two anti-Stokes lines approach the real axis at infinity.

We have that:

Theorem 1: *Let $V(x)$ be an even and periodic function with period $T = 2\pi$ satisfying Hypotheses 1 and 2. Then, for f small enough we have that:*

$$\Im E_0(f) = -\tfrac{1}{2} f e^{-2\rho_Z(f)}\left(1 + o(1)\right), \quad \textit{as } f \to 0 \tag{15}$$

where

$$\rho_Z(f) = \frac{1}{8f}\int_0^{2\pi} [V(x) - \langle V\rangle]^2 dx. \tag{16}$$

The proof of the Theorem will be given in the next section and it is split in several steps which we briefly explain now.

In Lemma 1, by means of the regular perturbation theory, we give the asymptotic behavior of the eigenvector $\zeta^\lambda = (\phi, \chi^\lambda)$ of H_f^λ associated to $E_0(f)$ as f goes to zero.

In Lemma 2 we give an explicit formula (equation (43) of the imaginary part of the resonance $E_0(f)$. It is obtained choosing $\omega = (\phi, 0)$ as a test vector in (46) (usual Rayleigh-Schrödinger formula is computed with the choice of the unperturbed eigenvector $\zeta_0 = (\phi_0, 0)$ as a test one) and using the results of appendix II in [17].

In Lemma 3, substituting in (43) the convergent perturbation series of the eigenvector, proving that the intraband coupling term $X^\lambda_{2,2}$ does not affect the leading term of the imaginary part of the resonance and using the explicit expression of the resolvent operator of the decoupled band approximation (where we can take into account only the most singular term of the kernel), we compute the contribution to the imaginary part of the resonances of any perturbative term. In such a way it turns out that all these terms have the same exponential and power behavior and an explicit expression is obtained using a computation performed by Berry in a different problem [7].

Finally, the numerical coefficient of the leading term of the imaginary part of the resonance is computed obtaining so (15).

3 Proof of Theorem 1

In order to prove Theorem 1 let us collect some results on the coupling term

$$X^\lambda = \begin{pmatrix} 0 & X^\lambda_{1,2} \\ X^\lambda_{2,1} & X^\lambda_{2,2} \end{pmatrix}. \tag{17}$$

The term $X^\lambda_{2,1}$ acts from $\mathcal{H}_1$ to $\mathcal{H}_2$ as

$$\left(X^\lambda_{2,1}\xi\right)(p) = e^{ip\lambda} X(p)\tilde{\xi}(p), \qquad \xi \in \mathcal{H}_1 \tag{18}$$

where $\tilde{\xi}$ is the periodical extension of ξ on R (in the following, for the sake of simplicity, let us denote both by ξ), and $X^\lambda_{1,2}$ acts from $\mathcal{H}_2$ to $\mathcal{H}_1$ as

$$\left(X^\lambda_{1,2}\eta\right)(k) = \sum_{L\in\mathsf{Z}} e^{-i(k+L)\lambda} \bar{X}(k+L)\eta(k+L), \qquad \eta \in \mathcal{H}_2. \tag{19}$$

The function $X(p)$ is defined as

$$X(p) = i\langle \mathrm{d}_p \mathbf{w}^p_1, \mathbf{w}^p_2\rangle_{\mathcal{K}} = i\sum_{K\in\mathsf{Z}} \frac{\mathrm{d}w^p_1(K)}{\mathrm{d}p}\overline{w^p_2(K)}, \qquad \mathcal{K} = \ell^2(\mathsf{Z}) \tag{20}$$

where $\varphi_j(x,p) = e^{ipx} u_j(x,p)$, $j = 1,2$, are the Bloch functions associated to the band functions, $u_j(x,p)$ are periodic functions with respect to x with period 2π and $\mathbf{w}_j^p = \{w_j^p(K)\} \in \ell^2(\mathsf{Z})$ where $w_j^p(K)$ are the Fourier coefficients of $u_j(x,p)$. We recall that $u_1(x,p)$ is a periodic function with respect to p with period 1 and that $u_2(x,p)$ admits the asymptotic behaviors (see (II.6) in [16]):

$$u_2(x,p) = 1 + \mathcal{O}(p^{-1}) \qquad \text{and} \qquad \frac{\partial u_2(x,p)}{\partial p} = \mathcal{O}(p^{-2}) \tag{21}$$

as p goes to infinity. Moreover, when the periodic potential is an even function

$$\mathcal{E}_j(-p) = \mathcal{E}_j(p), \quad \overline{\mathcal{E}_j(p)} = \mathcal{E}_j(\bar{p}), \quad j = 1,2 \tag{22}$$

and

$$\varphi_j(x,p) = \overline{\varphi_j(-x,p)} = \overline{\varphi_j(x,-p)}, \quad j = 1,2 \tag{23}$$

and so

$$\overline{X(p)} = -X(p). \tag{24}$$

The Bloch functions are analytic functions in the complex plane with branch point singularities of order $-\frac{1}{4}$ at the branch points k_c and $\bar{k}_c$. From this, (20) and (23) it follows that $X(p)$ is an analytic function in the complex plane with simple poles at k_c and $\bar{k}_c$ (see (A.40) in [17]) and branch point singularities of order $-\frac{1}{4}$ at $k_c + L$ and $\bar{k}_c + L$ for any $L \in \mathsf{Z}\backslash\{0\}$. Moreover, it has the following exponentially decreasing behavior as p goes to infinity (see Lemma 1 and Remark 5 in [16]): for any $\epsilon > 0$ there exists $c_\epsilon > 0$ such that

$$|X(p)| \le c_\epsilon |p| \exp[-(\lambda_0 - \epsilon)|p|] \quad \forall p \in \mathsf{R}. \tag{25}$$

Let us stress that this behavior can be easily extended to any strip $\mathcal{S}_\delta$, for $\delta > 0$ fixed, using the same arguments in [16] and from the boundedness of the Fourier coefficients $w_j^p(K)$ in $\mathcal{S}_\delta$.

The intraband term $X_{2,2}^\lambda$, which couples the infinite second band with itself, acts on $\mathcal{H}_2$ as

$$\left(X_{2,2}^\lambda \eta\right)(p) = \sum_{L \in \mathsf{Z}\backslash\{0\}} e^{-iL\lambda} X_{2,2}^L(p) \eta(p+L), \qquad \eta \in \mathcal{H}_2 \tag{26}$$

where the function $X^L_{2,2}(p)$ is defined as

$$X^L_{2,2}(p) = i\langle \mathrm{d}_p \mathbf{w}_2^{p+L}, T_L \mathbf{w}_2^p \rangle_K, \ (T_L \mathbf{a})(K) = a(K+L), \ \mathbf{a} = \{a(K)\}.$$

Therefore, $X^L_{2,2}(p)$ is an analytic function in the complex plane with branch point singularities of order $-\frac{1}{4}$ at k_c and $\bar{k}_c$ and branch point singularities of order $-\frac{5}{4}$ at $k_c - L$ and $\bar{k}_c - L$. Moreover, it has the following exponentially decreasing behavior as L goes to infinity: for any $\epsilon > 0$ there exists $c_\epsilon > 0$ independent of L such that

$$|X^L_{2,2}(p)| \leq c_\epsilon |L| \left(1 + |p|^{-2}\right) \exp[-(\lambda_0 - \epsilon)L] \quad \forall p \in \mathsf{R}, \ \forall L \in \mathsf{Z}, \tag{27}$$

which can be extended to any $\mathcal{S}_\delta$ as for $X(p)$.

Now, let

$$R^\lambda(z,f) = \begin{pmatrix} R_1^\lambda(z,f) & S^\lambda(z,f) \\ T^\lambda(z,f) & R_2^\lambda(z,f) \end{pmatrix}, \quad \Im z > f \Im \lambda \tag{28}$$

be the resolvent operator of H_f^λ defined on $\mathcal{H}$ where:

$$\begin{aligned} R_1^\lambda(z,f) &= \left[H_1 - z - f\mathcal{F}^\lambda(z)\right]^{-1} && (29) \\ T^\lambda(z,f) &= -\left[H_2^\lambda + fX_{2,2}^\lambda - z\right]^{-1} fX_{2,1}^\lambda R_1^\lambda(z,f) && (30) \\ \mathcal{F}^\lambda(z) &= X_{1,2}^\lambda f \left[H_2^\lambda + fX_{2,2}^\lambda - z\right]^{-1} X_{2,1}^\lambda. && (31) \end{aligned}$$

Let Σ be the clockwise contour surrounding E_0 with radius given by min $(f\pi, f\frac{1}{2}|\Im\lambda|)$. From the norm bound (Lemma 10 in [16])

$$\left\| \mathcal{F}^\lambda(z) \left[H_1 - z\right]^{-1} \right\|_{\mathcal{L}(\mathcal{H}_1)} \leq c, \tag{32}$$

for some $c > 0$ independent of f and $z \in \Sigma$ for f small enough, we have that the geometric series

$$R_1^\lambda(z,f) = \left[H_1 - z\right]^{-1} \sum_{n=0}^{\infty} f^n \left[\mathcal{F}^\lambda(z) \left[H_1 - z\right]^{-1}\right]^n \tag{33}$$

is uniformly convergent for any $z \in \Sigma$ for f small enough. Therefore, the resolvent operator $R^\lambda(z,f)$ is uniformly norm bounded (Theorem 9 in [16]) on Σ and, from the Kato-Rellich stability theorem, there exists one discrete eigenvalue $E_0(f)$ of H_f^λ in the strip $f\Im\lambda < \Im z < 0$ (i.e. a resonance of H_f) with associated eigenvector $\zeta^\lambda = (\phi^\lambda, \chi^\lambda) \in \mathcal{H}$ given by:

$$\phi^\lambda = -\frac{1}{2\pi i}\oint_\Sigma R_1^\lambda(z,f)\phi_0 \mathrm{d}z, \qquad \chi^\lambda = -\frac{1}{2\pi i}\oint_\Sigma T^\lambda(z,f)\phi_0 \mathrm{d}z \tag{34}$$

where $\zeta_0 = (\phi_0, 0) \in \mathcal{H}$, $\phi_0(k,f) = e^{\frac{i}{f}\int^k(\mathcal{E}_1(\tau)-E_0)\mathrm{d}\tau}$, is the unperturbed eigenvector of $H_f^{\mathrm{DB},\lambda}$ associated to E_0. Let us stress that ϕ^λ is actually independent of λ. This follows from (25) and because $\mathcal{F}^\lambda(z)$ is independent of λ (see App. II in [17]) for any z such that $\Im z > f\Im\lambda$. Hence, ϕ^λ will be simply denoted by ϕ in the following.

From (33) and (34) we obtain the convergent perturbation series for the eigenvector ζ^λ. In order to give an asymptotic behavior for ζ^λ we rewrite the perturbation series extracting the exponential term ϕ_0.

Let us write

$$\left[H_1 - z - f\mathcal{F}^\lambda(z)\right]^{-1}\phi_0 = \Phi_0\left[\widetilde{H}_1 - z - f\widetilde{\mathcal{F}}^\lambda(z)\right]^{-1}\mathrm{e}_1, \tag{35}$$

where $\mathrm{e}_1(k) \equiv 1$, Φ_0 is the multiplication operator defined by $\phi_0(k,f)$ and

$$\widetilde{\mathcal{F}}^\lambda(z) = X_{1,2}^\lambda f\left[\widetilde{H}_2^\lambda + fX_{2,2}^\lambda - z\right]^{-1}X_{2,1}^\lambda, \tag{36}$$

$$\widetilde{H}_1 = if\mathrm{d}_k + E_0, \qquad \widetilde{H}_2^\lambda = if\mathrm{d}_p + \Lambda(p) + E_0 + f\lambda. \tag{37}$$

Hence, we obtain

$$\phi = \Phi_0\nu \quad \text{and} \quad \chi^\lambda = \Phi_0\eta^\lambda$$

where:

Lemma 1: *For any $k \in \mathcal{B}$ and $p \in R$ it follows that:*

$$\nu(k,f) = 1 + \mathcal{O}(f) \quad \textit{and} \quad \eta^\lambda(p,f) = \mathcal{O}(f) \quad \textit{as } f \to 0. \tag{38}$$

Proof: From (34) and (35) we have

$$\nu = -\frac{1}{2\pi i}\oint_\Sigma\left[\widetilde{H}_1 - z - f\widetilde{\mathcal{F}}^\lambda(z)\right]^{-1}\mathrm{e}_1\mathrm{d}z$$

where the geometric series for $\nu(k,f)$, obtained from (32) and (33) adapted to $\widetilde{\mathcal{F}}^{\lambda}(z)$ and $[\widetilde{H}_1 - z]^{-1}$, converges for small f and any real k obtaining so

$$\nu(k,f) = 1 + \mathcal{O}(f) \qquad \text{as } f \to 0.$$

We consider now the asymptotic behavior of η^{λ} as f goes to zero. From (III.22) in [16] it follows that for any $\Im z > f \Im \lambda$ the geometric series

$$\left[\widetilde{H}_2^{\lambda} + f X_{2,2}^{\lambda} - z\right]^{-1} = \left[\widetilde{H}_2^{\lambda} - z\right]^{-1} \sum_{m=0}^{\infty} \left(-f X_{2,2}^{\lambda} \left[\widetilde{H}_2^{\lambda} - z\right]^{-1}\right)^m \tag{39}$$

is norm convergent for f small enough. Therefore, as f goes to zero we obtain that

$$\eta^{\lambda} = -\left[\widetilde{H}_2^{\lambda} - E_0\right]^{-1} f X_{2,1}^{\lambda} \mathrm{e}_1 \left(1 + \mathcal{O}(f)\right), \tag{40}$$

that is

$$\eta^{\lambda}(p,f) = -i e^{ip\lambda} \int_p^{+\infty} e^{\frac{i}{f}\int_q^p \Lambda(\tau)\mathrm{d}\tau} X(q)\mathrm{d}q \left(1 + \mathcal{O}(f)\right). \tag{41}$$

The asymptotic behavior of (41) for small f is given by two contributions: one given by the endpoint of integration p and the other one given by the saddle point coinciding with the branch point $\bar{k}_c$ where $X(q)$ has a simple pole with residue $-\frac{1}{4}$ (see, for instance, [15]). Therefore:

$$\eta^{\lambda}(p,f) = \left[f\frac{X(p)}{\Lambda(p)} - i\frac{\pi}{3} e^{-\frac{i}{f}\int_{k_c}^p \Lambda(\tau)\mathrm{d}\tau}\right] e^{ip\lambda} \left(1 + \mathcal{O}(f)\right) = \mathcal{O}(f) \tag{42}$$

as f goes to zero and for any real p because the contribution given by the saddle point is negligible with respect to the other one. $\triangle$

We give now the following Lemma:

Lemma 2: *The imaginary part of the resonance $E_0(f)$ is given by:*

$$\Im E_0(f) = -\frac{1}{2} f \left| \int_{R} [\psi(p,f)]^{-1} v(p,f) dp \right|^2 \left(1 + \mathcal{O}(f)\right) \qquad as\ f \to 0 \tag{43}$$

where

$$\psi(p,f) = \exp\left[\frac{i}{f}\int_0^p \Lambda(\tau) d\tau\right] \tag{44}$$

and

$$v=\left[\mathbb{1}+(fX_{2,2}+\Re E_0(f)-E_0)\left[\widetilde{H}_2-(E_0-i0^+)\right]^{-1}\right]^{-1}X_{2,1}\nu. \quad (45)$$

Proof: Let $\omega=(\phi,0)\in\mathcal{H}$, then

$$E_0(f)=\frac{\langle H_f^\lambda\zeta^\lambda,\omega\rangle_{\mathcal{H}}}{\langle\zeta^\lambda,\omega\rangle_{\mathcal{H}}}=\frac{\langle H_1\phi,\phi\rangle_{\mathcal{H}_1}}{\langle\phi,\phi\rangle_{\mathcal{H}_1}}+f\frac{\langle X_{1,2}^\lambda\chi^\lambda,\phi\rangle_{\mathcal{H}_1}}{\langle\phi,\phi\rangle_{\mathcal{H}_1}}. \quad (46)$$

Recalling that H_1 is symmetrical and $\chi^\lambda=-\left[H_2^\lambda+fX_{2,2}^\lambda-E_0(f)\right]^{-1}fX_{2,1}^\lambda\phi$ because $\zeta^\lambda=(\phi,\chi^\lambda)$ satisfies $[H_f^\lambda-E_0(f)]\zeta^\lambda=0$, we obtain:

$$\begin{aligned}\Im E_0(f) &= -\frac{f}{\langle\phi,\phi\rangle_{\mathcal{H}_1}}\Im\langle\mathcal{F}^\lambda(E_0(f))\phi,\phi\rangle_{\mathcal{H}_1} && (47)\\ &= -\frac{f}{\langle\nu,\nu\rangle_{\mathcal{H}_1}}\Im\langle\widetilde{\mathcal{F}}^\lambda(E_0(f))\nu,\nu\rangle_{\mathcal{H}_1} && (48)\end{aligned}$$

where

$$\langle\nu,\nu\rangle_{\mathcal{H}_1}=1+\mathcal{O}(f)\qquad\text{as } f\to 0 \quad (49)$$

from Lemma 1. Now, from the first resolvent formula (see (I.5.5) in [23]) it follows that

$$\Im E_0(f)=-f\frac{\Im\langle\widetilde{\mathcal{F}}^\lambda(\Re E_0(f))\nu,\nu\rangle_{\mathcal{H}_1}}{\langle\nu,\nu\rangle_{\mathcal{H}_1}(1+f^2\Re\mu)} \quad (50)$$

where

$$\mu=\langle X_{1,2}^\lambda\left[\widetilde{H}_2^\lambda+fX_{2,2}^\lambda-E_0(f)\right]^{-1}\left[\widetilde{H}_2^\lambda+fX_{2,2}^\lambda-\Re E_0(f)\right]^{-1}X_{2,1}^\lambda\nu,\nu\rangle_{\mathcal{H}_1}.$$

Because $E_0(f)=E_0+\mathcal{O}(f^2)$, $\Re E_0(f)=E_0+\mathcal{O}(f^2)$, (38) and using the same arguments we used to obtain (42) then it follows that $|\mu|\le c$ for some $c>0$ and any f small enough. Hence:

$$\Im E_0(f)=-f\Im\langle\widetilde{\mathcal{F}}^\lambda(\Re E_0(f))\nu,\nu\rangle_{\mathcal{H}_1}(1+\mathcal{O}(f)) \quad (51)$$

for small f. Recalling now that $\widetilde{\mathcal{F}}^\lambda(\Re E_0(f))$ is independent of λ and denoting

$$
\begin{aligned}
\widetilde{\mathcal{F}}^{+} &= \lim_{\lambda\to 0,\ \Im\lambda<0} \widetilde{\mathcal{F}}^{\lambda}(\Re E_0(f)) = \widetilde{\mathcal{F}}^{0}(\Re E_0(f)+i0^{+}) \\
&= X_{1,2} f \left[\widetilde{H}_2 + f X_{2,2} - (\Re E_0(f)+i0^{+})\right]^{-1} X_{2,1}
\end{aligned}
$$

and $\widetilde{\mathcal{F}}^{-} = \widetilde{\mathcal{F}}^{0}(\Re E_0(f) - i0^{+})$ the adjoint of $\widetilde{\mathcal{F}}^{+}$ then we obtain

$$
\begin{aligned}
\Im\langle \widetilde{\mathcal{F}}^{\lambda}(\Re E_0(f))\nu,\nu\rangle_{\mathcal{H}_1} &= \Im\langle \widetilde{\mathcal{F}}^{+}\nu,\nu\rangle_{\mathcal{H}_1} \\
&= \frac{1}{2i}\left\langle \left(\widetilde{\mathcal{F}}^{+} - \widetilde{\mathcal{F}}^{-}\right)\nu,\nu\right\rangle_{\mathcal{H}_1}.
\end{aligned}
$$

Let us notice that $\widetilde{\mathcal{F}}^{0}(z\pm i0^{+})$, for z real, is a bounded operator on $\mathcal{H}_1$ because (25). Now, let

$$
\left[\widetilde{H}_2 + f X_{2,2} - (\Re E_0(f)\pm i0^{+})\right]^{-1} = \left[\widetilde{H}_2 - (E_0 \pm i0^{+})\right]^{-1} A^{\pm},
$$

where

$$
A^{\pm} = \left[1 + A\left[\widetilde{H}_2 - (E_0\pm i0^{+})\right]^{-1}\right]^{-1},\ A = f X_{2,2} - \Re E_0(f) + E_0,
$$

and, let

$$
T = \left[\widetilde{H}_2 - (E_0 + i0^{+})\right]^{-1} - \left[\widetilde{H}_2 - (E_0 - i0^{+})\right]^{-1}. \tag{52}
$$

From the second resolvent formula we obtain:

$$
A^{+} - A^{-} = A^{+} A T A^{-},\quad \widetilde{\mathcal{F}}^{+} - \widetilde{\mathcal{F}}^{-} = X_{1,2} f W T A^{-} X_{2,1}
$$

where

$$
W = \mathbf{1} - \left[\widetilde{H}_2 + f X_{2,2} - (\Re E_0(f) + i0^{+})\right]^{-1} A.
$$

Hence:

$$
\begin{aligned}
\Im\langle\widetilde{\mathcal{F}}^{+}\nu,\nu\rangle_{\mathcal{H}_1} &= \frac{1}{2f}\int_{\mathbb{R}}[\psi(p,f)]^{-1}v(p,f)\mathrm{d}p\cdot\langle X_{1,2}fW\psi,\nu\rangle_{\mathcal{H}_1} \\
&= \frac{1}{2}\int_{\mathbb{R}}[\psi(p,f)]^{-1}v(p,f)\mathrm{d}p\cdot\int_{\mathbb{R}}\psi(p,f)\bar{u}(p,f)\mathrm{d}p
\end{aligned}
$$

from (A.35) in [17] and where $v = A^{-}X_{2,1}\nu$, $\psi(p,f) = \exp\left[\frac{i}{f}\int_0^p\Lambda(\tau)\mathrm{d}\tau\right]$ and

$$
\begin{aligned}
u &= \left(1 - A\left[\widetilde{H}_2 + fX_{2,2} - (\Re E_0(f) - i0^{+})\right]^{-1}\right)X_{2,1}\nu \\
&= A^{-}X_{2,1}\nu = v.
\end{aligned}
$$

It finally follows (43) because $\bar{\psi}(p) = \psi^{-1}(p)$ for real p. $\triangle$

Now, we are ready to prove Theorem 1. To this end we perform the asymptotic computation of the integral in (43) using the stationary phase method where the stationary phase point coincides with the branch point $\bar{k}_c$.

Now, let:

$$
\Im E_0(f) = -\tfrac{1}{2}f|I(f)|^2e^{-2\rho z(f)}\left(1+\mathcal{O}(f)\right) \tag{53}
$$

where

$$
I(f) = e^{\rho z(f)}\int_{\mathbb{R}}[\psi(p,f)]^{-1}v(p,f)\mathrm{d}p = \int_{\mathbb{R}}e^{-\frac{i}{f}\int_{\bar{k}_c}^{p}\Lambda(\tau)\mathrm{d}\tau}v(p,f)\mathrm{d}p. \tag{54}
$$

By means of the norm convergent geometric series (33) adapted to $[\widetilde{H}_1 - z - f\widetilde{\mathcal{F}}^{\lambda}(z)]^{-1}$ we obtain the following uniformly convergent series for f small enough:

$$
I(f) = \sum_{n=0}^{+\infty}I_n(f), \quad I_n(f) = f^n\int_{\mathbb{R}}e^{-\frac{i}{f}\int_{\bar{k}_c}^{p}\Lambda(\tau)\mathrm{d}\tau}v_n(p,f)\mathrm{d}p \tag{55}
$$

where v_n is the analytic function in the strip $\mathcal{S}_0$ given by $v_n = A^{-}X_{2,1}\nu_n$ and

$$
\nu_n = -\frac{1}{2\pi i}\oint_{\Sigma}[\widetilde{H}_1 - z]^{-1}\left[\widetilde{\mathcal{F}}^{\lambda}(z)[\widetilde{H}_1 - z]^{-1}\right]^n e_1\mathrm{d}z.
$$

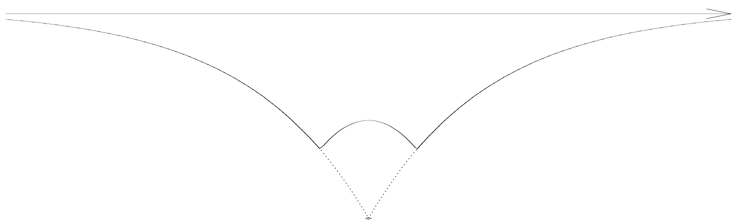

Figure 2: The path γ (full line) is along the anti-Stokes lines (broken lines) up to a small neighborhood of the branch point $\bar{k}_c$ where here it is indented.

We prove the following result:

Lemma 3: *Let $I_n(f)$ be the integral (55) obtained from (54) where we replace $v(p,f)$ with the $n-$th perturbation term. Then:*

$$I_n(f) = 2i\frac{(-1)^n \pi^{2n+1}}{(2n+1)!6^{2n+1}} + \mathcal{O}(f^{2/3}) \quad \textit{as } f \to 0.$$

Proof: We change the path of integration in I_n with the path γ along the two anti-Stokes lines and indented in a $f-$neighborhood (that is a neighborhood small for f small) of $\bar{k}_c$ (see Figure 2). From Hypothesis 2 and the Cauchy Theorem we obtain that:

$$I_n(f) = f^n \int_\gamma e^{-\frac{i}{f}\int_{\bar{k}_c}^p \Lambda(\tau)\mathrm{d}\tau} v_n(p,f)\mathrm{d}p. \tag{56}$$

Now, we put in evidence the most singular behavior of each term $v_n(p,f)$ for p near $\bar{k}_c$. To this end let us recall that the resolvent operator $\left[\widetilde{H}_1 - z\right]^{-1}$ acts on $\xi \in \mathcal{H}_1$ as [23]:

$$\left(\left[\widetilde{H}_1 - z\right]^{-1}\xi\right)(k) = \frac{1}{if\rho}\left\{\int_{\mathcal{B}} F_1(k,\rho,t)\xi(t)\mathrm{d}t + \int_0^k F_2(k,\rho,t)\xi(t)\mathrm{d}t\right\}$$

where

$$\rho = \frac{E_0 - z}{if}, \qquad F_1(k,\rho,t) = \frac{\rho}{1-e^{-\rho}} e^{\rho(t-k)}, \qquad F_2(k,\rho,t) = \rho e^{\rho(t-k)}.$$

From this and from the Cauchy residue theorem it follows that:

$$\begin{aligned} \nu_n(k,f) &= -\frac{1}{2\pi i} \oint_\Sigma \left(K^n(\rho)\mathrm{e}_1\right)(k) \frac{\mathrm{d}z}{(E_0 - z)^{n+1}} \\ &= \frac{1}{i^n f^n n!} \frac{\mathrm{d}^n}{\mathrm{d}\rho^n} \left[K^n(\rho)\mathrm{e}_1\right]_{\rho=0}(k) \end{aligned} \tag{57}$$

where $K(\rho) = K_1(\rho) + K_2(\rho)$ is the integral operator defined as:

$$(K_1(\rho)\xi)(k) = \int_{\mathcal{B}} F_1(k,\rho,k_1) \left(\widetilde{\mathcal{F}}^\lambda(z)\xi\right)(k_1)\mathrm{d}k_1, \qquad \xi \in \mathcal{H}_1$$

and

$$(K_2(\rho)\xi)(k) = \int_0^k F_2(k,\rho,k_1) \left(\widetilde{\mathcal{F}}^\lambda(z)\xi\right)(k_1)\mathrm{d}k_1, \qquad \xi \in \mathcal{H}_1.$$

Now, $(K_1(0)\xi)(k)$ (and its derivatives with respect to ρ) is independent of k and moreover, if $\xi \in \mathcal{H}_1$ is a test vector independent of f, we have that:

$$\begin{aligned} K_1(0)\xi &= f \int_{\mathcal{B}} \left(X_{1,2} \left[\widetilde{H}_2 + f X_{2,2} - (E_0 + i0^+) \right]^{-1} X_{2,1}\xi \right)(k_1)\mathrm{d}k_1 \\ &= -f \int_{\mathsf{R}} \frac{|X(p)|^2}{\Lambda(p)} \xi(p)\mathrm{d}p + \mathcal{O}(f^2) \qquad \text{as } f \to 0. \end{aligned}$$

On the other side, we have that $K_2(0) = \mathsf{0}$ and that the derivatives of $(K_2(0)\xi)(k)$, with respect to ρ, are actually dependent on k and they become singular at $\bar{k}_c$. Thus, we obtain that as f goes to zero

$$\begin{aligned} \nu_n(k,f) &= \frac{1}{i^n f^n} \left(\left[\frac{\mathrm{d}K_2(0)}{\mathrm{d}\rho} \right]^n \mathrm{e}_1 \right) (1 + \mathcal{O}(f)) \\ &= \frac{1}{i^n f^n} \left[\left(\prod_{m=1}^n \int_0^{k_{m-1}} \widetilde{\mathcal{F}}^0(E_0 + i0^+)\mathrm{d}k_m \right) \mathrm{e}_1 \right] (1 + \mathcal{O}(f)) \end{aligned}$$

where $k_0 = k$, since $K_2(0) = 0$ and $\widetilde{\mathcal{F}}^\lambda(E_0) = \widetilde{\mathcal{F}}^0(E_0 + i0^+)$ is independent of λ. Hence:

$$I_n(f) = \int_\gamma e^{-\frac{i}{f}\int_{\bar{k}_c}^{p}\Lambda(\tau)\mathrm{d}\tau} u_n(p,f)\mathrm{d}p + \mathcal{O}(f) \qquad \text{as } f \to 0 \tag{58}$$

where

$$u_n = (-i)^n A^- X_{2,1}\left[\left(\prod_{m=1}^{n}\int_0^{k_{m-1}} \widetilde{\mathcal{F}}^0(E_0+i0^+)\mathrm{d}k_m\right)\mathrm{e}_1\right]. \tag{59}$$

Now, in order to asymptotically evaluate the leading term of each integral (58), we show that the intraband term $X^\lambda_{2,2}$ can be neglected and that the term A^- can be replaced with the identity operator. To this end we introduce the (semiclassical action) variable

$$s(p) = \frac{1}{f}\int_{\bar{k}_c}^{p}\Lambda(\tau)\mathrm{d}\tau = \frac{2c}{3f}[p-\bar{k}_c]^{3/2}\left(1+\mathcal{O}(p-\bar{k}_c)\right), \tag{60}$$

where $c = |c|e^{-i\frac{\pi}{4}}$ because $\Lambda(p) = c[p-\bar{k}_c]^{1/2}\left(1+\mathcal{O}(p-\bar{k}_c)\right)$ and it is a positive real function for $p = \frac{1}{2} + ir$, $-r_c < r < r_c$ [13]. Then it appears that a singular term in $u_n(p,f)$ of the type $(p-\bar{k}_c)^{-\alpha}$ gives a contribution to (58) of order $f^{-\frac{2}{3}\alpha}$. By means of this argument we can prove that the term A^- in (59) and the intraband term $X_{2,2}$ do not contribute to the leading term of I_n.
Indeed, let us consider the geometric series

$$A^- = \sum_{m=0}^{+\infty}\left(-A[\widetilde{H}_2 - (E_0+i0^+)]^{-1}\right)^m \tag{61}$$

and (39) for $z = E_0$ and $\lambda = -i0^+$, that is:

$$\begin{aligned}&[\widetilde{H}_2 + fX_{2,2} - (E_0+i0^+)]^{-1} = \\ &\quad = [\widetilde{H}_2 - (E_0+i0^+)]^{-1}\textstyle\sum_{m=0}^{+\infty}\left(-fX_{2,2}[\widetilde{H}_2-(E_0+i0^+)]^{-1}\right)^m\end{aligned} \tag{62}$$

which are both norm convergent for f small enough from (25) and (III.22) in [16]. We formally compute now the contribution to I_n of each term of the series (62) when substituted in (59).
The zero-th order term of (62) gives a contribution to I_n of order $f^{-2/3}$; this follows from (19) and because $X(p)$ has a simple pole at $\bar{k}_c$ and $X(p+L)$ has a branch point singularity of order $-\frac{1}{4}$ at $\bar{k}_c$.

The first order term of (62) gives a contribution to I_n of order $f^{5/6}$ which is negligible with respect to the previous one. Indeed, let ξ be a test vector independent of f, then we have that the first order term of $\widetilde{\mathcal{F}}^0(E_0+i0^+)\xi$ is given by

$$X_{1,2}f[\widetilde{H}_2-(E_0+i0^+)]^{-1}\theta \tag{63}$$

where

$$\theta(p,f)=-f\left[X_{2,2}[\widetilde{H}_2-(E_0+i0^+)]^{-1}X_{2,1}\xi\right](p) \tag{64}$$

has a branch point singularity of order $-\frac{1}{4}$ at $\bar{k}_c$ because the sum in (25) is taken for $L\neq 0$.
The second order term of $\widetilde{\mathcal{F}}^0(E_0+i0^+)\xi$ is given by (63) where now

$$\begin{aligned}\theta(p,f) &= \left[\left(fX_{2,2}[\widetilde{H}_2-(E_0+i0^+)]^{-1}\right)^2 X_{2,1}\xi\right](p) \\ &= -\sum_{M,L\neq 0} X_{2,2}^L(p)\int_{p+L}^{+\infty} e^{\frac{i}{f}\int_q^{p+L}\Lambda(\tau)\mathrm{d}\tau} X_{2,2}^M(q)\mathrm{d}q\,\cdot \\ &\qquad \cdot\int_{q+M}^{+\infty} e^{\frac{i}{f}\int_r^{q+M}\Lambda(\tau)\mathrm{d}\tau} X(r)\xi(r)\mathrm{d}r.\end{aligned} \tag{65}$$

By the change of variable $q\to q-L$ and because the largest singular term is given for $M=-L$, where $X_{2,2}^{-L}(q+L)$ has a branch point singularity $q=\bar{k}_c$ of order $-\frac{5}{4}$, we obtain that the leading term of (65) for p near $\bar{k}_c$ is given by:

$$\begin{aligned}\theta(p,f) &\sim -\sum_{M\neq 0} X_{2,2}^{-M}(p)\int_p^{+\infty} e^{\frac{i}{f}\int_{q-M}^{p-M}\Lambda(\tau)\mathrm{d}\tau} X_{2,2}^M(q-M)\mathrm{d}q\,\cdot \\ &\qquad \cdot\int_q^{+\infty} e^{\frac{i}{f}\int_r^{q}\Lambda(\tau)\mathrm{d}\tau} X(r)\xi(r)\mathrm{d}r.\end{aligned}$$

where $\Lambda(q-M)\neq 0$ for q near $\bar{k}_c$ since $M\neq 0$. Thus, the contribution of the second order term to I_n is of order f^0.
In the same way one can prove that the $m-$th order term of (62), for $m\geq 2$, formally gives a contribution to I_n of order f^β for some $\beta\geq 0$. Hence:

$$I_n(f)=\int_\gamma e^{-is(p)}w_n(p,f)\mathrm{d}p+\mathcal{O}(f^{2/3}) \tag{66}$$

as f goes to zero, where

$$w_n = (-i)^n A^- X_{2,1} \left[\left(\prod_{m=1}^{n} \int_0^{k_{m-1}} X_{1,2} f [\widetilde{H}_2 - (E_0 + i0^+)]^{-1} X_{2,1} \mathrm{d}k_m \right) \mathrm{e}_1 \right].$$

In the same way, from (11) and (61), we have that we can replace in each I_n the operator A^- with the identity operator up to an error $\mathcal{O}(f^{2/3})$, that is:

$$I_n(f) = \int_\gamma e^{-is(p)} z_n(p,f) \mathrm{d}p + \mathcal{O}(f^{2/3})$$

as f goes to zero, where

$$\begin{aligned} z_n &= (-i)^n X_{2,1} \left[\left(\prod_{m=1}^{n} \int_0^{k_{m-1}} X_{1,2} f [\widetilde{H}_2 - (E_0 + i0^+)]^{-1} X_{2,1} \mathrm{d}k_m \right) \mathrm{e}_1 \right] \\ &= X_{2,1} \prod_{m=1}^{n} \int_0^{k_{m-1}} \bar{X}(p_m) e^{is(p_m)} \mathrm{d}p_m \int_{p_m}^{+\infty} X(k_m) e^{-is(k_m)} \mathrm{d}k_m. \end{aligned}$$

In the new variable s the above integral $I_n(f)$ takes the form

$$I_n(f) = \left(\frac{3f}{2c} \right)^{2/3} \int_{\gamma_s} \frac{e^{-is}}{s^{1/3}} z_n[p(s), f] \mathrm{d}s + \mathcal{O}(f^{2/3})$$

where γ_s is the path, starting from $-\infty$, clockwise surrounding the origin in the upper complex half-plane and then going to $+\infty$. Then, recalling that the residue of $X(p)$ at $\bar{k}_c$ is $-\frac{1}{4}$ and that $\bar{X}(p) = -X(p)$ we obtain:

$$\begin{aligned} I_n(f) &= \frac{(-1)^{n+1}}{6^{2n+1}} \int_{\mathsf{R}} \frac{e^{-is_0}}{s_0 + i0^+} \mathrm{d}s_0 \cdot \\ &\quad \cdot \prod_{m=1}^{n} \int_{-\infty}^{s_{m-1}} \frac{e^{ir_m}}{r_m + i0^+} \mathrm{d}r_m \int_{r_m}^{+\infty} \frac{e^{-is_m}}{s_m + i0^+} \mathrm{d}s_m + \mathcal{O}(f^{2/3}). \end{aligned}$$

Then, from formula (19) in Berry [7] we finally have

$$I_n(f) = -2\pi i \frac{(-1)^{n+1} \pi^{2n}}{(2n+1)! 6^{2n+1}} + \mathcal{O}(f^{2/3}) \quad \text{as } f \to 0 \tag{67}$$

proving so Lemma 3. $\triangle$

Finally, recalling that the series (55) is uniformly convergent with respect to f for f small enough, we have that:

$$\begin{aligned}
\lim_{f\downarrow 0} I(f) &= \sum_{n=0}^{\infty} \lim_{f\downarrow 0} I_n(f) \\
&= 2i \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{6^{2n+1}(2n+1)!} \\
&= 2i \sin \tfrac{1}{6}\pi \\
&= i
\end{aligned} \tag{68}$$

proving so Theorem 1.

4 Remarks

Remark 1: The same question about the numerical pre-factor $\frac{1}{3}\pi$ appears also in adiabatic problems and it was clearly understood by Berry [8] and rigorously solved in certain cases [9,19]. Moreover, a summation phenomenon which gives the correct numerical pre-factor was already observed for the computation of the reflection coefficient above an analytic barrier [6].

Remark 2: It is easy to show that Hypothesis 2 is satisfied for the class of Lamé potentials (3) with t near 1. Indeed, in the limit $t \uparrow 1$ one obtains the explicitly solvable problem with potential $V(x) = 2 - 2\cosh^{-2}(x)$ (see [14]) and boundary conditions given by the Sommerfeld and anti-Sommerfeld rule at plus and minus infinity respectively. Then the relation between the energy and the crystal momentum p is given by the conditions at infinity. The second band function is exactly $\mathcal{E}_2(p) = 2 + p^2$ and the first band function takes the constant value $\mathcal{E}_1(k) \equiv 1$. Let $z = (p - \bar{k}_c)$ and

$$S_t(z) = \int_{\bar{k}_c}^{p} \Lambda(\tau)\mathrm{d}\tau$$

where Λ is the difference between the two band functions. The limit function $S_1(z) = \frac{1}{3}(z^3 - 3iz^2)$ has imaginary part $\Im S_1(z) = -z^2 < 0$ for z real and different from zero, so that we expect $\Im S_t < 0$ for any $z \in \mathsf{R} - \{0\}$ and $1 - t$ small. In order to prove this we should consider the possible non uniformity

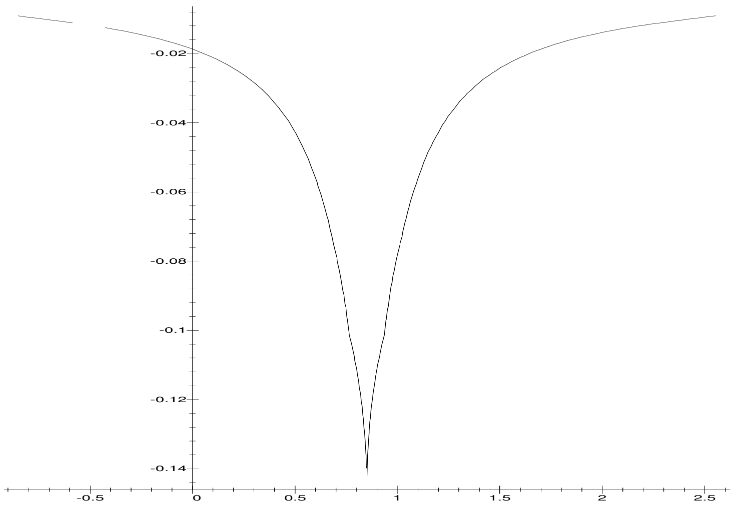

Figure 3: Anti-Stokes lines in the p complex plane for the Lamé potential with $t = 0.7$

of the $t \uparrow 1$ limit at $z = \infty$ and at $z = 0$. For what concern the large z regime we actually have uniformity of the limit since the infinite band $\mathcal{E}_2(p)$ has the free behavior $\mathcal{E}_2(p) \sim p^2$ for p large independently of t, and $\mathcal{E}_1(p)$ is bounded with $\mathcal{E}_1(p) \sim +1$. For small z the non uniformity of the limit does not destroy the result. Indeed, let us consider the holomorphic function $S_t^2(z)$; for real $z \neq 0$ and small enough we have:

$$\Re S_t^2(z) = -a_t z^3 - b_t z^4 + \mathcal{O}(z^5) < 0$$

because $a_t > 0$, $a_t \to 0$ and $b_t \to 1$ as $t \uparrow 1$. Then it implies $\Im S_t(z) \neq 0$, and so $\Im S_t(z) < 0$ by continuity.

Remark 3: For the class of Lamé potentials (3) with any $t \in (0,1)$ the two band functions $\mathcal{E}_1$ and $\mathcal{E}_2$ are given by

$$\mathcal{E}_{1,2}(p) = t^2 + \mathrm{dn}^2 \left[\mathsf{Z}^{-1} \left(-ip + i\pi/2K(t); t \right); t \right] \tag{69}$$

where $\mathrm{dn}(x;t)$ is an Jacobian elliptic function and Z^{-1} is the inverse (multi-valued) function of the Jacobian Zeta function [29]. Then one can numerically compute the two anti-Stokes lines (see for instance Figure 3). From this we obtain the numerical evidence of Hypothesis 2 for any t.

Remark 4: By assuming the relation $|\Im E_0(f)| = \frac{1}{2} f \nu p$, where $\nu = L^{-1} = 1$ is the frequency of the periodic motion in the band (see §119 [27]), from Theorem 1 follows the probability transition p between the two bands:

$$p \sim e^{-2\rho z(f)} \qquad \text{as } f \to 0$$

in agreement with the conjecture of Kane and Blount [22].

Remark 5: In [17] is discussed the crossing-anticrossing phenomenon of Wannier-Stark resonances under the conjecture that (15) holds for some numerical coefficient. Since Theorem 1 this assumption is justified.

Remark 6: From Lemma 3 and its proof (see in particular equation (66)) it follows that the intraband term $X_{2,2}$ does not contribute to the leading term of the imaginary part of the resonances. So that, it turns out:

$$\Im E_0(f) \sim -\tfrac{1}{2} f |\hat{I}(f)|^2 e^{-2\rho z(f)}$$

for small f where now $\hat{I}(f)$ is given by

$$\hat{I}(f) = \int_{\mathsf{R}} e^{-\frac{i}{f}\int_{k_c}^{p} \Lambda(\tau)\mathrm{d}\tau} \left(X_{2,1} \hat{\nu} \right)(p,f)\mathrm{d}p \tag{70}$$

where $(\hat{\nu}, \hat{\mu})$ is the solution of:

$$\begin{cases} if\hat{\nu}' + E_0\hat{\nu} + fX(p)\hat{\mu} = E_0(f)\hat{\nu} \\ if\hat{\mu}' + \Lambda(p)\hat{\mu} + E_0\hat{\mu} + f\bar{X}(p)\hat{\nu} = E_0(f)\hat{\mu} \end{cases}$$

with asymptotic condition $\hat{\nu}(p,f) = 1 + \mathcal{O}(f)$ and $\hat{\mu}(p,f) = \mathcal{O}(f)$, for real p and as f goes to zero, from Lemma 1. In the limit $f \to 0$, $p \to \bar{k}_c$ and the (semiclassical action) variable $s(p) = S(p)/f$ finite, the above system takes the form (see Joye [19]):

$$\begin{cases} i\hat{\nu}_s' + \frac{i}{6s}\hat{\mu} = 0 \\ i\hat{\mu}_s' - i\hat{\mu} - \frac{i}{6s}\hat{\nu} = 0 \end{cases} \tag{71}$$

because (20), $X(p)$ has residue $-\frac{1}{4}$ at $\bar{k}_c$ and $E_0(f) = E_0 + \mathcal{O}(f^2)$. We finally obtain the following equation for $\hat{\nu}(s)$:

$$\hat{\nu}''_{ss} + \left(\frac{1}{s} - i\right)\hat{\nu}'_s - \frac{a^2}{s^2}\hat{\nu} = 0 \qquad a = \tfrac{1}{6} \tag{72}$$

with the asymptotic condition $\hat{\nu}(+i\infty) = 1$. Let $\hat{\nu}(s) = s^a w(is)$ and $t = is$, then (72) becomes:

$$tw''(t) + (c-t)w'(t) - aw(t) = 0, \qquad c = 2a + 1 = \tfrac{4}{3} \tag{73}$$

where solutions of this equation are the confluent hypergeometric functions (see §6 in [11]) and in particular, from the asymptotic condition at $s = +i\infty$, we obtain the solution of (72)

$$\hat{\nu}(s) = (is)^{1/6}\Psi(\tfrac{1}{6}, \tfrac{4}{3}; is) \sim \textstyle\sum_{n=0}^{\infty}(-is)^{-n}\hat{\nu}_n, \;\; \hat{\nu}_n = \dfrac{\Gamma\left(n-\frac{1}{6}\right)\Gamma\left(n+\frac{1}{6}\right)}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(-\frac{1}{6}\right)n!} \tag{74}$$

as s goes to infinity (see (d.16) in [27]) where $\Psi(a, c; z)$ is the second confluent hypergeometric function. We stress that the asymptotic expansion (74) can be also obtained using the super-adiabatic expansion as in Berry [8] or using the Adams-Wannier expansion (see §II.2 in [17]).

Now, in order to compute the integral (70) we change the path of integration with the path γ along the two anti-Stokes lines and indented in a neighborhood of $\bar{k}_c$. Thus, in the (semiclassical action) variable and by means of the two bands approximation we obtain:

$$\hat{I}(f) \sim \frac{1}{6}\int_{-\infty-i0^+}^{+\infty-i0^+} e^{-is}\hat{\nu}(s)\frac{\mathrm{d}s}{s} \tag{75}$$

for small f. Substituting (74) in the above integral, from the change of variable $t = is$, and using formulas (10) in §6.10 [11], (14) in §2.1.3 [11] and (8) in §1.2 [11] we obtain:

$$\begin{aligned}
\hat{I}(f) &\sim i^{1/6}\frac{1}{6}\int_{-\infty-i0^+}^{+\infty-i0^+} e^{-is}s^{-5/6}\Psi(\tfrac{1}{6}, \tfrac{4}{3}; is)\mathrm{d}s \\
&= \frac{\pi}{3}\frac{1}{2\pi}\int_{0^+-i\infty}^{0^++i\infty} e^{-t}t^{-5/6}\Psi(\tfrac{1}{6}, \tfrac{4}{3}; t)\mathrm{d}t = i\tfrac{\pi}{3}\lim_{\beta\uparrow 1}\textstyle\sum_{n=0}^{\infty}\frac{\widetilde{\nu}_n}{n!}\beta^n \\
&= i\frac{\pi}{3}F\left(\tfrac{1}{6}, -\tfrac{1}{6}; 1; 1\right) \\
&= 2i\sin\pi/6 = i
\end{aligned}$$

for small f, in agreement with (68).

We stress that the regular perturbation theory gives the value of the numerical coefficient by means of the convergent series of the sine function (68) while the Adams-Wannier asymptotic expansion gives the same numerical coefficient by means of the Borel transform.

Remark 7: Following Berry [8,9] one can associate the two bands approximation [22] of an usual Wannier Stark-problem (with, say, all the gaps open) with a 2 levels periodic adiabatic problem and heuristically compute the probability transition during one period. The two bands approximation takes the form:

$$if\mathbf{a}' = \begin{pmatrix} \mathcal{E}_1 & fX \\ f\bar{X} & \mathcal{E}_2 \end{pmatrix} \mathbf{a}, \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{76}$$

where $\mathcal{E}_1(k)$ and $\mathcal{E}_2(k)$ are the first two band functions defined in the Brillouin zone $\mathcal{B}$ and where $\bar{X}(k) = -X(k)$ is purely imaginary for real k. For small f, this problem is similar to an usual adiabatic 2 levels problem, as Blount and Kane [22] already pointed out, but where the off-diagonal terms in (76) are dependent on f and singular at the branch points k_c and $\bar{k}_c$. Let $\mathbf{a}_1 = \mathcal{U}\mathbf{a}$, where $\mathcal{U} = \exp\left[\frac{i}{2f}\int^k [\mathcal{E}_1(\tau) + \mathcal{E}_2(\tau)]\mathrm{d}\tau\right]\mathbf{1}$, then (76) takes the form

$$if\mathbf{a}_1' = \left(-\tfrac{1}{2}\Lambda\sigma_3 + ifX\sigma_2\right)\mathbf{a}_1 \tag{77}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices. Now, by the further unitary transformation $\mathbf{a}_0 = \mathcal{V}\mathbf{a}_1$, $\mathcal{V} = \exp[-i\theta\sigma_2]$ where θ is such that $\theta' = iX$, (77) becomes

$$if\mathbf{a}_0' = -\tfrac{1}{2}\Lambda\left[\cos(2\theta)\sigma_3 + \sin(2\theta)\sigma_1\right]\mathbf{a}_0 \tag{78}$$

Let us notice that $\theta(k) = \theta(k+1)$ because the mean value of $X(k)$ is zero (for a symmetrical crystal). In such a way we have led the two bands approximation Wannier-Stark problem (76) to an usual analytic adiabatic multiple crossing (one for each period) problem (78). The transition between the two levels $\pm\frac{1}{2}\Lambda(k)$ occurs in an interval with center $\frac{1}{2}$, where the distance between the levels takes the minimum value, and amplitude (see [8])

$$\Delta k = \mathcal{O}(\sqrt{f}), \qquad \text{as } f \to 0. \tag{79}$$

Thus, in one period the transition between the two levels is essentially completed and one obtains that the leading term of the probability transition is given by $p \sim 4\sin^2(\pi\Omega/3)e^{-2\rho z(f)}$ where Ω is the residue of $2i\theta'(k)$ at the Kohn branch point [9]. Since we found $\Omega = 1/2$ for the Wannier-Stark problem, then we have the following value for the probability transition

$$p \sim e^{-2\rho z(f)} \tag{80}$$

in agreement with Remark 4.

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